Title: Mayer-Cluster Expansion of Instanton Partition Functions and Thermodynamic Bethe Ansatz

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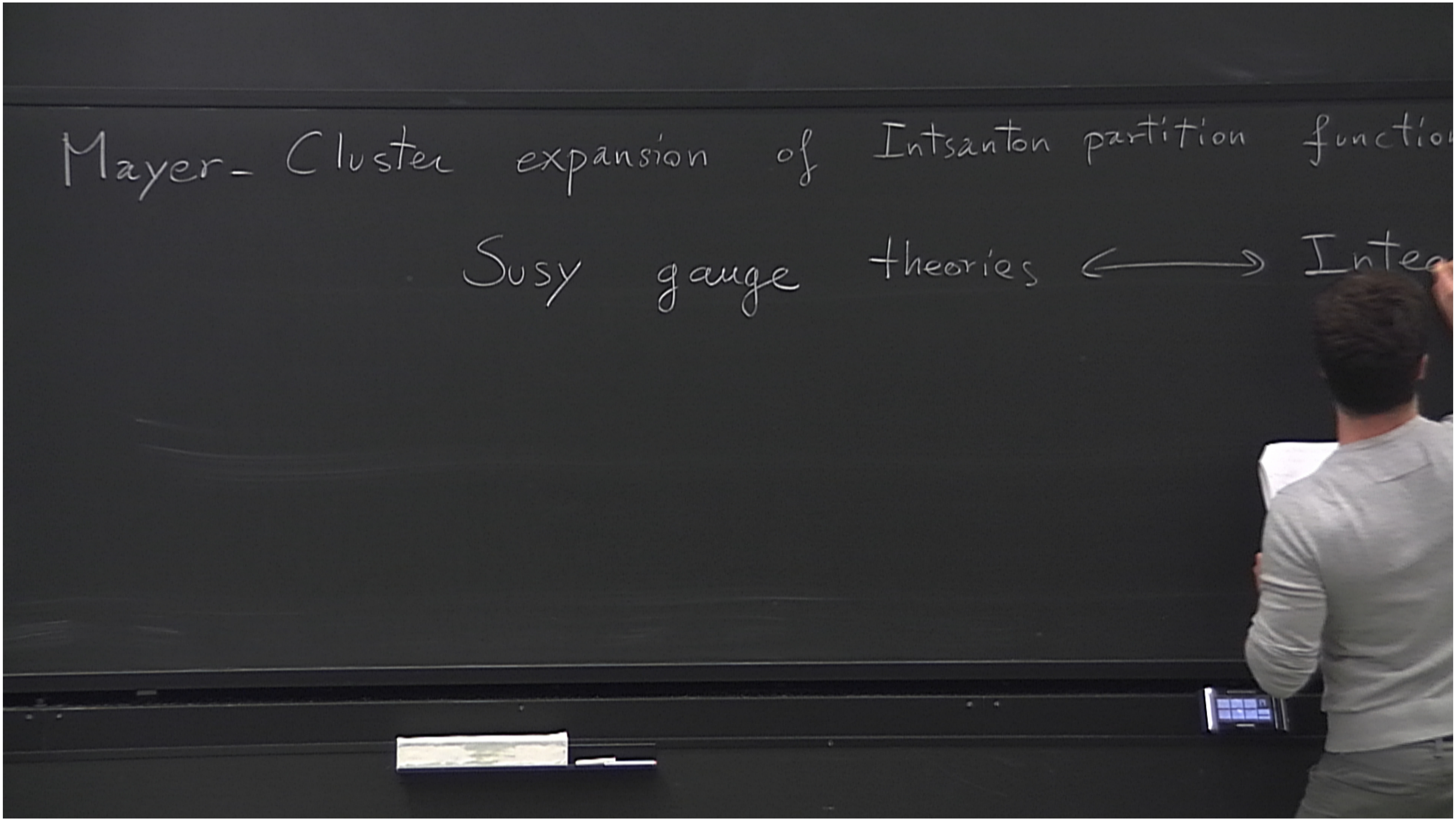
Abstract: In arXiv:0908.4052, Nekrasov and Shatashvili pointed out that the N=2 instanton partition function in a special limit of the Omega-deformation parameters is characterized by certain thermodynamic Bethe ansatz (TBA) like equations.In this talk I will present an explicit derivation of this fact as well as generalizations to quiver gauge theories. The TBA equations derived entirely within gauge theory have been proposed to encode the spectrum of a large class of quantum integrable systems. I will conclude with some remarks on this correspondence.

