Title: Introduction to Quantum Field Theory for Cosmology - Lecture 14

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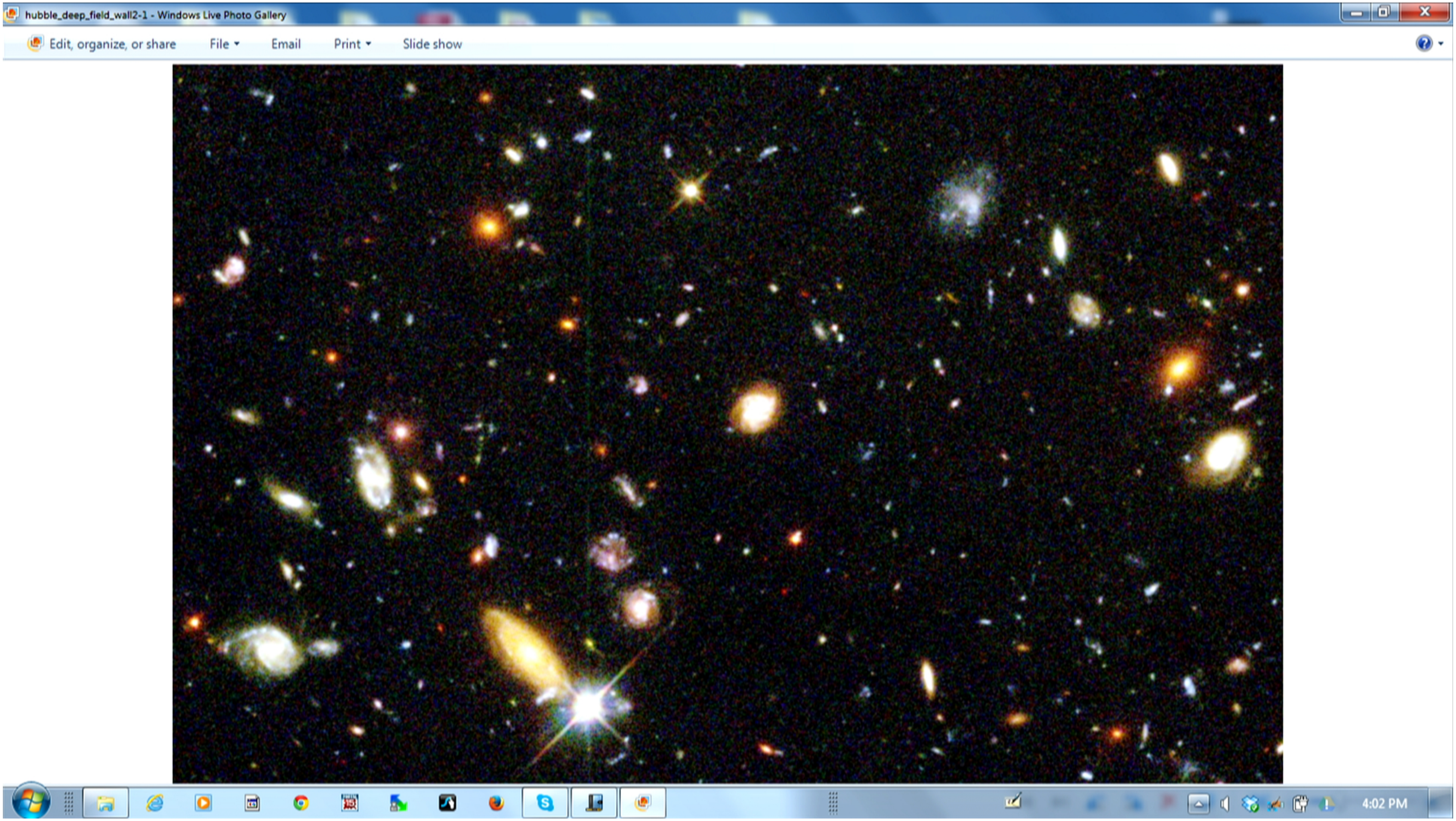
Abstract: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Note Title

QFT for Cosmology, Achim Kempf, Winter 14, Lecture 14

Quantum field theory on FRW spacetimes.

