Title: Black hole entropy and the case for self-dual loop quantum gravity

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Abstract: By focusing on aspects of black hole thermodynamics, I will present some evidences supporting the unexpected role of the complex self-dual variables in quantum gravity. This will also be the occasion of revisiting some aspects of three-dimensional gravity, and in particular the link between the BTZ black hole and the Turaev-Viro state sum model. Also the information on the website for next week needs to be modified: We will not have a seminar on Thursday (as Thursday is PI day). Instead, we will have the seminar on Wednesday, but actually would prefer a different time, namely 3.30 pm.

The case for self–dual loop quantum gravity

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January 29th 2014

Surprising interplay between various questions in LQG and spin foams:

- ⋆ Interpretation of the Barbero–Immirzi parameter γ: physically relevant parameter or regulator?
- ⋆ Lorentz symmetry and choice of gauge group.
- ⋆ Imposition of simplicity constraints and canonical Hamiltonian constraint.
- ⋆ Derivation of black hole entropy and fixation of $\gamma = \gamma_0$.

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Can we look at the self–dual theory ($\gamma = \mathrm{i}$) from a new perspective, and really see γ as a regulator?

- ⋆ We will motivate a positive answer by looking at 3d gravity and black holes.

1. The BI ambiguity in 3d gravity

2. BTZ black hole entropy
   - ⋆ Basic facts
   - ⋆ The Turaev–Viro model
   - ⋆ BTZ black hole entropy

3. 4d black holes in LQG
   - ⋆ Chern–Simons description
   - ⋆ Analytic continuation to $\gamma = \mathrm{i}$
   - ⋆ Radiation and temperature

4. Conclusion and outlook

Actions for $\mathrm{SL}(2,\mathbb{C})$ connection ω

$$S[B,\omega] = \int_{\mathcal{M}_3} \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}(\star + \gamma^{-1}) B^{IJ}_\mu F^{IJ}_{\mu\nu} + \phi^{\mu\nu} B^{IJ}_\mu B^{IJ}_\nu \tag{1}$$

$$S[e,n,\omega] = \int_{\mathcal{M}_3} \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}(\star + \gamma^{-1}) n^I e^J_\mu F^{IJ}_{\nu\rho} \tag{2}$$

- ⋆ γ disappears on half shell (zero torsion).
- ⋆ Spin foam quantization of (1) shows disappearance of γ if all the Plebanski constraints are imposed, and leads to the Ponzano–Regge model (MG, Noui, CQG 1112.1965 | Alexandrov, CQG 1202.5039).
- ⋆ What about canonical quantization? (Ben Achour, MG, Noui, Yu, 1306.3241)

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Two gauge choices

- ⋆ SU(1, 1) gauge: fix $n^I = (0, 0, 0, 1)$, then (2) becomes SU(1, 1) BF theory, i.e. 3d Lorentzian gravity. Known quantization, evidently γ–independent.

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Two gauge choices

- ⋆ $\mathrm{SU}(1,1)$ gauge: fix $n^I=(0,0,0,1)$, then (2) becomes $\mathrm{SU}(1,1)$ BF theory, i.e. 3d Lorentzian gravity. Known quantization, evidently γ–independent.
- ⋆ $\mathrm{SU}(2)$ gauge: fix $n^0=e^0_a=0$, then (2) leads to the canonical theory

$$\{A^i_a,E^b_j\}=\gamma\delta^i_j\delta^b_a, \qquad E^a_i := \varepsilon^{ab}(e_b\times n)_i, \qquad A^i_a := \Gamma^i_a+\gamma K^i_a \in \mathfrak{su}(2),$$

$$G_i = D_a E^a_i = 0, \qquad H^i = \mathcal{F}^i_{12} - (1+\gamma^{-2})(K_1\times K_2)^i = 0.$$

SU(2) kinematics

- ⋆ Discrete γ–dependent length spectrum: $\ell_j = 8\pi\gamma\ell_{\text{Pl}}\sqrt{j(j+1)}$.
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Quantization strategies

