Title: Introduction to Quantum Field Theory for Cosmology - Lecture 9

Date: Feb 04, 2014 04:00 PM

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Abstract:

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Plan today:

- Functional derivatives $\quad \dfrac{\delta F[g]}{\delta g(x)} = ?$

- Example use 1: to make the QFT Schrödinger equation well defined.

- Example use 2: to define the Functional Legendre transform.

- Use both to obtain the Lagrangian formulation of QFT

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Plan today:

- Functional derivatives $\quad \frac{\delta F[g]}{\delta g(x)} = ?$

- Example use 1: to make the QFT Schrödinger equation well defined.

- Example use 2: to define the Functional Legendre transform.

- Use both to obtain the Lagrangian formulation of QFT – which will be starting point for QFT on curved space.

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Functional differentiation

Recall:

a.) Differentiation of functions of one variable, $F(u)$:

$$\frac{dF(u)}{du} := \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon) - F(u)}{\varepsilon}$$

b.) Differentiation of functions of countably many variables, $F(\{u_j\}_{j=1,2,3,\dots})$:

a.) Differentiation of functions of one variable, $F(u)$:

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b.) Differentiation of functions of countably many variables, $F(\{u_j\}_{j=1,2,3,\ldots})$:

$$\frac{dF(\{u_j\}_{j=1,2,\ldots})}{du_i} := \lim_{\varepsilon \to 0} \frac{F(\{u_j + \varepsilon\,\delta_{ij}\}_{j=1,\ldots}) - F(\{u_j\}_{j=1,\ldots})}{\varepsilon}$$

$$= \lim \quad F(u_1, \ldots, u_i+\varepsilon, \ldots) - F(u_1, \ldots, u_i, \ldots$$

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b) Differentiation of functions of countably many variables, $F(\{u_j\}_{j=1,2,3,\ldots})$:

$$\frac{dF(\{u_j\}_{j=1,2,\ldots})}{du_i} := \lim_{\varepsilon\to 0} \frac{F(\{u_j+\varepsilon\delta_{ij}\}_{j=1\ldots}) - F(\{u_j\}_{j=1,\ldots})}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{F(u_1,\ldots,u_i+\varepsilon,\ldots) - F(u_1,\ldots,u_i,\ldots)}{\varepsilon}$$

Definition:

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Definition:

c.) Differentiation of functions of uncountably many variables, $F(\{u(x)\}_{x\in\mathbb{R}^n})$:

Note: Since the Dirac delta is not a function but a distribution, which is only defined relative to an integral, the full definition is more technical.

$$\frac{\delta F(\{u(x)\}_{x\in\mathbb{R}^n})}{\delta u(y)} := \lim_{\varepsilon\to 0} \frac{F(\{u(x)+\varepsilon\delta^n(x-y)\}_{x\in\mathbb{R}^n}) - F(\{u(x)\}_{x\in\mathbb{R}^n})}{\varepsilon}$$

→ Since F is a "functional", i.e. is mapping functions to numbers

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$$= \lim_{\varepsilon\to 0} \frac{F(u_1,\dots,u_i+\varepsilon,\dots) - F(u_1,\dots,u_i,\dots)}{\varepsilon}$$

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$$\frac{\delta F(\{u(x)\}_{x\in\mathbb{R}^n})}{\delta u(y)} := \lim_{\varepsilon\to 0} \frac{F(\{u(x)+\varepsilon\delta^n(x-y)\}_{x\in\mathbb{R}^n}) - F(\{u(x)\}_{x\in\mathbb{R}^n})}{\varepsilon}$$

→ Since F is a "functional", i.e. is mapping functions to numbers

$$F:\ u \to F[u] \in \mathbb{C}$$

function ↑ short for $\{u(x)\}_{x\in\mathbb{R}^n}$

we call $\frac{\delta F}{\delta u(x)}$ a functional derivative.

