Title: 13/14 PSI - Beyond the Standard Model - Lecture 7

Date: Feb 27, 2014 03:15 PM

URL: http://pirsa.org/14020059

Abstract:

TOPOLOGICAL ASPECT of ABELIAN + NON-ABELIAN

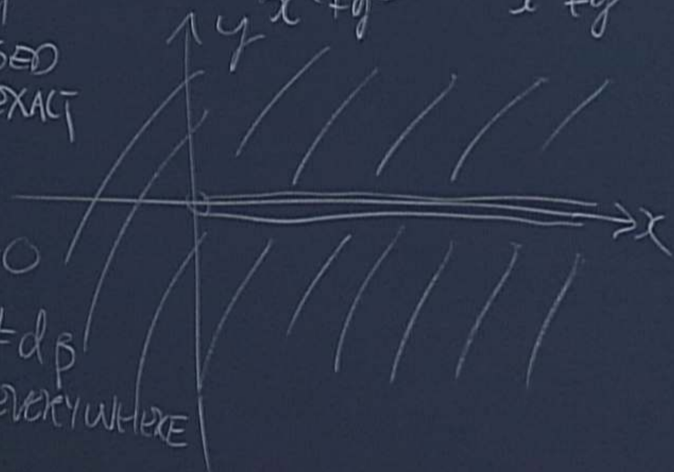
$\mathbb{R}-\{0\}$

$$\alpha = -\frac{y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

CLOSED
NO EXACT

NOT SINGLE VALUED on $\mathbb{R}-\{0\}$

OK on $\mathbb{R}-\mathbb{R}^+$



$d\alpha = 0$

$\alpha \neq d\beta$ EVERYWHERE

TOPOLOGICAL ASPECT of

ABELIAN + NON-ABELIAN

SOL

DEF

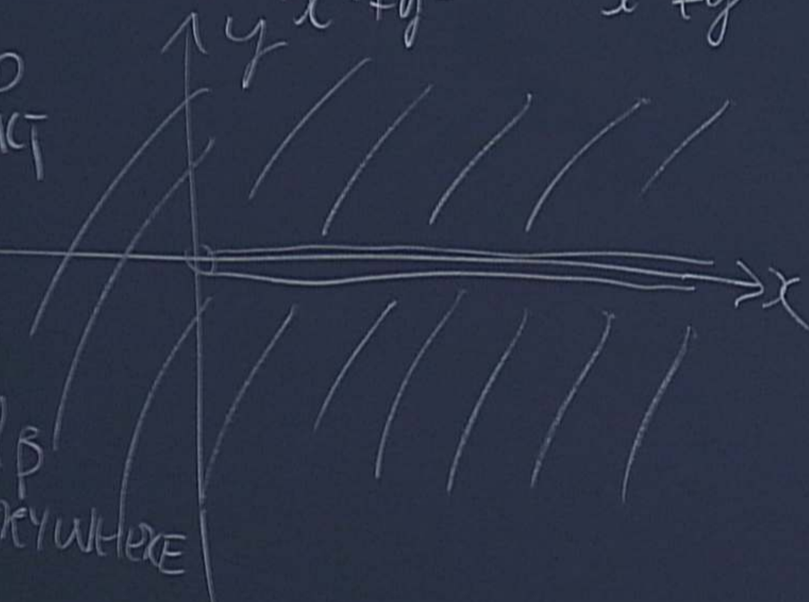
$\mathbb{R}^2-\{0\}$

$$\alpha = -\frac{y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

CLOSED
NO EXACT

NOT SINGLE VALUED on $\mathbb{R}-\{0\}$

OK on $\mathbb{R}^2-\mathbb{R}^+$



$d\alpha=0$

$\alpha\neq d\beta$
everywhere

SOL'N TO $d\alpha = 0$

DEFINE $X^\mu(\lambda, x) = \begin{cases} x^\mu & \lambda = 1 \\ y^\mu & \lambda = 0 \end{cases}$

CHOOSE TO VANISH

$$\alpha(x) = \int_0^1 \frac{\delta \alpha(X)}{\delta \lambda}\, d\lambda = \underline{\alpha(X(1,x))} - \cancel{\alpha(X(0,x))}$$

SOL'N TO $d\alpha = 0$

DEFINE $X^\mu(\lambda, x) = \begin{cases} x^\mu & \lambda = 1 \\ y^\mu & \lambda = 0 \end{cases}$

CHOOSE TO VANISH

$$\alpha(x) = \int_0^1 \frac{\delta \alpha(X)}{\delta \lambda}\, d\lambda = \underline{\alpha(X(1,x))} - \cancel{\alpha(X(0,x))}$$

$$\frac{\delta \alpha}{\delta \lambda} = (p+1)\, d\left[\frac{\delta X^\tau}{\delta \lambda}\, \alpha_{\tau \nu_1 \cdots \nu_p}\, dX^{\nu_1} \cdots dX^{\nu_p}\right]$$

$\left(\frac{y}{x}\right)\Big)$
NGLE
on $\mathbb{R}-\{0\}$
on
$\mathbb{R}^+$

SOL'N TO $d\alpha = 0$

DEFINE $X^\mu(\lambda, x) = \begin{cases} x^\mu & \lambda = 1 \\ y^\mu & \lambda = 0 \end{cases}$

