Title: Introduction to Quantum Field Theory for Cosmology - Lecture 11

Date: Feb 13, 2014 04:00 PM

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Abstract:

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Recall the strategy:

SR, 1st Q Hamiltonian formalism

step 1
Legendre transform
(equivalence)

SR, 1st Q. Lagrangian formalism

we are here

step 2 allow curvature

GR, 1st Q Lagrangian formalism

step 3
Legendre transform
(equivalence)

GR, 1st Q Hamiltonian formalism

step 4

GR, 2nd Q Hamiltonian formalism

Dyson Schwinger eqns are same
(equivalence)

GR, 2nd Q Lagrangian formalism (Path integral of QFT)

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Step 2 so far:

- We started with the Klein Gordon action in special relativity:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} \eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - V(\phi, \ell_i)\, d^4x \qquad (a)$$

This formulation is correct for any inertial observer using a rectangular coordinate system, and only for those observers.

Remark: Here, $V(\phi, \ell_i)$ is a potential. It describes how the ϕ field interacts with other fields $\{\ell_i\}$ and with itself. The very early universe is assumed to have been

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Remark: Here, $V(\phi, \varphi_i)$ is a potential. It describes how the ϕ field interacts with other fields $\{\varphi_i\}$ and with itself. The very early universe is assumed to have been dominated by a scala field ϕ which interacted mainly with itself, i.e., one had effectively: $V(\phi) = \lambda_1 \phi^3 + \lambda_2 \phi^4 + \ldots$

- We obtained a formulation of the Klein Gordon action in special relativity which is the same in all coordinate systems:

$\sqrt{\det(g_{\mu\nu})}$

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- We obtained a formulation of the Klein Gordon action in special relativity which is the same in all coordinate systems:

$$S_{KG}[\phi] := \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - V(\phi,\ell_i)\right)\overbrace{\sqrt{-\det(g_{\mu\nu})}}^{\sqrt{|g|}}\, d^4x \qquad (b)$$

- If an inertial observer sets up a rectangular coordinate system, then the metric tensor $g_{\mu\nu}(x)$ is of course given by this constant matrix

$$g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

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But we saw that in generic (i.e. arbitrarily chosen) coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x)$, the metric tensor $\tilde{g}_{\mu\nu}(\tilde{x})$ is given by:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta} \qquad (c)$$

$\Rightarrow$ In special relativity, in arbitrary coordinates, the metric $g_{\mu\nu}$ is a position-dependent matrix of the form (c).

* We notice that $g_{\mu\nu}(x)$ is always symmetric $g_{\mu\nu}(x) = g_{\nu\mu}(x)$

$\rightsquigarrow$ Key Question:

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Key Question:

Do all metrics arise from $\eta_{\mu\nu}$ by a change of coordinates?

I.e.: Can any arbitrary function obeying $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ arise from $\eta_{\mu\nu}$ by changing coordinates according to (c)?

Answer: No! The others describe curved spacetimes.

⇒ A given spacetime can be described by any one of an equivalence class of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates

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Answer: No! The others describe curved spacetimes.

⇒ A given spacetime can be described by any one of an equivalence class of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

⇝ Q: How many independent degrees of freedom D (i.e. independent functions) describe a spacetime fully?

→ ∞ # of choices of coordinate functions $Y^\mu = Y^\mu(x)$

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⇒ A given spacetime can be described by any one of an equivalence class of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

⇝ Q: How many independent degrees of freedom D (i.e. independent functions) describe a spacetime fully?

A: $D = \underbrace{n(n+1)/2}_{\text{\# of indep elements of a symmetric } n\times n \text{ matrix } g_{\mu\nu}} - \underbrace{n}_{\text{\# of change of coordinate functions } y^{\mu} = y^{\mu}(x)}$

Examples: For $n = 1+3$, have $D = 6$. For $n = 2$, have $D = 1$.

Introduction of symmetries:

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⇒ A given spacetime can be described by any one of an equivalence class of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

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Examples: For $n = 1+3$, have $D = 6$. For $n = 2$, have $D = 1$.

Introduction of curvature:

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Introduction of curvature:

- We continue to postulate the coordinate system-independent Klein Gordon action of above:

$$S_{KG}[\phi] := \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - V(\phi,\ldots)\right)\sqrt{|g|}\,d^4x$$

- We will allow **almost** arbitrary metric tensors $g_{\mu\nu}(x)$, even those for which there do not exist coordinates $\tilde{x}$ in which:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} \quad \text{for all } x$$

- But we must have that, at least locally, special relativity holds!

