Title: Noncontextuality without determinism and admissible (in)compatibility relations: revisiting Specker's parable.

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Abstract: The purpose of this talk is twofold: First, following Spekkens, to motivate noncontextuality as a natural principle one might expect to hold in nature and introduce operational noncontextuality inequalities motivated by a contextuality scenario first considered by Ernst Specker. These inequalities do not rely on the assumption of outcome-determinism which is implicit in the usual Kochen-Specker (KS) inequalities. We argue that they are the appropriate generalization of KS inequalities, serving as a test for the possibility of noncontextual explanations of experimental data. This is very much in the spirit of Bell inequalities, which provide theory-independent tests for local hidden variable explanations of experimental data without relying on the assumption of outcome-determinism. The second purpose is to point out a curious feature of quantum theory, motivated by the connections between (in)compatibility and (non)contextuality: namely, that it admits all conceivable (in)compatibility relations between observables.

Noncontextuality without determinism and admissible (in)compatibility relations: revisiting Specker's parable.

Ravi Kunjwal

January 14, 2014

The Institute of Mathematical Sciences ('Matscience'), Chennai.

Note

Important references:

- Liang, Spekkens, and Wiseman (LSW): "Specker's parable of the overprotective seer: A road to contextuality, nonlocality and complementarity" (Physics Reports 506.1 (2011): 1-39; arXiv:1010.1273 [quant-ph]).
- Ernst Specker's 1960 work on the logic of quantum theory. An English translation of the original German paper by Ernst Specker, "The logic of non-simultaneously decidable propositions" (1960), due to M.P. Seevinck, can be found at arXiv:1103.4537 [physics.hist-ph].
- Rob's talks! (PIRSA:10100060 and PIRSA:10120065)

Operational theories and Ontological models

Noncontextuality

Specker's scenario

Odd n-cycle scenarios

Joint measurability

Takeaway

Motivation

To pin down the classical/nonclassical divide operationally. The specific notion of classicality we study here is noncontextuality.

Questions

- What is an *operational theory*, its *ontological model*, and how do the two *fit* together? Definitions.
- What does it mean for an ontological model of an operational theory to be noncontextual? *Noncontextuality*: ontological differences must imply operational differences.
- Does the operational statistics of an experiment admit a noncontextual model? Noncontextuality (NC) inequalities.

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- Does the operational statistics of an experiment admit a noncontextual model? Noncontextuality (NC) inequalities.

Noncontextuality

A methodological principle: If two experimental procedures are operationally indistinguishable then they must also be ontologically indistinguishable—the ontological identity of operational indiscernables. [1]

[1] Rob Spekkens, Phys. Rev. A 71, 052108 (2005)

Noncontextuality

Preparation noncontextuality (PNC) is expressed as the following inference from the operational theory to its ontological model:

$$\begin{aligned} & p(k|P,M) = p(k|P',M), \forall (M,\mathcal{K}_M) \in \mathcal{M} \text{ or } P \simeq P' \\ \Rightarrow \quad & \mu_P(\lambda) = \mu_{P'}(\lambda), \forall \lambda \in \Lambda. \end{aligned} \tag{2}$$

That is, two preparations P and P' which are operationally indistinguishable are represented by identical distributions in the ontological model. Note that $P \simeq P'$ denotes the operational equivalence of preparation procedures P and P'.

Outcome-determinism

Outcome-determinism (OD) is the assumption that the ontological response functions for every measurement procedure $(M, \mathcal{K}_M) \in \mathcal{M}$ in the operational theory are deterministic: $\xi(k|M, \lambda) \in \{0, 1\}$ for all $k \in \mathcal{K}_M, \lambda \in \Lambda$.

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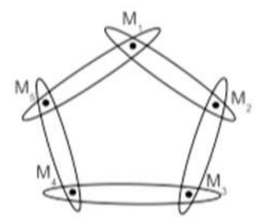
Kochen-Specker theorem

In a Hilbert space of dimension 3 or higher, a measurement noncontextual and outcome-deterministic (or KS-noncontextual) model of quantum theory is impossible, e.g., the KCBS inequality:



$$R_5(P) \leq \frac{4}{5} = 0.8, \tag{4}$$

where

$$R_5(P) \equiv \frac{1}{5} \sum_{(ij)} Pr(X_i \neq X_j | M_{ij}; P). \tag{5}$$

Assumptions:

- MNC: the probability, $\xi(X_i|M_i, \lambda)$, is independent of the joint measurement context.
- OD: $\xi(X_i|M_i, \lambda) \in \{0, 1\}$.

