

Title: 13/14 PSI - Quantum Gravity Review - Lecture 2

Date: Feb 19, 2014 10:15 AM

URL: <http://pirsa.org/14020062>

Abstract:

n-beins

$$e^i_\mu e^j_\nu \delta_{ij} = g_{\mu\nu}$$

$$R^k_\mu e^i_\mu R^l_\nu e^j_\nu \delta_{kl} = g_{\mu\nu} \quad \underline{SO(4)\text{-freedom}}$$

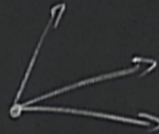
- degenerate metrics $\{e's\} \neq \{g\}$

Spin-connection

- Conditions uniquely specifying Levi-Civita connections

n-beins

$$e^i_\mu e^j_\nu \delta_{ij} = g_{\mu\nu}$$



$$R^k_\mu e^i_\mu R^l_\nu e^j_\nu \delta_{kl} = g_{\mu\nu} \quad \underline{SO(n)\text{-freedom}}$$

- degenerate metrics $\{e's\} \nleftrightarrow \{g\}$

$$\det(g) \neq 0$$

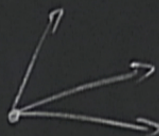
Spin-connection

- Conditions uniquely specifying Levi-Civita connections

n-beins

$$e_{\mu}^{\boxed{i}} e_{\nu}^{\boxed{j}} \delta_{ij} = g_{\mu\nu}$$

← internal indices



$$R^k_{\mu} e_{\mu} R^{\rho}_{\nu} e_{\nu} \delta_{k\rho} = g_{\mu\nu}$$

SO(4)-freedom

• degenerate metrics $\{e's\} \nleftrightarrow \{g\}$

$$\det(g) \neq 0$$

Spin-connection

- Conditions uniquely specifying Levi-Civita connections

Spin - connection

- Conditions uniquely specifying
Levi-Civita connections



$\Gamma_{\mu\sigma}^{\nu}$

- torsion-free $\Gamma_{\mu\sigma}^{\nu} = \Gamma_{\sigma\mu}^{\nu}$

- metricity: $\nabla_\mu g_{\nu\sigma} = 0$

spin-connection $\omega_{\mu k}^j$, D_μ

$$D_\mu \phi^j = \partial_\mu \phi^j + \omega_{\mu k}^j \phi^k$$

$$D_\mu \phi_j = \partial_\mu \phi_j - \omega_{\mu j}^k \phi_k$$

$$D_\mu v_v^j = \partial_\mu v_v^j + \omega_{\mu k}^j v_v^k - \Gamma_{\mu\nu}^\sigma v_\sigma^j$$

• How do we define ω^2 compatibility.

$$D_\mu e^j_\nu = 0 \Rightarrow$$

$$\Rightarrow 0 = D_\mu e^j_\nu = \partial_\mu e^j_\nu + \overbrace{\omega^j_{\mu k} e^k_\nu} - \Gamma^j_{\mu\nu} e^j_\nu$$

$$\boxed{\omega^j_{\mu k} = -e^j_k \nabla_\mu e^j_\nu = e^j_k \Gamma^j_{\mu\nu} e^j_\nu - e^j_k \partial_\mu e^j_\nu}$$

$$\omega^j_{\mu k} [\Gamma^j_{\mu\nu}, e]$$

• How do we define $\omega^?$
compatibility.

$$D_\mu e^j_\nu = 0 \Rightarrow$$

$$\Rightarrow 0 = D_\mu e^j_\nu = \partial_\mu e^j_\nu + \overbrace{\omega^j_{\mu k}} e^k_\nu - \Gamma^j_{\mu\nu} e^j_\nu$$

$$\boxed{\omega^j_{\mu k} = -e^j_k \nabla_\mu e^j_\nu = e^j_k \Gamma^j_{\mu\nu} e^j_\nu - e^j_k \partial_\mu e^j_\nu}$$

$$\omega^j_{\mu k} [\Gamma^j_{\mu\nu}, e] \Rightarrow \Gamma[\omega, e]$$

• How do we define ω^2 compatibility.

$$D_\mu e^j_\nu = 0 \Rightarrow$$

$$\Rightarrow 0 = D_\mu e^j_\nu = \partial_\mu e^j_\nu + \overbrace{\omega^j_{\mu k}} e^k_\nu - \Gamma^j_{\mu\nu} e^j_\nu$$

$$\boxed{\omega^j_{\mu k} \tilde{\Gamma}^{(*)}_{\nu} - e^j_\nu \nabla_\mu e^j_\nu = e^j_\nu \Gamma^j_{\mu\nu} e^j_\nu - e^j_\nu \partial_\mu e^j_\nu}$$

$$\omega^j_{\mu k} [\Gamma^j_{\mu\nu}, e] \Rightarrow \Gamma[\omega, e]$$

• How do we define ω^2 compatibility.