- ⋆ Quantize the 3d Ashtekar–Barbero Hamiltonian constraint using 4d techniques.
- → This could help understand some of the 4d ambiguities, since the 3d physical states are known (c.f. QSD IV).
- ⋆ Map the canonical theory back to a BF theory, i.e. $\mathbf{F}(\mathbf{A}) = 0 = \mathbf{G}(\mathbf{E}, \mathbf{A})$. One finds

$$\mathfrak{sl}(2,\mathbb{C}) \ni \mathbf{A} = \Big[-u \times \mathrm{d}u \cdot (\vec{L} \pm \gamma^{-1}\vec{P})\Big] + \Big[(A \cdot u)(u \cdot \vec{L}) \mp \gamma^{-1}(A \times u) \cdot (\vec{P} \times u)\Big],$$

with $u = u(n)$, rotations $\vec{L}$ and boosts $\vec{P}$.

NB: the self–dual connection is the shifted connection of Alexandrov,

$$\mathcal{A}_a = \mathbf{A}_a \pm \frac{\mathrm{i}\gamma^{-1} \mp 1}{|E|}\varepsilon_{ab}(E^b \cdot G)x.$$

Reality conditions on $\mathbf{A}$

- Physical motivation: the underlying theory is $\mathrm{SU}(1,1)$ BF.
- Two possibilities:

$$\vec{L} \pm \gamma^{-1}\vec{P} = 0 \quad \Rightarrow \quad \mathbf{A} = A^i L_i, \text{ the } \mathfrak{su}(2) \text{ Ashtekar–Barbero connection,}$$

$$\vec{G} \pm \gamma^{-1}\vec{F} = 0 \quad \Rightarrow \quad \mathbf{A} = \pm \mathrm{i}\gamma^{-1} A^i L_i, \text{ an } \mathfrak{su}(1,1) \text{ connection,}$$

where $\vec{L}$ spans $\mathfrak{su}(2)$, $\vec{F}$ spans $\mathfrak{su}(1,1)$, and $\vec{P}$ and $\vec{G}$ the complements in $\mathfrak{sl}(2,\mathbb{C})$ (Conrady, Hnybida, 1002.1959).

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Length operator and physical states

- ⋆ 3d geometric operators induced from the 4d ones.
- ⋆ For the $\mathfrak{su}(1,1)$ connection $\{E^a_i, \mathbf{A}^j_b\} = \pm \mathrm{i}\delta^a_b\delta^j_i$, so γ–independent spectrum

$$(\hat{X}_e)^2 \triangleright \mathbf{D}^{(s)}(h_{e'}) = \ell^2_{\mathrm{Pl}}\delta_{e,e'}Q^{(s)}\mathbf{D}(h_e),$$

  and $Q^{(s)} = F_1^2 + F_2^2 - F_0^2$ positive for the continuous series $j = \mathrm{i}s - 1/2$, $s \in \mathbb{R}$. Consistent with (Freidel, Livine, Rovelli, 0212077).
- ⋆ Discrete spectra and γ–dependent theory if we work with the $\mathfrak{su}(2)$ connection.
- ? However, what happens if we quantize the $\mathrm{SU}(2)$ theory is still unclear.

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Why does 3d gravity look simple?

- ⋆ In 3d, the Weyl tensor is identically vanishing:
  - → $R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$,
  - → all solutions are spaces of constant curvature, locally Minkowski, dS or AdS, depending on sign(Λ).
- ⋆ No local degrees of freedom, no gravitational waves, no gravitons.
- ⋆ 6 variables (q_{ab}, π^{ab}) and 3 first class constraints.
- ⋆ In first order formulation, on–shell diffeomorphisms = internal gauge transformations
  - → topological theory.