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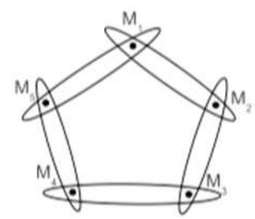

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Kochen-Specker theorem

In quantum theory, the maximum value of $R_5(P)$ is given by

$$R_5(P) = \frac{2}{\sqrt{5}} \approx 0.89442. \quad (6)$$

Noncontextual model

If the assumption of outcome-determinism is relaxed, a noncontextual model,

$$\xi(X_i = 0, X_j = 1|M_{ij};\lambda) = \xi(X_i = 1, X_j = 0|M_{ij};\lambda) = \frac{1}{2}$$

for all $\lambda \in \Lambda$, would achieve perfect anticorrelation:

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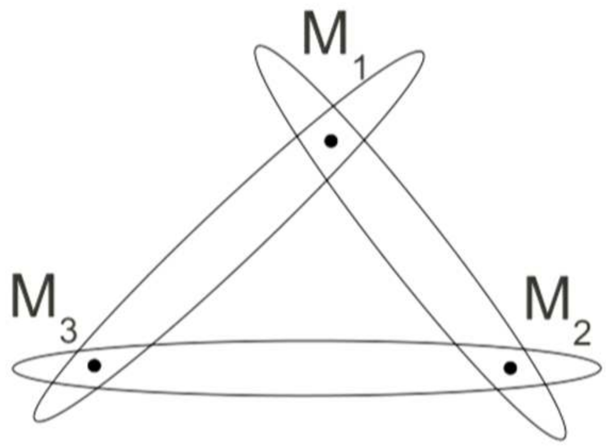
Noncontextuality (NC) inequalities

- If the operational theory of interest is quantum theory, it can be shown that $PNC \Rightarrow OD$ for sharp (projective) measurements (ODSM).
- However, $PNC \Rightarrow ODSM$ has not been established without assuming quantum theory. The problem is twofold: first, define an operational notion of sharpness, and second, show $PNC \Rightarrow$ ODSM for any operational theory.
- We circumvent this problem by the time-honoured method of ignoring it.
- Instead, we ask: Is it possible to prove operational noncontextuality inequalities using only the assumptions of PNC and MNC, under some operational equivalences that can be verified experimentally?
- Our answer: Yes! Noncontextuality without determinism does impose nontrivial constraints on the operational statistics.

Specker's scenario



Three binary-outcome measurements, denoted M_1, M_2, and M_3, each with outcome space $\{0, 1\}$, for which every pair is jointly measurable.

Specker's scenario

For every pair $\{M_i, M_j\}$ where $(i,j) \in \{(1,2),(2,3),(3,1)\}$, there is a four-outcome measurement M_{ij}, such that the operational statistics of measurements M_i and M_j are recovered as marginals of the operational statistics of M_{ij}. We denote the outcome of M_{ij} by (X_i, X_j) and let $M_i^{(j)}$ ($M_j^{(i)}$) denote the coarse-graining over X_j (X_i) of M_{ij}:

$$p(X_i|M_i^{(j)}, P) \equiv \sum_{X_j} p(X_i, X_j|M_{ij}, P), \tag{8}$$

$$p(X_j|M_j^{(i)}, P) \equiv \sum_{X_i} p(X_i, X_j|M_{ij}, P). \tag{9}$$

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Specker's scenario

We can express the assumption of pairwise joint measurability of M_1, M_2 and M_3 as the following operational equivalences

$$M_1^{(2)} \simeq M_1^{(3)} \simeq M_1, \tag{10}$$

$$M_2^{(1)} \simeq M_2^{(3)} \simeq M_2, \tag{11}$$

$$M_3^{(1)} \simeq M_3^{(2)} \simeq M_3. \tag{12}$$

We also consider a second set of measurements, M'_{12}, M'_{23}, and M'_{31}, where M'_{ij} acheives a joint measurement of M_i and M_j. Consequently, we also have