⇒ Consider only $g_{\mu\nu}(x)$ for which for each x there exists a change of coordina...

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$\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu}$ for all x

- But we must have that, at least locally, special relativity holds!

⟹ Consider only $g_{\mu\nu}(x)$ for which for each x_0, there exists a change of coordinates

$$x \to \tilde{x}$$

so that: $\tilde{g}_{\mu\nu}(\tilde{x}_0) = \eta_{\mu\nu}$

- This requirement is <u>The Equivalence Principle</u>:
  - * We postulate that gravity can always locally be eliminated:
  - * We assume that if a freely falling observer in a sm[illegible]

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- This requirement is **The Equivalence Principle**:

  * We postulate that gravity can always locally be eliminated:

  * We assume that if a freely falling observer in a small region sets up a rectangular coordinate system the observer will see arbitrarily small gravity effects if the region is made arbitrarily small.

  * For this to be true, any body's gravitational mass must be equal to its inertial mass, i.e. all bodies must fall equally. (Else the notion of freely falling observer

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Identifying Curvature:

* Assume we are given a metric tensor

$$g_{\mu\nu}(x)$$

as an explicit matrix-valued function, in some coordinates.

* How can we determine whether or not this is the metric tensor of flat space-time, i.e., whether or not there exist coordinates $\tilde{x}$ in which $\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu}$?

* This problem is solved in differential geometry:

Define: The "Christoffel symbol functions":

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* This problem is solved in differential geometry:

Define: The "Christoffel symbol functions":

$$\Gamma^{\mu}_{\alpha\beta}(x) := \frac{1}{2} g^{\mu\nu}(x) \left(g_{\alpha\nu,\beta}(x) + g_{\beta\nu,\alpha}(x) - g_{\alpha\beta,\nu}(x) \right)$$

Define: The "Riemann Curvature Tensor":

$$R^{i}_{jk\ell}(x) := \Gamma^{i}_{\ell jk}(x) - \Gamma^{i}_{kj\ell}(x) + \Gamma^{s}_{\ell j}(x)\Gamma^{i}_{ks}(x) - \Gamma^{s}_{kj}(x)\Gamma^{i}_{\ell s}(x)$$

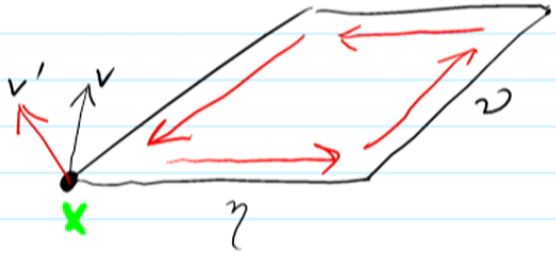
Meaning: A space is called curved at x if the parallel

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Meaning: A space is called curved at x if the parallel transport of a vector v along an infinitesimal parallelogram returns the vector v' to x, but v' is rotated by some amount. $R^{\nu}{}_{\nu\rho}$ tells by how much:



$$(v'-v)^a = \eta^b \xi^c R^a{}_{bcd}(x)\, v^d$$

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Proposition:

Assume that, in a region, A, of space-time:

$$R^{\rho}{}_{\nu\sigma\varepsilon}(x) = 0 \quad \text{for all } x \in A$$

Then and only then there exist coordinates $\tilde{x}$ so that:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} \quad \text{for all } x \in A$$

The dynamics of space-time

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The dynamics of space-time

Problem: What are the equations of motion for spacetime's curvature?

Which are the degrees of freedom of curvature for which we have to find an equation of motion?

- We saw that the curvature of space-time is encoded in the matrix-valued metric function:

$$g_{\mu\nu}(x)$$

- However, if $g_{\mu\nu}(x)$ looks nontrivial, this can

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for which we have to find an equation of motion?

- We saw that the curvature of space-time is encoded in the matrix-valued metric function:

$$g_{\mu\nu}(x)$$

- However, if $g_{\mu\nu}(x)$ looks nontrivial, this can be for <u>two</u> different reasons:

1. Spacetime has little or no curvature and $g_{\mu\nu}(x)$

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1. Spacetime has little or no curvature and $g_{\mu\nu}(x)$ is nontrivial just because of an unlucky choice of coordinates.

2. Spacetime is curved, i.e., we can not make $g_{\mu\nu}(x)$ take the form $\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu}$ for all $\tilde{x}$ no matter which coordinates we choose.