The BTZ black hole

- ⋆ Solution of 3d gravity with $\Lambda = -1/\ell_c^2 < 0$.
- ⋆ Euclidean metric (by Wick rotation of the Lorentzian solution):

$$\mathrm{d}s^2 = N^2\mathrm{d}\tau^2 + N^{-2}\mathrm{d}r^2 + r^2\left(\mathrm{d}\phi + N^\phi \mathrm{d}\tau\right)^2,$$

with

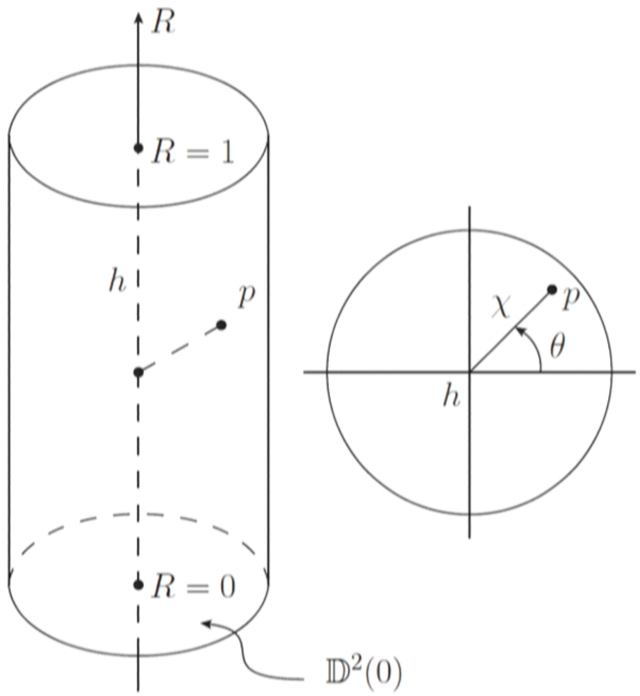
$$N = \left(-8GM + \frac{r^2}{\ell_c^2} - \frac{16G^2J^2}{r^2}\right)^{1/2}, \qquad N^\phi = -\frac{4GJ}{r^2},$$

$$r_\pm^2 = 4GM\ell_c^2\left(1 \pm \sqrt{1 + \left(\frac{J}{M\ell_c}\right)^2}\right).$$

- ⋆ No curvature singularity at the origin, but event horizon and inner horizon (if $J \neq 0$).

The BTZ black hole

- ⋆ Global geometry of the Euclidean solution is a solid torus (Bañados, Henneaux, Teitelboim, Zanelli 1992):



boundary: $\mathbb{S}^1 \times \mathbb{S}^1$ at $\chi = \pi/2$

horizon: $\mathbb{S}^1$ at $\chi = 0$

Further motivations

- ⋆ The statistical mechanics of the BTZ black hole has been extensively studied, using a huge variety of methods (see Carlip, hep-th/9806026).

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- ⋆ The statistical mechanics of the BTZ black hole has been extensively studied, using a huge variety of methods (see Carlip, hep-th/9806026).
- ⋆ However, no LQG or spin foam description so far, because the gauge groups involved are non–compact:

	Euclidean	Lorentzian
$\Lambda = 0$	$\mathfrak{isu}(2)$	$\mathfrak{isu}(1,1)$
$\Lambda > 0$	$\mathfrak{so}(4)$	$\mathfrak{so}(3,1)$
$\Lambda < 0$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(2,2)$

  The only spin foam model under control is the Euclidean $\Lambda > 0$.
- ⋆ Idea: define $\mathcal{O}_{\mathrm{E}}(\Lambda > 0)$, and then switch to $\mathcal{O}_{\mathrm{E}}(\Lambda < 0)$ (MG, Noui, 1312.1696 | Frodden, MG, Noui, Perez, JHEP 05 (2013) 139).
  The Euclidean spin foam model with $\Lambda > 0$ is well understood.