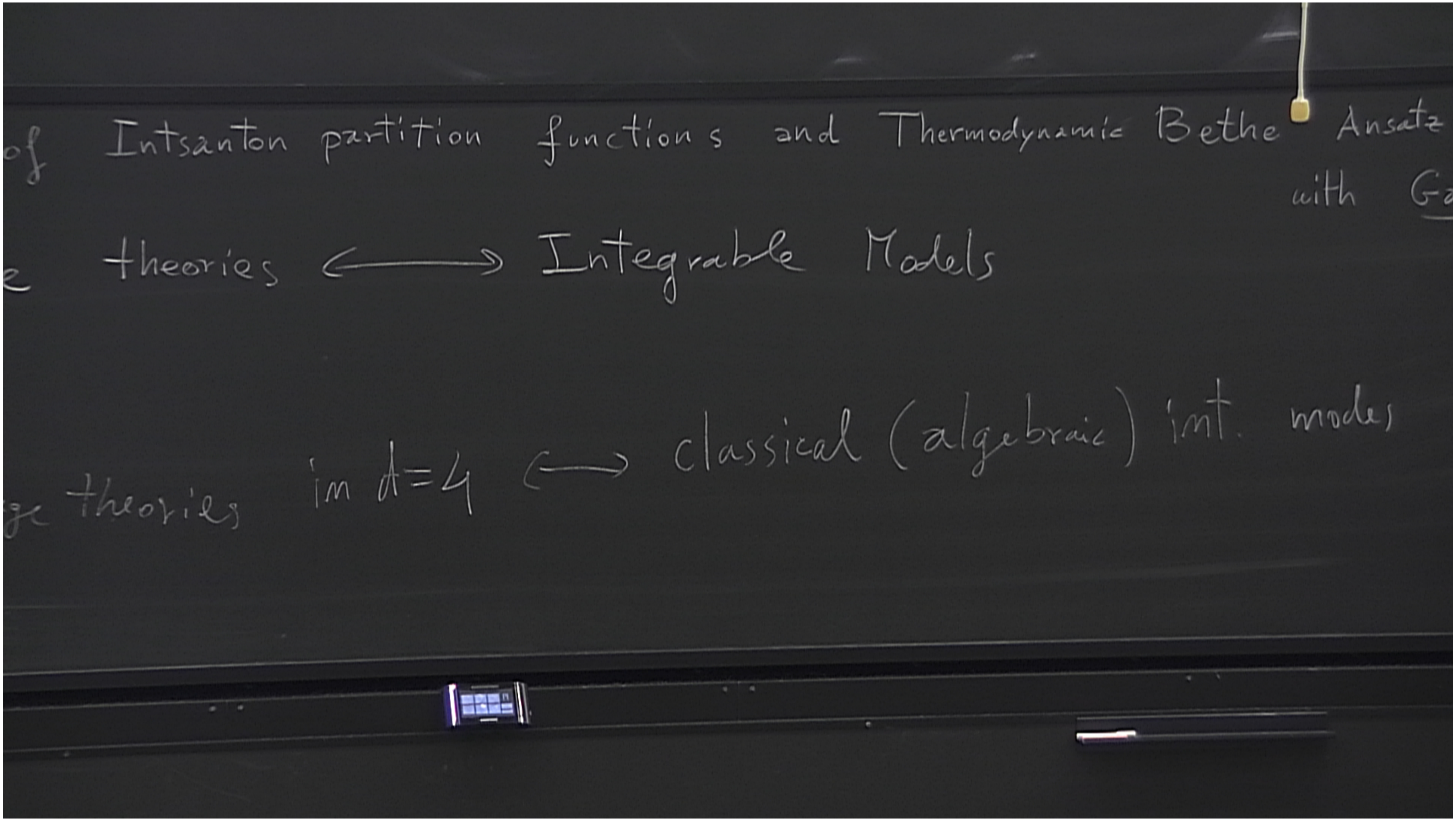

Mayer- Cluster expansion of Intsanton partition functio
Susy gauge theories ⟷ Inte

Mayer-Cluster expansion of Intsanton partition function

Susy gauge theories $\longleftrightarrow$ Integr

- Main Example $(*)$

  $\mathcal{N}=2$ susy gauge theories in $d=4$


of Intsanton partition functions and Thermodynamic Bethe Ansatz
with
theories ⟷ Integrable Models
classical (algebraic) int. modes
theories in d=4 ⟷

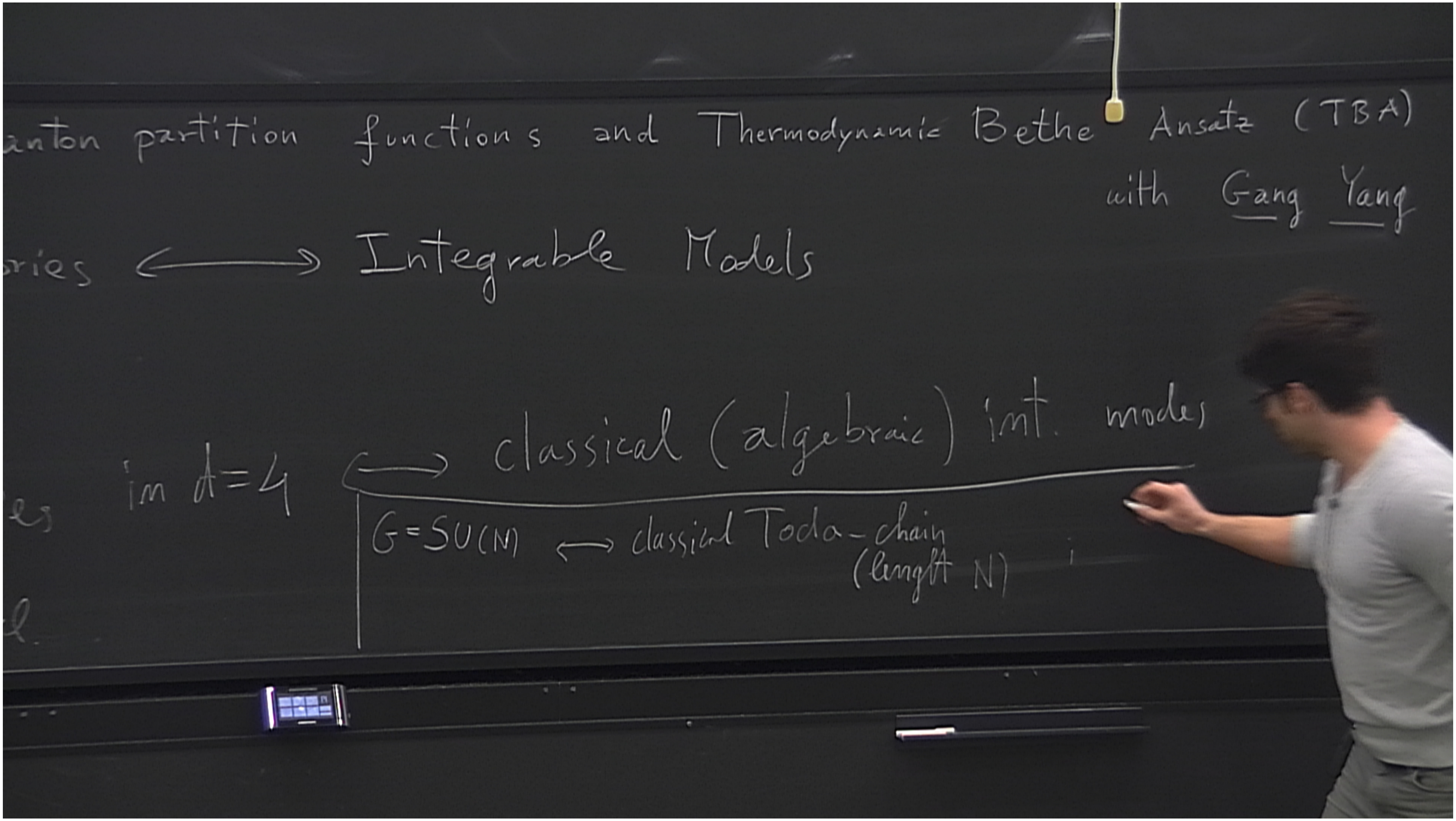
ayer - Cluster expansion of Intsanton partition functions a

Susy gauge theories $\longleftrightarrow$ Integrable

- Main Example (*)