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Then:

$$\frac{\delta F}{\delta u(y)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\int_{\mathbb{R}} \cos(x) \big(u(x) + \varepsilon \delta(x-y) \big)^2 - \int_{\mathbb{R}} \cos(x)\, u(x)^2 \, dx \right]$$

Distribution theory would be needed. But it drops out anyway

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \cos(x) \Big(\cancel{u(x)^2} + \varepsilon\, 2\, u(x) \delta(x-y) + \overbrace{\varepsilon^2 \delta^2(x-y)} - \cancel{u(x)^2} \Big)\, dx$$

$$= \lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}} 2 u(x)\, \delta(x-y) \cos(x)\, dx$$

$$= 2 \cos(y)\, u(y)$$

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Similarly, one obtains: $\frac{\delta}{\delta u(y)} \int_{\mathbb{R}} f(x)\, u(x)^n\, dx = f(y)\, n\, u(y)^{n-1}$

$\Rightarrow$ Functional derivatives act on polynomials (and suitable power series) in u by removing the integral and reducing the power in u by one, as expected from ordinary derivatives.

Remark: * Worked with $u(x)$.

* Would obtain same result if we used any other continuous or discrete basis of L^2.

* E.g. other basis (continuous): e^{ixp}, i.e. use $\tilde{u}(p)$

* E.g. other basis (countable): $H_n(x)e^{-x^2}$, i.e. use $\check{u}_n$

↑ Hermite polynomials

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Example application 1:

Schrödinger equation of QFT now well defined:

QM: $\hat{q}_j$ $\hat{p}_j$ j t

QFT: $\hat{\phi}(x)$ $\hat{\pi}(x)$ x t

Plays rôle of $V(\hat{q},t)$ although the first term is usually not considered to be part of the QFT's potential.

QM: $$\hat{H}(t) = \sum_{j=1}^{n} \frac{\hat{p}_j^2}{2} + V(\hat{q},t)$$

↑ all $\hat{q}_j$

QFT: $$\hat{H}(t) = \int_{\mathbb{R}^3} \frac{\hat{\pi}(x)^2}{2} + \frac{1}{2}\hat{\phi}(x)(m^2 - \Delta)\hat{\phi}(x) + W(\hat{\phi},t)\, d^3x$$

↑ Example: $W(\hat{\phi}) = \frac{1}{4!}\hat{\phi}^4(x,t)$

In general: $W(\hat{\phi})$ also contains other fields

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↳ Example: $W(\hat{\phi}) = \frac{1}{4!}\hat{\phi}^4(x,t)$

In general: $W(\hat{\phi})$ also contains other fields

QM: Example of complete set of commuting s.adj. operators: $\{\hat{q}_j\}_{j=1}^{m}$

QFT: Example of complete set of commuting s.adj. operators: $\{\hat{\phi}(x)\}_{x\in\mathbb{R}^3}$

QM: The joint eigenbasis $\{|\{q_j\}_{j=1}^{m}\rangle\}$ of the $\{\hat{q}_j\}_{j=1}^{m}$ obeys:

$$\hat{q}_i |\{q_j\}_{j=1}^{m}\rangle = q_i |\{q_j\}_{j=1}^{m}\rangle$$

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QM: The joint eigenbasis $\{|\{q_j\}_{j=1}^m\rangle\}$ of the $\{\hat{q}_j\}_{j=1}^m$ obeys:

$$\hat{q}_i\,|\{q_j\}_{j=1}^m\rangle = q_i\,|\{q_j\}_{j=1}^m\rangle$$

QFT: The joint eigenbasis $\{|\{\phi(x)\}_{x\in\mathbb{R}^3}\rangle\}$ of the $\{\hat{\phi}(x)\}_{x\in\mathbb{R}^3}$ obeys:

$$\hat{\phi}(y)\,|\{\phi(x)\}_{x\in\mathbb{R}^3}\rangle = \phi(y)\,|\{\phi(x)\}_{x\in\mathbb{R}^3}\rangle$$

QM: Wave function of a state $|\psi(t)\rangle \in \mathcal{H}$ in position eigenbasis:

$$\psi(\{q_j\}_{j=1}^m, t) = \langle\{q_j\}_{j=1}^m|\psi(t)\rangle \quad (\text{like } \psi(q) = \langle q|\psi\rangle)$$

Hilbert space of QFT, of course

QFT: Wave function of a state $|\Psi(t)\rangle \in \mathcal{H}$ in field eigenbasis:

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QM: Wave function of a state $|\psi(t)\rangle \in \mathcal{H}$ in position eigenbasis:

$$\psi(\{q_i\}_{i=1}^{n}, t) = \langle \{q_i\}_{i=1}^{n} | \psi(t) \rangle \quad (\text{like } \psi(q) = \langle q | \psi \rangle)$$

QFT: Wave functional of a state $|\Psi(t)\rangle \in \mathcal{H}$ in field eigenbasis:

Hilbert space of QFT, of course

$$\Psi[\{\phi(x)\}_{x \in \mathbb{R}^3}, t] = \langle \{\phi(x)\}_{x \in \mathbb{R}^3} | \Psi(t) \rangle$$

Probability amplitude for finding function $\phi(x)$ when measuring $\hat{\phi}(x)$ at t.

