Title: Introduction to Quantum Field Theory for Cosmology - Lecture 13

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Abstract:

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QFT for Cosmology, Achim Kempf, Winter 2014, Lecture 13

Note Title

Recall: The free Klein Gordon quantum field (i.e. with $V(\hat{\phi})=0$) in a generic curved space-time must obey:

$$\hat{\phi}^+(x,t) = \hat{\phi}(x,t), \qquad \hat{\pi}^+(x,t) = \hat{\pi}(x,t) \qquad \text{(HC)}$$

$$i\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}(t)], \quad i\dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}(t)] \qquad \text{(EoM)}$$

which can be written in this form:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0, \quad \hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \qquad \text{(EoM)}$$

And: On all spacelike hypersurfaces, Σ, the CCRs must hold:

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And: On all spacelike hypersurfaces, Σ, the CCRs must hold:

$$[\hat{\phi}(x,t),\hat{\phi}(x',t)]=0, \quad [\hat{\pi}(x,t),\hat{\pi}(x',t)]=0, \quad [\hat{\phi}(x,t),\hat{\pi}(x',t)]=i\delta^3(x,x') \qquad \text{(CCR)}$$

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We want to show: The following ansatz for $\hat{\phi}(x,t)$ succeeds:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+ \quad , \text{with } [a_k, a_{k'}^+] = \delta_{k,k'}$$

number-valued solutions to K.G. eqn.

at least if the spacetime is globally hyperbolic.

So far we showed:

- The HC and EoM obeyed at all time.
- In a fixed coordinate system, CCRs are obeyed $\forall t$ if $\{u_k\}$ obey $\forall t$:

$$\boxed{\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\, \delta^3(x-x')}$$

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We showed this by using Darboux's theorem for symplectic for....

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  We showed this by using Darboux's theorem for symplectic forms.

Remark: Non-conservation of the CCRs would imply non-unitary time evolution:

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- It allows one to express the time evolution of the observables, such as field operators, through:

$$\hat{\phi}(x,t) = \hat{U}(t,t_0)\,\hat{\phi}(x,t_0)\,\hat{U}^{+}(t,t_0)$$

$$\hat{\pi}(x,t) = \hat{U}(t,t_0)\,\hat{\pi}(x,t_0)\,\hat{U}^{+}(t,t_0)$$

- Thus: $[\hat{\phi}(x,t), \hat{\pi}(x',t)] = [\hat{U}\hat{\phi}(x,t_0)\hat{U}^{-1}, \hat{U}\hat{\pi}(x',t_0)\hat{U}^{-1}]$

$$= \hat{U}\,[\hat{\phi}(x,t_0), \hat{\pi}(x',t_0)]\,\hat{U}^{-1}$$

$$= \hat{U}\, i\,\delta^3(x-x')\,\hat{U}^{-1} = i\,\delta^3(x-x')$$

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Problem: If we change coordinate system, and therefore the choice of $\{\Sigma\}$, would the CCRs still hold on every spacelike hypersurface Σ?

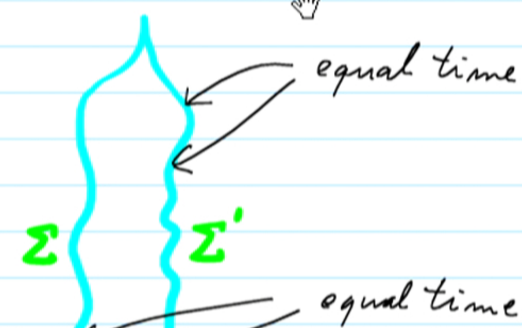
Proposition: Yes: if CCRs hold in one coordinate system, then they hold in all: The CCRs keep holding when deforming a Σ.