$$\nabla_\mu (g_{\alpha\beta}) = D_\mu (e'_\alpha e'_\beta)$$

$$D_\mu e'_\nu = 0 \Rightarrow$$

$$\Rightarrow 0 = D_\mu e'_\nu = \partial_\mu e'_\nu + \overbrace{\omega^j_{\mu k} e'_k} - \Gamma^s_{\mu\nu} e'_s$$

$$\omega^j_{\mu k} \tilde{\Gamma}^j_{(*)} - e'_k \nabla_\mu e'_\nu = e'_k \Gamma^s_{\mu\nu} e'_s - e'_k \partial_\mu e'_\nu$$

$$\omega^j_{\mu k} [\Gamma^s_{\mu\nu}, e] \Rightarrow \Gamma[\omega, e]$$

• What are the conditions on ω so that $\Gamma[\omega, e]$ is
 Levi-Civita?

• metric $\nabla_\mu (e_k^v e_j^s g_{ss}) \stackrel{(*)}{=} -\omega_{\mu jk} - \omega_{\mu k j}$

metricity $\Leftrightarrow$ anti-symmetry. $\omega_{\mu jk} = -\omega_{\mu k j}$

• torsion-free.

• What are the conditions on ω so that $\Gamma[\omega, e]$ is
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metricity $\Leftrightarrow$ anti-symmetry. $\omega_{\mu jk} = -\omega_{\mu k j}$

• torsion-free: $0 = D_\mu e_\nu^j - D_\nu e_\mu^j = \partial_\mu e_\nu^j - \partial_\nu e_\mu^j + \omega_{\mu\nu}^j e_j^k - \omega_{\nu\mu}^j e_\mu^k$
 $+ (\Gamma_{\mu\nu}^s - \Gamma_{\nu\mu}^s) e_s^j$

$$T^j_{\mu\nu} = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu + \omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu$$

$$= (\Gamma^j_{\mu\nu} e^j_s - \Gamma^j_{\nu\mu} e^j_s)$$

$$T^j_{\mu\nu} = d\omega^j_{\mu\nu} e^j_\mu$$

on-shell-free: $T^j_{\mu\nu} = 0$

$$S = \int R_{\mu\nu} g^{\mu\nu} \sqrt{g} d^D x$$

$$\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) e^j_s \sigma_j$$

$$S = \int R_{\mu\nu} g^{\mu\nu} \sqrt{g} d^D x$$

$$\begin{aligned} R_{\mu\nu} \sigma^\nu \sigma^\mu &= (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \sigma^\mu = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) (e^\mu_\sigma \sigma^\sigma) \\ &= e^\mu_\sigma \left[\partial_\mu \omega_{\nu j k} - \partial_\nu \omega_{\mu j k} + \omega_{\mu j l} \omega_{\nu k}^l - \omega_{\nu j l} \omega_{\mu k}^l \right] e^{\sigma k} \sigma^\sigma \\ &=: F \end{aligned}$$

$$S = \int R_{\mu\nu} g^{\mu\nu} \sqrt{g} d^D x$$

$$\begin{aligned} R_{\mu\nu} \sigma^\nu \sigma^\mu &= (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \sigma^\mu = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) (e^\mu_j \sigma^j) \\ &= e^\mu_j \left[\partial_\mu \omega_{\nu j k} - \partial_\nu \omega_{\mu j k} + \omega_{\mu j l} \omega_{\nu k}^l - \omega_{\nu j l} \omega_{\mu k}^l \right] e^{\sigma k} \sigma_k \\ &=: F_{\mu\nu jk} \end{aligned}$$

3 D :

$l = 1, 2, 3$

$(SO(3) \text{ Lie})$

$\omega_{\mu, jk}$

$=$

$\in \mathfrak{so}(3)$

$j = 1, 2, 3$

$F_{\mu\nu}$

$F_{\mu\nu}$

• degenerate metrics $\{e^i\} \leftrightarrow \{g_{ij}\}$ $\det(g) \neq 0$

• metricity. $\nabla_\mu g_{\nu\sigma} = 0$

3 D.

$$\omega_{\mu} \underbrace{j_k}_{j=1,2,3} = \omega_{\mu} \quad \begin{matrix} \rho \leftarrow \\ \ell=1,2,3 \text{ su}(2) \\ (SO(3) \text{ Lie-} \\ \text{algebra}) \end{matrix}$$

