

Title: Introduction to Quantum Field Theory for Cosmology - Lecture 15

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Abstract:

# QFT for Cosmology, Achim Kempf, Winter 14, Lecture 15

Note Title

## Solving the quantized K.G. eqns. on FRW spacetimes

□ Recall:

1.) We obtain the solution  $\hat{\phi}(x,t)$  through the ansatz

$$\hat{\phi}(x,t) = \sum_k u_k(x,t) a_k + u_k^*(x,t) a_k^*$$

\* if we use operators  $a_k$  obeying  $[a_k, a_{k'}^*] = \delta^3(k-k')$  and

\* if we find classical solutions  $\{u_k(x,t)\}$  of the K.G.

□ Recall:

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\* if we use operators  $a_k$  obeying  $[a_k, a_{k'}^*] = \delta^3(k-k')$  and

\* if we find classical solutions  $\{u_k(x,t)\}$  of the K.G. eqn., called mode functions, which obey:

$$\sqrt{g} \sum_k \left( u_k(x,t) \frac{\partial}{\partial x} u_k^*(x,t) - u_k^*(x,t) \frac{\partial}{\partial x} u_k(x,t) \right) = i \delta^3(\vec{x} - \vec{x}') \quad (5)$$

2.) Then, we can use the  $\{a_k\}$  to build a convenient basis in the Hilbert space:

□ Namely:  $|0\rangle$  is the vector obeying  $a_k|0\rangle = 0$

□ The other basis vectors are:

$$a_k^+|0\rangle, \dots \frac{1}{\sqrt{n!}}(a_n^+)^n|0\rangle, \dots a_k^+a_j^+, |0\rangle, \dots$$

$$\frac{1}{\sqrt{n_1!}} \dots \frac{1}{\sqrt{n_m!}} (a_{n_1}^+)^{n_1} \dots (a_{n_m}^+)^{n_m}|0\rangle, \dots \text{etc.}$$

3.) Choosing a different set of classical solutions  $\{\tilde{u}_k(x,t)\}$  which obey (G) yields the same  $\hat{\phi}(x,t)$ , namely,

$$\hat{\phi}(x,t) = \sum \tilde{u}_k(x,t) \tilde{a}_k + \tilde{u}_k^*(x,t) \tilde{a}_k^*$$

◻ Namely:  $|0\rangle$  is the vector obeying  $a_k|0\rangle = 0$

◻ The other basis vectors are:

$$a_k^+|0\rangle, \dots \frac{1}{\sqrt{n!}}(a_n^+)^n|0\rangle, \dots a_n^+a_1^+|0\rangle, \dots$$

$$\frac{1}{\sqrt{n_1!}} \dots \frac{1}{\sqrt{n_m!}} (a_{n_1}^+)^{n_1} \dots (a_{n_m}^+)^{n_m}|0\rangle, \dots \text{etc.}$$

3.) Choosing a different set of classical solutions  $\{\tilde{u}_k(x,t)\}$   
which obey (C) yields the same  $\hat{\phi}(x,t)$ , namely,

$$\hat{\phi}(x,t) = \sum_k \tilde{u}_k(x,t) \tilde{a}_k + \tilde{u}_k^*(x,t) \tilde{a}_k^*$$

but the basis of vectors  $|\tilde{0}\rangle, \tilde{a}_n^+|\tilde{0}\rangle, \tilde{a}_n^+\tilde{a}_m^+|\tilde{0}\rangle \dots$   
is a different basis.

## Application to FRW spacetime

- For convenience (namely, to avoid a "friction"-type term) we aim to solve not for  $\hat{\phi}(x, t)$  directly, but instead for:

$$\hat{x}(\gamma, x) := a(\gamma) \hat{\phi}(\gamma, x)$$

- In terms of  $\hat{x}(\gamma, x)$  the quantum K.G. eqn. reads:

$$\hat{x}''(\gamma, x) - \Delta \hat{x}(\gamma, x) + \left( m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)} \right) \hat{x}(\gamma, x) = 0$$

- Note: This is a time-dependent eqn.

aim to solve not for  $\phi(x,t)$  directly, but instead for:

$$\hat{\chi}(\gamma, x) := a(\gamma) \hat{\phi}(\gamma, x)$$

□ In terms of  $\hat{\chi}(\gamma, x)$  the quantum K.G. eqn. reads:

$$\hat{\chi}''(\gamma, x) - \Delta \hat{\chi}(\gamma, x) + \left( m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)} \right) \hat{\chi}(\gamma, x) = 0$$

□ Note: This is a partial differential equation because both time and space derivatives occur.