CHOOSE TO VANISH

$$\alpha(x) = \int_0^1 \frac{\delta \alpha(X)}{\delta \lambda} \, d\lambda = \underline{\alpha(X(1,x))} - \cancel{\alpha(X(0,x))}$$

$$\frac{\delta\alpha}{\delta\lambda} = (p+1)\, d\left[\frac{\delta X^\tau}{\delta\lambda}\, \alpha_{\nu_1 \ldots \nu_p, \tau}\, dX^{\nu_1} \cdots dX^{\nu_p}\right]$$

$$\alpha = \alpha_{\nu_1 \ldots \nu_p}\, dX^{\nu_1} \cdots dX^{\nu_p}$$

ECT of
RIAN

$d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

NOT SINGLE VALUED on $\mathbb{R}-\{0\}$

OK on $\mathbb{R}^2-\mathbb{R}^+$

SOL'N TO $d\alpha = 0$

DEFINE $X^\mu(\lambda, x) = \begin{cases} x^\mu & \lambda = 1 \\ y^\mu & \lambda = 0 \end{cases}$

$$\alpha(x) = \int_0^1 \frac{\delta \alpha(X)}{\delta \lambda}\, d\lambda = \alpha(X(1,x)) - \cancel{\alpha(X(0,x))}$$

CHOOSE TO VANISH

$$\frac{\delta \alpha}{\delta \lambda} = (p+1)\, d\left[\frac{\delta X^\tau}{\delta \lambda} \alpha_{\tau \nu_1 \ldots \nu_p}\, dX^{\nu_1} \cdots dX^{\nu_p}\right]$$

$$(p+1)!\, \alpha = \alpha_{\nu_1 \ldots \nu_{p+1}}\, dX^{\nu_1} \cdots dX^{\nu_{p+1}} \longrightarrow \beta$$

$d\alpha = 0 \Rightarrow \alpha = d\beta$ LOCALLY

GAUGE FIELDS

$$A_\lambda = A(\lambda) \equiv A(X(\lambda, x)) = \lambda A = \lambda A_\mu dx^\mu$$

$$F_\lambda = dA_\lambda + \underbrace{ig_1}_{\text{SET to } 1} A_\lambda^2 = \lambda[dA + \lambda A^2]$$

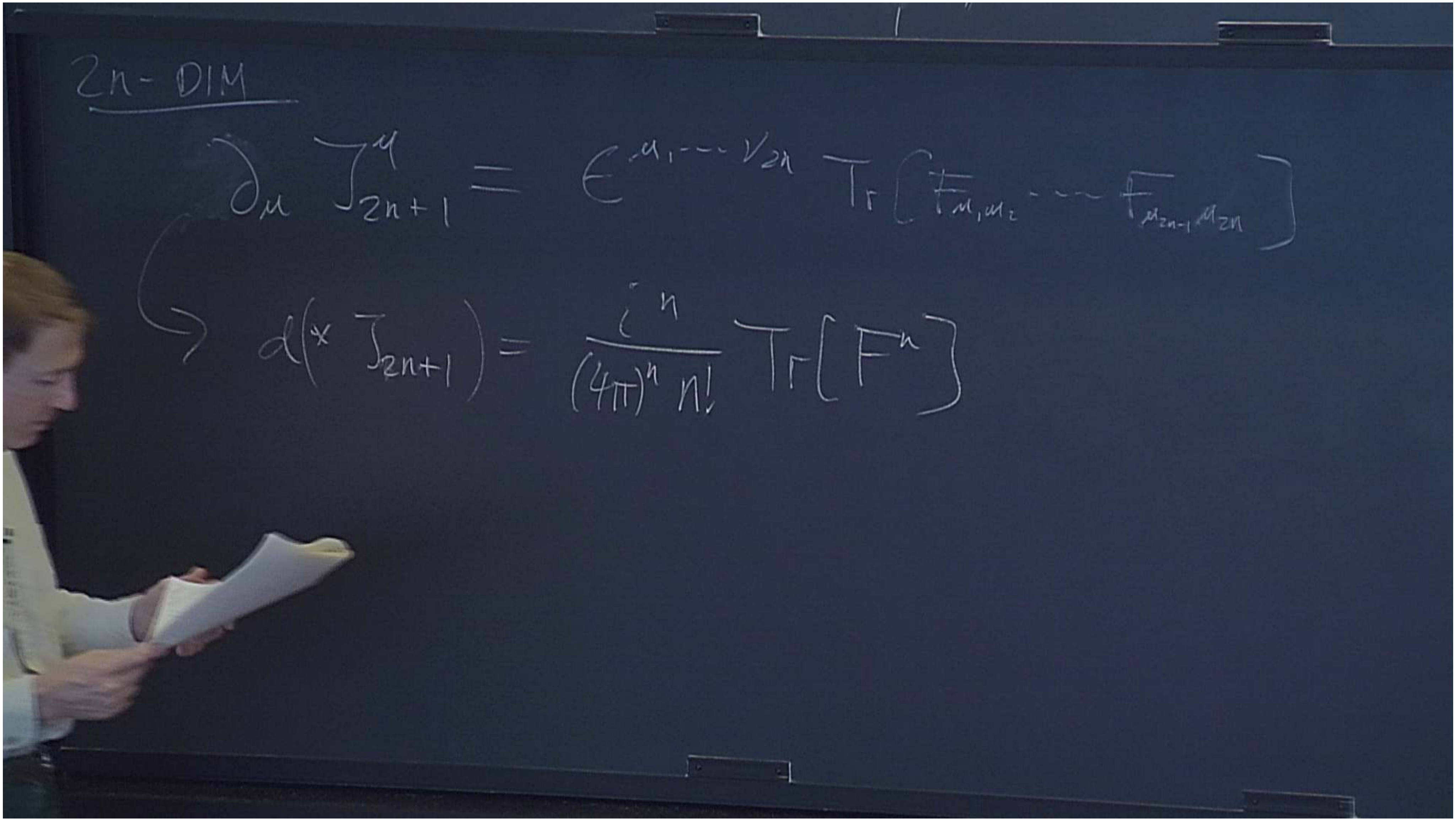
$$\Rightarrow \frac{\delta}{\delta\lambda}\left[\mathrm{Tr}(F_\lambda^2)\right] = 2\, d\left(\mathrm{Tr}\left[\frac{\delta A_\lambda}{\delta\lambda} F_\lambda\right]\right)$$