Friedmann Robertson Walker (FRW) spacetimes:

- While galaxies are causing some curvature locally, the universe is found to be spatially very flat on larger scales.

  (E.g., for triangles of billions of light years in size, Pythagoras' theorem holds with very good precision, the angles add up to 180°, etc)

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- Thus, one often considers the simplifying approximation that spacetime has no spatial curvature at all.

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- Thus, one often considers the simplifying approximation that spacetime has no spatial curvature at all.

- This leaves the possibility that spacetime may contract or expand, and potentially differently in different directions.

- The experimental evidence so far supports the simplifying assumption that the cosmic expansion is isotropic.

Remark: It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic

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- This leaves the possibility that spacetime may contract or expand, and potentially differently in different directions.

- The experimental evidence so far supports the simplifying assumption that the cosmic expansion is isotropic.

Remark: It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic spacetimes, see, e.g., the text by Wainwright & Ellis.

- With these assumptions, let us choose these coordinates:

* Time:

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- With these assumptions, let us choose these coordinates:

  * Time:

  - Galaxies mostly move away from another according to the $\underbrace{\text{Hubble flow}}_{\text{general spacetime expansion}}$.

  - The peculiar velocity is the "small" extra random velocity that galaxies can possess relative to the general Hubble flow.

→ As the time coordinate, t, let us use the proper time, t, of a freely streaming

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- The peculiar velocity is the "small" extra random velocity that galaxies can possess relative to the general Hubble flow.

→ As the time coordinate, t, let us use the proper time, t, of a freely streaming observer who has no peculiar velocity.

(to a good approximation, you can use your wrist watch on earth)

* Space:

- It is convenient to use "comoving

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* Space :

- It is convenient to use "comoving coordinates", x_1, x_2, x_3 :

- At one time, t_0, (say today) we measure the distances (say between all galaxies) and record those distances.

- At a later time, the galaxies will have moved away from another due to the Hubble flow (we neglect peculiar velocities).

- If we let our spatial coordinate syst...

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- Any galaxy will always have the same numerical coordinates this way.

The metric: In these coordinates, $g_{\mu\nu}(x)$ must read:

Recall: $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ is "proper 4-distance".

because we use wrist watch "proper" time

$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & \\ & & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix}$$

because our coordinate system's unit of length means over time a larger and larger proper length.

The "scale factor":

- The scale factor function $a(t)$ is needed to take into account the expansion when calculating distances

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The "scale factor":

- The scale factor function $a(t)$ is needed to take into account the expansion when calculating distances.

- Example: The proper distance d between two galaxies with comoving distance $(\Delta x_1, \Delta x_2, \Delta x_3)$ at proper time t is:

$$d = \sqrt{|g_{\mu\nu}(t)\,\Delta x^\mu \Delta x^\nu|}$$

$$= a(t)\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}$$

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when calculating distances.

- **Example:** The proper distance d between two galaxies with comoving distance $(\Delta x_1, \Delta x_2, \Delta x_3)$ at proper time t is:

$$d = \sqrt{|g_{\mu\nu}(t_0)\,\Delta x^\mu \Delta x^\nu|}$$

$$= a(t)\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}$$

Note: $\Delta x_0 = t_0 - t_0 = 0$ since we are looking at the distance between the galaxies at equal time.

Dynamics of $a(t)$:

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Dynamics of $a(t)$:

- The function $a(t)$ is calculated this way:

    1. Calculate the energy momentum tensor $T_{\mu\nu}(t,\vec{x})$ contributions of at least the most important fields, say $\varphi_i(x,t)$.

    2. Solve, simultaneously:

        * The equations of motion for the fields φ_i

        * The Einstein equation for $g_{\mu\nu}$, while setting $g_{\mu\nu}(t,x) = \begin{pmatrix} 1 & & \\ & -a^2 & \\ & & -a^2 \end{pmatrix}$:

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o We can do this classically, but not quantum mechanically:

* Can quantize only φ_i, not $g_{\mu\nu}$. So: need to make quantum $T_{\mu\nu}(t,\vec{x})$ classical to use it in the classical Einstein equation.