□ Observation: The derivatives  $\frac{\partial}{\partial x_i}$  become multiplication operators  $ik_i$  under spatial Fourier transform.

□ Idea: Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more manageable set of ordinary DEs.

□ Define:  $\hat{x}_k(\gamma) := \int \frac{1}{(2\pi)^{3/2}} \hat{x}(\gamma, x) e^{-ikx} d^3x$

i.e.:  $\hat{x}(\gamma, x) = \int \frac{1}{(2\pi)^{3/2}} \hat{x}_k(\gamma) e^{ikx} d^3k$



□ Analogously:

into a more manageable set of ordinary DEs.

□ Define:  $\hat{x}_k(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{x}(\eta, x) e^{-ikx} d^3x$

i.e.:  $\hat{x}(\eta, x) = \int \frac{1}{(2\pi)^{3/2}} \hat{x}_k(\eta) e^{ikx} d^3k$

□ Analogously:

$$\hat{\Pi}_x^{(x)}(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{\Pi}^{(x)}(\eta, x) e^{-ikx} d^3x$$



□ Thus, in terms of  $\hat{x}_k(\eta)$ , the K.G. eqn. reads:

□ Thus, in terms of  $\hat{x}_k(\gamma)$ , the K.G. eqn. reads:

$$\hat{x}_k''(\gamma) + \left( k^2 + m^2 \alpha^2(\gamma) - \frac{\alpha''(\gamma)}{\alpha(\gamma)} \right) \hat{x}_k(\gamma) = 0 \quad (\text{EqM})$$

$\Rightarrow$  for each Fourier mode  $k$  the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{x}_k''(\gamma) + \omega_k^2(\gamma) \hat{x}_k(\gamma) = 0$$

$$\hat{x}_k''(\gamma) + \left( k^2 + m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)} \right) \hat{x}_k(\gamma) = 0 \quad (\text{EoM})$$

$\Rightarrow$  for each Fourier mode  $k$  the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{x}_k''(\gamma) + \omega_k^2(\gamma) \hat{x}_k(\gamma) = 0$$

with:  $\omega_k(\gamma) := \sqrt{k^2 + m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)}}$

## □ Notice:

The frequency  $\omega_n(\gamma)$  may become imaginary, namely if  $a''(\gamma)$  is large enough, i.e., if the expansion is rapid enough. Note that the discriminant also depends on  $k$ , i.e., some modes may have imaginary frequencies while others don't.

## □ Exercise:

\* Show that  $\hat{x}_k^+(\gamma) = \hat{x}_{-k}(\gamma)$ ,  $\hat{\pi}_k^{(cc)}(\gamma) = \hat{\pi}_{-k}^{(cc)}(\gamma)$  (HC)

\* Show that

$$[\hat{x}_k(\gamma), \hat{\pi}_{k'}(\gamma)] = i\delta^3(k+k') \quad (\text{CC}^{\text{cl}})$$

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\* Show that  $\hat{x}_k^+(\gamma) = \hat{x}_{-k}(\gamma)$ ,  $\hat{\pi}_k^{(x)}(\gamma) = \hat{\pi}_{-k}^{(x)}(\gamma)$  (HC)

\* Show that

$$[\hat{x}_k(\gamma), \hat{\pi}_{k'}(\gamma)] = i \delta^3(k+k') \quad (\text{CCR})$$

i.e.  $[\hat{x}_k(\gamma), \hat{\pi}_{k'}^+(\gamma)] = i \delta^3(k-k')$

□ In order to solve EoM, HG, CCR for  $x_k(\gamma)$ , we make this ansatz:

convenient later

□ In order to solve EoM, HC, CCR for  $x_k(\eta)$ , we make this ansatz:

$$\hat{x}_k(\eta) := \frac{1}{\sqrt{2}} \left( v_k^*(\eta) a_k + v_{-k}(\eta) a_k^* \right) \quad (\text{A})$$

convenient later

□ Exercise: What are the mode functions  $u_k(\eta, x)$  in terms of the functions  $v_k(\eta)$ ?