$$F_{\mu\nu jk} =$$

$$R = e^{\mu}_j e^{\nu}_k \epsilon^{jkl} F_{\mu\nu l}$$

• degenerate metrics $\{e^s\} \leftrightarrow \{g_{\mu\nu}\}$ $\det(g) \neq 0$

• metricity. $\nabla_\mu g_{\nu\sigma} = 0$

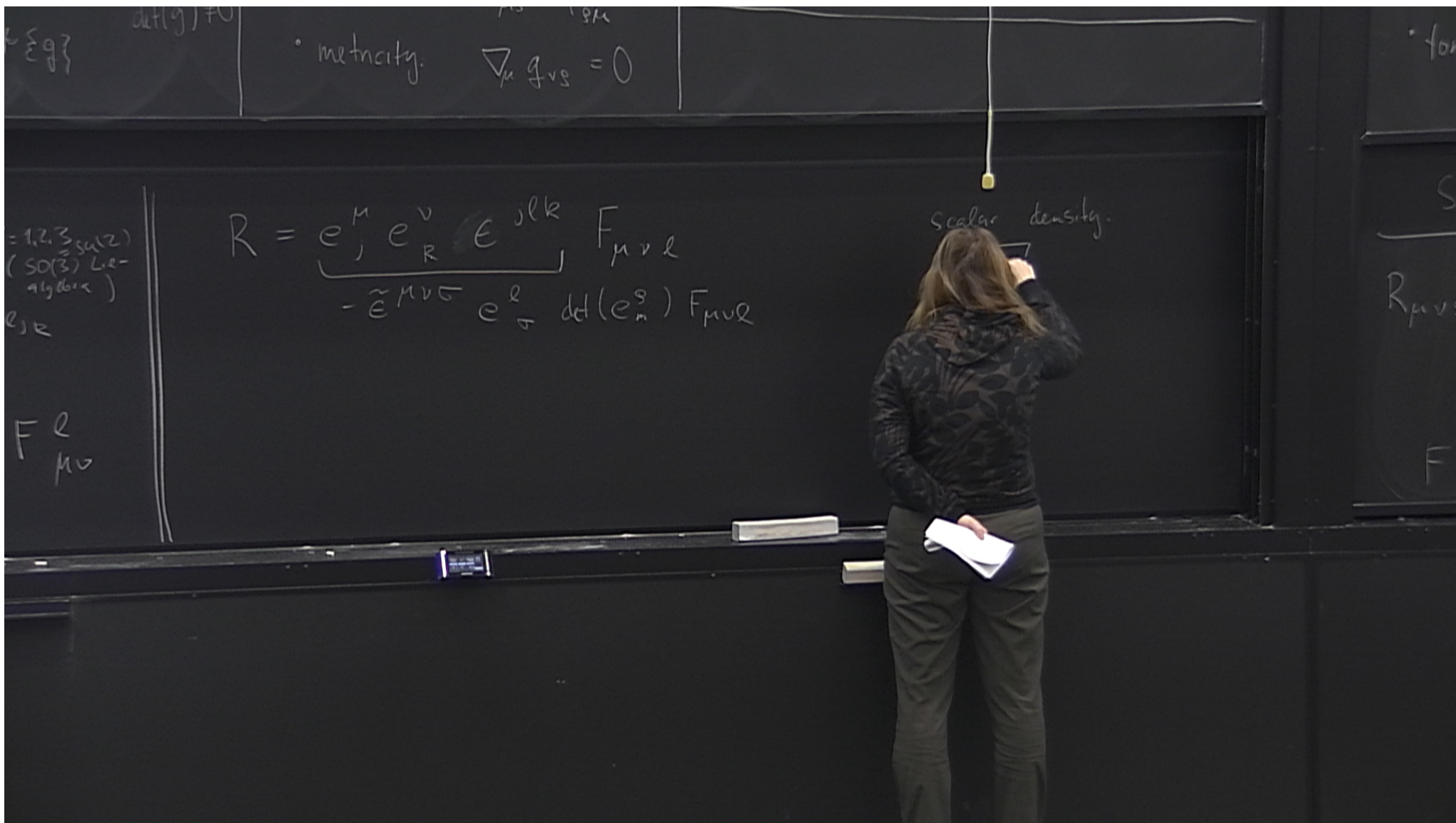
3 D.