"Semiclassical approximation" $\rightarrow$ One uses $\overline{T}_{\mu\nu}(x) := \langle\Omega|T_{\mu\nu}(t,\vec{x})|\Omega\rangle$.

Problem: Energy & Momentum are naturally nonlocal because of uncertainty principle.

o We will return to the dynamics of $a(t)$.

o **Remark:** $\dot{a}(t)$ is related to curvature between space & time.

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For now, we will assume that the expansion's scale factor function $a(t)$ is given.

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For now, we will assume that the expansion's scale factor function $a(t)$ is given.

Definition: The conformal time coordinate.

- Recall that:

$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & 0 \\ & -a^2(t) & & \\ & & -a^2(t) & \\ 0 & & & -a^2(t) \end{pmatrix}$$

- It would be convenient if $g_{\mu\nu}$ were proportional to $\eta_{\mu\nu} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$

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$$(\Delta t)^2 = a^2(t_0)(\Delta \eta)^2$$

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$$\frac{d\eta}{dt} = \frac{1}{a}$$

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and therefore $\eta(t) = \int_{t_0}^{t} \frac{1}{a(t')}\, dt'$

t_0 ← yields arbitrary integration constant

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- The variable η is called the "conformal time".

(... because it shows that the FRW spacetime is equivalent to Minkowski space up to time-dependent conformal, i.e., angle-preserving, i.e. scale-factor-only transformations)

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(... because it shows that the FRW spacetime is equivalent to Minkowski space up to time-dependent conformal, i.e., angle-preserving, i.e. scale-factor-only transformations)

- Using conformal time and comoving spatial coordinates the metric reads:

do not mix up

$$g_{\mu\nu}(\eta,\vec{x}) = a^2(\eta)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = a^2(\eta)\,\eta_{\mu\nu}$$

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- This also implies:

$$g^{\mu\nu}(\eta,\vec{x}) = a^{-2}(\eta)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = a^{-2}(\eta)\,\eta^{\mu\nu}$$

Recall: $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$, i.e., $g_{\mu\nu}$ and $g^{\mu\nu}$ are inverse to another.

- We easily obtain the integral measure needed for the action:

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We easily obtain the integral measure needed for the action:

$$\sqrt{|g|} = \sqrt{|\det(g_{\mu\nu}(\eta,\vec{x}))|} = a^{4}(\eta)$$

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The Klein Gordon field in FRW spacetimes

- Neglecting a potential $V(\phi)$ for now, we obtain the action of the "free K.G. field on the FRW background":

$$S_{KG} = \int \left(\frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) \sqrt{|g|}\, d^4x$$

$$\overset{\text{here}}{=} \int \left(\frac{1}{2} a^{-2}(\eta)\, \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) a^4\, d\eta\, d^3x$$

- Thus, the general Euler Lagrange equation

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- Thus, the general Euler Lagrange equation

$$\left(\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} g^{\mu\nu} \sqrt{|g|} \frac{\partial}{\partial x^\nu} + m^2 \right) \phi(x) = 0$$

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$$= \int \left(\frac{1}{2} a \; (\eta) \, \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) a^4 \, d\eta \, d^3x$$

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now takes this form:

$$\phi''(\eta,\vec{x}) + 2 \frac{a'(\eta)}{a(\eta)} \phi'(\eta,\vec{x}) - \Delta \phi(\eta,\vec{x}) + a^2(\eta) m^2 \phi(\eta,\vec{x}) = 0$$

Here, we use the notation: $f' := \frac{\partial f}{\partial \eta}$

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Here, we use the notation: $f' := \frac{\partial f}{\partial \eta}$

Exercise: verify.

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We notice that the equation of motion above has this general form:

$$\phi'' + \ast\,\phi' + \ast\,\phi = 0$$

a time-dependent friction-like term that is entirely new.

a term that also occurs in the usual harmonic oscillator. Notice though that it is now time-dependent.

Strategy:

We try to change from $\phi(\eta,\vec{x})$ to a new field variable, say $\chi(\eta,\vec{x})$, so that the equation of motion for χ has no "friction" term...