Therefore, it is difficult to pinpoint in the matrix function $g_{\mu\nu}(x)$ the curvature degrees of freedom.

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- Therefore, it is difficult to pinpoint in the matrix function $g_{\mu\nu}(x)$ the curvature degrees of freedom.
- And: Even the entries $R^{\nu}{}_{\nu\sigma\varepsilon}$ of the curvature tensor are coordinate system dependent.

Strategy:

- Use the degrees of freedom of curvature to build a scalar and therefore coordinate system independent

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Strategy:

- Use the degrees of freedom of curvature to build a scalar and therefore coordinate system independent function $S_{grav}[g]$
- Then, use this function as the action for gravity.
- The equations of motion for gravity should follow from the action principle (and they do).

→ Need to define a scalar that encodes curvature!

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→ Need to define a scalar that encodes curvature!

We begin by going from a 4-tensor to a 2-tensor:

Definition: The "Ricci tensor":

$$R_{\mu\nu}(x) := R^{i}{}_{\mu i\nu}(x)$$

↙ Recall: $\sum_{i=0}^{3}$ is implied

Note: Other index contractions would vanish because of antisymmetries of $R^{\rho}{}_{\nu\sigma\varepsilon}(x)$ that are implied by the definition of $R^{\rho}{}_{\nu\sigma\varepsilon}(x)$.

Remark:

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Recall: $\sum_{i=0}$ is implied

$$R_{\mu\nu}(x) := R^{i}{}_{\mu i\nu}(x)$$

Note: Other index contractions would vanish because of antisymmetries of $R^{\rho}{}_{\nu\sigma\varepsilon}(x)$ that are implied by the definition of $R^{\rho}{}_{\nu\sigma\varepsilon}(x)$.

Remark:

$R_{\mu\nu}(x)$ carries strictly less information than the full Riemann curvature tensor:

* If $R_{\mu\nu}(x) = 0$ it is still possible that $R^{\rho}{}_{\nu\sigma\varepsilon}(x) \neq 0$!
* This happens to be the case, e.g., for gravitational waves.

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Definition: The curvature scalar (or Ricci scalar)

$$R(x) := g^{\mu\nu}(x)\, R_{\mu\nu}(x)$$

Other curvature scalars:

* The simplest scalar that can be formed from the metric alone is $g^{\mu\nu}(x)\, g_{\nu\mu}(x) = 4$.

* The next simplest scalar that can be formed is the Ricci scalar $R(x)$.

* All other scalars made out of g only are composed of higher powers of the Riemann tensor $R^{\mu}{}_{\nu\sigma\varepsilon}(x)$:

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* The simplest scalar that can be formed from the metric alone is $g^{\mu\nu}(x)\, g_{\nu\mu}(x) = 4$.

* The next simplest scalar that can be formed is the Ricci scalar $R(x)$.

* All other scalars made out of g only are composed of higher powers of the Riemann tensor $R^{\mu}{}_{\nu\sigma\varepsilon}(x)$:

$$R_{\mu\nu} R^{\mu\nu}, \quad R^{\cdots} R^{\cdots} R^{\cdots} \text{ etc.}$$

The gravitational action

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The gravitational action

- A priori, the full action now reads:

other "matter" fields for e^-, quarks, photons etc

$$S_{tot}[g, \phi, e_i] = S_{KG} + S_{other} + S'_{grav}$$

S_{KG}: Klein Gordan action

with: $$S'_{grav}[g] := \int \left(c_0 + c_1 R(x) + c_2 R_{\mu\nu}(x) R^{\mu\nu}(x) + c_3 R \cdot\cdot R \cdot\cdot R + \dots \right) \sqrt{|g|}\, d^4x$$

- Comparison with experiment shows evidence only for the first two terms:

$$S_{...}[g] = -\frac{1}{\dots} \int (2\Lambda + R(x)) \sqrt{|g(x)|}\, d^4x$$

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- A priori, the full action now reads:

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- Comparison with experiment shows evidence only for the first two terms:

$$S_{grav}[g] = -\frac{1}{16\pi G} \int (2\Lambda + R(x)) \sqrt{|g(x)|}\, d^4x$$

"Einstein action" — Newton's constant — "Cosmological constant"

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The equations of motion

The action principle is to require that the action be extremal with respect to all degrees of freedom:

A) $\frac{\delta S_{tot}}{\delta \ell_i(x)} = 0$ B) $\frac{\delta S_{tot}}{\delta g_{\mu\nu}(x)} = 0$ C) $\frac{\delta S_{tot}}{\delta \phi(x)} = 0$

A) Require: $\frac{\delta S_{tot}[g, \ell, \phi]}{\delta \ell_i(x)} = 0$

This yields the general relativistically covariant field equations for all "other" fields. (We will ignore the $\ell_i(x)$ for now.)