□ Proposition: The ansatz (A) solves

1.) the hermiticity condition (HC) by construction.

we make this ansatz:

$$\vec{x}_k(\eta) := \frac{1}{\sqrt{2}} \left( v_k^*(\eta) \alpha_k + v_{-k}(\eta) \alpha_k^+ \right) \quad (A)$$

↙ convenient later

□ Exercise: What are the mode functions  $u_k(\eta, x)$  in terms of the functions  $v_k(\eta)$ ?

□ Proposition: The ansatz (A) solves

- 1.) the hermiticity condition (HC) by construction.
- 2.) the (EoM), if the  $v_k(\eta)$  each solve (EoM) as (complex!) number-valued functions:

2) the (EoM), if the  $v_k(\gamma)$  each solve (EoM)  
as (complex!) number-valued functions:

*Note: The equation depends only on  $|k|$ , not on the direction of  $k$ . Thus if  $v_k(\gamma)$  is a solution for one  $k$  then it is solution for all  $k'$  with  $|k'| = |k|$ .  $\Rightarrow$  We can and will choose  $v_k(\gamma) = V_k(\gamma)$*

$$V_k''(\gamma) + \left( k^2 + m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)} \right) V_k(\gamma) = 0 \quad (n)$$

3.) the commutation relations (CCR) if  
the  $V_k$  are chosen such that they also obey:

$$V_k'(\gamma) V_k^*(\gamma) - V_k(\gamma) V_k^*(\gamma)' = 2i \quad (w)$$

□ Exercise:

3.) the commutation relations (CCR) if  
the  $v_k$  are chosen such that they also obey:

$$v_k'(\gamma) v_k^*(\gamma) - v_k(\gamma) v_k^*(\gamma)' = 2i \quad (W)$$

□ Exercise:

a) Prove the proposition.

b.) Assume that  $v_k(\gamma)$  is any solution of (EoM).  
Show that if (W) holds at one time  
then it holds at all time.

(Note: The LHS of (W) is the "Wronskian" of the ODE)

## Conclusion:

In order to obtain the solution  $\hat{\phi}(y, x)$ , we do:

- A) Find for each mode  $k \in \mathbb{R}^3$  a solution  $v_k(y)$  to (M), i.e., a solution to the classical harmonic oscillator with time-dependent frequency.
- B) Make sure  $v_k(y)$  obeys (W), if need be by multiplying with a constant. (Recall exercise b))
- C) Build a basis in the Hilbert space:

In order to obtain the solution  $\hat{\phi}(y, x)$ , we do:

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- B) Make sure  $v_k(y)$  always (W), if need be by multiplying with a constant. (Recall exercise b))
- C) Build a basis in the Hilbert space:  
 $a_k |0\rangle = 0$ ,  $a_k^\dagger |0\rangle$ ,  $a_k^\dagger a_n^\dagger |0\rangle$ , etc ...

## Choice of mode solutions $\{V_k(z)\}$

□ For each choice, say  $\{V_k(z)\}_{k \in \mathbb{N}_0}$  or  $\{\tilde{V}_k(z)\}_{k \in \mathbb{N}_0}$ , we obtain the same  $\hat{\phi}(x, t)$  but the bases

$$|0\rangle, a_k^+ |0\rangle, a_k^+ a_k^+, |0\rangle, \dots$$

and

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_k^+, |\tilde{0}\rangle, \dots$$

will of course be different.

□ We will often find it convenient to use the basis  $|0\rangle, a_k^+ |0\rangle, a_k^+ a_k^+, |0\rangle, \dots$  that comes with one set of mode functions  $\{V_k(z)\}$

□ For each choice, say  $\{v_k(y)\}_{k \in \mathbb{N}}$  or  $\{\tilde{v}_k(y)\}_{k \in \mathbb{N}}$ , we obtain the same  $\hat{\phi}(x, t)$  but the bases

$$|0\rangle, a_k^+ |0\rangle, a_k^+ a_k^+ |0\rangle, \dots$$

and

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots$$

will of course be different.