$$M_1'^{(2)} \simeq M_1'^{(3)} \simeq M_1, \tag{13}$$

$$M_2'^{(1)} \simeq M_2'^{(3)} \simeq M_2, \tag{14}$$

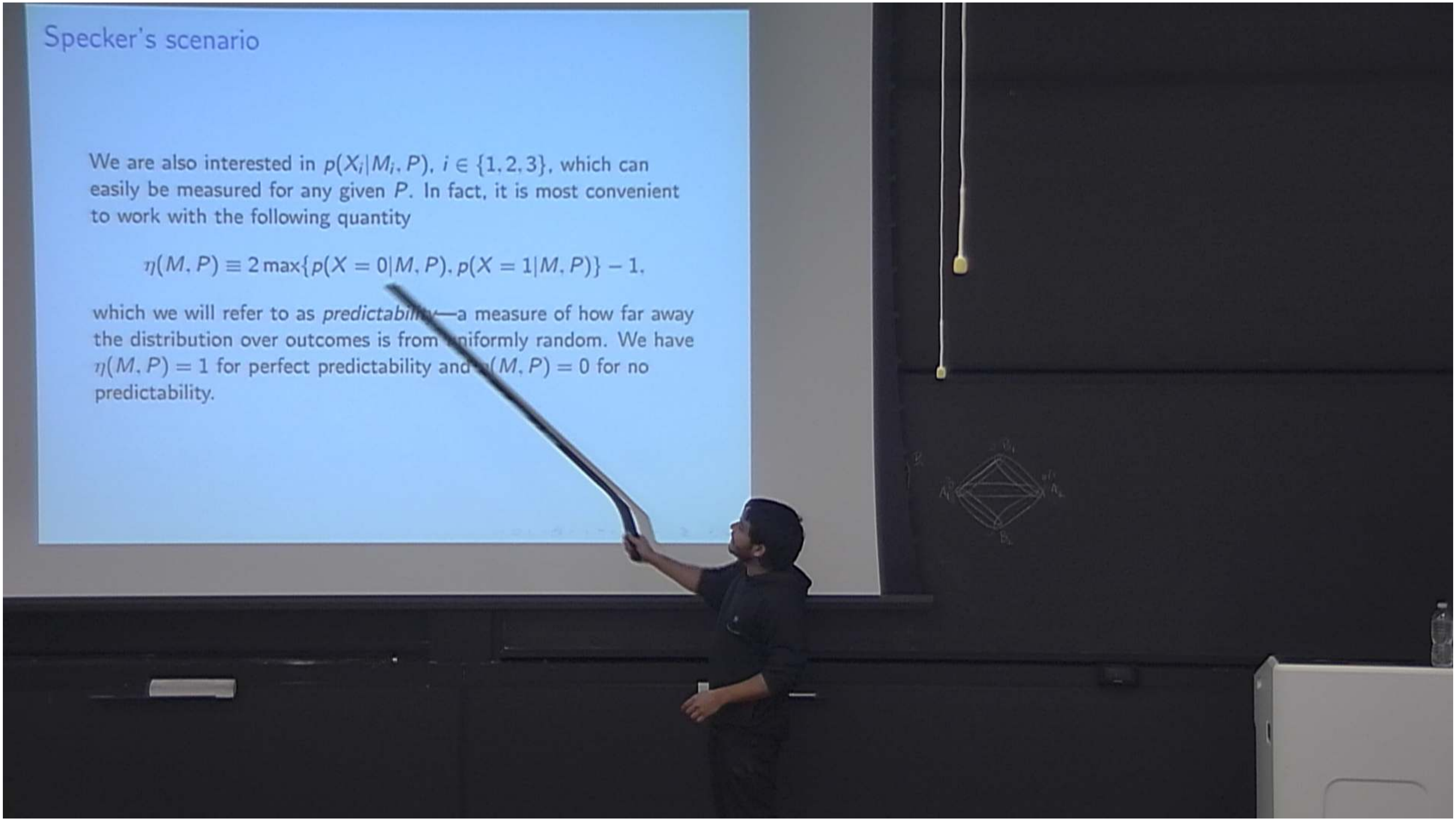
$$M_3'^{(1)} \simeq M_3'^{(2)} \simeq M_3. \tag{15}$$

Specker's scenario

We are also interested in $p(X_i|M_i, P)$, $i \in \{1, 2, 3\}$, which can easily be measured for any given P. In fact, it is most convenient to work with the following quantity

$$\eta(M, P) \equiv 2\max\{p(X = 0|M, P), p(X = 1|M, P)\} - 1,$$

which we will refer to as *predictability*—a measure of how far away the distribution over outcomes is from uniformly random. We have $\eta(M, P) = 1$ for perfect predictability and $\eta(M, P) = 0$ for no predictability.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Noncontextuality inequality for Specker's scenario

If the ontological model for Specker's scenario is noncontextual (PNC and MNC), the following inequality on operational quantities will also hold:

$$p(\mathrm{anti}|M_*, P_*) + p(\mathrm{anti}|M'_*, P_*^{\perp}) \leq 2 - \frac{1}{9}\sum_{i=1}^{3}\left(\eta(M_i, P_i) + \eta(M_i, P_i^{\perp})\right). \qquad (19)$$

This inequality quantifies the tradeoff between the achievable anticorrelation and the predictabilities of the three measurements.

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- The corresponding KS-inequality for this scenario is $p(\mathrm{anti}|M_*, P_*) + p(\mathrm{anti}|M'_*, P_*^\perp) \leq 4/3$ which would follow from the assumption of outcome-determinism. In our treatment, however, this inequality is justified if and only if $\eta(M_i, P_i) = \eta(M_i, P_i^\perp) = 1$ for all $i \in \{1, 2, 3\}$.
- If, for example, there is no predictability, i.e., $\eta(M_i, P_i) = \eta(M_i, P_i^\perp) = 0$ for all $i \in \{1, 2, 3\}$, it is possible to achieve perfect anticorrelation in a noncontextual model.
- In general, for $\eta(M_i, P_i), \eta(M_i, P_i^\perp) \in (0, 1]$, the assumption of noncontextuality constrains the achievable anticorrelation nontrivially.

