Title: Introduction to Quantum Field Theory for Cosmology - Lecture 12

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Abstract:

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Recall strategy:

SR, 1st Q Hamiltonian formalism

step 1
Legendre transform
(equivalence)

SR, 1st Q. Lagrangian formalism

step 2 allow curvature

GR, 1st Q Lagrangian formalism

we are here

step 3
Legendre transform
(equivalence)

GR, 1st Q Hamiltonian formalism

step 4

GR, 2nd Q Hamiltonian formalism

Dyson Schwinger eqns are same
(equivalence)

GR, 2nd Q Lagrangian formalism (Path integral of QFT)

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Recall:

General relativistic covariant Klein Gordon theory in the Lagrangian formulation (neglecting the potential):

* The Klein Gordon eqn:

$$\frac{1}{\sqrt{|g(x)|}} \frac{\partial}{\partial x^\nu} \left(g^{\mu\nu}(x) \sqrt{|g(x)|}\, \phi_{,\nu}(x) \right) + m^2 \phi(x) = 0 \qquad (KG)$$

arises via $\frac{\delta S_{KG}}{\delta \phi} = 0$ from the action:

$$S = \int \left(\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} m^2 \phi^2 \right) \sqrt{|g|}\, d^4$$

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the Lagrangian formulation (neglecting the potential):

* **The Klein Gordon eqn:**

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arises via $\frac{\delta S_{KG}}{\delta \phi} = 0$ from **the action**:

$$S_{KG} = \frac{1}{2} \int_{\mathbb{R}^4} \left(g^{\nu\nu}(x)\, \phi_{,\nu}(x)\, \phi_{,\nu}(x) - m^2 \phi^2(x) \right) \sqrt{|g|}\, d^4x$$

↖ we assume that the coordinate system is such, for simplicity.

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$$S_{KG} = \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}(x)\,\phi_{,\mu}(x)\,\phi_{,\nu}(x) - m^2\phi^2(x)\right)\sqrt{|g|}\,d^4x$$

↖ we assume that the coordinate system is such, for simplicity.

* We read off the Lagrangian:

$$L_{KG}(t) = \frac{1}{2}\int_{\mathbb{R}^3}\left(g^{\mu\nu}(x,t)\,\phi_{,\mu}(x,t)\,\phi_{,\nu}(x,t) - m^2\phi^2(x,t)\right)\sqrt{|g|}\,d^3x$$

Step 3: Legendre transform back to the Hamiltonian form

* The transform:

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$$L_{\mathrm{KG}}(t) = \frac{1}{2}\int_{\mathbb{R}^3}\left(g^{\mu\nu}(x,t)\,\phi_{,\mu}(x,t)\,\phi_{,\nu}(x,t) - m^2\phi^2(x,t)\right)\sqrt{|g|}\,d^3x$$

Step 3: Legendre transform back to the Hamiltonian form

* The transform:

$$H(\phi,\pi,t) \xleftarrow[\text{Legendre transform}]{\pi(x,t) := \frac{\delta L}{\delta \phi_{,0}(x,t)}} L(\phi,\phi_{,0},t)$$

* Thus, the canonically conjugate field $\pi(x,t)$ reads:

$$\pi(\quad) = \frac{\delta(\quad)}{\delta(\quad)} = \quad$$

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$$L_{KG}(t) = \frac{1}{2}\int_{\mathbb{R}^3}\left(g^{\mu\nu}(x,t)\,\phi_{,\mu}(x,t)\,\phi_{,\nu}(x,t) - m^2\phi^2(x,t)\right)\sqrt{|g|}\,d^3x$$

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$$\pi(x,t) = \frac{\delta L}{\delta \phi_{,0}(x,t)} = \sqrt{|g(x,t)|}\; g^{0\mu}(x,t)\,\phi_{,\mu}(x,t)$$

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$$\pi(x,t) = \frac{\delta L}{\delta \phi_{,0}(x,t)} = \sqrt{|g(x,t)|}\; g^{0\nu}(x,t)\, \phi_{,\nu}(x,t)$$