$$(d*J_5) = \mathrm{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}] = \int_0^1 \frac{\delta}{\delta\lambda}\left(\mathrm{Tr}(F_\lambda^2)\right) d\lambda \qquad F_\lambda^2 = F_\lambda \wedge F_\lambda$$

$$= 2\int_0^1 d\lambda\, \mathrm{Tr}\left[\lambda A(dA + \lambda A^2)\right]$$

$$(d*J_5) = \mathrm{Tr}\left[A\left(dA + \frac{2}{3}A^2\right)\right]$$

$$F^2 \equiv F\wedge F = F_{\mu\nu}\tilde{F}^{\mu\nu} = F\tilde{F}$$


2n-DIM
$\partial_\mu J^\mu_{2n+1} = \epsilon^{\mu_1 \cdots \nu_{2n}} \mathrm{Tr}[F_{\mu_1 \mu_2} \cdots F_{\mu_{2n-1} \mu_{2n}}]$
$d(* J_{2n+1}) = \frac{i^n}{(4\pi)^n n!} \mathrm{Tr}[F^n]$

$\neq d\beta$ everywhere

$\mathbb{R} = \mathbb{R}'$

$\delta\lambda = (p+1)\, d \left[\delta\lambda \, \alpha_{(\nu_1 \cdots \nu_p} \, dA \cdots d\lambda \right]$

$(p+1)\ \alpha = \alpha_{\nu_1 \cdots \nu_{p+1}} dX^{\nu_1} \cdots dX^{\nu_{p+1}} \longrightarrow \beta$

2n- DIM

$$\partial_\mu J^\mu_{2n+1} = \epsilon^{\mu_1 \cdots \nu_{2n}} \, \mathrm{Tr}\left[F_{\mu_1 \mu_2} \cdots F_{\mu_{2n-1} \mu_{2n}} \right]$$

SYMMETRACE

$$d\left({}^* J_{2n+1} \right) = \frac{i^n}{(4\pi)^n n!} \mathrm{Tr}\left[F^n \right] = \frac{i^n}{(4\pi)^n (n!)} \, n \, d \left\{ \int_0^1 d\lambda \, \lambda^{n-1} \, \mathrm{STr}\left[A, (dA + \lambda A^2)^{n-1} \right] \right\}$$

$$\mathrm{STr}\left[T^{a_1} \cdots T^{a_n} \right] \equiv \frac{1}{n!} \mathrm{Tr}\left(T^{(a_1} T^{a_2} \cdots T^{a_n)} \right)$$

$\alpha \neq d\beta$ EVERYWHERE

$\mathbb{R} = \mathbb{R}$

$(p+1)\ \alpha = \alpha_{\nu_1 \cdots \nu_{p+1}} dX^{\nu_1} \cdots dX^{\nu_{p+1}} \longrightarrow \beta$

2n- DIM

$$\partial_\mu J^\mu_{2n+1} = \epsilon^{\mu_1 \cdots \nu_{2n}} \mathrm{Tr}\left[F_{\mu_1 \mu_2} \cdots F_{\mu_{2n-1} \mu_{2n}}\right]$$

SYMMETRACE

$$d(* J_{2n+1}) = \frac{i^n}{(4\pi)^n n!} \mathrm{Tr}\left[F^n\right] = \frac{i^n}{(4\pi)^n (n!)} n\, d\left[\int_0^1 d\lambda\, \lambda^{n-1} \mathrm{STr}\left[A, (dA + \lambda A^2)^{n-1}\right]\right]$$

$$\mathrm{STr}\left[T^{a_1} \cdots T^{a_n}\right] \equiv \frac{1}{n!} \mathrm{Tr}\left(T^{(a_1} T^{a_2} \cdots T^{a_n)}\right)$$

eg $n=3$ $$\mathrm{STr}\left[T^a T^b T^c\right] = \frac{1}{3!} \mathrm{Tr}\left[T^{(a} T^b T^{c)}\right] = \frac{1}{2} \mathrm{Tr}\left[T^a \{T^b, T^c\}\right]$$