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A) **Require:** $\frac{\delta S_{tot}[g, e, \phi]}{\delta e_i(x)} = 0$

This yields the general relativistically covariant field equations for all "other" fields. (We will ignore the $e_i(x)$ for now.)

Quantization: Legendre transform $\rightarrow H(e_i, \pi_{e_i}) \rightarrow$ impose $[e_i, \pi_{e_j}] = i\delta$ etc.

B) **Require:** $\frac{\delta}{\delta g_{\mu\nu}(x)} S_{tot}[\phi, e_i, g] = 0$

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B) **Require:** $\frac{\delta}{\delta g_{\mu\nu}(x)} S_{tot}[\phi, \ell_i, g] = 0$

This yields the equation of motion for the dynamics of curvature, i.e., the Einstein equation:

(See exercise in Mukhanov's text) →

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = + 8\pi G \, T_{\mu\nu}(x)$$

$$\sim \frac{\delta S'_{grav}}{\delta g_{\mu\nu}(x)} \qquad \sim - \frac{\delta (S_{other} + S'_{KG})}{\delta g_{\mu\nu}(x)}$$

* Here, $T_{\mu\nu}(x)$ is the "Energy Momentum Tensor".

Neglecting the contribution by the $\ell_i(x)$, one obtains:

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* Here, $T_{\mu\nu}(x)$ is the "Energy Momentum Tensor".

Neglecting the contribution by the $\ell_i(x)$, one obtains:

$$T^{(k.G.)}_{\mu\nu}(x) = \phi_{,\mu}(x)\,\phi_{,\nu}(x) - g_{\mu\nu}(x)\left(\frac{1}{2} g^{\alpha\beta}(x)\,\phi_{,\alpha}(x)\,\phi_{,\beta}(x) - V(\phi(x))\right)$$

(mass term $m^2\phi^2$ included)

* **Quantization:** To quantize the Einstein equation is difficult for many reasons:

  - For example, it is difficult to separate the

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* **Quantization:** To quantize the Einstein equation is difficult for many reasons:

  - For example, it is difficult to separate the curvature degrees of freedom from mere artifacts of the choice of the coordinate system.
  - Also, the Einstein equation is highly nonlinear.
  - So far, all attempts have run into severe difficulties, even perturbative approaches.

⇝ **This course:** 1.) We will here initially consider known classical

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of curvature, i.e., the Einstein equation:

(See exercise in Mukhanov's text) →

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = + 8\pi G \, T_{\mu\nu}(x)$$

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* Here, $T_{\mu\nu}(x)$ is the "Energy Momentum Tensor".

Neglecting the contribution by the $\ell_i(x)$, one obtains:

$$T^{(k.6.)}_{\mu\nu}(x) = \phi_{,\mu}(x)\,\phi_{,\nu}(x) - g_{\mu\nu}(x)\left(\frac{1}{2} g^{\alpha\beta}(x)\,\phi_{,\alpha}(x)\,\phi_{,\beta}(x) - V(\phi(x))\right)$$

(mass term $m^2\phi^2$ included)

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This course: 1.) We will here initially consider known classical solutions $g_{\mu\nu}(x)$ and quantize only $\phi(x)$.

2.) Then, we will quantize linear perturbations of the metric.

c) Require: $\frac{\delta}{\delta \phi(x)} S_{tot}[g, \phi] = 0$

- Since ϕ occurs only in S_{KG} we have, equivalently:

$$\frac{\delta S_{KG}}{\delta \phi(x)} = 0$$

- Recall S_{KG} :

Expands as a potential

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c) Require: $\frac{\delta}{\delta \phi(x)} S_{tot}[g, \phi] = 0$

- Since ϕ occurs only in S_{KG} we have, equivalently:

$$\frac{\delta S_{KG}}{\delta \phi(x)} = 0$$

- Recall S_{KG}:

$$S_{KG}[\phi] = \frac{1}{2} \int_{\mathbb{R}^4} \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 - \lambda \phi^4 \right) \sqrt{|g|}\, d^4x$$

(Example of a potential)

- Apply the Euler Lagrange equations:

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- Recall S_{KG} :

$$S_{KG}[\phi] = \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - \underbrace{\lambda\phi^4}_{\text{Example of a potential}}\right)\sqrt{|g|}\,d^4x$$