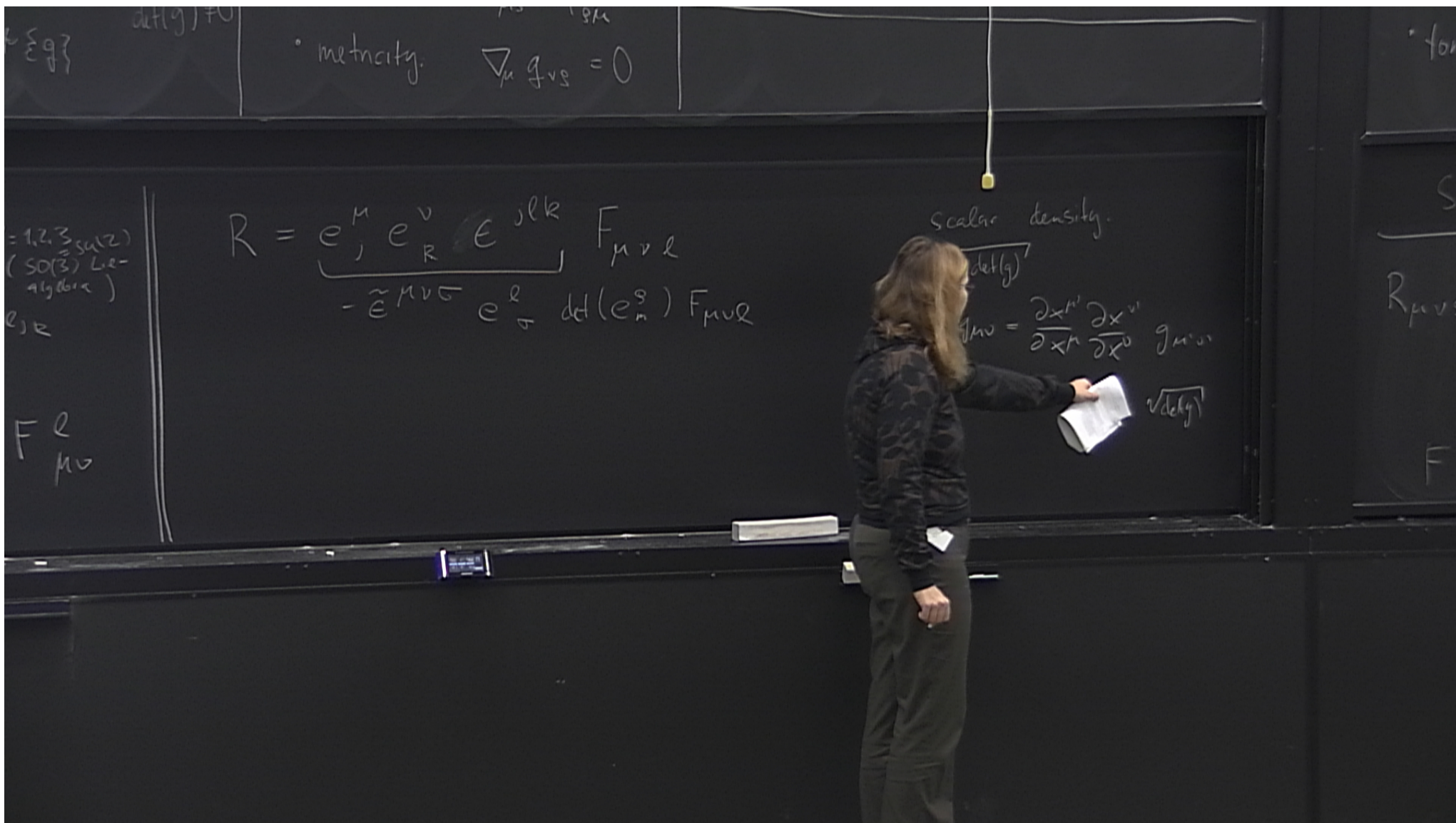
$\omega_{\mu j}$ $\omega_\mu \in e_{j\ell}$ $\ell=1,2,3 \text{ su}(2)$
($SO(3)$ Lie-algebra)