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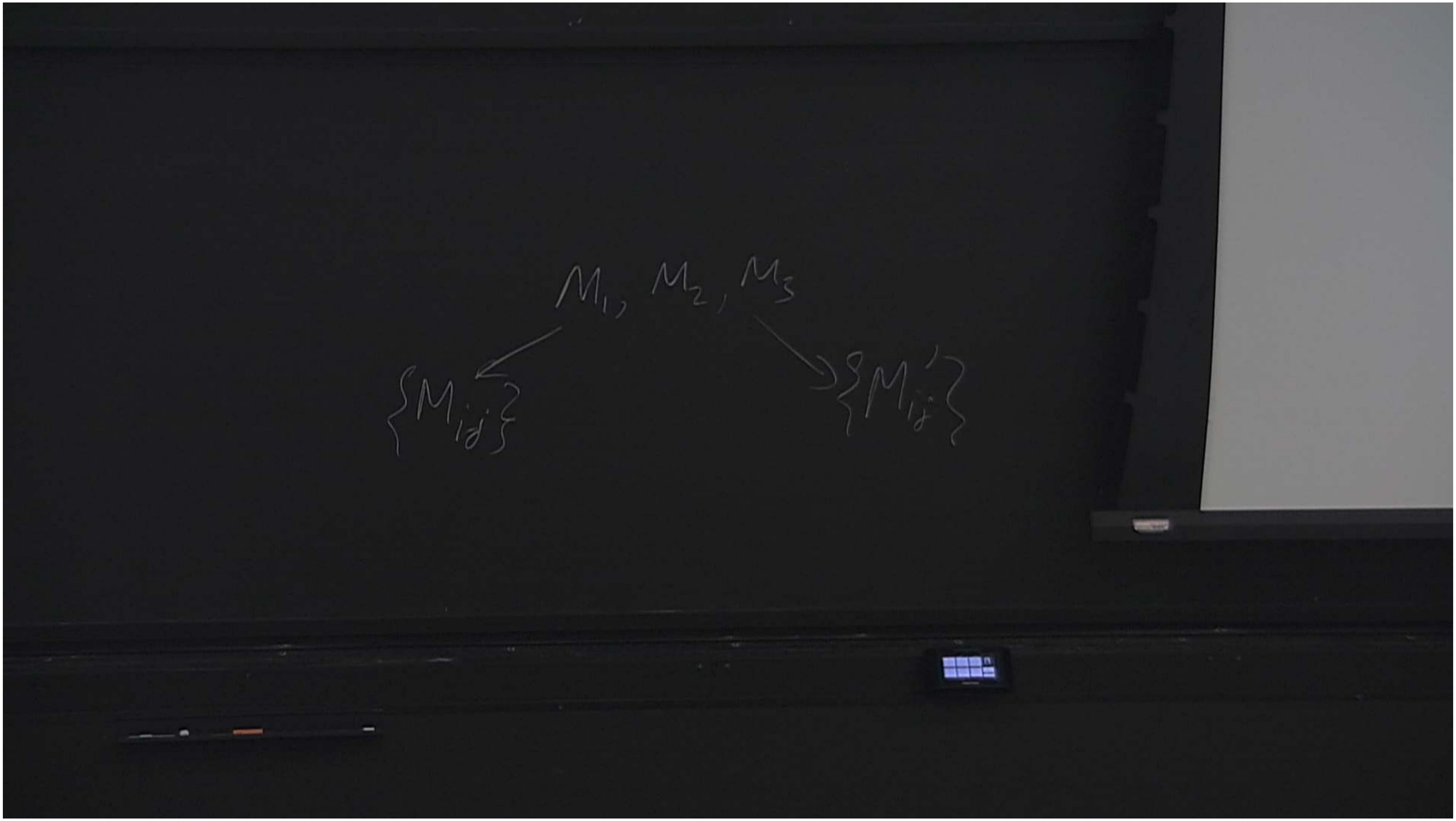
M_1, M_2, M_3
$\{M_{ij}\}$
$\{M'_{ij}\}$

Quantum violation

We take $\{M_1, M_2, M_3\}$ to be three qubit measurements which are pairwise jointly measurable: M_i is associated with the qubit POVM $\{E_0^{(i)}, E_1^{(i)}\}$, where $E_b^{(i)}$ corresponds to the outcome $X_i = b$. In particular, we assume

$$E_0^{(i)} \equiv \frac{1}{2}I + \frac{1}{2}\eta_0\vec{\sigma}\cdot\hat{n}_i,$$
$$E_1^{(i)} \equiv \frac{1}{2}I - \frac{1}{2}\eta_0\vec{\sigma}\cdot\hat{n}_i, \tag{20}$$

where $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ is the vector of qubit Pauli matrices, and I is the identity matrix: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The three measurements $\{M_1, M_2, M_3\}$ correspond to three choices of measurement directions $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ in the Bloch sphere.