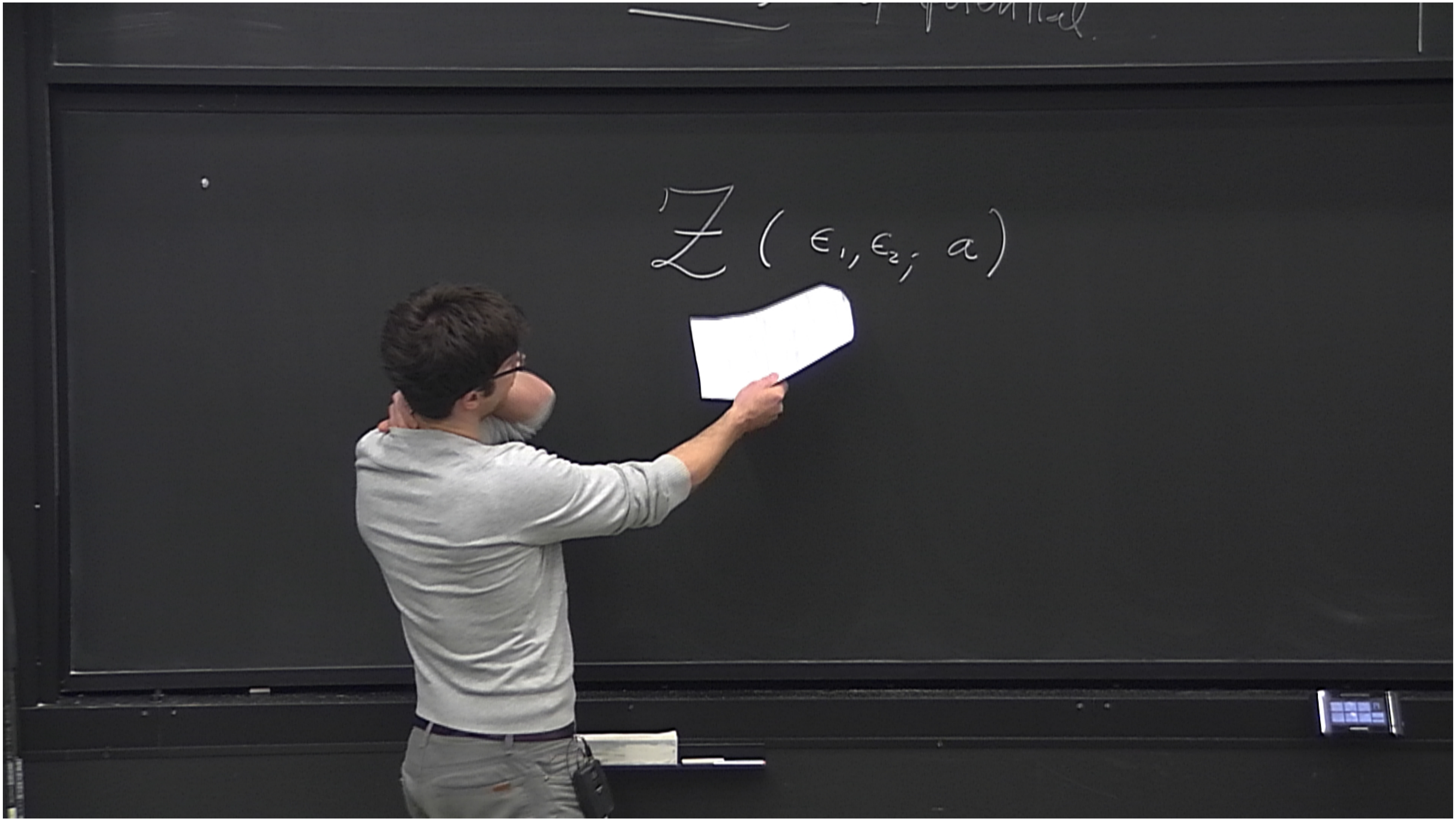
$\mathcal{N}=2$ susy gauge theories in $d=4$ $\longleftrightarrow$ classi

cartan sub. of Lie(G) $\underline{F(a)}$ superpotential


anton partition functions and Thermodynamic Bethe Ansatz (TBA)
with Gang Yang
ories ⟷ Integrable Models
classical (algebraic) int. models
es in d=4 ⟷
G=SU(N) ⟷ classical Toda-chain
(length N)
l.

$\longleftrightarrow$ classical (algebraic) int. modes

$G = SU(N) \longleftrightarrow$ classical Toda-chain (length N) , $H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} e^{x_i - x_{i+1}}$


$Z(\epsilon_1, \epsilon_2; a)$

$$F(a) \doteq \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \epsilon_1 \epsilon_2 \log Z(\epsilon_1, \epsilon_2; a)$$

• [NS '09]

a

a)

• [NS '09] a quantum version of (*)

$$\mathcal{W}(\epsilon_1, a) = \lim_{\epsilon_2 \to 0} \epsilon_1 \log Z(\epsilon_1, \epsilon_2, a)$$ twisted superpotential.

$\parallel$ $\hbar$

1) Eigenstates of the quantum int. systems.

$$\exp\left(\frac{\partial \mathcal{W}}{\partial a_\lambda}\right) =$$

$\lambda = 1, \ldots,$ ra

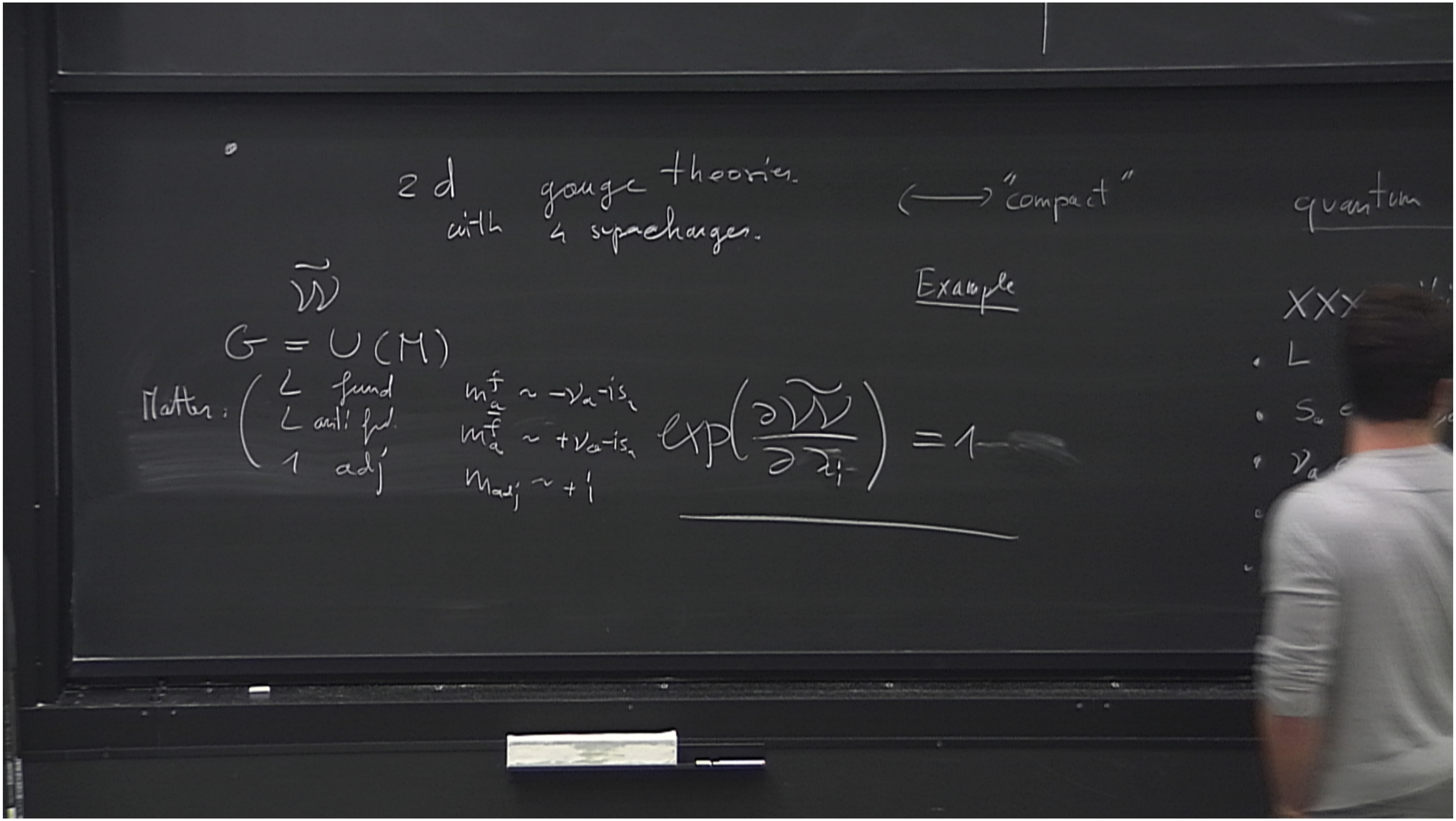
quantum integrable models (finite dim Hilbert space)

XXX Heisenberg chain

- L lenght
- $s_a \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ $\quad a = 1, \dots L$
- $\nu_a \in \mathbb{C}$ $\quad$ - - -
- fix $S_z = \sum_a s_a - M$
- q : twist parameter.