Proof: We only need to show that the value of the symplectic form

Recall: $f, g \in V$ are solutions of KG eqn.

$$(f,g) := \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu} \left(f \partial_\nu g - g \partial_\nu f\right)$$

is independent of Σ:



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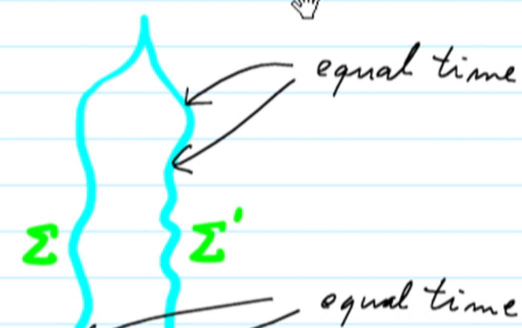

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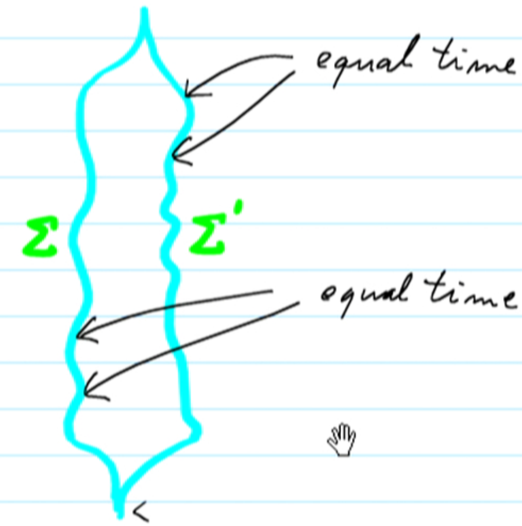

Analyze the integrand:

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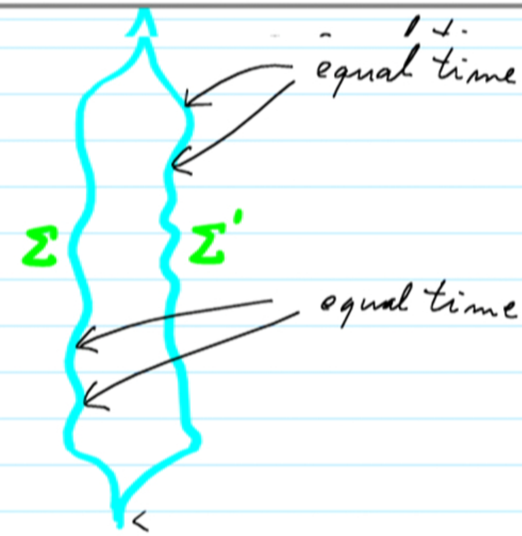

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$$j^\mu(x,t) := g^{\mu\nu}\left(f\,\partial_\nu g - g\,\partial_\nu f\right)$$

It is a contravariant vector field.

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- To integrate it over 3-dim submanifolds, Σ, need to turn it into a 3-form:

inner derivation

$$\tilde{j} := i_j\,\Omega$$

Volume 4-form $\sqrt{|g|}\,d^4x$

3-form

- We obtain: $(f,g) := \int \tilde{j}$

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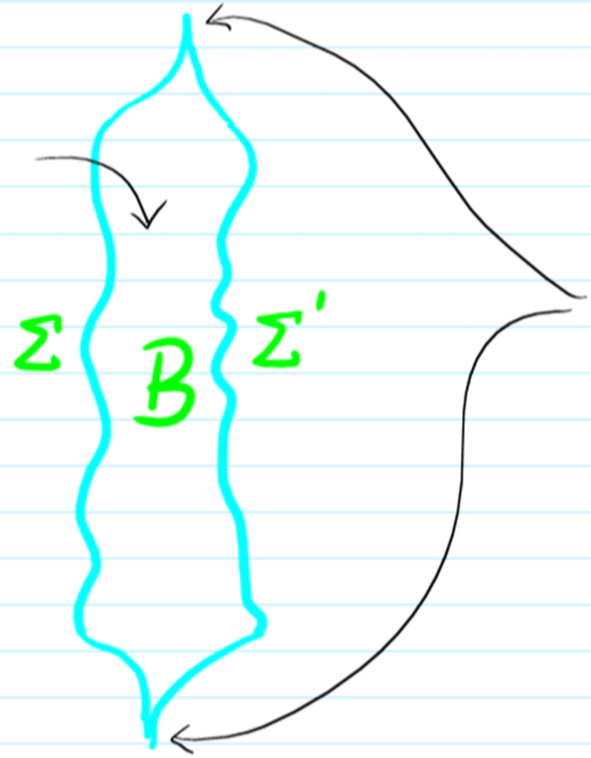
- Now integrate over both Σ and Σ':

Σ and Σ' enclose the 4-dim. volume B



We close the hyper-surfaces arbitrarily far out: in the limit at spatial infinity.