$d\alpha = 0$

$\alpha \neq d\beta$ everywhere

OR ∂M

$\mathbb{R}^2 - \mathbb{R}^+$

$$\frac{\delta \alpha}{\delta \lambda} = (p+1)\, d\left[\frac{\delta X^\tau}{\delta \lambda} \alpha_{\tau \nu_1 \cdots \nu_p} dX^{\nu_1} \cdots dX^{\nu_p} \right]$$

$$(p+1)!\, \alpha = \alpha_{\nu_1 \cdots \nu_{p+1}} dX^{\nu_1} \cdots dX^{\nu_{p+1}} \longrightarrow \beta$$

2n- DIM

$$\partial_\mu J^\mu_{2n+1} = \epsilon^{\mu_1 \cdots \nu_{2n}} \mathrm{Tr}\left[F_{\mu_1 \mu_2} \cdots F_{\mu_{2n-1} \mu_{2n}} \right]$$

SYMMETRACE

$$d(*J_{2n+1}) = \frac{i^n}{(4\pi)^n n!} \mathrm{Tr}\left[F^n\right] = \frac{i^n}{(4\pi)^n (n!)}\, n\, d\left[\int_0^1 d\lambda\, \lambda^{n-1} \mathrm{STr}\left(A (dA + \lambda A^2)^{n-1} \right) \right]$$

$$\rightarrow A(\ )^{n-1} + (\ )A(\ )^{n-2} + (\ )^2 A(\ )^{n-3} - \cdots$$

$$\mathrm{STr}\left[T^{a_1} \cdots T^{a_n}\right] \equiv \frac{1}{n!} \mathrm{Tr}\left(T^{(a_1} T^{a_2} \cdots T^{a_n)} \right)$$

e.g. $n=3$ $\quad \mathrm{STr}\left[T^a T^b T^c\right] = \frac{1}{3!} \mathrm{Tr}\left[T^{(a} T^b T^{c)}\right] = \frac{1}{2} \mathrm{Tr}\left[T^a \{T^b, T^c\}\right]$

$$\Rightarrow \frac{\delta}{\delta\lambda}\left[\mathrm{Tr}\left(F_\lambda^2\right)\right] = 2\,d\left(\mathrm{Tr}\left[\frac{\delta A_\lambda}{\delta\lambda}F_\lambda\right]\right)$$

$$F \equiv F\wedge F = F_{\mu\nu}F \ldots = FF$$

IF $\Lambda_i = \alpha_i^a T^a$

$$\mathrm{STr}\left[\Lambda_1\cdots\Lambda_k\right]$$

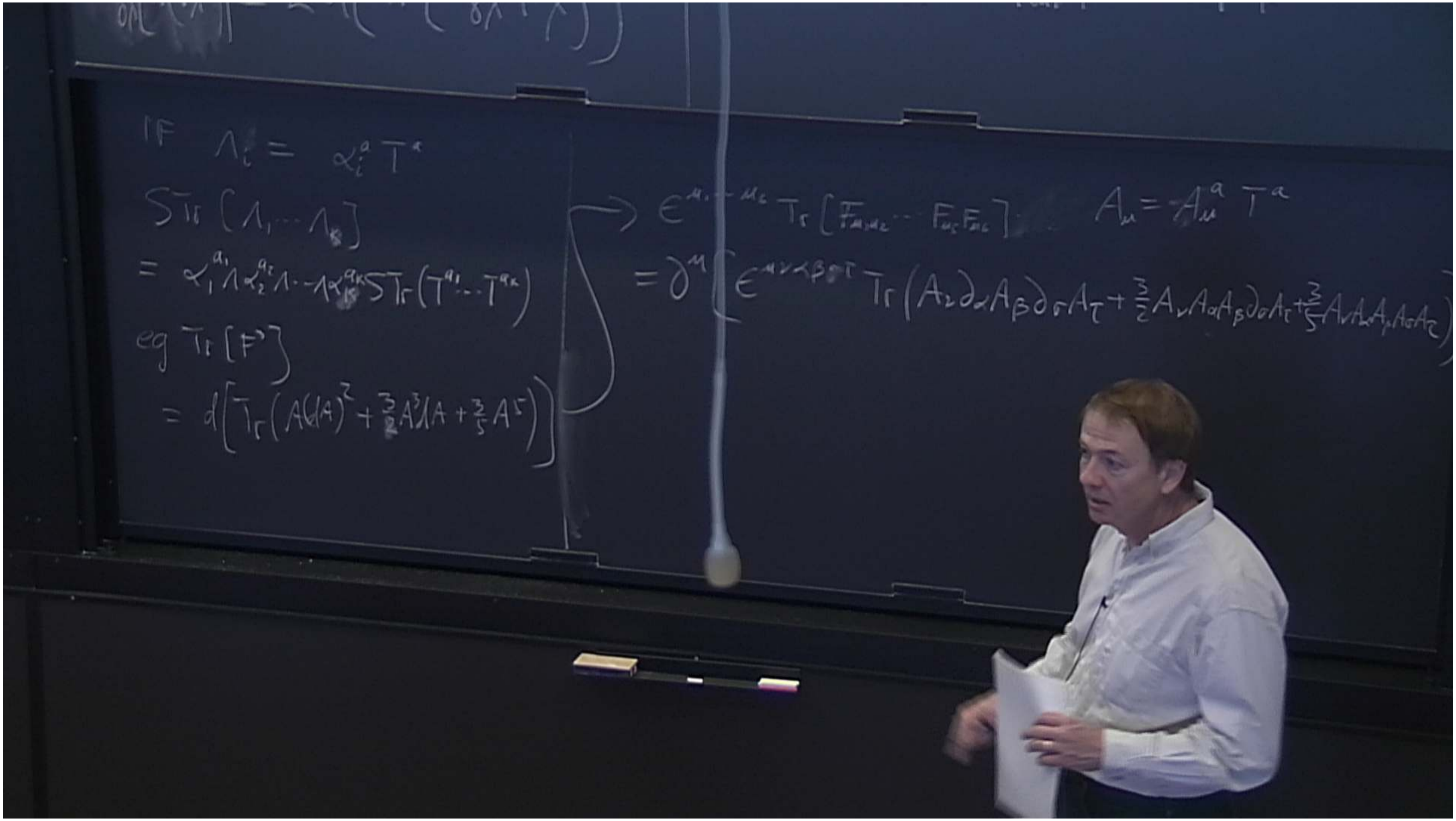
$$= \alpha_1^{a_1}\wedge\alpha_2^{a_2}\wedge\cdots\wedge\alpha_k^{a_k}\,\mathrm{STr}\left(T^{a_1}\cdots T^{a_k}\right)$$