Quantum violation

Furthermore, we consider two sets of pairwise joint measurements denoted by $\{M_{12}, M_{23}, M_{31}\}$ and $\{M'_{12}, M'_{23}, M'_{31}\}$, where M_{ij} is associated with the qubit POVM $\{E^{(ij)}_{00}, E^{(ij)}_{01}, E^{(ij)}_{10}, E^{(ij)}_{11}\}$ and M'_{ij} is associated with the qubit POVM $\{E'^{(ij)}_{00}, E'^{(ij)}_{01}, E'^{(ij)}_{10}, E'^{(ij)}_{11}\}$. These POVMs are given by:

$$
\begin{aligned}
E^{(ij)}_{00} &= \frac{1}{2}\left[\frac{\alpha_{ij}}{2} I + \vec{\sigma}.\frac{1}{2}\left(\eta_0(\hat{n}_i + \hat{n}_j) - \vec{a}_{ij}\right)\right] \\
E^{(ij)}_{01} &= \frac{1}{2}\left[\left(1 - \frac{\alpha_{ij}}{2}\right) I + \vec{\sigma}.\frac{1}{2}\left(\eta_0(\hat{n}_i - \hat{n}_j) + \vec{a}_{ij}\right)\right] \\
E^{(ij)}_{10} &= \frac{1}{2}\left[\left(1 - \frac{\alpha_{ij}}{2}\right) I + \vec{\sigma}.\frac{1}{2}\left(\eta_0(-\hat{n}_i + \hat{n}_j) + \vec{a}_{ij}\right)\right] \\
E^{(ij)}_{11} &= \frac{1}{2}\left[\frac{\alpha_{ij}}{2} I + \vec{\sigma}.\frac{1}{2}\left(\eta_0(-\hat{n}_i - \hat{n}_j) - \vec{a}_{ij}\right)\right]
\end{aligned}
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Furthermore, we consider two sets of pairwise joint measurements denoted by $\{M_{12}, M_{23}, M_{31}\}$ and $\{M'_{12}, M'_{23}, M'_{31}\}$, where M_{ij} is associated with the qubit POVM $\{E_{00}^{(ij)}, E_{01}^{(ij)}, E_{10}^{(ij)}, E_{11}^{(ij)}\}$ and M'_{ij} is associated with the qubit POVM $\{E_{00}^{\prime(ij)}, E_{01}^{\prime(ij)}, E_{10}^{\prime(ij)}, E_{11}^{\prime(ij)}\}$. These POVMs are given by:

$$
\begin{aligned}
E_{00}^{(ij)} &= \frac{1}{2}\left[\frac{\alpha_{ij}}{2} I + \vec{\sigma}.\frac{1}{2}\left(\eta_0(\hat{n}_i + \hat{n}_j) - \vec{a}_{ij}\right)\right] \\
E_{01}^{(ij)} &= \frac{1}{2}\left[\left(1 - \frac{\alpha_{ij}}{2}\right) I + \vec{\sigma}.\frac{1}{2}\left(\eta_0(\hat{n}_i - \hat{n}_j) + \vec{a}_{ij}\right)\right] \\
E_{10}^{(ij)} &= \frac{1}{2}\left[\left(1 - \frac{\alpha_{ij}}{2}\right) I + \vec{\sigma}.\frac{1}{2}\left(\eta_0(-\hat{n}_i + \hat{n}_j) + \vec{a}_{ij}\right)\right] \\
E_{11}^{(ij)} &= \frac{1}{2}\left[\frac{\alpha_{ij}}{2} I + \vec{\sigma}.\frac{1}{2}\left(\eta_0(-\hat{n}_i - \hat{n}_j) - \vec{a}_{ij}\right)\right]
\end{aligned}
$$

One can easily verify that for $b \in \{0, 1\}$

$$\begin{aligned} E_{b0}^{(ij)} + E_{b1}^{(ij)} &= E_b^{(i)} \\ E_{0b}^{(ij)} + E_{1b}^{(ij)} &= E_b^{(j)} \\ E_{b0}'^{(ij)} + E_{b1}'^{(ij)} &= E_b^{(i)} \\ E_{0b}'^{(ij)} + E_{1b}'^{(ij)} &= E_b^{(j)} \end{aligned} \tag{21}$$

and therefore the operational equivalences of Eq. (17) hold for these choices.