Group theoretical data

- ⋆ Choose an integer $\mathrm{k} \geq 1$, define $q = \exp\left(\frac{i\pi}{\mathrm{k}+2}\right)$, and

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \sin\left(\frac{\pi}{\mathrm{k}+2}n\right)\sin^{-1}\left(\frac{\pi}{\mathrm{k}+2}\right).$$

- ⋆ Consider the algebra $\mathrm{U}_q(\mathfrak{su}(2))$, and its unitary irreducible representations

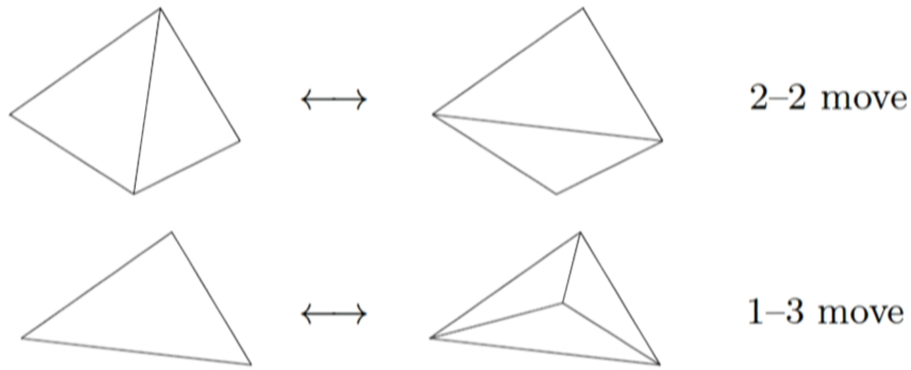
$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{\mathrm{k}}{2}, \qquad \text{with} \qquad \dim(\mathcal{H}_j) = 2j+1.$$

Assignment of amplitudes

- ⋆ vertices $v \longleftrightarrow$
- ⋆ edges $e \longleftrightarrow$
- ⋆ tetrahedra $\tau \longleftrightarrow$

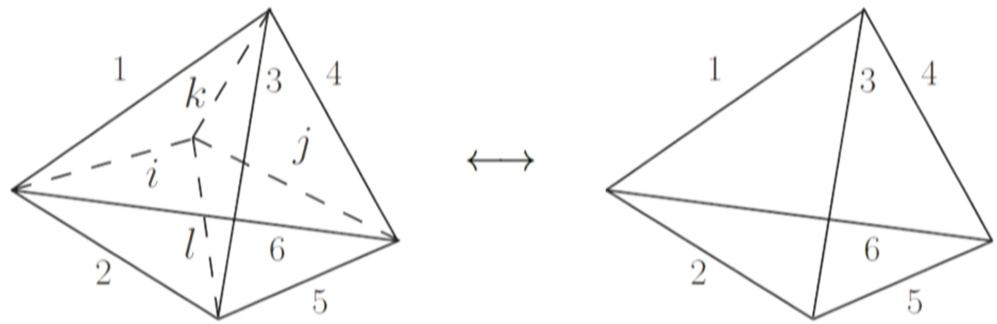
Topological invariance

⋆ Triangulations of d–dimensional manifolds can be related by elementary moves (Alexander 1930, Pachner 1978):
- in $d = 2$ (i.e. on triangles)



Topological invariance

- For example, the 1–4 move



is encoded in the algebraic relation

$$\sum_{i,j,k,l} \omega_i^2\omega_j^2\omega_k^2\omega_l^2 \begin{vmatrix} 1 & 2 & 3 \\ l & k & i \end{vmatrix}_q \begin{vmatrix} 3 & 4 & 5 \\ j & l & k \end{vmatrix}_q \begin{vmatrix} 1 & 4 & 6 \\ j & i & k \end{vmatrix}_q \begin{vmatrix} 2 & 5 & 6 \\ j & i & l \end{vmatrix}_q$$

$$= \quad \omega^2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \end{vmatrix}_q .$$

Link with 3d gravity

Ingredients

We want to describe the entropy of a Euclidean BTZ black hole with the Turaev–Viro state sum. For this we need:

1) A notion of observable.
2) A proper discretization of the solid torus.
3) Understand how to return to $\Lambda < 0$.