eg $\mathrm{Tr}\left[F^3\right]$

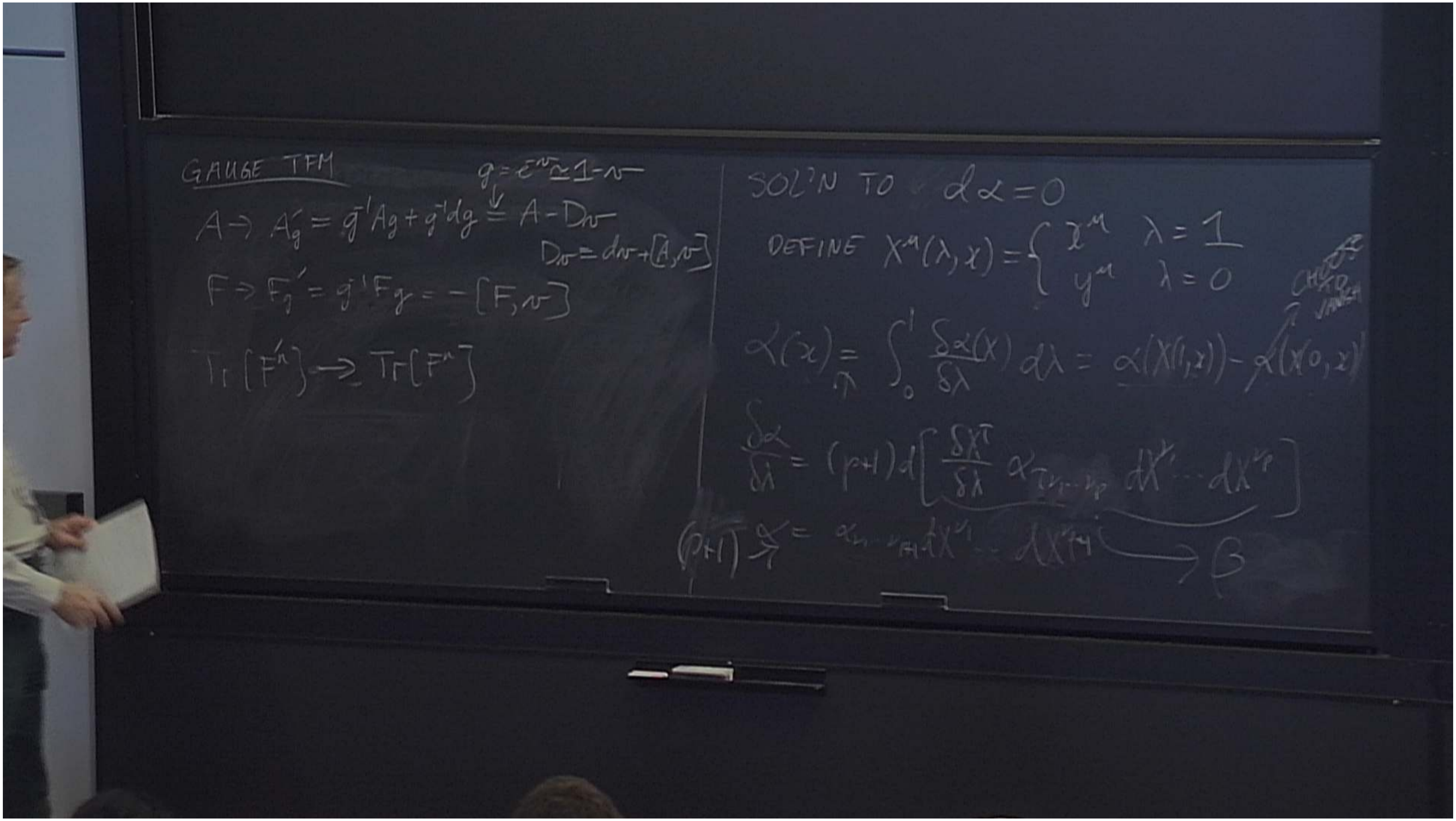
$$= d\left[\mathrm{Tr}\left(A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5\right)\right]$$