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Strategy:

We try to change from $\phi(\eta,\vec{x})$ to a new field variable, say $\chi(\eta,\vec{x})$, so that the equation of motion for χ has no "friction"-type term.

- This simple ansatz succeeds:

$$\chi(\eta,\vec{x}) := a(\eta)\,\phi(\eta,\vec{x})$$

Indeed:

* we have: $\phi' = \frac{\partial}{\partial\eta}\frac{1}{a}\chi = -\frac{a'}{a^2}\chi + \frac{1}{a}\chi'$

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and: $\phi_{,i} = \frac{\partial}{\partial x^i}\frac{1}{a(\eta)}\chi(\eta,\vec{x}) = \frac{1}{a}\chi_{,i}$ for $i=1,2,3$

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and: $\phi_{,i} = \frac{\partial}{\partial x^i} \frac{1}{a(\eta)} \chi(\eta, \vec{x}) = \frac{1}{a} \chi_{,i}$ for $i = 1, 2, 3$

* Using these, the action in terms of χ reads:

$$S_{KG} = \int \frac{1}{2} \left(\chi'^2 - \sum_{i=1}^{3} \chi_{,i}^2 - \underbrace{\left(m^2 a^2 - \frac{a''}{a} \right)}_{} \chi^2 \right) d\eta \, d^3x$$

Note that this term is
like a time-dependent

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* Using these, the action in terms of χ reads:

$$S'_{KG} = \int \frac{1}{2}\left(\chi'^2 - \sum_{i=1}^{3} \chi_{,i}^2 - \underbrace{\left(m^2a^2 - \frac{a''}{a}\right)}_{\text{Note that this term is like a time-dependent mass term } m_{eff}^2(\eta)} \chi^2\right) d\eta\, d^3x$$

Exercise: verify

Equation of motion:

* Do

$$\frac{\delta S'}{\delta \phi(\eta,\vec{x})} = 0 \quad \text{and} \quad \frac{\delta S'}{\delta \chi(\eta,\vec{x})} = 0$$

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Equation of motion:

* Do

$$\frac{\delta S'}{\delta \phi(\eta,\vec{x})} = 0 \quad \text{and} \quad \frac{\delta S'}{\delta \chi(\eta,\vec{x})} = 0$$

yield equivalent equations of motion?

* Yes, because:

$$0 = \frac{\delta S'}{\delta \phi} = \frac{\delta S}{\delta \chi}\frac{\delta \chi}{\delta \phi}$$

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* Yes, because:

$$0 = \frac{\delta S'}{\delta \phi} = \frac{\delta S}{\delta \chi}\frac{\delta \chi}{\delta \phi}$$

if $\delta S/\delta \chi$ vanishes then also $\delta S/\delta \phi$ vanishes.

* Thus, we may calculate the equation of motion directly in terms of χ from $S[\chi]$, to obtain:

Exercise: verify!

$$\chi'' - \Delta \chi + \left(m^2 a^2 - \frac{a''}{a}\right)\chi = 0 \qquad (EoM1)$$

Remark:

We could have obtained this equation of motion directly from that of ϕ by change

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Preparation for quantization:

* We need the canonically conjugate field

$$\pi^{(x)}(\eta, \vec{x})$$

to the field $x(\eta, \vec{x})$, i.e., the Legendre transform of x':

* To this end, we consider the Lagrangian:

$$L = \int \frac{1}{2}\left(x'^2 - \sum_{i=1}^{3} x_{,i}{}^2 - \left(m^2 a^2 - \frac{a''}{a}\right) x\right) d^3x$$

* Thus, the Legendre transformed variable reads:

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* Thus, the Legendre transformed variable reads:

$$\pi^{(x)}(\eta,\vec{x}) := \frac{\delta L}{\delta x'(\eta,\vec{x})} = x'(\eta,\vec{x}) \qquad \text{(EoM 2)}$$

* Which is the field that is conjugate to ϕ ?

$$S_{K.G.} = \int \left(\frac{1}{2} a^{-2}(\eta)\, \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) a^4 \, d\eta \, d^3x$$