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State sum observable

- ⋆ Choose a graph Γ consisting of a subset of n edges of Δ, and then define

$$\mathcal{Z}_n(M,\Gamma) = \sum_{j|_\Gamma} \mathcal{Z}(M,\Delta).$$

- ⋆ This object is triangulation–independent away from Γ, and

$$\sum_{j_\Gamma} \mathcal{Z}_n(M,\Gamma) = \mathcal{Z}(M).$$

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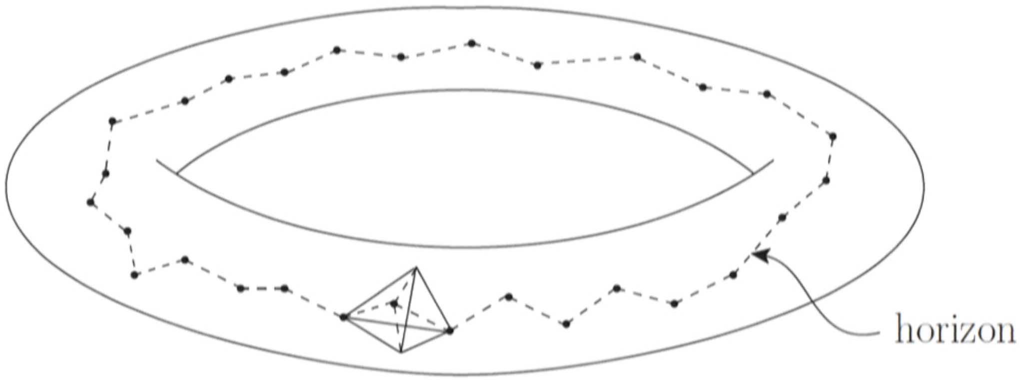
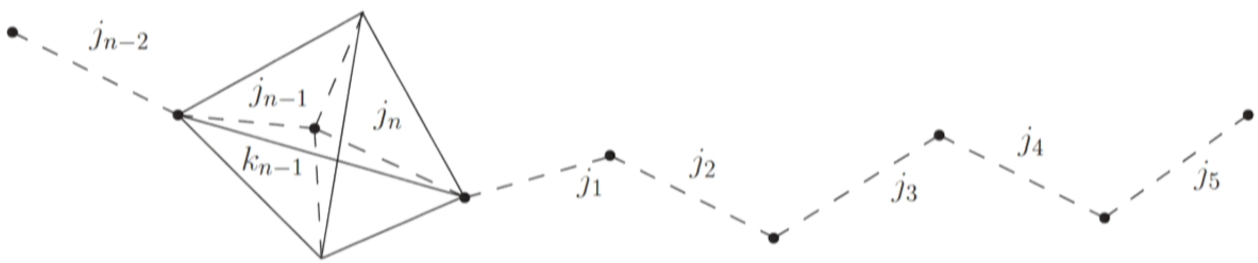
 BTZ black hole entropy

Triangulation of $\mathbb{D}^2 \times \mathbb{S}^1$



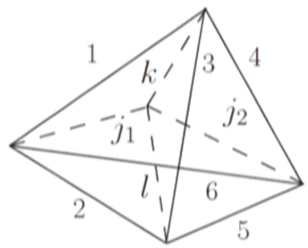


 BTZ black hole entropy

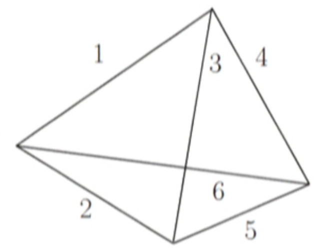
Computation of the observable partition function

- Use recursively the partial Pachner 4–1 move

$$\sum_{k,l} \omega_k^2 \omega_l^2 \omega_{j_1}^2 \omega_{j_2}^2$$



$$= \omega_{j_1}^2 \omega_{j_2}^2 \omega_6^{-2} Y(j_1, j_2, 6)$$



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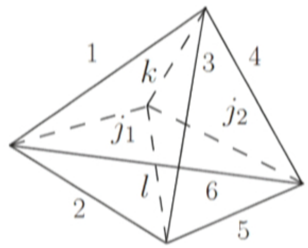