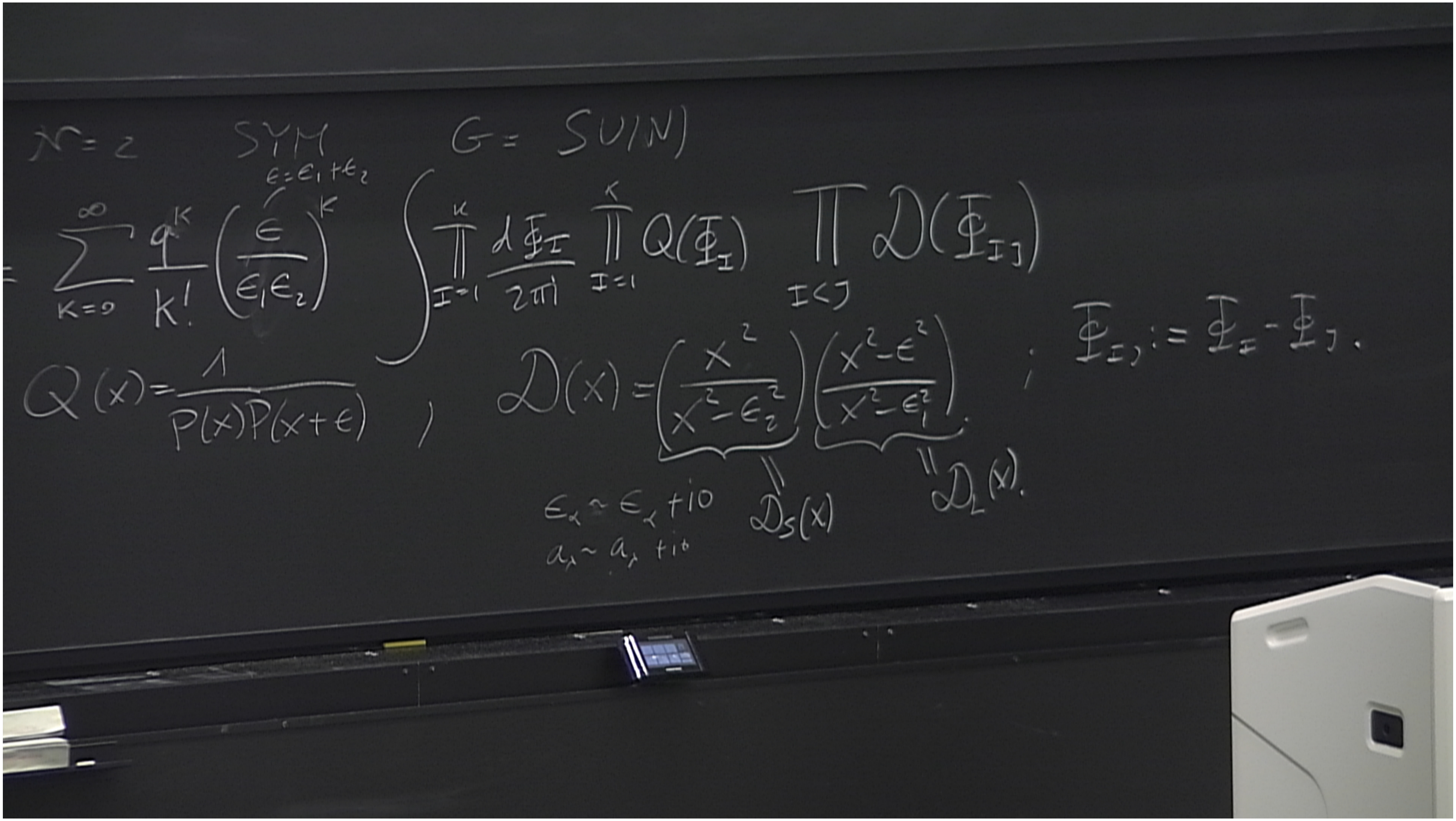
$$\to \prod_{a=1}^{L} \frac{\lambda_k - \nu_a - i s_a}{\lambda_k - \nu_a + i s_a} = -q \prod_{j=1}^{M} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}$$