* Explicitly:

$$\pi(x,t) = \sqrt{|g|}\, g^{00} \phi_{,0} + \sum_{i=1}^{3} \sqrt{|g|}\, g^{0i} \phi_{,i}$$

* Thus, we can also express $\phi_{,0}(x,t)$ in terms of $\phi(x,t)$ and $\pi(x,t)$ (as will be necessary after the Legendre transform):

(of course, $g^{\mu\nu}$ depends on x,t too)

$$\phi_{,0}(x,t) = \frac{\pi(x,t)}{g^{00}\sqrt{|g|}} - \sum_{i=1}^{3} \frac{g^{0i}}{g^{00}} \phi_{,i}(x,t) \qquad (V)$$

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$i=1$

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* The Hamiltonian:

$$H = \int_{\mathbb{R}^3} \pi(x,t)\,\phi_{,0}(x,t)\,d^3x - \underbrace{\frac{1}{2}\int_{\mathbb{R}^3} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)\sqrt{|g|}\,d^3x}_{= L(\phi,\phi_{,0}(\phi,\pi),t)}$$

Why don't we need a factor of $\sqrt{|g|}$ for covariance here? Because π has it built in!

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* The equations of motion:

We know from the general properties of the Legendre transform that the equations of motion now take the form:

$$\frac{d}{dt}\phi(x,t) = \frac{\delta H(\phi,\pi,t)}{\delta \pi(x,t)}\,, \quad \frac{d}{dt}\pi(x,t) = -\frac{\delta H(\phi,\pi,t)}{\delta \phi(x,t)}$$

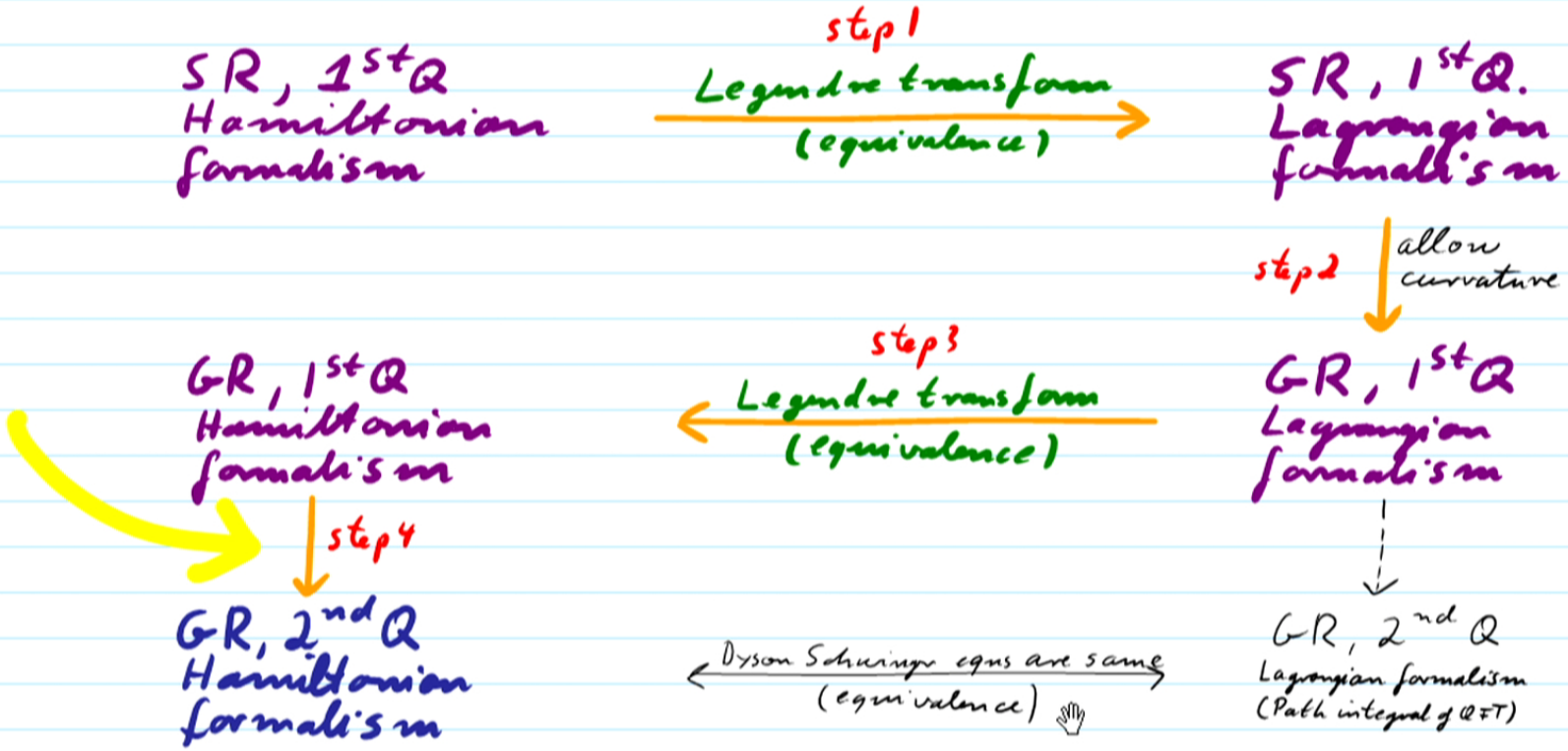
* Exercise: Verify that these eqns are equivalent to (KG).