$$\rightarrow \epsilon^{\mu_1\cdots\mu_6}\,\mathrm{Tr}\left[F_{\mu_1\mu_2}\cdots F_{\mu_5\mu_6}\right]$$

$$= \partial^\mu\left[\epsilon^{\mu\nu\alpha\beta\sigma\tau}\,\mathrm{Tr}\left(A_\nu\partial_\alpha A_\beta\partial_\sigma A_\tau + \frac{3}{2}A_\nu A_\alpha A_\beta\partial_\sigma A_\tau + \frac{3}{5}A_\nu A_\alpha A_\beta A_\sigma A_\tau\right)\right]$$



IF $\Lambda_i = \alpha_i^a T^a$
$STr[\Lambda_1 \cdots \Lambda_k]$
$= \alpha_1^{a_1} \wedge \alpha_2^{a_2} \wedge \cdots \wedge \alpha_k^{a_k} STr(T^{a_1} \cdots T^{a_k})$
eg $Tr[F^3]$
$= d[Tr(A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5)]$
$\rightarrow \epsilon^{\mu_1 \cdots \mu_6} Tr[F_{\mu_1\mu_2} \cdots F_{\mu_5\mu_6}]$
$A_\mu = A_\mu^a T^a$
$= \partial^\mu [\epsilon^{\mu\nu\alpha\beta\sigma\tau} Tr(A_\nu \partial_\alpha A_\beta \partial_\sigma A_\tau + \frac{3}{2} A_\nu A_\alpha A_\beta \partial_\sigma A_\tau + \frac{3}{5} A_\nu A_\alpha A_\beta A_\sigma A_\tau)]$


GAUGE TFM
g = e^v ≃ 1 - v
A → A'_g = g^{-1}Ag + g^{-1}dg = A - Dv
Dv = dv + [A, v]
F → F'_g = g^{-1}Fg = -[F, v]
Tr[F'^n] → Tr[F^n]
SOL'N TO dα = 0
DEFINE X^μ(λ, x) = { x^μ λ = 1 ; y^μ λ = 0
CHOOSE
α(x) = ∫_0^1 δα(X)/δλ dλ = α(X(1,x)) - α(X(0,x))
δα/δλ = (p+1) d[δX^τ/δλ α_{τν_1…ν_p} dX^{ν_1} … dX^{ν_p}]
(p+1) α = α_{ν_1…ν_{p+1}} dX^{ν_1} … dX^{ν_{p+1}}
β

GAUGE TFM

$$g = e^{-v} \simeq 1 - v$$

$$A \to A_g = g^{-1} A g + g^{-1} dg \doteq A - Dv$$

$$Dv \equiv dv + [A, v]$$

$$F \to F_g = g^{-1} F g = -[F, v]$$

$$\mathrm{Tr}[F_g^n] = \mathrm{Tr}[F^n]$$

$$0 = \delta_v\left(\mathrm{Tr}(F^n)\right) = \delta_v\left[d\omega^0_{2n-1}(A)\right]$$

$$= d\left[\delta_v\left(\omega^0_{2n-1}(A)\right)\right]$$

$$\Rightarrow \delta_v\left(\omega^{0}_{2n-1}(A)\right) \overset{\text{LOCALLY}}{=} d\left(\omega^{1}_{2n-2}(v, A)\right)$$

$dv + [A, v]$

$$\delta_v F_\lambda = -[F_\lambda, v] - (\lambda - 1)\{A_\lambda, dv\}$$

COMMUTATOR

$$\delta_v\left(\omega^{0}_{2n-1}\right) = -n\int_0^1 d\lambda\, d\left\{ STr\left(v, F_\lambda^{n-1}\right) + \underbrace{(n-1)(\lambda-1)\, STr\left(A, [A, v], F_\lambda^{n-2}\right)} \right\}$$

$$\mathrm{STr}\left(A_\lambda, [A, v], F_\lambda^{\,n-2}\right)$$

LONG CALC

$$\downarrow$$

$$= \mathrm{STr}\left(v, \{A, A_\lambda\}, F_\lambda^{\,n-2}\right) + \mathrm{STr}\left(v, A, [F_\lambda^{\,n-2}, A_\lambda]\right)$$

$$= d\left[\delta_{v}\left(\omega^{0}_{2n-1}(A)\right)\right]$$

$$\mathrm{STr}\left(A_\lambda, [A, v], F_\lambda^{n-2}\right)$$

LONG CALC

$$\overset{\downarrow}{=} \mathrm{STr}\left(v, \{A, A_\lambda\}, F_\lambda^{n-2}\right) + \mathrm{STr}\left(v, A, [F_\lambda^{n-2}, A_\lambda]\right) = \mathrm{STr}\left(v, \left(\frac{\delta F_\lambda}{\delta\lambda} - dA\right), F_\lambda^{n-2}\right) + \mathrm{STr}\left(v, A, dF_\lambda^{n-2}\right)$$