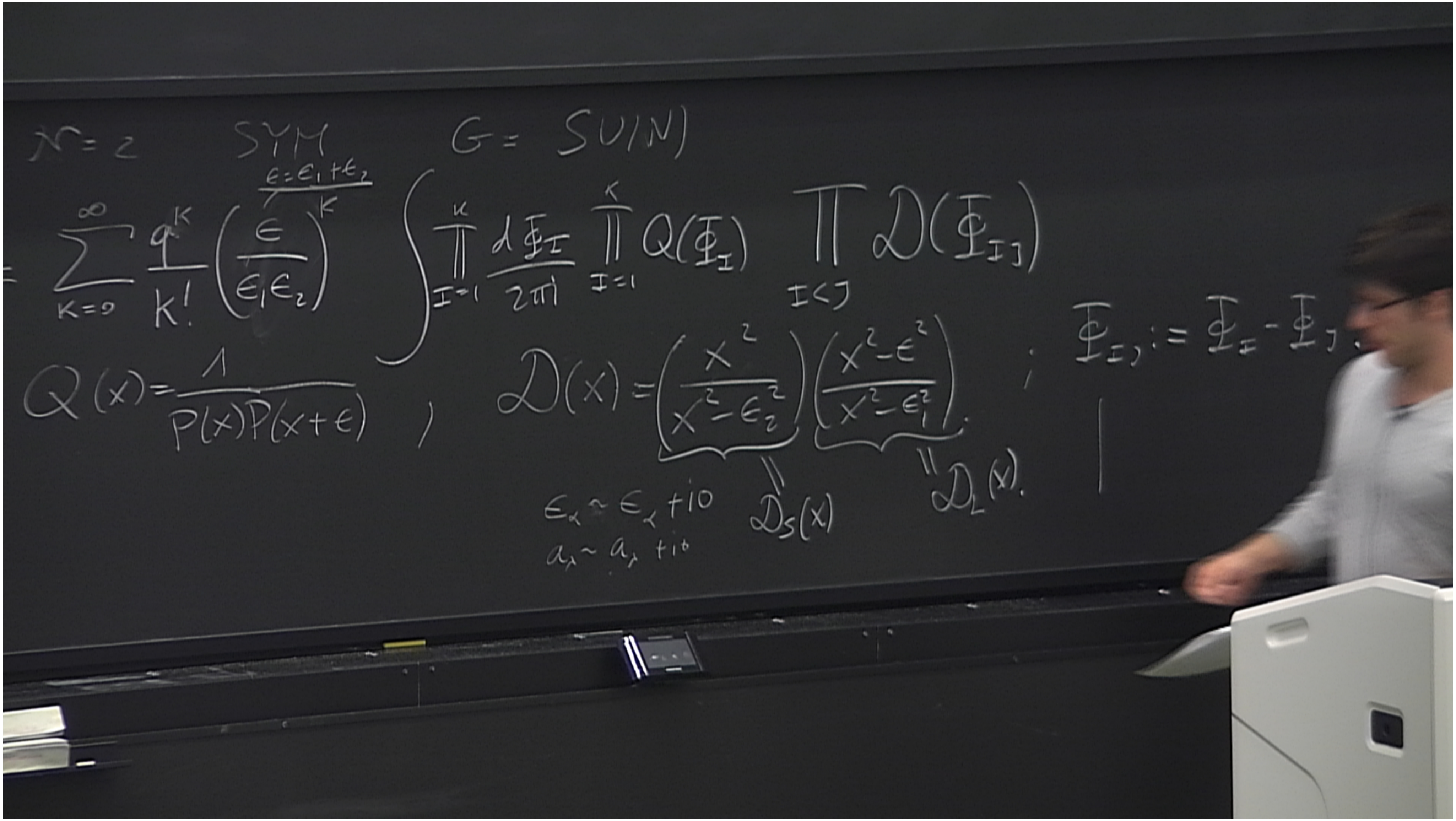
$K = 1, \dots, M$


2d gauge theories
with 4 supercharges.
⟷ "compact"
quantum
W̃
G = U(M)
Matter: (L fund
L anti fund
1 adj
m_a^f ~ −ν_a − is_a
m_a^f̄ ~ +ν_a − is_a
m_adj ~ +i
exp(∂W̃/∂σ_i) = 1
Example
XXX
L
S_a
ν_a

Main Example: Pure $\mathcal{N}=2$ SYM $G=SU(N)$

$$Z_{inst} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\frac{\overbrace{\epsilon}^{\epsilon=\epsilon_1+\epsilon_2}}{\epsilon_1 \epsilon_2} \right)^k \int \prod_{I=1}^{k} \frac{d\Phi_I}{2\pi i} \prod_{I=1}^{k} Q(\Phi_I)$$


$\mathcal{N}=2$ SYM $G = SU(N)$
$\epsilon = \epsilon_1 + \epsilon_2$
$= \sum_{k=0}^{\infty} \frac{q^k}{k!} \left(\frac{\epsilon}{\epsilon_1 \epsilon_2}\right)^k \int \prod_{I=1}^{k} \frac{d\Phi_I}{2\pi i} \prod_{I=1}^{k} Q(\Phi_I) \prod_{I<J} \mathcal{D}(\Phi_{IJ})$
$\Phi_{IJ} := \Phi_I - \Phi_J$.
$Q(x) = \frac{1}{P(x)P(x+\epsilon)}$,
$\mathcal{D}(x) = \underbrace{\left(\frac{x^2}{x^2-\epsilon_2^2}\right)}_{= \mathcal{D}_S(x)} \underbrace{\left(\frac{x^2-\epsilon^2}{x^2-\epsilon_1^2}\right)}_{= \mathcal{D}_L(x)}$;
$\epsilon_\alpha \sim \epsilon_\alpha + i0$
$a_\alpha \sim a_\alpha + i0$



$\mathcal{N}=2$ SYM $\quad G = SU(N)$
$\epsilon = \epsilon_1 + \epsilon_2$
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$\Phi_{IJ} := \Phi_I - \Phi_J$
$Q(x) = \frac{1}{P(x)P(x+\epsilon)}$, $\quad \mathcal{D}(x) = \underbrace{\left(\frac{x^2}{x^2-\epsilon_2^2}\right)}_{= \mathcal{D}_S(x)} \underbrace{\left(\frac{x^2-\epsilon^2}{x^2-\epsilon_1^2}\right)}_{= \mathcal{D}_L(x)}$;
$\epsilon_\alpha \sim \epsilon_\alpha + i0$
$a_\alpha \sim a_\alpha + i0$

$Z_{Long/short} \neq Z|_{\omega = \omega_{L/S}}$;

Z_{Long} ; <u>Mayer expansion</u> ; $Z = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int \prod_{I=1}^{k} d\mu(\Phi_I) \prod_{I<J} (1 + \ldots$

Short range interactions

$$\log Z_{\text{short}} = \frac{1}{\epsilon_2} \int \frac{d\Phi_0}{2\pi i}\, \mathrm{Li}_2\left(q\, Q(\Phi_0)\right) + \mathcal{O}(\epsilon_2^0).$$