□ We will often find it convenient to use the basis  $|0\rangle, a_k^+ |0\rangle, a_k^+ a_k^+ |0\rangle, \dots$  that comes with one set of mode functions  $\{v_k(y)\}$  at one time (say initially) and then the basis  $|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots$  of some  $\{\tilde{v}_k(y)\}$  later.

Q Why?

We may choose a set  $\{v_x\}$  whose vector  $|0\rangle$  happens to be the vacuum state at one time and we may choose another set  $\{\tilde{v}_x\}$  whose vector  $|\tilde{0}\rangle$  happens to be the vacuum state at another time.

Recall: In the Heisenberg picture, the system's state vector is always the same Hilbert space vector. Therefore, since things do change, of course, the meaning of each Hilbert space vector

□ How many possible choices of

$$\left\{ v_k(\gamma) \right\}_{k \in \mathbb{N}^3}, \left\{ \tilde{v}_k(\gamma) \right\}_{k \in \mathbb{N}^3}, \left\{ \hat{v}_k(\gamma) \right\}_{k \in \mathbb{N}^3}, \dots$$

do exist?

□ We can consider each mode,  $k$ , separately:

□ The solution space of (M), for fixed  $k$ ,

$$v_k''(\gamma) + \left( k^2 + m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)} \right) v_k(\gamma) = 0$$

has of course 2 complex dimensions.

— Note: Every solution obeying (M) must be complex-valued. Why?

□ If  $v_k$  is a complex-valued solution, then



of  $v_k, v_k^*$ , i.e., there must exist  $\alpha, \beta \in \mathbb{C}$ , so that:

$$\tilde{v}_k(\gamma) = \alpha_k v_k(\gamma) + \beta_k v_k^*(\gamma)$$

□ The actual dimensionality is 3 !

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions

$v_k, \tilde{v}_k$ , etc must also obey (W), i.e.:

$$v_k'(\gamma) v_k^*(\gamma) - v_k(\gamma) v_k^*(\gamma)' = 2i \quad (W)$$

(i.e.  $\text{Im}(v'v^*) = 1$ , which is only one real equation)

□ If  $v_k$  is a complex-valued solution, then

$v_k$  and  $v_k^*$

form a basis in the solution space.



Every solution,  $\tilde{v}_k$ , is a linear combination  
of  $v_k, v_k^*$ , i.e., there must exist  $\alpha, \beta \in \mathbb{C}$ , so  
that:

$$\tilde{v}_k(\gamma) = \alpha_k v_k(\gamma) + \beta_k v_k^*(\gamma)$$

□ The actual dimensionality is 3 !

□ Proposition:

Assume  $v_k$  always (W). Then,  $\tilde{v}_k$  defined through

$$\tilde{v}_k(\gamma) = \alpha_k v_k(\gamma) + \beta_k v_k^*(\gamma) \quad (\text{B})$$

also always (W), iff the coefficients  $\alpha_k, \beta_k$  always:

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$



□ Proof: Exercise

⇒ we easily obtain:  $\hat{x}_n(\gamma) = \frac{1}{\sqrt{2}} (v_k^*(\gamma) a_k + v_k(\gamma) a_{-k}^*)$

1 ... n ~ ... 1 } (F)

1 ... n ~ ... 1

14

14 / 18

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also always (W), iff the coefficients  $\alpha_k, \beta_k$  always:

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

□ Proof: Exercise

$$\Rightarrow \text{we easily obtain: } \hat{x}_k(\gamma) = \frac{1}{\sqrt{2}} (v_k^*(\gamma) \alpha_k + v_k(\gamma) \alpha_{-k}^*) \\ = \frac{1}{\sqrt{2}} (\tilde{v}_k^*(\gamma) \hat{\alpha}_k + \tilde{v}_k(\gamma) \hat{\alpha}_{-k}^*) \quad \left. \right\} (\text{P})$$

= ...