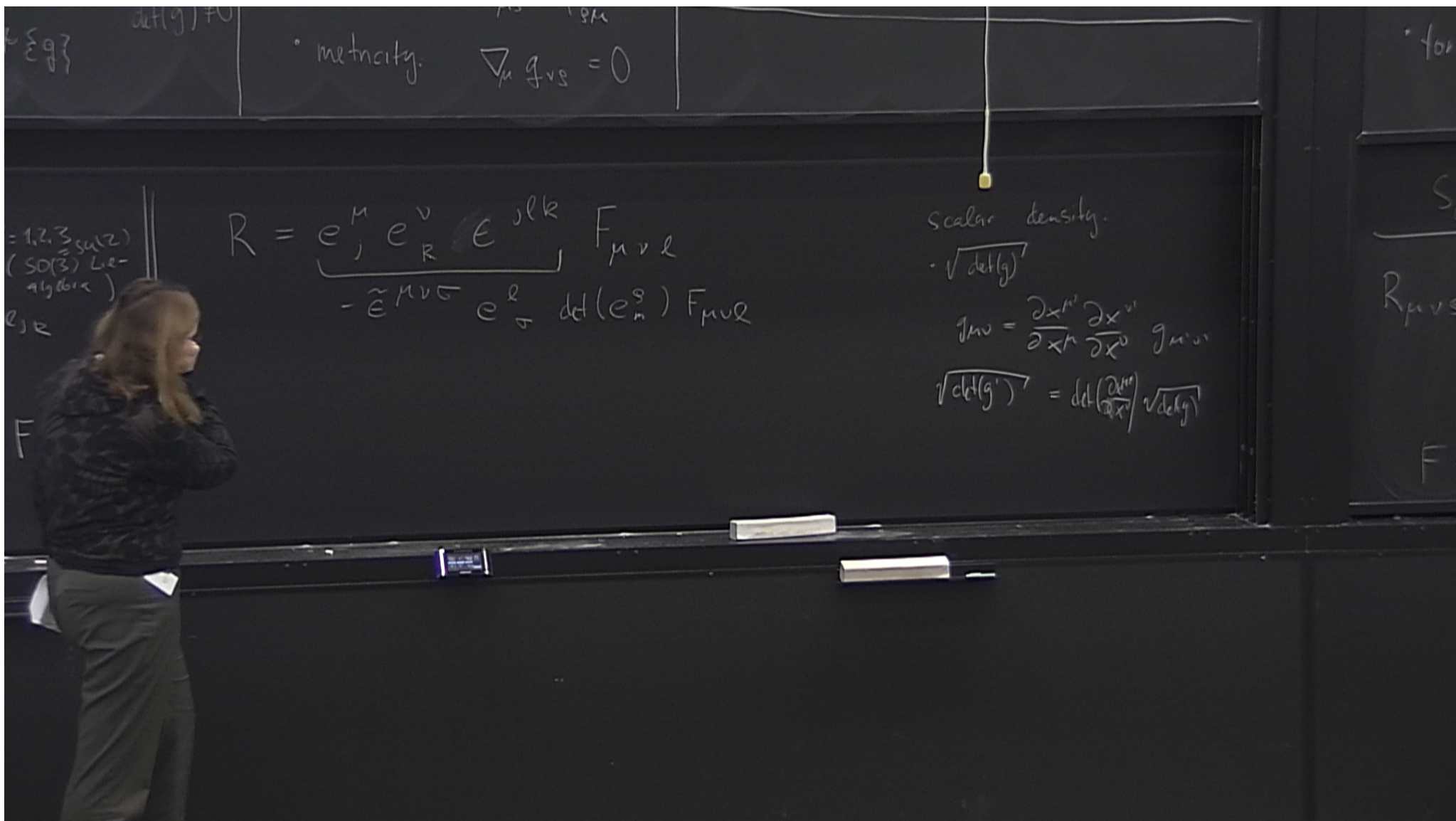
$j=$

$\epsilon_{j\ell k}$ $F_{\mu\nu}^\ell$

$$R = \underbrace{e^\mu_j e^\nu_k \epsilon^{j\ell k}}_{-\tilde{\epsilon}^{\mu\nu\sigma} e^\ell_\sigma \det(e^s_m)} F_{\mu\nu\ell}$$







$\{g\}$
 $\det(g) \neq 0$

metricity. $\nabla_\mu g_{\nu\sigma} = 0$

$= 1, 2, 3$ $\text{SO}(3)$ Lie algebra
 $e_i \in$

$F_{\mu\nu}$

$$R = \underbrace{e^\mu_\mu e^\nu_\nu e^{\rho\sigma}_{\rho\sigma}}_{-\tilde{\epsilon}^{\mu\nu\sigma} e^\rho_\sigma \det(e^\mu_\mu)} F_{\mu\nu\rho}$$

$$\boxed{\epsilon^{a_1 \dots a_{n-1}} A_{a_1 b_1} \dots A_{a_{n-1} b_{n-1}} = \det(A) (A^{-1})^{b_n}_{a_n} \epsilon_{b_1 \dots b_n}}$$

scalar density.
 $\cdot \sqrt{\det(g)}$

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} g_{\mu'\nu'}$$

$$\sqrt{\det(g')} = \det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) \sqrt{\det(g)}$$