Short range interactions

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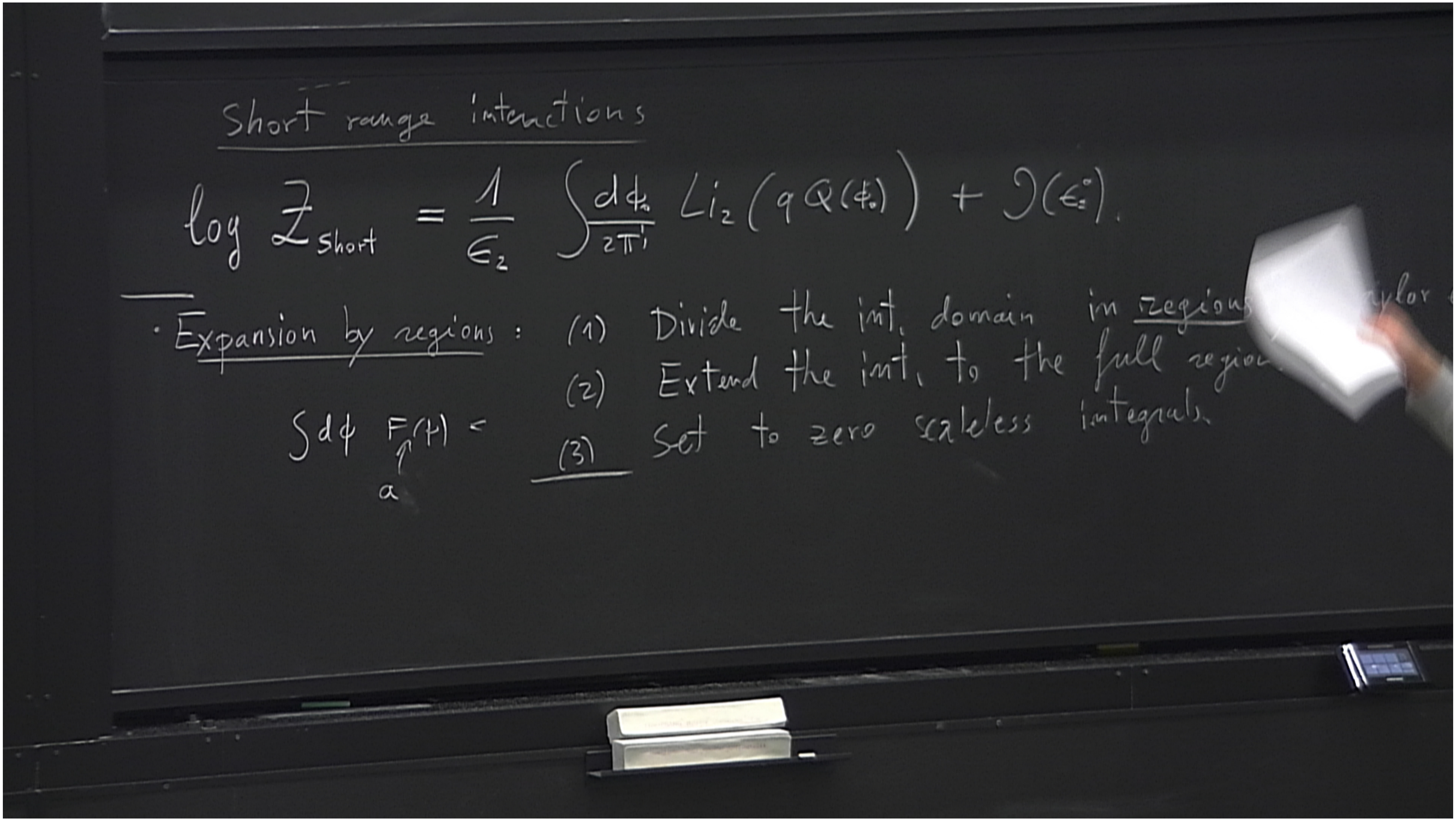
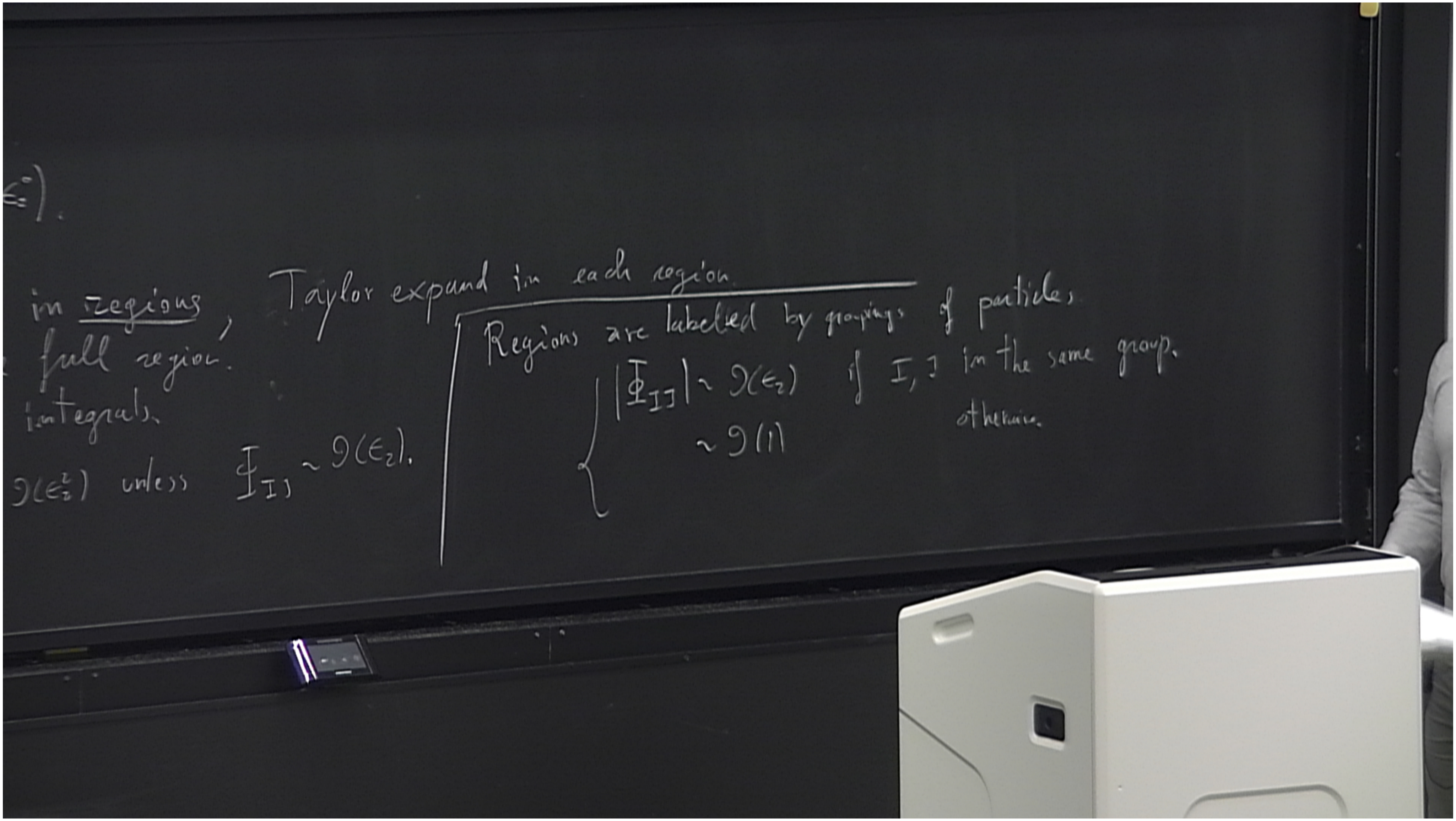
- Expansion by regions : (1) Divide the int. domain in regions, Taylor ex