We are now ready to 2nd quantize:

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Solving the quantized theory is to solve:

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Solving the quantized theory is to solve:

1.) Commutation relations:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x')$$

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$$

$$[\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0$$

} (CCRs)

2.) Hermiticity:

$$\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t), \qquad \hat{\pi}^\dagger(x,t) = \hat{\pi}(x,t) \qquad \text{(HC)}$$

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(CCRs)

2.) Hermiticity:

$$\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t), \qquad \hat{\pi}^\dagger(x,t) = \hat{\pi}(x,t) \qquad \text{(HC)}$$

which is needed so that the expectation values are real.

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3.) Equations of motion:

In the Heisenberg picture, they are formally unchanged:

$$\frac{d}{dt}\hat{f}(\phi,\pi) = \frac{1}{i\hbar}[\hat{f}, \hat{H}] \qquad \text{for } \hat{f} = \hat{\phi},\ \hat{f} = \hat{\pi},\ \text{etc}$$

Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad \text{(E.o.M.)}$$

and:

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and:

$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \qquad \text{(EOM2)}$$

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How to solve the CCR, HC and EoM equations?

- **Strategy:** Try an ansatz which
  - * ensures hermiticity, HC, by construction,
  - * separates the CCR and EoM problems:

- **Ansatz:**

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger$$

$$\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)$$

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* The 1st equation of motion:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad \text{(Eom1)}$$

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (Eom1).

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This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

the a_k are constant operators while the functions $u_k(x,t)$ are assumed to solve (E.o.M 1).

* The 2nd equation of motion:

This equation hold by the way we define $\hat{\pi}(x,t)$.

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Checking the CCRs:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x')$$

- Express $\hat{\phi}$ in terms of the ansatz:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\,a_k + u_k^*(x,t)\,a_k^\dagger$$

- Express $\hat{\pi}$ in terms of the ansatz:

$$\hat{\pi}(x,t) := \sqrt{|g|}\,g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \quad \Rightarrow$$

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$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta(x-x')$$

- Express $\hat{\phi}$ in terms of the ansatz:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\,a_k + u_k^*(x,t)\,a_k^\dagger$$

- Express $\hat{\pi}$ in terms of the ansatz:

$$\hat{\pi}(x,t) := \sqrt{|g|}\,g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \quad \Rightarrow$$

$$\hat{\pi}(x,t) = \sqrt{|g|}\,g^{0\nu}\sum_k\left[\left(\frac{\partial}{\partial x^\nu}u_k(x,t)\right)a_k + \left(\frac{\partial}{\partial x^\nu}u_k^*(x,t)\right)a_k^\dagger\right]$$