It follows that

$$p(\mathrm{anti}|M_*, P_*) + p(\mathrm{anti}|M'_*, P_*^{\perp})$$
$$= 2 - \frac{1}{3} \sum_{(i,j)} (\alpha_{ij} - \hat{n}_* \cdot \vec{a}_{ij})\,. \qquad (27)$$

We choose our measurements in an equatorial plane of the Bloch sphere, say the Z-X plane, and the vector $\hat{n}_*$ (associated with preparation P_*) perpendicular to this plane:

$$\begin{aligned}\hat{n}_1 &\equiv (0,0,1),\\ \hat{n}_2 &\equiv (\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}),\\ \hat{n}_3 &\equiv (-\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}).\\ \hat{n}_* &\equiv (0,1,0).\end{aligned} \tag{28}$$

We choose all of the $\vec{a}_{ij}$ to be collinear with $\hat{n}_*$

$$\forall(i,j) : \vec{a}_{ij} = \sqrt{1 + \eta_0^4(\hat{n}_i \cdot \hat{n}_j)^2 - 2\eta_0^2}\,\hat{n}_*$$

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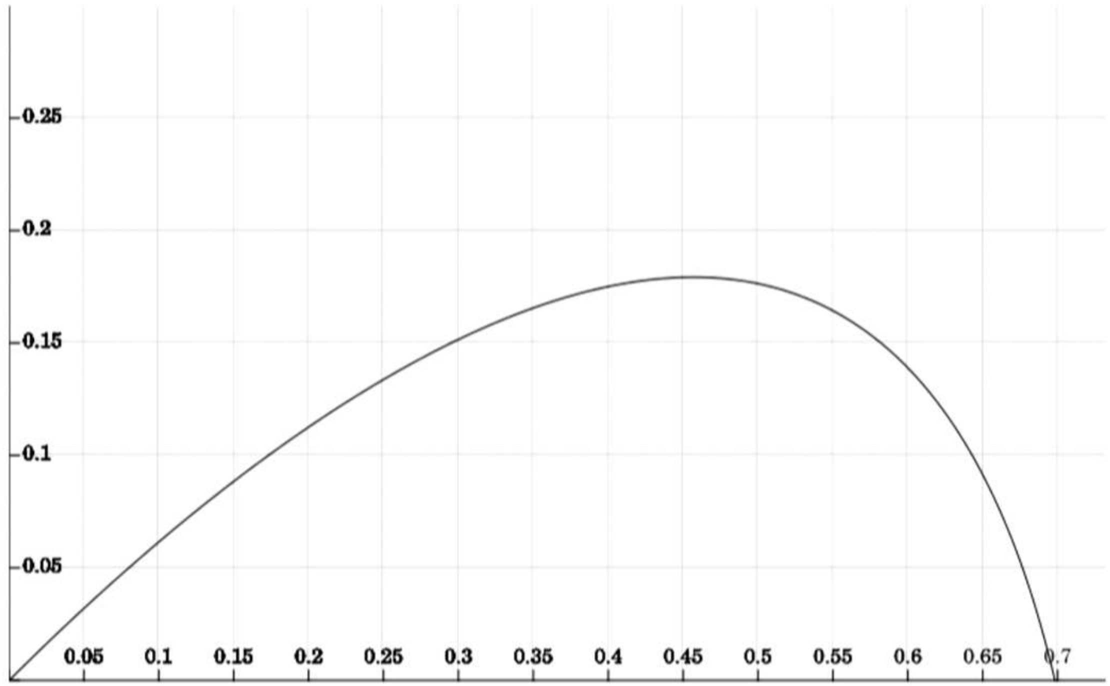


The maximum quantum violation from our construction is 0.1793 for $\eta_0 \approx 0.4566$.