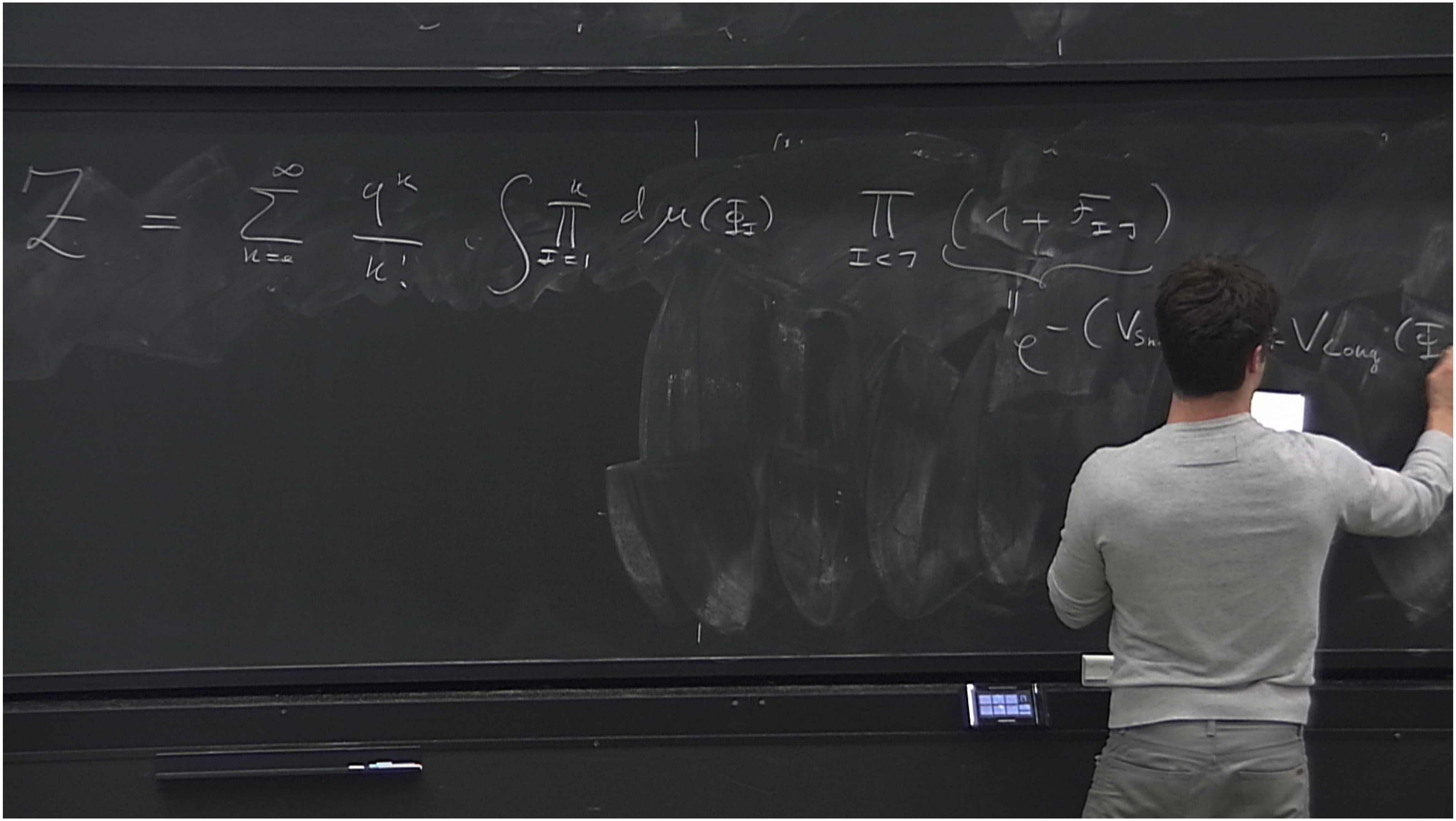

Short range interactions
$\log Z_{\text{short}} = \frac{1}{\epsilon_2} \int \frac{d\phi_0}{2\pi i} \, Li_2(q\, Q(\phi_0)) + O(\epsilon_2^0)$.
· Expansion by regions :
(1) Divide the int. domain in regions
(2) Extend the int. to the full region
(3) Set to zero scaleless integrals.
$\int d\phi \; F(\phi)$
a


in regions, Taylor expand in each region.
full region.
integrals.
$\mathcal{O}(\epsilon_2^2)$ unless $\Phi_{IJ} \sim \mathcal{O}(\epsilon_2)$.
Regions are labeled by groupings of particles.
$|\Phi_{IJ}| \sim \mathcal{O}(\epsilon_2)$ if I, J in the same group.
$\sim \mathcal{O}(1)$ otherwise.

$$\log Z_{short} = \frac{1}{\epsilon_2}$$

$$\log Z_{\text{short}} = \frac{1}{\epsilon_2} \sum_{k=1}^{\infty} \frac{q^k}{k!} \int d\Phi_0 \; Q^k(\Phi_0) \; J_k \; + \; \mathcal{O}(\epsilon_2^0)$$

$$J_k = \frac{1}{\epsilon_2^{k-1}} \int \prod_I \frac{d\Phi_I}{2\pi i} \; \delta\Big(\frac{1}{k} \sum_I \Phi_I$$



$Z = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int \prod_{I=1}^{k} d\mu(\Phi_I) \prod_{I<J} (1 + f_{IJ})$
$= e^{-(V_{Sh}$
$V_{Long}(\Phi$

$$Z = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int \prod_{I=1}^{k} d\mu(\Phi_I) \prod_{I<J} \underbrace{(1 + f_{IJ})}_{= e^{-(V_{short}}}$$

$$\log Z = \sum_{k=1}^{\infty} \frac{q^k}{k!} \sum_{\substack{\{X_1, X_2, \ldots, X_\ell\} \\ \cup X_i = \{1, \ldots, k\}}} \int \quad S_{\{X_1, \ldots, X_\ell\}} \; i$$

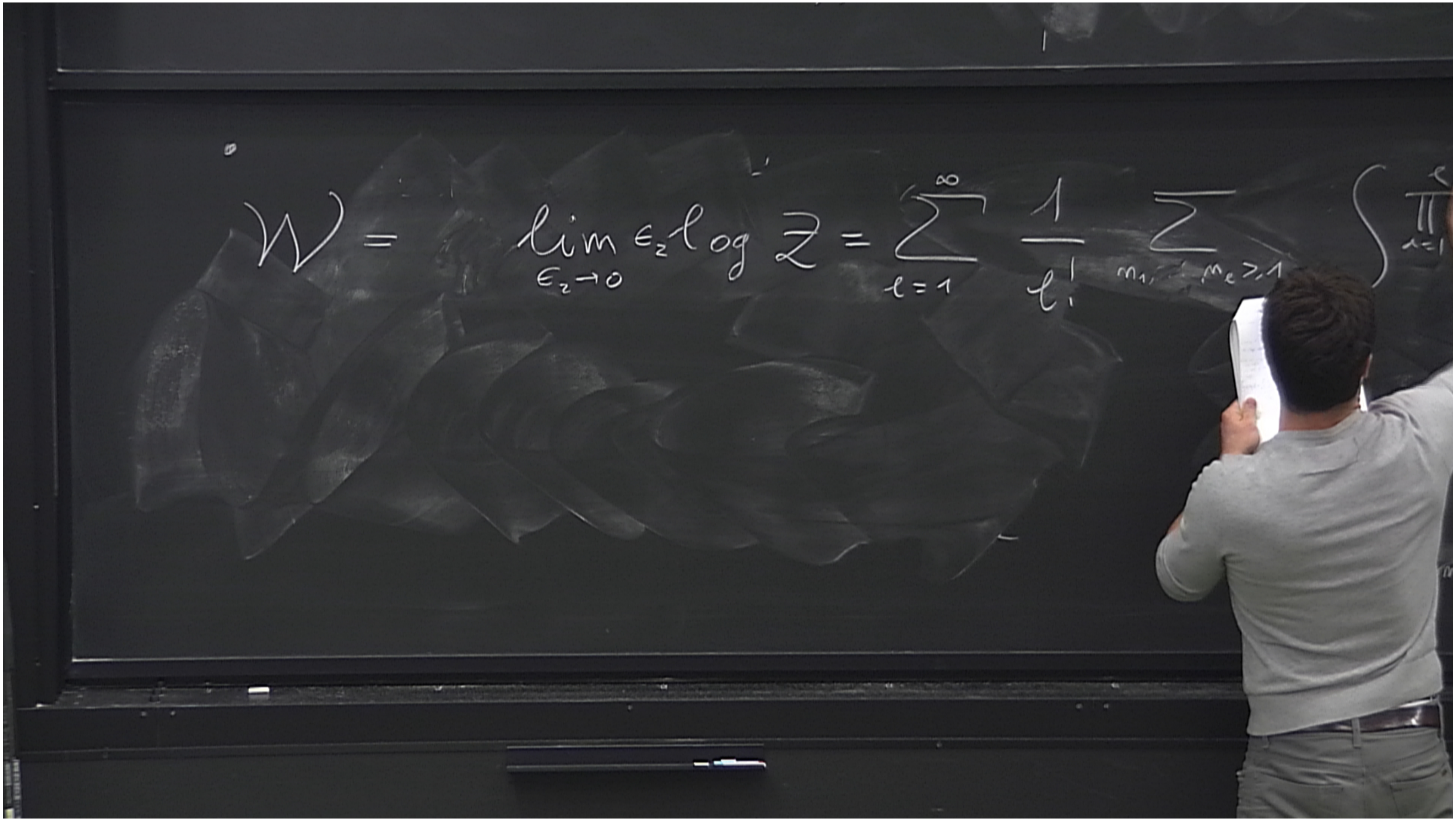
$$\int \prod_{I=1}^{k} d\mu(\Phi_I) \; \prod_{I<J} \underbrace{\left(1 + F_{IJ}\right)}_{= e^{-\left(V_{short}(\Phi_{IJ}) + V_{long}(\Phi_{IJ})\right)}}$$