$\{g\}$
 $\det(g) \neq 0$

metricity: $\nabla_\mu g_{\nu\sigma} = 0$

$= 1, 2, 3$ $\text{SO}(3)$ Lie algebra
 e_i
 $F^{\ell}_{\mu\nu}$

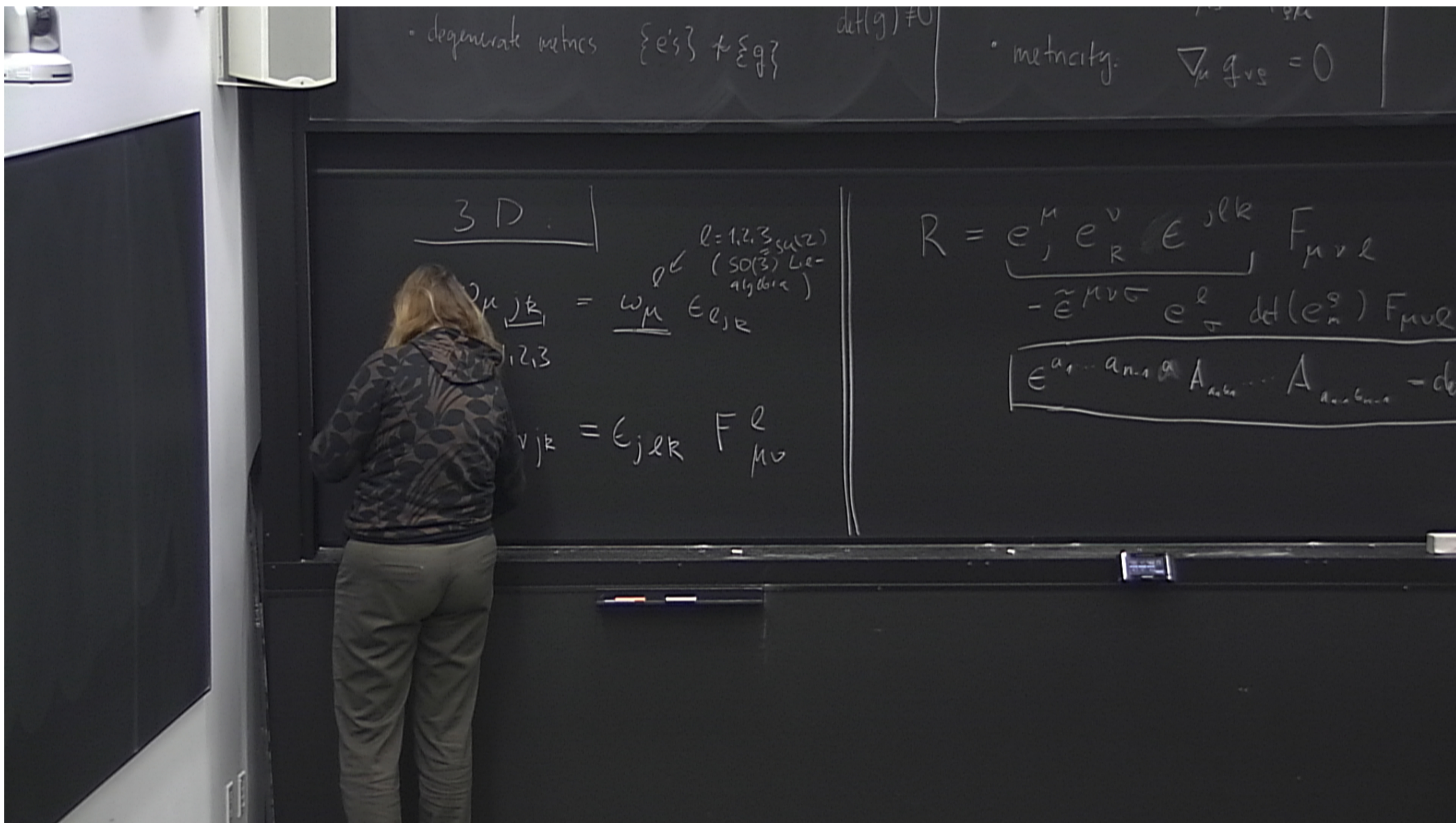
$$R = \underbrace{e^\mu_\mu e^\nu_\nu e^{\rho\sigma}}_{-\tilde{\epsilon}^{\mu\nu\rho\sigma} e^\ell_\ell \det(e^\mu_\mu)} F_{\mu\nu\rho\sigma}$$

$$\epsilon^{a_1 \dots a_{n-1} a} A_{a_1 b_1} \dots A_{a_{n-1} b_{n-1}} = \det(A) (A^{-1})^{ba} \epsilon_{b_1 \dots b_{n-1} b}$$

scalar density.
 $\sqrt{\det(g)}$

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} g_{\mu'\nu'}$$

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• degenerate metrics $\{e^i_s\} + \{g^i_j\}$

$$\det(g) \neq 0$$

• metricity.