- Use Stokes' theorem:

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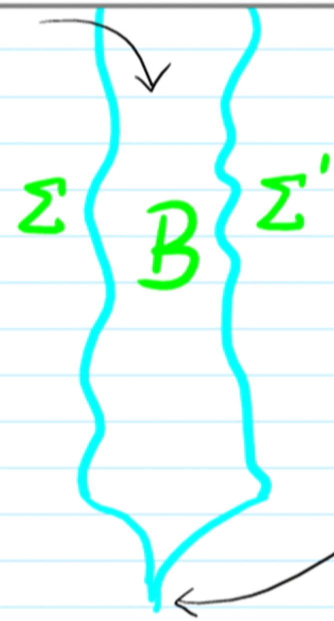
the 4-dim. volume B



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$$\int_{\Sigma \cup \Sigma' = \partial B} \tilde{j} = \int_B d\tilde{j}$$

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$$\Sigma \cup \Sigma' = \int_{\partial B} \tilde{j} = \int_B d\tilde{j}$$

Notice:

If we can show $d\tilde{j} = 0$ we are done!

That's because then:

$$0 = \int_{\Sigma \cup \Sigma'} \tilde{j} = \int_{\Sigma} \tilde{j} + \int_{\Sigma'} \tilde{j} = -\int_{\Sigma} \tilde{j} + \int_{\Sigma'} \tilde{j}$$

Both j pointing

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Both j pointing out of B, i.e. one to the future one to the past.

Both j future pointing.

$\Rightarrow \int_{\Sigma} \tilde{j}$ is indeed indep. of choice of Σ,

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▫ Indeed:

$$d\tilde{j} = d(i_j \Omega) = \mathrm{div}_\Omega\, j = \left(\sqrt{|g|}\, j^\mu\right)_{,\mu} d^4x$$

Here:

$$\left(\sqrt{|g|}\, j^\mu\right)_{,\mu} = \Big(\sqrt{g}\, \overbrace{g^{\mu\nu}(f\partial_\nu g - g\partial_\nu f)}^{\text{(is definition of } j\text{)}}\Big)_{,\mu}$$

$$= \cancel{\sqrt{|g|}\, g^{\mu\nu}\partial_\nu g\, \partial_\mu f} + f\,\overbrace{\left(\sqrt{|g|}\, g^{\mu\nu}\partial_\nu g\right)_{,\mu}}^{=-m^2 g\sqrt{|g|}}$$

$$-\cancel{\sqrt{|g|}\, g^{\mu\nu}\partial_\mu g\, \partial_\nu f} - g\,\overbrace{\left(\sqrt{|g|}\, g^{\mu\nu}\partial_\nu f\right)_{,\mu}}^{=-m^2 f\sqrt{|g|}}$$

Recall:

$(\Box + m^2)\phi = 0$ reads:

$$\frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\, g^{\mu\nu}\partial_\nu \phi\right)_{,\mu} + m^2\phi = 0$$

and the f and g are

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⇝ We finally proved that, for globally hyperbolic spacetimes, there always exist mode functions $\{u_k(x,t)\}$ so that our ansatz for $\hat{\phi}$ and $\hat{\pi}$ also obeys the CCRs at all time and indeed $\forall \Sigma$:

$$\sqrt{|g|}\, g^{0\nu} \int \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) d^3k = i\delta^3(x-x') \qquad (R1)$$

Example: For Minkowski space, we had found this solution for the noninteracting Klein Gordon field:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_k\, e^{-i\omega_k t + ikx} + a_k^+ e^{i\omega_k t - ikx} \right) d^3k$$

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We read off: $u_k(x,t) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}}\, e^{-i\omega_k t + ikx}$

Now: Verify the CCR condition, (R1):

Γ Here: $\sqrt{|g|} = 1$ and $g^{0\nu} = \delta_{0\nu}$.