Noncontextual inequalities for odd n-cycle scenarios

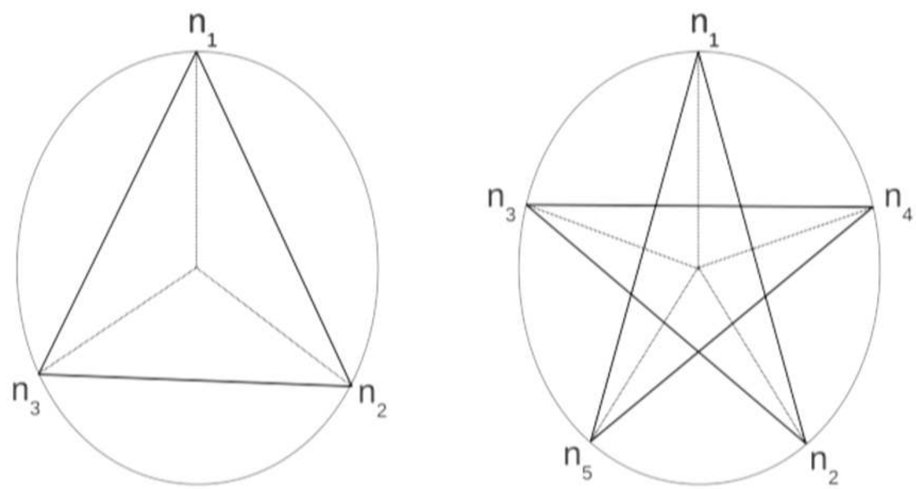
Let $\{M_1, \ldots, M_n\}$ be measurements each with outcome space $\{0,1\}$, and let M_{ij}, as well as M'_{ij}, where $i \in \{1, \ldots, n\}$ and $j = i + 1 \mod n$, be measurements each with outcome space $\{0,1\} \times \{0,1\}$. Let M_* be the measurement that is the equal mixture of M_{ij} and let M'_* be the measurement that is the equal mixture of M'_{ij}. Let P_*, $P_*^{\perp}$, P_i, $P_i^{\perp}$, $i \in \{1, \ldots, n\}$, be preparation procedures.

If the ontological model is noncontextual (PNC and MNC), the following inequality on operational quantities will also hold:

$$p(\mathrm{anti}|M_*, P_*) + p(\mathrm{anti}|M'_*, P_*^\perp) \leq 2 - \frac{1}{n^2}\sum_{i=1}^{n}\left(\eta(M_i, P_i) + \eta(M_i, P_i^\perp)\right). \quad (33)$$

Quantum violation

We choose a set of measurement directions in an equatorial plane of the Bloch sphere, say the Z-X plane, such that $\hat{n}_i$ and $\hat{n}_j$, $i \in \{1, \ldots, n\}$ and $j = i + 1 \mod n$, are at an angle of $\frac{n-1}{n}\pi$ relative to each other: $\hat{n}_i.\hat{n}_j = \cos \frac{n-1}{n}\pi$, where $i \in \{1, \ldots, n\}$ and $j = i + 1 \mod n$.



We tabulate the value of the quantum violation for a few values of n and point out that in the limit $n \to \infty$, we have: $\frac{n-1}{n}\pi \to \pi$ and the optimal $\eta_0 \to 1$ since the quantum violation is given by $2\frac{\eta_0}{n} \to 0$.

n	Quantum Violation	Optimal η_0
3	0.1793	0.4566
5	0.1393	0.5412
7	0.1164	0.5990
9	0.1007	0.6403
11	0.0889	0.6715
13	0.0798	0.6960
99	0.0160	0.8881
199	0.0086	0.9211

Joint measurability of POVMs. A finite set of POVMs

$$\{M_1, \ldots, M_N\},$$

where measurement M_i has outcome set X_i, is said to be *jointly measurable* or *compatible* if there exists a POVM M with outcome set $X_1 \times X_2 \times \cdots \times X_N$ that marginalizes to each M_i with outcome set X_i, meaning that

$$M_i(x_i) = \sum_{x_1,\ldots,\not{x_i},\ldots,x_N} M(x_1, \ldots, x_N)$$

for all outcomes $x_i \in X_i$.

Commutativity vs. Compatibility

- Commutativity $\Rightarrow$ Compatibility.
- Commutativity $\Leftrightarrow$ Compatibility for sharp (projective) measurements. Pairwise commutativity/compatibility is a necessary and sufficient condition for a set of projective measurements to be compatible.
- For unsharp measurements (POVMs), Compatibility $\not\Rightarrow$ Commutativity.