$$\nabla_\mu g_{\nu\sigma} = 0$$

3 D.

$\ell = 1, 2, 3$ $so(3)$ Lie algebra

$$\omega^{\ell}_{jk} = \omega^{\ell}_{\mu} \epsilon^{\mu}_{jk}$$

$j, k = 1, 2, 3$

$$\omega^{\ell}_{jk} = \epsilon_{j\ell k} F^{\ell}_{\mu\nu}$$

$$-\partial_\nu \omega^{\ell}_{\mu} + \epsilon^{\ell}_{jk} \omega^j_{\mu} \omega^k_{\nu}$$

$$R = \underbrace{e^{\mu}_i e^{\nu}_j \epsilon^{ijk}}_{- \tilde{\epsilon}^{\mu\nu\sigma} e^{\ell}_{\sigma} dt(e^{\ell}_\mu) F_{\mu\nu\ell}}$$

$$\epsilon^{a_1 \dots a_n} A_{a_1 b_1} \dots A_{a_n b_n} = \det(A)$$

$$\{e^s\} + \{g\}$$

$$\det(g) \neq 0$$

$$\text{metricity: } \nabla_\mu g_{\nu\sigma} = 0$$

$$\omega_\mu \in \mathfrak{so}(3) \text{ Lie-algebra}$$

$$R = \underbrace{e^\mu_\mu e^\nu_\nu \epsilon^{jkl}}_{-\tilde{\epsilon}^{\mu\nu\sigma}} F_{\mu\nu l}$$

$$\left[\epsilon^{a_1 \dots a_n} A_{a_1 \dots a_{n-1} b} = \det(A) (A^{-1})^{ba} \epsilon_{a_1 \dots a_{n-1} b} \right]$$

$$\int R \sqrt{g} d^3x$$

scalar density.

$$\sqrt{\det(g)}$$

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} g_{\mu'\nu'}$$

$$\sqrt{\det(g')} = \det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) \sqrt{\det(g)}$$

$$\{e^s\} + \{g\}$$

$$\det(g) \neq 0$$

metricity: $\nabla_\mu g_{\nu\sigma} = 0$

$$R = \underbrace{e^\mu_\mu e^\nu_\nu e^{\rho\sigma}}_{-\tilde{e}^{\mu\nu\rho\sigma}} \underbrace{F_{\mu\nu\rho\sigma}}_{\det(e^s_\mu) F_{\mu\nu\rho\sigma}}$$

$$[e^a_{\mu} \dots e^a_{\mu-n} \cdot A_{a \dots a-n, \mu \dots \mu-n} = \det(A) (A^{-1})^{\mu a} e_{a, \mu \dots \mu-n}]$$

$$\left(\int R \sqrt{g} \right) = \int \tilde{e}^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} d^4x$$

scalar density.

$$-\sqrt{\det(g)}$$

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} g_{\mu'\nu'}$$

$$\sqrt{\det(g')} = \det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) \sqrt{\det(g)}$$

$$\omega_\mu \in \mathfrak{so}(3) \text{ Lie-algebra}$$

$$F_{\mu\nu} = \epsilon_{jkr} F_{\mu\nu}^r$$

$$\{e^i_s\} \neq \{g^i_j\}$$

$$\det(g) \neq 0$$

$$\text{metricity: } \nabla_\mu g_{\nu\sigma} = 0$$

$$\begin{aligned} & \ell = 1, 2, 3 \text{ su(2)} \\ & \omega_\mu \in \mathfrak{so}(3) \text{ Lie-algebra} \\ & \omega_\mu \in \mathfrak{e}_{ijk} \end{aligned}$$

$$R = \underbrace{e^M_\mu e^N_\nu \epsilon^{jkl}}_{-\tilde{\epsilon}^{\mu\nu\sigma} e^l_\sigma \det(e^s_m)} F_{\mu\nu l}$$

$$\left[\epsilon^{a_1 \dots a_{n-1} a} A_{a, b_1} \dots A_{a, b_{n-1}} = \det(A) (A^{-1})^{ba} \epsilon_{b, b_1 \dots b_{n-1}} \right]$$