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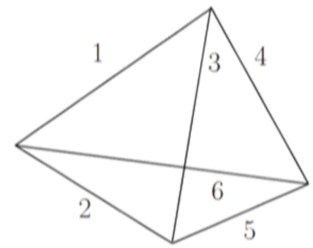

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⋆ Upshot: $$\mathcal{Z}_n = \omega^{-2} \sum_{k_{n-1}} \omega_{j_{n-1}}^2 \omega_{j_n}^2 \omega_{k_{n-1}}^{-2} Y(j_{n-1}, j_n, k_{n-1}) \mathcal{Z}_{n-1} = \cdots$$

$$\propto \sum_{k_1,\ldots,k_n} \delta_{k_1,0} \delta_{k_{n+1},0} \prod_{i=1}^{n} Y(j_i, k_{i+1}, k_i).$$

⋆ This is precisely the dimension N of the Chern-Simons Hilbert space

$$\mathrm{Inv}_{\mathrm{U}_q(\mathfrak{su}(2))}(j_1 \otimes \cdots \otimes j_n), \qquad \text{with dimension} \quad \xrightarrow{\mathrm{k}\to\infty} \prod_{e=1}^{n} (2j_e + 1)$$

(Kaul, Majumdar, 9801080 | Engle, Noui, Perez, Pranzetti, 1103.2723).

Analytic continuation to $\Lambda < 0$

- Since

$$k = \frac{\ell_c}{\ell_{\rm Pl}} = \frac{1}{\hbar G\sqrt{\Lambda}},$$

working with $\Lambda < 0$ amounts to choosing a purely imaginary level $k = i\lambda$.

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- It was shown in (Frodden, MG, Noui, Perez, 1212.4060) that for $\mathrm{k} = \mathrm{i}\lambda$, and in the double scaling limit $j_e \to \infty$ and $\ell_{\mathrm{Pl}} \to 0$ with $\ell_{\mathrm{Pl}} j_e$ finite, the dimension N behaves as
$$\log N \approx \frac{L}{4\ell_{\mathrm{Pl}}} + \circ \log(L),$$
where the length of the horizon is $L = 8\pi\ell_{\mathrm{Pl}} \sum_{e=1}^{n} \sqrt{j_e(j_e+1)}$.

(more details in the next slides. . .)

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- Notice that there is no γ in the length spectrum, as opposed to attempts by (Suneeta, Kaul, Govindarajan, 9811071 | García-Islas, 0804.2082).

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4. Conclusion and outlook

SU(2) state counting

- Counting the number of states compatible with a given macroscopic area A_{H} by

$$\mathcal{N}(A_{\mathrm{H}}) = \sum_{p=1}^{\infty} \sum_{j_1,\dots,j_n} \delta\left(A_{\mathrm{H}} - 8\pi\gamma\ell_{\mathrm{Pl}}^2 \sum_{e=1}^{n} \sqrt{j_e(j_e+1)} \right) N(j_1,\dots,j_n).$$

- This is a sum over distinguishable puncture labels.
- Assuming that $\gamma = \gamma_0 \approx 0.274067$,

$$S = \log \mathcal{N}(A_{\mathrm{H}}) = \frac{A_{\mathrm{H}}}{4\ell_{\mathrm{Pl}}^2} - \frac{3}{2}\log\left(\frac{A_{\mathrm{H}}}{4\ell_{\mathrm{Pl}}^2}\right) + \circ(1),$$

and small spins dominate.

SU(2) state counting

- Counting the number of states compatible with a given macroscopic area A_{H} by

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- This is a sum over distinguishable puncture labels.
- Assuming that $\gamma = \gamma_0 \approx 0.274067$,

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- ⋆ Why?