- Apply the Euler Lagrange equations :

$$\frac{\delta S_{KG}[\phi,\{\phi_{,\mu}\},g]}{\delta\phi(x,t)} = \partial_\mu\frac{\delta S[\phi,\{\phi_{,\mu}\},g]}{\delta(\phi_{,\mu}(x,t))}$$

- $\Rightarrow$ Klein Gordon equation in general relativity :

$$\left(-\frac{1}{2}m^2 2\phi(x) - \frac{1}{2}\lambda 4\phi^3(x)\right)\sqrt{|g(x)|} = \partial_\mu\left(\frac{2}{2}g^{\mu\nu}(x)\phi_{,\nu}(x)\sqrt{|g(x)|}\right)$$

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$\Rightarrow$ Klein Gordon equation in general relativity:

$$\left(-\frac{1}{2}m^2 2\phi(x) - \frac{1}{2}\lambda 4\phi^3(x)\right)\sqrt{|g(x)|} = \partial_\mu\left(\frac{2}{2}g^{\mu\nu}(x)\,\phi_{,\nu}(x)\sqrt{|g(x)|}\right)$$

Thus:

$$\boxed{\frac{1}{\sqrt{|g(x)|}}\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}(x)\sqrt{|g(x)|}\,\phi_{,\nu}(x)\right) + m^2\phi^2(x) + 2\lambda\phi^3(x) = 0}$$

Definition: The "d'Alembert operator", $\Box$

$$\Box := \frac{1}{\sqrt{|g(x)|}}\frac{\partial}{\partial x^\mu}\,g^{\mu\nu}(x)\sqrt{|g(x)|}\frac{\partial}{\partial x^\nu}$$

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$$\left(-\frac{1}{2}m^2 2\phi(x) - \frac{1}{2}\lambda 4\phi^3(x)\right)\sqrt{|g(x)|} = \partial_\mu\left(\frac{2}{2}g^{\mu\nu}(x)\,\phi_{,\nu}(x)\sqrt{|g(x)|}\right)$$

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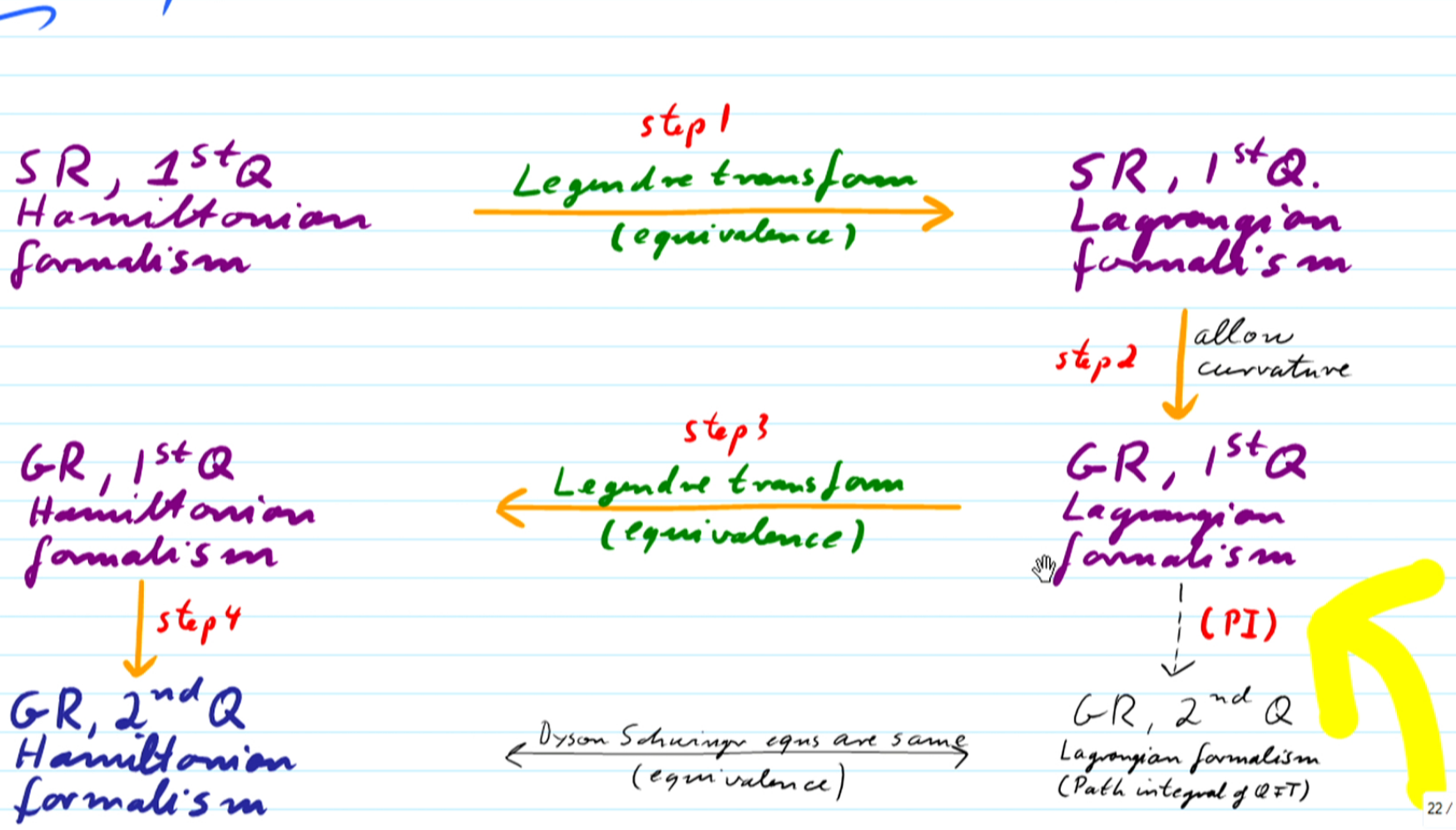
Thus:

$$\boxed{\Box\phi(x) + m^2\phi^2(x) + 2\lambda\phi^3(x) = 0}$$

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Next: Step 3 in



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Comment on step (PI): 2nd quantization with path integral

- Assume a fixed spacetime is chosen and we are given its metric $g_{\mu\nu}(x)$ in some arbitrary coordinate system.
- Then, for each field $\phi(\vec{x},t)$ we can calculate its action $S_{WC}[\phi,g]$:

$$S_{WC}[\phi,g] = \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - \lambda\phi^4\right)\sqrt{|g|}\,d^4x$$

- Following Feynman, we obtain "probability amplitudes":

$$\frac{i}{\hbar}S[\phi,g]$$

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- Assume a fixed spacetime is chosen and we are given its metric $g_{\mu\nu}(x)$ in some arbitrary coordinate system:
- Then, for each field $\phi(\vec{x},t)$ we can calculate its action $S_{KG}[\phi, g]$:

$$S_{KG}[\phi, g] \equiv \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - \lambda\phi^4 \right) \sqrt{|g|}\, d^4x$$

- Following Feynman, we obtain "probability amplitudes":

$$\text{prob. ampl.}[\phi] := e^{\frac{i}{\hbar} S_{KG}[\phi, g]}$$

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- Consider, e.g., the vacuum expectation value of $\phi(\vec{x},t)\,\phi(\vec{x}',t)$, i.e., the correlation function of field amplitudes:

$$G(\vec{x},t,\vec{x}',t) := \langle 0|\, \hat{\phi}(\vec{x},t)\,\hat{\phi}(\vec{x}',t)\,|0\rangle$$

- We will later see how to calculate it using commutation relations etc.

- With Feynman we also get it from the path integral:

Advantages:

1) Quick derivation of Feynman rules

2) Manifestly covariant.

$$G(\vec{x}-\vec{x}',t) = N \int \phi(\vec{x},t)\,\phi(\vec{x}',t)\, e^{\frac{i}{\hbar} S_{WW}[\phi,g]}\, \mathcal{D}[\phi]$$

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of field amplitudes:

$$G(\vec{x},t,\vec{x}',t) := \langle 0|\, \hat{\phi}(\vec{x},t)\, \hat{\phi}(\vec{x}',t)\,|0\rangle$$

- We will later see how to calculate it using commutation relations etc.

- With Feynman we also get it from the path integral:

$$G(\vec{x},t,\vec{x}',t) = N \int_{\text{all } \phi} \phi(\vec{x},t)\,\phi(\vec{x}',t)\, e^{\frac{i}{\hbar} S_{\text{tot}}[\phi,g]}\, \mathcal{D}[\phi]$$

↑ "Path Integral" over a function space

Advantages:

1) Quick derivation of Feynman rules
2) Manifestly covariant.

Problems:

1) Ill defined due to uncountable number of integrations and related divergencies.
2) Even when these are temporarily regularized, the identification of the vacuum state is ambiguous.
→ This issue is better handled in canonical formalism.

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