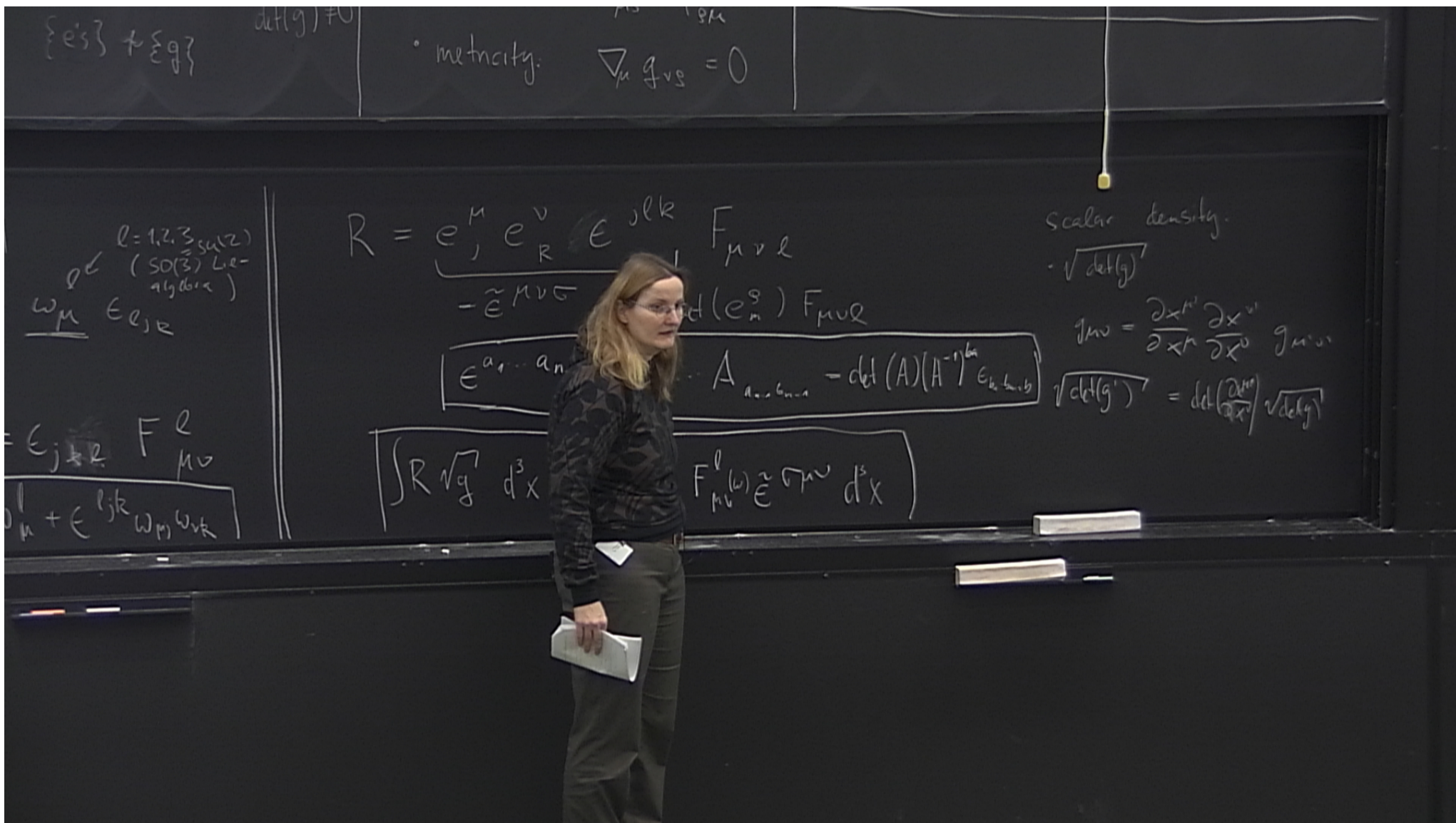
$$\left[\int R \sqrt{g} d^3x = - \int e_{\sigma i} F^i_{\mu\nu} \tilde{\epsilon}^{\sigma\mu\nu} d^3x \right]$$

scalar density.

$$\sqrt{\det(g)}$$

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} g_{\mu'\nu'}$$

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$$\{e^s\} + \{g\}$$

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$$\left[\epsilon^{a_1 \dots a_{n-1} a} A_{a_1 a_2} \dots A_{a_{n-1} a_n} = \det(A) (A^{-1})^{ba} \epsilon_{b_1 \dots b_{n-1} b} \right]$$

$$\left[\int R \sqrt{g} d^3x = - \int e_{\sigma\mu} F_{\mu\nu}^{\rho} \tilde{\epsilon}^{\sigma\mu\nu} d^3x \right]$$

scalar density.

$$-\sqrt{g}$$

$$\frac{\partial x^\mu}{\partial x^\nu} g_{\mu\nu}$$

$$F$$

$$\omega_\mu \in \mathfrak{so}(3)$$

$$\epsilon_{jkl} F_{\mu\nu}^{\rho}$$