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$$=\frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}\Big[e^{-i\omega_k t+ikx}(i\omega_k)e^{i\omega_k t-ikx'}$$

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$$= \frac{1}{(2\pi)^3}\int\frac{2i\omega_k}{2\omega_k}\, e^{ik(x-x')}\, d^3k \overset{\text{Fourier}}{=} i\,\delta^3(x-x') \quad \checkmark$$

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Summary so far:

- To solve the QFT of a free KG field on curved spacetime is to solve the HC, EoM and CCRs.

- Make solution ansatz:

or integral, e.g., if no IR cutoff

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^\dagger \qquad (A)$$

- We showed that at least if spacetime is globally hyperbolic:

- There exists a set of solutions of the KG eqn, $\{u_k\}$, so that ansatz (A) solves HC, EoM and CCR for all time.

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Q: Does there exist only one such set of solutions?

A: No, there exist many other such sets of solutions: $\{\bar{u}_k\}, \{\bar{\bar{u}}_k\} \ldots$

How to see this non-uniqueness?

- Recall symplectic form for $f, g \in V$:

$$(f,g) := \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu} (f \partial_\nu g - g \partial_\nu f)$$

- Darboux: There exists a basis $\{v_n\}$ of V in which the form $(\ ,\ )$ reads:

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$$\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & \multicolumn{2}{c}{0} \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & 0 & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$$

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□ From the v_n we constructed the $u_n := v_{2n} + i\, v_{2n+1}$

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- **However:** Darboux bases are not unique!

- **Example:** 2-dim. solution subspace.

Assume: $v_1, v_2 \in V$ are a Darboux basis, so

so that $(\,,\,)$ reads $\begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}$.

Q: Can we change basis, $v_i = B\bar{v}_i$, so that $(\,,\,)$ keeps that matrix form? Is there a matrix B so that

$$B^t \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix} B = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix} \quad ?$$

A: Yes, any change of basis $B = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

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Non-uniqueness of the solution (A):

- Clearly, this means that there are infinitely many solutions (A) to HC, EoM and CCRs:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

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- Correspondingly, we obtain different Fock vectors:

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$$\hat{\phi}(x,t) := \sum_k \bar{\bar{u}}_k(x,t)\, \bar{\bar{a}}_k + \bar{\bar{u}}_k^*(x,t)\, \bar{\bar{a}}_k^+$$ etc, etc...

- Correspondingly, we obtain different Fock vectors:

Either: $a_k|0\rangle = 0 \qquad |n_k\rangle := \frac{1}{\sqrt{n!}}(a_k^+)^n|0\rangle$

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solutions (A) to HC, EoM and CCRs:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

$$\hat{\phi}(x,t) := \sum_k \bar{u}_k(x,t)\, \bar{a}_k + \bar{u}_k^*(x,t)\, \bar{a}_k^+$$

$$\hat{\phi}(x,t) := \sum_k \bar{\bar{u}}_k(x,t)\, \bar{\bar{a}}_k + \bar{\bar{u}}_k^*(x,t)\, \bar{\bar{a}}_k^+ \quad \text{etc, etc...}$$

- Correspondingly, we obtain different Fock vectors:

Either: $a_k|0\rangle = 0 \qquad |n_k\rangle := \frac{1}{\sqrt{n!}}(a_k^+)^n|0\rangle$

Or: $\bar{a}_k|\bar{0}\rangle = 0 \qquad |\bar{n}_k\rangle := \frac{1}{\sqrt{n!}}(\bar{a}_k^+)^n|\bar{0}\rangle$ etc, etc...

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- **Q:** Do these solutions of the QFT
  - describe different physics, or
  - do they differ by a mere change of basis in Fock space and so describe the same physics?

- **A:** It depends!

1) Assume first we can impose IR and UV cutoffs with negligible consequences

* This means we truncate to a finite (though large) number of independent mode oscillators, a_n, a^+_n.

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□ A: It depends!

1) Assume first we can impose IR and UV cutoffs with negligible consequences

* This means we truncate to a finite (though large) number of independent mode oscillators, a_n, a_n^+.