□ Terminology: Such transformations from one choice

$\{v_k\}$ ,  $a_k$  and corresponding basis

$|0\rangle$ ,  $a_k^+|0\rangle$ ,  $a_k^+a_n^+|0\rangle$ , ...

to some  $\{\tilde{v}_k\}$ ,  $\tilde{|0\rangle}$ ,  $\tilde{a}_k$  and their basis

$|\tilde{0}\rangle$ ,  $\tilde{a}_k^+|\tilde{0}\rangle$ ,  $\tilde{a}_k^+\tilde{a}_n^+|\tilde{0}\rangle$ , ...

is called a "Bogoliubov transformation".

Strategy: We have two tasks now:

Strategy: We have two tasks now:

\* Make Bogoliubov Hilbert basis transforms explicit.

(E.g. so that  $|0\rangle$  is, at least at one time, the vacuum.)

\* Find out when which choice of  $\{\alpha_k\}$  is convenient.

## Bogoliubov transformations of Hilbert bases

□ How can we express the basis vectors

$$|\tilde{0}\rangle, \tilde{\alpha}_i^\dagger |\tilde{0}\rangle, \frac{1}{\sqrt{2!}} \tilde{\alpha}_k^2 |\tilde{0}\rangle, \dots, \tilde{\alpha}_k^\dagger \tilde{\alpha}_n^\dagger |\tilde{0}\rangle, \dots$$

as linear combinations of the basis

# Bogoliubov transformations of Hilbert bases

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as linear combinations of the basis

$$|0\rangle, \alpha_k^+ |0\rangle, \frac{1}{\sqrt{2!}} \alpha_k^+ \alpha_{k'}^+ |0\rangle, \dots, \alpha_k^+ \alpha_{k'}^+ |0\rangle, \dots ?$$

□ Proposition: Equations (B) & (P) yield:

Proof: Exercise.

$$\alpha_k = \alpha_k^* \tilde{\alpha}_k + \beta_k \tilde{\alpha}_{-k}^*$$

□ Now we observe that  $\alpha_k |0\rangle = 0$  becomes:

# Bogoliubov transformations of Hilbert bases

□ How can we express the basis vectors

$$|\tilde{0}\rangle, \tilde{a}_+^\dagger |\tilde{0}\rangle, \frac{1}{\sqrt{2!}} \tilde{a}_+^2 |\tilde{0}\rangle, \dots, \tilde{a}_+^\dagger \tilde{a}_+^\dagger |\tilde{0}\rangle, \dots$$

as linear combinations of the basis

$$|0\rangle, a_+^\dagger |0\rangle, \frac{1}{\sqrt{2!}} a_+^2 |0\rangle, \dots, a_+^\dagger a_+^\dagger |0\rangle, \dots ?$$

□ Proposition: Equations (B) & (P) yield:

Proof: Exercise.

$$a_+ = \alpha_+^* \tilde{a}_+ + \beta_+ \tilde{a}_+^*$$

□ Now we observe that  $a_+ |0\rangle = 0$  becomes:

$$\tilde{V}_n(\gamma) = \alpha_n V_n(\gamma) + \beta_n V_n^*(\gamma) \quad (\text{B})$$

also always (W), iff the coefficients  $\alpha_n, \beta_n$  obey:

$$|\alpha_n|^2 - |\beta_n|^2 = 1$$

□ Proof: Exercise

$$\Rightarrow \text{we easily obtain: } \hat{x}_n(\gamma) = \frac{1}{\sqrt{2}} (V_n^*(\gamma) a_n + V_n(\gamma) a_{-n}^*) \quad \left. \begin{array}{l} \\ \end{array} \right\} (\text{P})$$

$$= \frac{1}{\sqrt{2}} (\tilde{V}_n^*(\gamma) \tilde{a}_n + \tilde{V}_n(\gamma) \tilde{a}_{-n}^*)$$

= ...

□ How can we express the basis vectors

$$|\tilde{0}\rangle, \tilde{a}_+^* |\tilde{0}\rangle, \frac{1}{\sqrt{2!}} \tilde{a}_+^{*2} |\tilde{0}\rangle, \dots, \tilde{a}_+^* \tilde{a}_n^* |\tilde{0}\rangle, \dots$$

as linear combinations of the basis

$$|0\rangle, a_+^* |0\rangle, \frac{1}{\sqrt{2!}} a_+^{*2} |0\rangle, \dots, a_+^* a_n^* |0\rangle, \dots ?$$

□ Proposition: Equations (B) & (P) yield:

$$a_n = d_n^* \tilde{a}_n + f_n \tilde{a}_{-n}$$

Proof: Exercise.