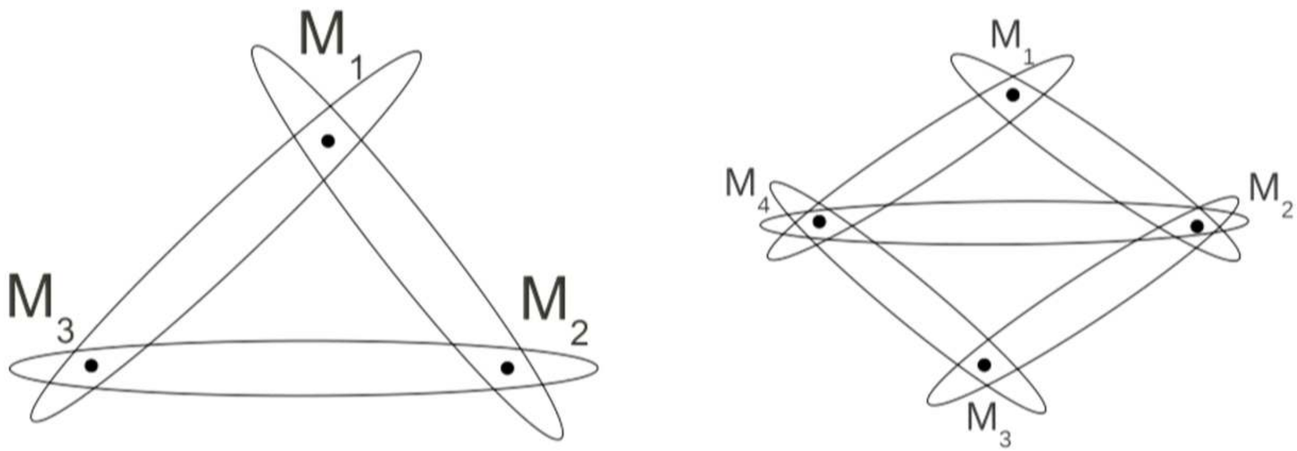


Figure: Joint measurability hypergraphs for $N = 3$ and $N = 4$.

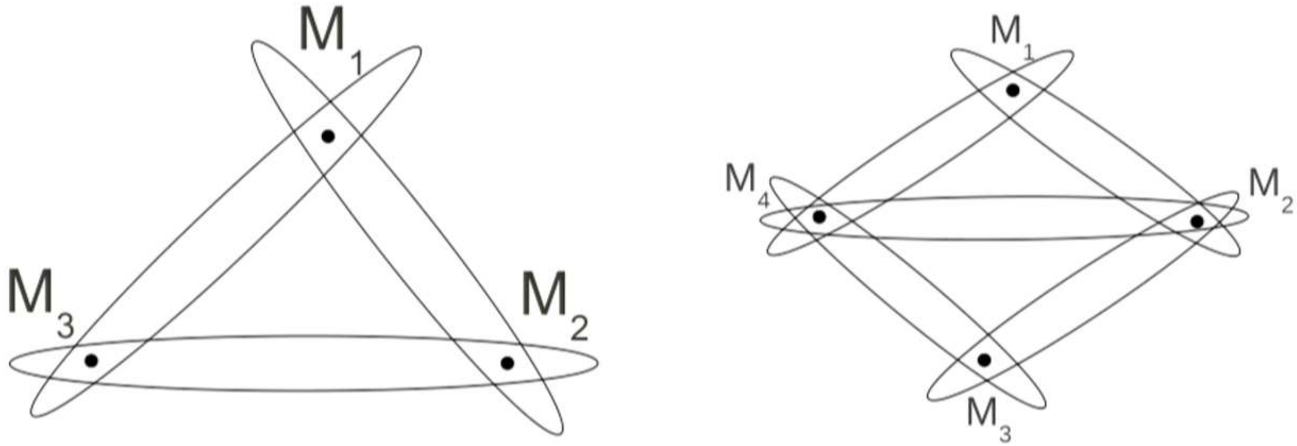


Figure: Joint measurability hypergraphs for $N = 3$ and $N = 4$.

Every set of POVMs on $\mathcal{H}$ has an associated joint measurability hypergraph. Figuring out this joint measurability hypergraph is the problem of deciding compatibility of observables. We solve the converse problem:

Theorem

Every joint measurability hypergraph admits a quantum realization with POVMs.

Takeaway – Some answers

- Noncontextuality without determinism allows us to give theory-independent criteria—NC-inequalities—for deciding whether operational statistics obtained in an experiment admits a classical explanation.
- Noncontextuality inequalities provide a more rigorous benchmark for nonclassicality than KS-inequalities.
- Quantum theory admits all conceivable (in)compatibility relations between observables.
- Unquestioning adherence to the 'Church of the larger Hilbert space' can forbid the faithful from asking new questions. The questions we ask do not arise for projective measurements.

Takeaway – Open questions

- Could one argue for operational equivalences from a physical principle independent of quantum theory, in a manner similar to the no-signalling condition from relativity for Bell scenarios? This would obviate the need to verify operational equivalences in an experiment.
- Could one devise new noncontextuality inequalities for contextuality scenarios that arise out of joint measurability hypergraphs?
- Application: What would be some information-theoretic tasks in which these results could be exploited in a manner not previously possible?

The noncontextual bound is given by

$$\begin{aligned} & p(\mathrm{anti}|M_*, P_*) + p(\mathrm{anti}|M'_*, P_*^{\perp}) \\ & \leq 2 - \frac{1}{n^2}\sum_{i=1}^{n}\Big(\eta(M_i, P_i) + \eta(M_i, P_i^{\perp})\Big) \\ & = 2 - 2\frac{\eta_0}{n}. \end{aligned} \tag{34}$$

The quantum violation of the noncontextual bound is given by

$$\sqrt{1 + \eta_0^4(\cos\frac{n-1}{n}\pi)^2 - 2\eta_0^2} - \eta_0^2\cos\frac{n-1}{n}\pi + 2\frac{\eta_0}{n} - 1.$$

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