* Then, the following theorem implies that all solutions to HC, EoM, CCR differ merely by a change of basis:

Theorem (Stone and von Neumann):

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Theorem (Stone and von Neumann):

* Assume in a Hilbert space, $\mathcal{H}$, the operators $\hat{x}_i, \hat{p}_j$ obey:

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \qquad [\hat{x}_i, \hat{x}_j] = 0 = [\hat{p}_i, \hat{p}_j] \qquad i,j \in \{1,\dots,N\}$$

* Assume that in that Hilbert space other operators $\tilde{x}_i, \tilde{p}_j$ also obey:

$$[\tilde{x}_i, \tilde{p}_j] = i\delta_{ij} \qquad [\tilde{x}_i, \tilde{x}_j] = 0 = [\tilde{p}_i, \tilde{p}_j] \qquad i,j \in \{1,\dots,N\}$$

* Assume that the representations are irreducible.

(and exclude pathological cases)

* Then, there exists a unitary operator $\hat{U}$ so that:

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* Remark:

The pathological cases can be avoided by requiring representations of the CCRs of the (bounded and therefore better behaved) operators

$$e^{i\alpha_i \hat{x}_i}, \quad e^{i\beta_i \hat{p}_i}$$

* Application to our QFT:

Consider $\hat{x}_n := \frac{1}{\sqrt{2}}(a_n + a_n^+)$, $\hat{p}_n := \frac{-i}{\sqrt{2}}(a_n - a_n^+)$

and $\bar{x}_n := \frac{1}{\sqrt{2}}(\bar{a}_n + \bar{a}_n^+)$, $\bar{p}_n := \frac{-i}{\sqrt{2}}(\bar{a}_n - a_n^+)$ etc.

The theorem of Stone & v. Neumann implies that

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and $\bar{X}_n := \frac{1}{\sqrt{2}}(\bar{a}_n + \bar{a}^+_n)$, $\bar{p}_n := \frac{-i}{\sqrt{2}}(\bar{a}_n - a^+_n)$ etc.

The theorem of Stone & v. Neumann implies that

$$a_n = \hat{U}\,\bar{a}_n\,\hat{U} \quad \text{with } \hat{U} \text{ unitary.}$$

$\Rightarrow$ All solutions are the same up to a mere change of basis.

2.) Consider now the possibility that we cannot truncate to a finite number of degrees of freedom.

Q: When would this happen?

A: E.g., phase transitions formally need systems with

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Then: The QFT can have unitarily non-equivalent solutions, that differ physically: different "phases".

Underlying math of non-equivalent representations?

Assume $\langle a|b\rangle = \alpha$ with $0<\alpha<1$, i.e., not $\perp$

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A: E.g., phase transitions formally need systems with an infinite number of degrees of freedom.

Then: The QFT can have unitarily non-equivalent solutions, that differ physically: different "phases".

Underlying math of non-equivalent representations?

Assume $\langle a|b\rangle = \alpha$ with $0<\alpha<1$, i.e., not $\perp$

Then $(\langle a|\langle a|\langle a|\ldots\langle a|)\overbrace{(|b\rangle|b\rangle|b\rangle\ldots|b\rangle)}^{N} = \alpha^N$, i.e., not $\perp$

But for $N=\infty$ have $|a\rangle|a\rangle\ldots|a\rangle \perp |b\rangle|b\rangle\ldots|b\rangle$, so that then can no longer use $|a\rangle|a\rangle\ldots|a\rangle$ to help linearly combine, e.g., $|b\rangle|b\rangle\ldots|b\rangle$

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From now on: We will assume IR & UV cutoffs are possible and that Stone v. Neumann therefore applies.

Therefore:

- No matter which set of suitable mode functions

$$\{u_n(x,t)\} \text{ or } \{\bar{u}_n(x,t)\} \text{ or } \{\bar{\bar{u}}_n(x,t)\}, \dots$$

we choose, we obtain the same solution

$$\hat{\phi}(x,t) = \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

$$= \sum_k \bar{u}_k(x,t)\, \bar{a}_k + \bar{u}_k^*(x,t)\, \bar{a}_k^+$$

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Therefore:

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$$= \sum_k \bar{\bar{u}}_k(x,t)\, \bar{\bar{a}}_k + \bar{\bar{u}}_k^*(x,t)\, \bar{\bar{a}}_k^+ \quad = \ldots$$

up to a change of basis.