□ Now we observe that  $a_n |0\rangle = 0$  becomes:

$$(d_n^* \tilde{a}_n + f_n \tilde{a}_{-n}) |0\rangle = 0$$

□ Now we observe that  $a_n |0\rangle = 0$  becomes:

$$(d_n^* \tilde{a}_n + \beta_n \tilde{a}_{-n}^*) |0\rangle = 0$$

□ Try to solve for  $|0\rangle$  using ansatz:  $|0\rangle := \left( \prod_k f_k(\tilde{a}_k^*, \tilde{a}_{-k}^*) \right) |\tilde{0}\rangle$

□ Proposition:

$$|0\rangle = \left[ \prod_k \frac{1}{|d_k|^{\frac{1}{2}}} e^{-\frac{\beta_k}{2d_k^*} \tilde{a}_k^* \tilde{a}_{-k}^*} \right] |\tilde{0}\rangle \quad (\text{T})$$

↑  
needed for  
normalization
↙

□ Proof: Exercise.

$2a^+$     $2a^+$     $\dots$     $2a^+$

$$(d_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^*) |0\rangle = 0$$

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□ Proof: Exercise.

Hint: Use  $a e^{\lambda a^*} = e^{\lambda a^*} a + \lambda e^{\lambda a^*} \dots$

□ Proof: exercise.

Hint: Use  $a e^{2a^\dagger} = e^{2a^\dagger} a + \lambda e^{2a^\dagger} \dots$

## Interpretation of $(T)$ :

- Assume, e.g., that  $|0\rangle$  and  $|\tilde{0}\rangle$  are those Hilbert space vectors which happen to be the vacuum state vectors at the times  $\eta_1$  and  $\eta_2$  respectively.

(We will soon explore how to identify the vacuum state at any given time)

- Assume at time  $\eta_1$ , the system's state  $|1\Omega\rangle$  is the vacuum state (in the sense of no particle state). Then

□ Proposition:

$$|0\rangle = \left[ \prod_k \frac{1}{|\alpha_k|^{1/2}} e^{-\frac{\beta_k}{2\alpha_k^2} \hat{a}_k^+ \hat{a}_k^-} \right] |\tilde{0}\rangle (T)$$

↑  
needed for  
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□ Proof: Exercise.

Hint: Use  $a e^{2a^+} = e^{2a^+} a + \lambda e^{2a^+} \dots$

Interpretation of (T):

□ Assume, e.g., that  $|0\rangle$  and  $|\tilde{0}\rangle$  are those Hilbert space vectors which happen to be the vacuum state vectors at the times  $\gamma_1$  and  $\gamma_2$  respectively.

## Interpretation of $\langle T \rangle$ :

- Assume, e.g., that  $|0\rangle$  and  $|\tilde{0}\rangle$  are those Hilbert space vectors which happen to be the vacuum state vectors at the times  $\eta_1$  and  $\eta_2$  respectively.

(We will soon explore how to identify the vacuum state at any given time)

- Assume at time  $\eta_1$ , the system's state  $|\Omega\rangle$  is the vacuum state (in the sense of no particle state). Then it is convenient to choose the mode functions  $\{v_n\}$  so that:

$$|\Omega\rangle = |0\rangle$$

At a later time,  $\eta_2$ , the system is still in this state:

the vacuum state (in the sense of no particle state). Then it is convenient to choose the mode functions  $\{v_n\}$  so that:

$$|\Omega\rangle = |0\rangle$$

At a later time,  $\eta_2$ , the system is still in this state but the vacuum is then some other vector  $|\tilde{\Omega}\rangle$ , which obeys  $\tilde{a}_n |\tilde{\Omega}\rangle = 0$  with mode functions  $\tilde{v}_n(t)$ .

- Thus, from (T) we see that the system's state,  $|\Omega\rangle$ , is at  $\eta_2$  a state with many particles.
- Note: the particles have been created in  $k, -k$  pairs.
- Intuitively: The expansion rips virtual particle + anti-particle pairs out