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Now check the CCR:

$$\left[\hat{\phi}(x',t),\ \hat{\pi}(x,t)\right]$$

$$= \left[\sum_k u_k(x',t)\,a_k + u_k^*(x',t)\,a_k^\dagger,\ \sqrt{|g|}\,g^{0\nu}\sum_{k'}\left(\left(\frac{\partial}{\partial x^\nu}u_{k'}(x,t)\right)a_{k'} + \left(\frac{\partial}{\partial x^\nu}u_{k'}^*(x,t)\right)a_{k'}^\dagger\right)\right]$$

$$= \sqrt{|g|}\,g^{0\nu}(x,t)\sum_{k,k'}\left(u_k(x',t)\frac{\partial}{\partial x^\nu}u_{k'}^*(x,t) - u_{k'}^*(x',t)\frac{\partial}{\partial x^\nu}u_k(x,t)\right)\delta_{k,k'}$$

$$= \sqrt{|g|}\,g^{0\nu}\sum_k\left(u_k(x',t)\frac{\partial}{\partial x'^\nu}u_k^*(x,t) - u_k^*(x',t)\frac{\partial}{\partial x'^\nu}u_k(x,t)\right)$$

$$\overset{!}{=} i\,\delta^3(\vec{x}-\vec{x}')$$

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$$\left[\hat{\phi}(x',t),\ \hat{\pi}(x,t)\right]$$

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$$= \sqrt{|g|}\,g^{0\nu}\sum_k\left(u_k(x',t)\frac{\partial}{\partial x'^\nu}u_k^*(x,t) - u_k^*(x',t)\frac{\partial}{\partial x'^\nu}u_k(x,t)\right)$$

$$\overset{!}{=} i\,\delta^3(\vec{x}-\vec{x}')$$

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Conclusion so far:

Our ansatz

$$\hat{\phi}(x,t) := \sum u_n(x,t)\, a_n + u_n^*(x,t)\, a_n^\dagger$$

solves the QFT, i.e., HC, EoM and CCR

if we can find a set of number-valued solutions

$$\{u_n(x,t)\}$$

of the Klein Gordon equation that obeys:

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if we can find a set of number-valued solutions

$$\{u_n(x,t)\}$$

of the Klein Gordon equation that obeys:

(RI)
$$\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\,\delta^3(x-x')$$

When do such $\{u_k(x,t)\}$ exist? I.e., when does the ansatz succee

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When do such $\{u_n(x,t)\}$ exist? I.e., when does the ansatz succeed?

Proposition:

- Assume spacetime is "globally hyperbolic", i.e., that it is topologically of the form:

$$\mathbb{R} \times M$$

↑ any 3-dim differentiable manifold

- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on so called "Cauchy surfaces" determine the solution everywhere.

- Then, such a set of functions $\{u_n\}$ can be shown to exi...

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- In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on so called "Cauchy surfaces" determine the solution everywhere.

- Then, such a set of functions $\{u_n\}$ can be shown to exist.

- In fact there are many such sets $\{\tilde{u}_n\}$!

(And we will have to address which set to choose to solve the theory.)

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Proof:

- Strategy:
  - Study space of solutions to KG eqn.
  - Define on it: a hermitian form (s.adjoint operator).
  - Diagonalize the s.adj. operator to obtain Hilbert basis.
  - Show that (RI) is a result of the identity.

- Consider the vector space, V, of all real-valued solutions of the Klein Gordon equations.

- We define a bi-linear form $(\ ,\ )$ on V. For all $f, g \in V$:

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- Consider the vector space, V, of all real-valued solutions of the Klein Gordon equations.

- We define a bi-linear form $(\ ,\ )$ on V. For all $f, g \in V$:

$$(f,g) := \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu} (f \partial_\nu g - g \partial_\nu f)$$

↑ any spacelike hypersurface
i.e. set of points of equal time.

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