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- For example, using the $\{u_k\}$, we are led to span the Hilbert space of states $\mathcal{H}$ using this ON basis:

$$|0\rangle \text{ where } a_k|0\rangle = 0 \quad \forall k$$

$$a^+_k|0\rangle, \quad \frac{1}{\sqrt{n!}}(a^+_k)^n|0\rangle$$

$$\frac{1}{\sqrt{\cdots}}(a^+_k)^{\cdot\cdot}\cdots(a^+_{k'})^{\cdot\cdot}|0\rangle, \text{ etc} \ldots$$

- Or, using other mode functions, say $\{\bar{u}_k\}$, we may span the same Hilbert space, $\mathcal{H}$, using this ON basis:

$$|\bar{0}\rangle \text{ where } \bar{a}_k|\bar{0}\rangle = 0 \quad \forall k$$

$$\bar{a}^+_k|\bar{0}\rangle, \quad \frac{1}{\sqrt{n!}}(\bar{a}^+_k)^n|\bar{0}\rangle$$

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$\frac{1}{\sqrt{\cdots}}(\bar{a}_k^+)^{\cdot\cdot}\cdots(\bar{a}_{k'}^+)^{\cdot\cdot}|\bar{0}\rangle$, etc ...

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Does the choice of mode functions matter?

- In principle, it does not:

Any state of the system, say $|\Psi\rangle$, can be expanded in each basis.

- In practice, however:

It is convenient, whenever we know which state is the no-particle (i.e., vacuum) state, say $|\Omega\rangle$, to choose the mode functions $\{u_k\}$ such that the corresponding $|0\rangle$ is $|\Omega\rangle$, i.e., such that

$$|0\rangle = |\Omega\rangle, \text{ i.e., such that } a_k|\Omega\rangle = 0$$

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Then, conveniently, states like $\frac{1}{\sqrt{n!}}(a_k^+)^n|0\rangle$ are the multi-particle states.

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Outlook: (only a rough sketch)

- Say we know the system's state, $|\Psi\rangle$, is the vacuum initially.
- ⇝ We choose $\{u_n\}$ appropriately, so that $|0\rangle_{in} = |\Psi\rangle$.
- After some evolution (e.g. the universe expands) the vacuum state may be a different state, say $|\chi\rangle$.
- ⇝ We choose $\{\bar{u}_n\}$ appropriately, so that $|\bar{0}\rangle_{out} = |\chi\rangle$
- At late times, since we work in the Heisenberg picture, the system is still in the state $|0_{in}\rangle$, but this is then an excited state!

⇝ Description of particle production due to cosmic expansion.

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Exercise: ▫ Recall that with respect to the hermitean bi-linear form $\langle\, ,\, \rangle$ of Lecture 12, the mode functions $\{u_n\}$ obey:

$$\left.\begin{aligned} \langle u_n, u_m \rangle &= \delta_{nm} \\ \langle u_n^*, u_m^* \rangle &= -\delta_{nm} \\ \langle u_n, u_m^* \rangle &= 0 = \langle u_n^*, u_m \rangle \end{aligned}\right\} \quad (*)$$

▫ Now consider an invertible change of basis (in the space of complex-number-valued solutions of the K.G. eqn) to new mode functions:

$$\bar{u}_n := \sum_m \left(A_{nm}\, u_m + B_{nm}\, u_m^* \right)$$

▫ Show that for the $\{\bar{u}_n\}$ to qualify as mode functions, i.e., fo...

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$$\left.\begin{array}{l} \langle u_n, u_m \rangle = \delta_{nm} \\ \langle u_n^*, u_m^* \rangle = -\delta_{nm} \\ \langle u_n, u_m^* \rangle = 0 = \langle u_n^*, u_m \rangle \end{array}\right\} (*)$$

- Now consider an invertible change of basis (in the space of complex-number-valued solutions of the K.G. eqn) to new mode functions:

$$\bar{u}_n := \sum_m \left(A_{nm} u_m + B_{nm} u_m^* \right)$$

- Show that for the $\{\bar{u}_n\}$ to qualify as mode functions, i.e., for them to obey (*), i.e., $\langle \bar{u}_n, \bar{u}_m \rangle = \delta_{nm}$ etc, A, B must obey:

$$A^+ A - B^t B^* = 1 \quad \text{and} \quad A^+ B - B^t A^* = 0$$

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