$$\left.\begin{aligned} dF_\lambda &= [F_\lambda, A_\lambda] \Rightarrow dF_\lambda^{n-2} = [F_\lambda^{n-2}, A_\lambda] \\ \frac{\delta F_\lambda}{\delta\lambda} &= dA + \{A, A_\lambda\} \end{aligned}\right\}$$

$$= d\left[\mathrm{STr}\left(W^{0}_{2n-1}(A) \right) \right] \qquad = d\left(\omega^{1}_{2n-2} \right)$$

$$\mathrm{STr}\left(A_\lambda, [A, v], F_\lambda^{n-2} \right)$$

LONG CALC

$$\overset{\downarrow}{=} \mathrm{STr}\left(v, \{A, A_\lambda\}, F_\lambda^{n-2} \right) + \mathrm{STr}\left(v, A, [F_\lambda^{n-2}, A_\lambda] \right) = \mathrm{STr}\left(v, \left(\frac{\delta F_\lambda}{\delta \lambda} - dA \right), F_\lambda^{n-2} \right) + \mathrm{STr}\left(v, A, dF_\lambda^{n-2} \right)$$

$$\left. \begin{aligned} & dF_\lambda = [F_\lambda, A_\lambda] \Rightarrow dF_\lambda^{n-2} = [F_\lambda^{n-2}, A_\lambda] \\ & \frac{\delta F_\lambda}{\delta \lambda} = dA + \{A, A_\lambda\} \end{aligned} \right\}$$

$$\omega^1_{2n-2} = n(n-1)\int_0^1 d\lambda\,(\lambda-1)\, S\,\mathrm{Tr}\left(v, d\left(A, F_\lambda^{\,n-2}\right)\right)$$

$$\left(A, F_\lambda^{\,n-2}\right) = A F_\lambda^{\,n-2} + F_\lambda A \ldots$$

eg
$n=3$

$$\omega^2_4 = 3\times 2\int_0^1 d\lambda(\lambda-1)\, S\,\mathrm{Tr}\left(v, d\left(A\left(\lambda dA + \lambda^2 A^2\right)\right)\right)$$

$$= -\frac{3\times 2}{6}\, S\,\mathrm{Tr}\left(v\, d\left(A dA + \tfrac{1}{2}A^3\right)\right)$$

NON-ABELIAN ANOMALY

SUMMARY

$$\text{AB ANOMALY} \propto \mathrm{Tr}(F^n) = d\left(\omega^0_{2n-1}(A)\right) \qquad \text{since } d(\mathrm{Tr}(F^n)) = 0$$

$$\delta_v \Big(\underbrace{\omega^0_{2n-1}(A)}_{\text{CHERN-SIMONS FORM}} \Big) = d\left(\omega^1_{2n-2}(v, A)\right) = d\left(\begin{array}{c}\text{NON-AB}\\ \text{ANOMALY}\end{array}\right)$$

ANOTHER VIEWPT

ANOMALY

$$D_\mu \underbrace{\left(\frac{\delta \Gamma(A)}{\delta A^a_\mu}\right)}_{\text{CURRENT}} \neq 0$$

DEFINE

$$X^a \equiv \left(D_\mu \frac{\delta}{\delta A^a_\mu}\right) \Rightarrow [X^a, X^b] = f^{abc} X^c$$

ANOTHER VIEWPT

ANOMALY

$$\underbrace{D_\mu \left(\frac{\delta \Gamma(A)}{\delta A_\mu^a}\right)}_{\text{CURRENT}} \neq 0 \Longrightarrow X_a(x)\,\Gamma[A] \equiv G_a[A(x)]$$

DEFINE

$$X^a \equiv \left(D_\mu \frac{\delta}{\delta A_\mu^a}\right) \Longrightarrow [X^a, X^b] = f^{abc} X^c$$

$$X_a(x)\,\Gamma[A] \equiv G_a[A(x)]$$

$$X^b] = f^{abc} X^c$$

$$\Rightarrow \left[X_a(x), G_b[A(y)]\right] = f_{ab}{}^{c}\, G_c[A]\, \delta(x-y)$$