Simplified notation:

QM: $\psi(q,t) = \langle q | \psi(t) \rangle$

QFT: $\Psi[\phi, t] = \langle \phi | \Psi(t) \rangle$

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Simplified notation:

QM: $\psi(q,t) = \langle q | \psi(t) \rangle$

QFT: $\Psi[\phi,t] = \langle \phi | \Psi(t) \rangle$

QM: Representation of $\hat{q}_i, \hat{p}_j$ obeying $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$ in $\hat{q}$ eigenbasis:

$$\hat{q}_j: \quad \psi(q,t) \to q_j\, \psi(q,t)$$

$$\hat{p}_j: \quad \psi(q,t) \to -i\frac{\partial}{\partial q_j} \psi(q,t)$$

QFT: Representation of $\hat{\phi}(x), \hat{\pi}(y)$ obeying $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^3(x-y)$ in $\hat{\phi}$ eigenbasis:

$$\hat{\phi}(x): \quad \Psi[\phi,t] \to \phi(x)\, \Psi[\phi,t]$$

Exercise:

Verify that $\hat{\phi}(x)$, $\hat{\pi}$,

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$$\hat{\pi}(x): \quad \Psi[\phi,t] \to -i \frac{\delta}{\delta\phi(x)} \Psi[\phi,t]$$

Exercise: Verify that $\hat{\phi}(x), \hat{\pi}(x)$ obey the CCRs.

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QM: Schrödinger equation:

$$i\frac{d}{dt}\Psi(q,t) = \sum_{j=1}^{n} -\frac{1}{2}\frac{\partial^2}{\partial q_j^2}\Psi(q,t) + V(q,t)\Psi(q,t)$$

Recall: It is to be solved for all q

QFT: Schrödinger equation:

$$\boxed{i\frac{d}{dt}\Psi[\phi,t] = \int_{\mathbb{R}^3}\left(-\frac{1}{2}\frac{\delta^2}{\delta\phi(x)} + \frac{1}{2}\phi(x)(m^2-\Delta)\phi(x) + W(\phi(x),t)\,d^3x\right)\Psi[\phi,t]}$$

Recall: It is to be solved for all ϕ

Remark: With W it can be solved only perturbatively.

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$$i\frac{d}{dt}\Psi[\phi,t] = \int_{\mathbb{R}^3}\left(-\frac{1}{2}\frac{\delta^2}{\delta\phi(x)} + \frac{1}{2}\phi(x)(m^2-\Delta)\phi(x) + W(\phi(x),t)\,d^3x\right)\ \Psi[\phi,t]$$

Recall: It is to be solved for all ϕ

Remark: With W it can be solved only perturbatively.

Exercise: Set $W=0$. Fourier transform to k variables in box regularization. Verify that the wave functional Ψ_0 of the vacuum state obtained before does obey the Schr. eqn.

Example application 2: The functional Legendre transform!

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Example application 2: The functional Legendre transform!

- **Motivation?** We will need to determine in curved space:

  What becomes of: $\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$?

- **Problem?** Time is preferred coordinate in Hamiltonian formalism.

  * But the formalism must be coordinate system independent to fit general relativity (GR).

  * Now, for example, $\hat{\pi}(x,t) = \frac{d}{dt}\hat{\phi}(x,t)$ is not the same as $\hat{\pi}(x,\tau) = \frac{d}{d\tau}\hat{\phi}(x,\tau)$ for arbitrary $\tau(t)$

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What becomes of: $\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$?

Problem? Time is preferred coordinate in Hamiltonian formalism.

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$$\hat{\pi}(x,\tau) = \frac{d}{dt}\hat{\phi}(x,\tau(t)) = \frac{d}{d\tau}\hat{\phi}(x,\tau(t))\left(\frac{d\tau}{dt}\right) \neq \frac{d}{d\tau}\hat{\phi}(x,\tau)$$

!