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- This corresponds in principle to $SL(2,\mathbb{C})$ Chern–Simons theory (or just one chiral copy) (Witten, 1001.2933).
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- When expanding the $\mathrm{SL}(2,\mathbb{C})$ representations $|(\rho,k)\rangle$ in terms of the $\mathrm{SU}(1,1)$ subgroup, the representations of the continuous series are self–dual i.e. satisfy

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Rewriting of the analytically–continued partition function

- When the level is large, we have

$$\mathcal{Z}\approx\frac{2\sinh^2\pi}{\lambda}\prod_{e=1}^{n}\mathcal{Z}_e,\qquad\text{with}\qquad\mathcal{Z}_e=\sum_{m=0}^{2j_e}\exp\big(-\beta E_e(m)\big),\qquad\text{where}$$

$$\beta=\frac{2\pi}{a}\qquad\text{is the inverse Unruh temperature with acceleration } a,$$

$$E_e(m)=am+E_e(0)\qquad\text{is the discrete “energy spectrum” with } E_e(0)=-aj_e.$$

- Partition function $\mathcal{Z}_e$ for a bath of particles in equilibrium at Unruh temperature.
- Matches the near–horizon energy of (Frodden, Gosh, Perez, 1110.4055) seen by a stationary observer:

$$\delta E_{\text{local}}=\frac{\delta A}{8\pi G}a.$$

- Rewrite $\mathcal{Z}_e=\sum_{m=0}^{2j_e}g_e\exp(-\beta am)$, with puncture degeneracy $g_e=\exp\big(A_e/4\ell_{\text{Pl}}^2\big)$ responsible for the entropy (Gosh, Noui, Perez, 1309.4563).

Self–dual decomposition of a state

- Consider the (anti) self–dual generators $\vec{J}^{\pm} = (\vec{L} \pm \mathrm{i}\vec{P})/2$, and the helicity (or unitary spinor) basis $|(\ell^+, \ell^-), \mu, \nu\rangle$ of $\mathrm{SL}(2,\mathbb{C})$, where L_3 and P_3 are diagonal:

$$L_3|\mu,\nu\rangle = \mu|\mu,\nu\rangle, \qquad P_3|\mu,\nu\rangle = \nu|\mu,\nu\rangle, \qquad \text{with} \qquad \mu \in \mathbb{Z}/2,\ \nu \in \mathbb{R},$$

$$J_3^{\pm}|\mu,\nu\rangle = m^{\pm}|\mu,\nu\rangle, \qquad \text{with} \qquad m^{\pm} = \frac{1}{2}(\mu \pm \mathrm{i}\nu),$$

$$(\vec{J}^{\pm})^2|\mu,\nu\rangle = \ell^{\pm}(\ell^{\pm}+1)|\mu,\nu\rangle, \qquad \text{with} \qquad \ell^{\pm} = \frac{1}{2}(k \pm \mathrm{i}\rho - 1).$$

- One can compute the overlap coefficients for the change of basis (Huszár, 1971)

$$|\mu,\nu\rangle = \sum_{j,n}\langle j,n|\mu,\nu\rangle|j,n\rangle, \qquad |j,n\rangle = \int \mathrm{d}\nu \sum_{\mu}\langle \mu,\nu|j,n\rangle|\mu,\nu\rangle.$$

- According to the dynamics of spin foams, a state is labelled

$$|0(j)\rangle_{\mathrm{SF}} := |(\gamma(j+1),j),j,j\rangle \qquad \text{for EPRL/FK,}$$
$$|0(\rho)\rangle_{\mathrm{SF}} := |(\rho,0),0,0\rangle \qquad \text{for BC.}$$

3d theory

- ⋆ The SU(2) Ashtekar–Barbero formulation of Lorentzian gravity is also available in $(2+1)$ dimensions, and presents the same Barbero–Immirzi and connection ambiguity as in 4d.
- ⋆ Turaev–Viro model (Euclidean, $\Lambda > 0$) to describe the entropy of a BTZ black hole.
- ⋆ No Barbero–Immirzi parameter needed in the length spectrum.
- ? Should proceed with the quantization of the Hamiltonian constraint.
- ? How to describe the Lorentzian black hole?
- ? Link with other counting methods, in particular CFT?
- ? How to treat the boundary in the state sum model, and link with log corrections?

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- ? Can we make more rigorous the analytic continuation to $\gamma = \mathrm{i}$?
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Conclusion and outlook

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