$\Rightarrow$ The field $\pi^{(\phi)}$ which is conjugate to ϕ reads:

$$\pi^{(\phi)} := \frac{\delta L}{\delta \phi'} = a^2 \phi'$$

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$\Rightarrow$ The field $\pi^{(\phi)}$ which is conjugate to ϕ reads:

$$\pi^{(\phi)} := \frac{\delta L}{\delta \phi'} = a^2 \phi'$$

* Compare:

$$\begin{aligned} \pi^{(x)} &= x' \\ &= (a\phi)' \\ &= a\phi' + a'\phi \\ &= \frac{1}{a}\pi^{(\phi)} + a'\phi \end{aligned}$$

, i.e., $\pi^{(\phi)}, \pi^{(x)}$ are different!

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$$[\hat{\phi}(\eta,\vec{x}), \hat{\pi}^{(\phi)}(\eta,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')$$

$$[\hat{\phi}(\eta,\vec{x}), \hat{\phi}(\eta,\vec{x}')] = 0$$

$$[\hat{\pi}^{(\phi)}(\eta,\vec{x}), \hat{\pi}^{(\phi)}(\eta,\vec{x}')] = 0$$

□ Proposition:

In terms of the fields $\hat{\chi} := a\hat{\phi}$, $\hat{\pi}^{(\chi)} := \hat{\chi}'$, these commutation relations become:

$$[\hat{\chi}(\eta,\vec{x}), \hat{\pi}^{(\chi)}(\eta,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')$$

$$[\hat{\chi}(\eta,\vec{x}), \hat{\chi}(\eta,\vec{x}')] = 0$$

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□ **Proof** Only the first CCR is nontrivial to check:

$$[\hat{\chi}(\eta,\vec{x}), \hat{\pi}^{(\chi)}(\eta,\vec{x}')] = [a(\eta)\hat{\phi}(\eta,\vec{x}), \frac{1}{a(\eta)}\hat{\pi}^{(\phi)}(\eta,\vec{x}')+a'(\eta)\hat{\phi}(\eta,\vec{x}')]$$

$$= [\hat{\phi}(\eta,\vec{x}), \hat{\pi}^{(\phi)}(\eta,\vec{x}')]$$

$$= i\delta^3(\vec{x}-\vec{x}')$$

□ Thus, the change from ϕ to χ is fairly trivial.

Notice, however:

$$L \xrightarrow[\text{L.T.}]{\phi' \text{ replaced by } \pi^{\phi}} H^{(\phi)} := \left\{ \int \phi' \pi^{(\phi)} d^3x - L \right\}$$

L.T. χ' replaced

they have no reason to be the same

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- Thus, the change from ϕ to x is fairly trivial.

Notice, however:

$$L \xrightarrow[\phi' \text{ replaced by } \pi^{\phi}]{\text{L.T.}} H^{(\phi)} := \int \phi' \pi^{(\phi)} d^3x - L \Big\}$$

$$L \xrightarrow[x' \text{ replaced by } \pi^{x}]{\text{L.T.}} H^{(\pi)} := \int x' \pi^{(x)} d^3x - L \Big\}$$

they have no reason to be the same!

- **Question:**

How can both be valid generators of time evolution,

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Question:

How can both be valid generators of time evolution,

i.e., how can we have:

$$i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(\phi)}] \quad \text{and} \quad i\,\hat{x}' = [\hat{x}, \hat{H}^{(x)}]$$

and yet $\hat{H}^{(\phi)} \neq \hat{H}^{(x)}$?

- Should there not be one Hamiltonian for all variables?

- **Answer:** Yes, and it is, of course $\hat{H}^{(\phi)}$.

This extra term is there if the variable $\hat{Q}$ has also explicit time-
dependence, e.g. $\hat{Q} = \cos(at+bt^2)\hat{a} + c\hat{n}$ or here: $\hat{x} = \frac{1}{[illegible]}\hat{\phi}$.

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This extra term is there if the variable $\hat{Q}$ has also explicit time-dependence, e.g., $\hat{Q} = \cos(at+bt^2)\hat{q} + c\hat{p}$, or here: $\hat{x} = \frac{1}{a}\hat{\phi}$.