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Strategy:

1. Transform to coordinate-independent Lagrange formalism.
2. Move from special to general relativity (GR).
3. Transform GR result back to Hamilton formalism.
4. Apply 2nd quantization.

$$\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{l}\text{SR, }1^{st}\text{Q}\\ \text{Hamiltonian}\\ \text{formalism}\end{array} \arrow[r, "\text{“Legendre transform”}", "\text{equivalence}"'] & \begin{array}{l}\text{SR, }1^{st}\text{Q.}\\ \text{Lagrangian}\\ \text{formalism}\end{array} \arrow[d, "\begin{array}{l}\text{allow}\\ \text{curvature}\end{array}"] \\
\begin{array}{l}\text{GR, }1^{st}\text{Q}\\ \text{Hamiltonian}\\ \text{formalism}\end{array} \arrow[d, "\begin{array}{l}\text{as outlined}\\ \text{already}\end{array}"] & \begin{array}{l}\text{GR, }1^{st}\text{Q}\\ \text{Lagrangian}\\ \text{formalism}\end{array} \arrow[l, "\text{Legendre transform}"', "\text{equivalence}"] \arrow[d, dashed] \\
 &
\end{tikzcd}$$

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SR, 1^{st}Q
Hamiltonian
formalism

"Legendre transform"
equivalence
$\longrightarrow$

SR, 1^{st}Q.
Lagrangian
formalism

$\downarrow$ allow curvature

GR, 1^{st}Q
Hamiltonian
formalism

Legendre transform
equivalence
$\longleftarrow$

GR, 1^{st}Q
Lagrangian
formalism

$\downarrow$ as outlined already

GR, 2^{nd}Q
Hamiltonian
formalism

$\longleftrightarrow$
Dyson Schwinger eqns are same
equivalence

GR, 2^{nd} Q
Lagrangian formalism
(Path integral of QFT)

The Legendre transform (LT):

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The Legendre transform (LT):

- Assume given a function, $F(u)$.

- Define a new variable $w(u)$:

$$w(u) := \frac{dF}{du} \qquad \text{(I)}$$

- Assume that (I) can be solved to obtain:

$$u(w)$$

(that's ok if F is convex, say $F''(u) > 0$ for all u)

- The Legendre transform of F is a new function, G, of w:

$$F(u) \xrightarrow{LT} G(w)$$

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- Namely: $\boxed{G(w) := w\, u(w) - F(u(w))}$

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- Namely: $\boxed{G(w) := w\, u(w) - \mathcal{F}(u(w))}$

Proposition:

$$(LT)^2 = id$$

Proof:

- Define a new variable: $v(w) := \dfrac{\partial G(w)}{\partial w}$
- In fact:

$$v(w) = \frac{\partial}{\partial w}\Big(w\, u(w) - \mathcal{F}(u(w))\Big)$$

$$= u(w) + w\, \cancel{\partial u(w)} - \cancel{\partial \mathcal{F}(u(w))\, \partial u(w)}$$

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$$= u(w) + \cancel{w\,\frac{\partial u(w)}{\partial w}} - \cancel{\underbrace{\frac{\partial F(u(w))}{\partial u}}_{=\,w}\,\frac{\partial u(w)}{\partial w}}$$

$$= u\ !$$

- Therefore LT^2 yields $F(u) \xrightarrow{LT} G(w) \xrightarrow{LT} H(v)$ with:

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$$H = vw - G = \underbrace{v}_{= u \text{ from just above}}w - (wu - F) = F \quad \checkmark$$

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$(LT)^2 = 1d$

Proof:

- Define a new variable: $v(w) := \frac{\partial G(w)}{\partial w}$
- In fact:

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$$= u(w) + \cancel{w\,\frac{\partial u(w)}{\partial w}} - \cancel{\underbrace{\frac{\partial F(u(w))}{\partial u}}_{= w}\frac{\partial u(w)}{\partial w}}$$
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$$H = vw - G = \underset{\substack{\| \\ u \text{ from just above}}}{v}w - (wu - F) = F \quad \checkmark$$

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Example:

* Consider $f(a,b,c) := a e^{bc}$

* Find LT with respect to b (i.e. while treating a, c as "spectator variables":

$$f(a,b,c) \xrightarrow[b\to\beta]{LT} g(a,\beta,c)$$

* Define $\beta(a,b,c) := \frac{\partial f}{\partial b} = a c e^{bc}$

* Invert: $b(a,\beta,c) = \frac{1}{c} \ln \frac{\beta}{ac}$

* Legendre transform: $f(a,b,c) \xrightarrow{LT} g(a,\beta,c)$

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* Legendre transform: $f(a,b,c) \xrightarrow{LT} g(a,\beta,c)$

$$g(a,\beta,c) := \beta\, b(a,\beta,c) - f(a, b(a,\beta,c), c)$$

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$$f(a,b,c) \xrightarrow[b\to\beta]{LT} g(a,\beta,c)$$

* Define $\beta(a,b,c) := \frac{\partial f}{\partial b} = a c e^{bc}$

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* Legendre transform: $f(a,b,c) \xrightarrow{LT} g(a,\beta,c)$

$$g(a,\beta,c) := \beta\, b(a,\beta,c) - f(a, b(a,\beta,c), c)$$

$$g(a,\beta,c) = \frac{\beta}{c} \ln \frac{\beta}{ac} - a e^{\frac{c}{c} \ln \frac{\beta}{ac}} = \frac{\beta}{c} \ln \frac{\beta}{ac} - \frac{\beta}{c}$$

Case of countably many variables:

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Case of countably many variables:

- How to define

$$F(\{u_j\}) \xrightarrow{LT} G(\{w_j\}) \ ?$$

- Define: $w_j := \frac{\partial F}{\partial u_j}$

- Assume we can invert to obtain:

$$u_j(\{w_i\})$$

- Define:

$$G(\{w_j\}) := \sum_i w_i u_i(\{w_i\}) - F(\{u_j(\{w_i\})\})$$

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- Assume we can invert to obtain:

$$u_j(\{w_i\})$$

- Define:

$$G(\{w_j\}) := \sum_j w_j u_j(\{w_i\}) - \mathcal{F}(\{u_j(\{w_i\})\})$$

(we may also allow for spectator variables)

Case of uncountably many variables:

- How to define

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Case of uncountably many variables:

- How to define

$$F[\{u(x)\}_{x\in\mathbb{R}^n}] \xrightarrow{LT} G[\{w(x)\}_{x\in\mathbb{R}^n}] \;?$$

- Define:

$$w(x) := \frac{\delta F}{\delta u(x)}$$

- Assume we can solve to obtain:

$$u(x, \{w(x')\}_{x\in\mathbb{R}^n})$$

- Define:

$$G[\{w(x)\}_{x\in\mathbb{R}}] := \int w(x)\, u(x, \{w(x')\}_{x'\in\mathbb{R}^n})\, d^n x - F[\{u(x, \{w(x')\})\}$$

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- Define:

$$w(x) := \frac{\delta F}{\delta u(x)}$$

- Assume we can solve to obtain:

$$u\left(x, \{w(x')\}_{x' \in \mathbb{R}^n}\right)$$

- Define:

$$G[\{w(x)\}_{x\in\mathbb{R}}] := \int_{\mathbb{R}^n} w(x)\, u(x, \{w(x')\}_{x'\in\mathbb{R}^n})\, d^n x - F[\{u(x, \{w(x')\})\}]$$

- Note: We still have that $LT \circ LT = id$.

← classical mechanics

Application to CM:

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↙ classical mechanics

Application to CM:

* Assume the Hamiltonian $H(q,p)$ is given.

* Hamilton equations for arbitrary $f(q,p)$:

Recall: Poisson bracket $\{q,p\}=1$

$$\dot{f}(q,p) = \{f(q,p), H(q,p)\}$$

See my notes to AMATH 673:

Dirac showed: Quantization consists in keeping the Poisson bracket definition and the Hamilton equations unchanged while allowing q,p noncommutativity in such a way that the Poisson algebra structure stays. This fixes noncommutativity to be $\hat{q}\hat{p}-\hat{p}\hat{q}=i\hbar$ and $\{\hat{f},\hat{g}\}=\frac{1}{i\hbar}\{\hat{f},\hat{g}\}$

* From this, one can prove the eqns of motion for q,p:

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}, \qquad \dot{p} = -\frac{\partial H(q,p)}{\partial q} \qquad \text{(EoM)}$$

* Legendre transform: ⟵ The "Lagrangian"

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$$\dot{f}(q,p) = \{f(q,p), H(q,p)\}$$

See my notes to AMATH 673:

Dirac showed: Quantization consists in keeping the Poisson bracket definition and the Hamilton equations <u>unchanged</u> while allowing q,p noncommutativity in such a way that the Poisson algebra structure stays. This fixes noncommutativity to be $\hat{q}\hat{p}-\hat{p}\hat{q}=i\hbar$ and $\{\hat{f},\hat{g}\}=\frac{1}{i\hbar}[\hat{f},\hat{g}]$

* From this, one can prove the eqns of motion for q,p:

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}, \qquad \dot{p} = -\frac{\partial H(q,p)}{\partial q} \qquad (EoM)$$

* Legendre transform:

The "Lagrangian"

$$H(q,p) \xrightarrow{LT} L(q,b) \qquad (q \text{ is spectator})$$

* Example: $H(q,p) := \frac{p^2}{2} + V(q)$

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* Example: $H(q,p) := \frac{p^2}{2} + V(q)$

$$b = \frac{\partial H(q,p)}{\partial p} = \dot{q}$$ ← Notice: This arose due to $\frac{p^2}{2}$ term.

$$\Rightarrow \quad L(q,b) = L(q,\dot{q}) = \dot{q}\, p(q,\dot{q}) - H(q, p(q,\dot{q}))$$

Proposition:

The equations of motion (EoM) now take the form:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$ (Euler Lagrange equation)

Proof: Exercise

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Example: $H = \frac{p^2}{2m} + \frac{\omega^2}{2} q^2 \xleftrightarrow{LT} L = \frac{1}{2}\dot q^2 - \frac{\omega^2}{2} q^2$

$\dot q = p, \ \dot p = -\omega^2 q \qquad\qquad -\omega^2 q = \ddot q$

classical (not conformal) field theory

Application to CFT:

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Application to CFT:

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Application to CFT:

- Assume Hamiltonian $H(\phi, \pi)$ is given.
- Hamilton equation for arbitrary $f(\phi, \pi)$:

$$\dot{f}(\phi, \pi, x, t) = \{ f(\phi, \pi, x, t), H(\phi, \pi) \}$$

with: $\{ \phi(x,t), \pi(x',t) \} = \delta^3(x - x')$

- This yields the eqns of motion:

$$\dot{\phi}(x,t) = \frac{\delta H}{\delta \pi(x,t)} \qquad \dot{\pi}(x,t) = -\frac{\delta H}{\delta \phi(x,t)} \qquad \text{(EoM)}$$

- Legendre Transform:

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- Legendre Transform:

spectator

$$H(\phi,\pi) \xrightarrow{LT} L(\phi,\dot{\phi})$$

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Legendre Transform:

$$H(\phi,\pi) \xrightarrow{LT} L(\phi, \xi)$$

spectator

Example: $H := \int \frac{1}{2}\pi(x,t)^2 + V(\phi(x))\, d^3x$

$$\xi(x,t) := \frac{\delta H}{\delta \pi(x,t)}$$

$$= \dot{\phi}(x,t)$$

← Notice: this is because of the particular π^2 term in H. On curved space it will be different.

Thus:

$$L(\phi,\pi) = L(\phi,\dot{\phi})$$

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← Notice: this is because of the particular π^2 term in H. On curved space it will be different.

Thus:

$$L(\phi,\pi) = L(\phi,\dot\phi)$$

$$= \int_{\mathbb{R}^3} \dot\phi(x,t)\,\pi(\phi,\dot\phi,x,t)\,d^3x - H(\phi,\pi(\phi,\dot\phi,x,t))$$

Proposition: The eqns of motion (EoM) are equivalent to:

$$\frac{\delta L}{\delta\phi(x,t)} = \frac{d}{dt}\frac{\delta L}{\delta\dot\phi(x,t)}$$

Exercise: Check

Euler Lagrange eqn.

Example:

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Example:

$$H(\phi,\pi) = \int_{\mathbb{R}^3} \frac{\pi^2(x,t)}{2} + \frac{1}{2}\phi(x,t)\left(m^2-\Delta\right)\phi(x,t)\, d^3x$$

yields: $\dot{\phi}(x,t) = \pi(x,t)$ $\quad \dot{\pi}(x,t) = (-m^2+\Delta)\phi(x,t)$

i.e: $\ddot{\phi} - \Delta\phi + m^2\phi = 0$ K.G. eqn.

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yields directly: $-(m^2-\Delta)\,\phi - \ddot{\phi}$

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