$$\circledast \iff \delta_u G[v,A] - \delta_v G(u,A) = G([u,v],A]$$

$$\delta_u \equiv \int d^{2n}x\, u^a X_a$$

$$G[u,A] \equiv \int d^{2n}x\, u^a G_a[A(x)]$$

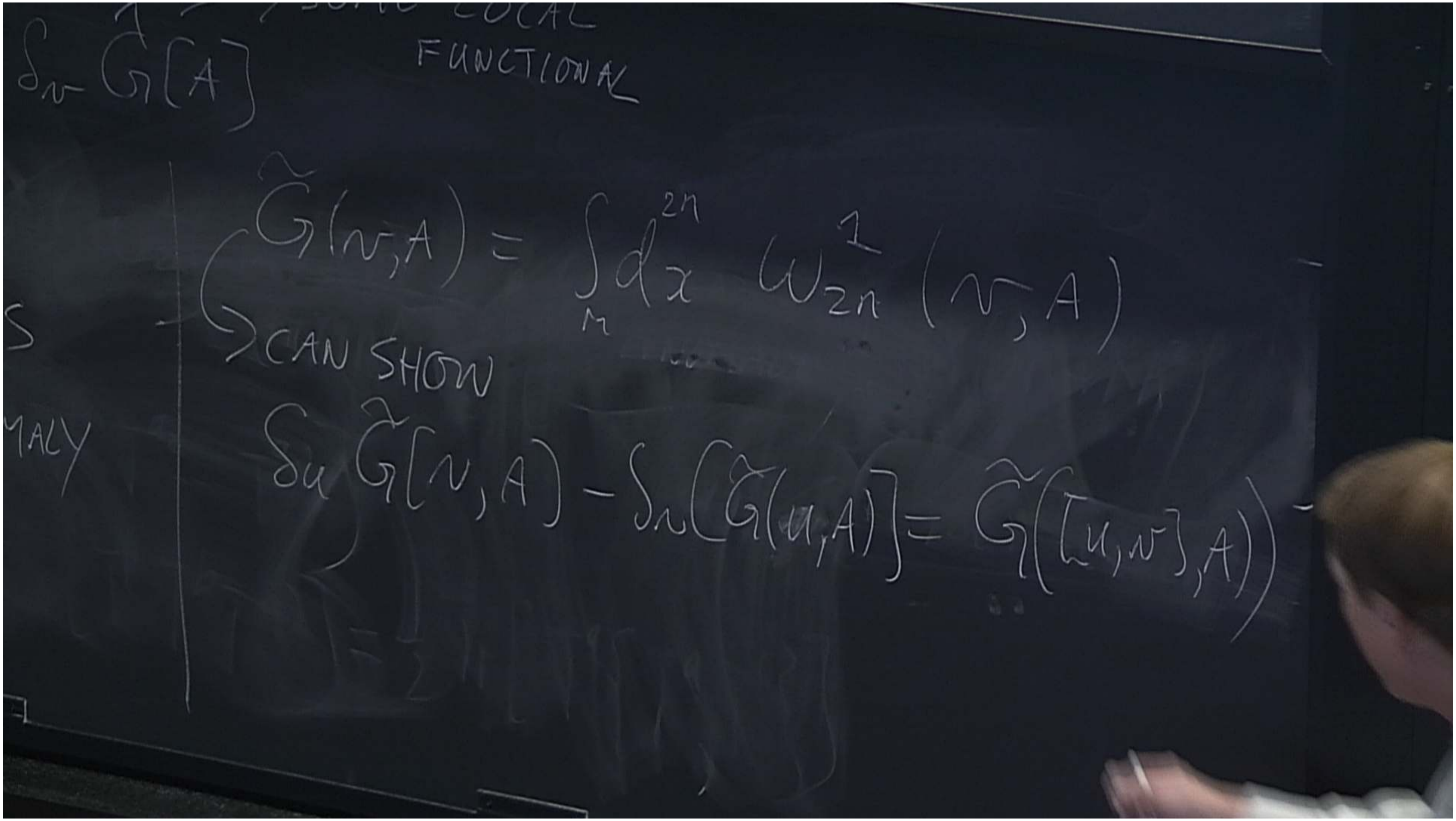
$$\Rightarrow \left[X_a(x), G_b[A(y)] \right] = f_{ab}{}^{c}\, G_c[A]\, \delta(x-y)$$

$$\Leftrightarrow \boxed{\delta_u G[v,A] - \delta_v G[u,A] = G[[u,v],A]}$$

WESS-ZUMINO CONDITIONS

$$\delta_u \equiv \int d^{2n}x\, u^a X_a$$

$$G[u,A] \equiv \int d^{2n}x\, u^a G_a[A(x)]$$



FUNCTIONAL
$\delta_v \mathcal{G}[A]$
$\tilde{\mathcal{G}}(v,A) = \int_M d^{2n}x \; \omega^1_{2n}(v,A)$
CAN SHOW
$\delta_u \tilde{\mathcal{G}}[v,A) - \delta_v[\tilde{\mathcal{G}}(u,A)] = \tilde{\mathcal{G}}([u,v],A))$

$= d\left[\delta_v\left(\omega^0_{2n-1}(A)\right)\right]$

$= d\left(\omega^1_{2n-2}\right)$

WHAT are the SOL'NS to ⊗ → SOME LOCAL FUNCTIONAL

ONE SOL'N ⇒ $G(v,A) = \delta_v \hat{G}[A]$

NOT MOST GENERAL SOL'N

IF $\exists$ A SOL'N THAT IS NOT $\delta_v\left[\hat{G}\right]$ ⇒ ANOMALY

$\tilde{G}(v,A) = \int_M d^{2n}x\ \omega^1_{2n}(v,A)$

CAN SHOW

$\delta_u \tilde{G}[v,A) - \delta_v\left[\tilde{G}(u,A)\right] = \tilde{G}([u,v],A)$