Recall that in QM: $i\hat{Q}' = [\hat{Q}, \hat{H}] + i\frac{\partial}{\partial t}\hat{Q}$

- Explicitly:

  * From $\hat{x} = a\hat{\phi}$ and $i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(\phi)}]$ we obtain:

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* From $\hat{x} = a\hat{\phi}$ and $i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(\phi)}]$ we obtain:

$$i\left(\frac{1}{a}\hat{x}\right)' = \frac{1}{a}[\hat{x}, \hat{H}^{(\phi)}]$$

$$\Rightarrow \quad i\frac{1}{a}\hat{x}' - i\frac{a'}{a^2}\hat{x} = \frac{1}{a}[\hat{x}, \hat{H}^{(\phi)}]$$

$$\Rightarrow \quad i\hat{x}' = [\hat{x}, \hat{H}^{(\phi)}] + i\frac{a'}{a}\hat{x}$$

* But we also have:

$$i\hat{x}' = [\hat{x}, \hat{H}^{(x)}]$$

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* From $x = a\varphi$ and $i\varphi' = [\varphi, H^{(\varphi)}]$ we obtain:

$$i\left(\frac{1}{a}\hat{x}\right)' = \frac{1}{a}\left[\hat{x}, \hat{H}^{(\varphi)}\right]$$

$$\Rightarrow \quad i\frac{1}{a}\hat{x}' - i\frac{a'}{a^2}\hat{x} = \frac{1}{a}\left[\hat{x}, \hat{H}^{(\varphi)}\right]$$

$$\Rightarrow \quad i\hat{x}' = \left[\hat{x}, \hat{H}^{(\varphi)}\right] + i\frac{a'}{a}\hat{x}$$

* But we also have:

$$i\hat{x}' = \left[\hat{x}, \hat{H}^{(x)}\right]$$

$\Rightarrow$ We must have: $\quad \hat{H}^{(x)} \neq \hat{H}^{(\varphi)}$

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Since there are multiple Hamiltonians, which, if anyone, is the energy?

- One usually defines the energy as the generator of time evolution. We saw that in the presence of gravity this is ambiguous: one can define many different Hamiltonians for the same theory (same action).

- Therefore, with Einstein, we define the energy (density) not as the generator of time evolution but as a generator of curvature:

- **Recall:** The Einstein equation

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- Therefore, with Einstein, we define the energy (density) not as the generator of time evolution but as a generator of curvature:

- **Recall:** The Einstein equation

$$\underbrace{R_{\mu\nu}(x) - \tfrac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x)}_{\text{curvature}} = \underbrace{8\pi G\, T_{\mu\nu}(x)}_{\text{"energy momentum"}}$$

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- **Recall:** The K.G. field's energy-momentum tensor

$$T_{\mu\nu}(\eta,\vec{x}) = \frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\tfrac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}m^2\phi^2\right]$$

- Consider $T_{00}(\eta,\vec{x})$, which is called the "energy density":

Note: In differential geometry, there is also another use of the term "density": For every tensor, say $Q_{\mu\nu}$, there is a so-called "tensor density" $\tilde{Q}_{\mu\nu}$, defined as $\tilde{Q}_{\mu\nu} := Q_{\mu\nu}\sqrt{g}$, which absorbs the obligatory measure factor in integrations.

$$T_{00}(\eta,\vec{x}) = a^{-4}\frac{1}{2}\pi^{(\phi)\,2} + \frac{1}{2}\sum_{i=1}^{3}\phi_{,i}^{\;2} + \frac{a^2}{2}m^2\phi^2 \qquad (T)$$

- **Exercises:**

a) Verify (T).

b) Calculate $H^{(\phi)}$. Notice that $H^{(\phi)}$ is not a scalar.

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Exercises:

a) Verify (T).

b) Calculate $H^{(\phi)}$.

Notice that $H^{(\phi)}$ is not a scalar.

c) Show that $H^{(\phi)}(\eta) = \int_{\mathbb{R}^3} T^0_{\;0}(\eta,\vec{x})\sqrt{g}\;d^3x$.

d) Calculate $H^{(\chi)}(\eta)$.