$$S_{\{X_1,\ldots,X_\ell\}} \;;\quad S_{\{X_1,\ldots X_\ell\}} = \left(\prod_{a=1}^{\ell} h_{X_a}\right) \sum_{\substack{g \in \text{connected} \\ \text{graphs} \\ \text{on } \{1,\ldots,\ell\}}}$$

$$\underbrace{\left(1+F_{IJ}\right)}_{=\, e^{-\left(V_{Short}(\Phi_{IJ})+V_{Long}(\Phi_{IJ})\right)}}$$

$$\left(1+f^{Long}_{X_a,X_b}\right)=\prod_{\substack{I\in X_a\\ J\in X_b}} e^{-V_{Long}\left(\Phi_I-\Phi_J\right)}$$

$$\ldots, X_e\} \quad ; \quad S_{\{X_1,\ldots X_e\}} = \left(\prod_{a=1}^{e} h_{X_a}\right) \sum_{\substack{g\in \text{connected}\\ \text{graphs}\\ \text{on } \{1,\ldots,e\}}} \prod_{e(a,b)\in g} f^{Long}_{X_a,X_b}$$


$W = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{n_1, \ldots, n_l \geq 1} \int \prod_{i=1}^{l}$

$$\lim_{\epsilon_2 \to 0} \epsilon_2 \log Z = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{n_1, \dots, n_l \geq 1} \int \prod_{a=1}^{l} \frac{d\phi_a}{2\pi i} \cdots \frac{q^{n_a} Q^{n_a}(\phi_a)}{n_a^2} \sum_{g \in T_c}$$

$$\sum_{m_e \geq 1} \int \prod_{a=1}^{e} \frac{d\phi_a}{2\pi i} \; \frac{q^{m_a} Q^{m_a}(\phi_a)}{m_a^2} \sum_{g \in \mathcal{T}_c^{(e)}} \prod_{\ell(a,b) \in g} m_a m_b \, G(\phi_{ab})$$

$$\mathcal{W} = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z = \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{n_1, \ldots, n_\ell \geq 1} \int \prod_{i=1}^{\ell}$$

$$\mathcal{W} = \int dx \left(-\frac{1}{2} \varphi(x) \operatorname{Log}\left(1 - q\, Q(x) e^{-\varphi(x)}\right) + \operatorname{Li}_2 \right($$

$$\varphi(x) = \int dy\, G(x-y) \operatorname{Log}\left(1 - q\, Q(y) e^{-\varphi(y)}\right)$$

$$\log Z = \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{n_1,\dots,n_\ell \geq 1} \int \prod_{a=1}^{\ell} \frac{d\phi_a}{2\pi i} \frac{q^{n_a} Q^{n_a}(\phi_a)}{n_a^2} \sum_{g \in \mathcal{T}_c^{(\ell)}} \prod_{\ell(a,b)\in}$$

$$\varphi(x) \operatorname{Log}\left(1 - q Q(x) e^{-\varphi(x)}\right) + \operatorname{Li}_2\left(q Q(x) e^{-\varphi(x)}\right)$$

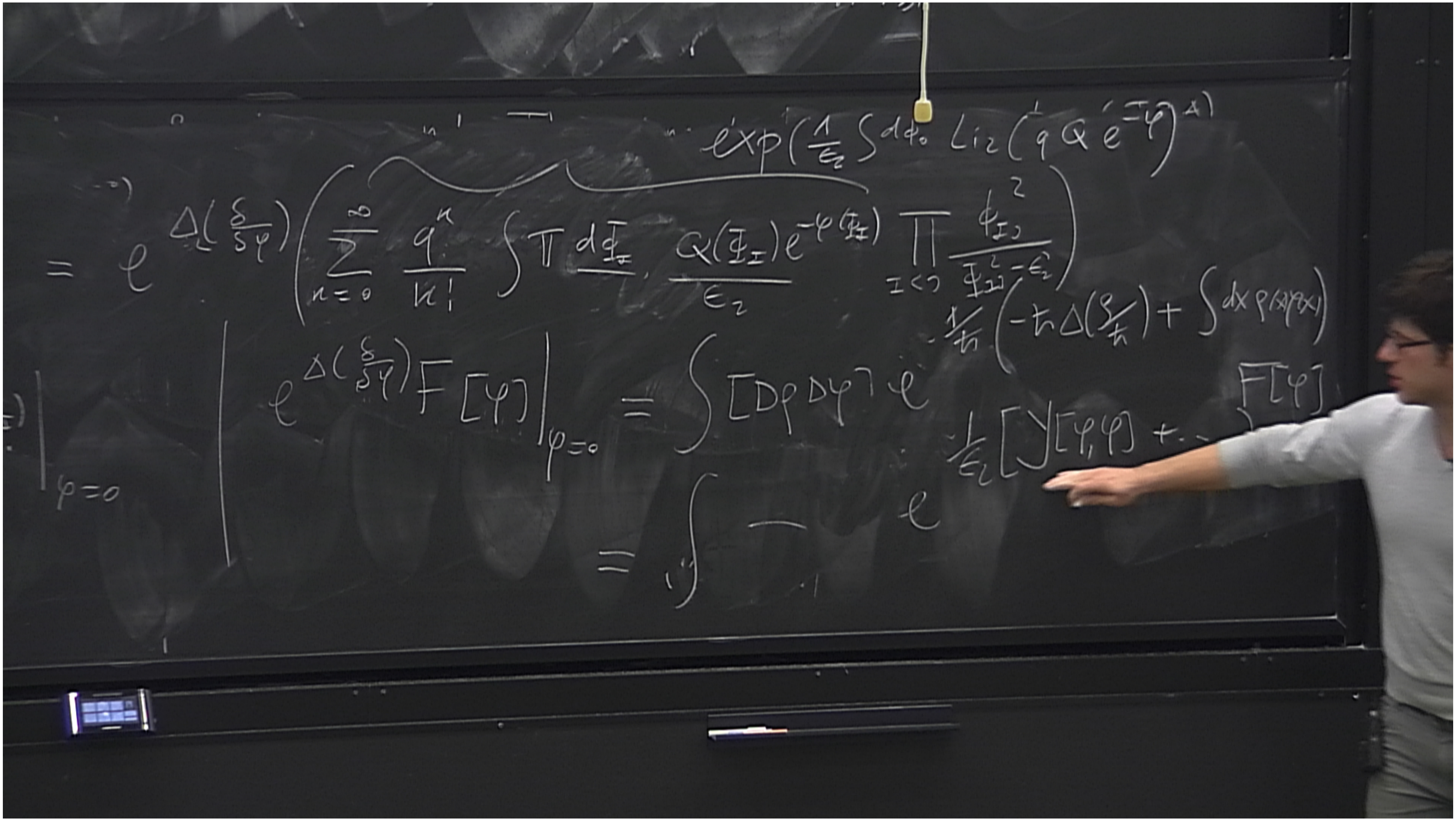
$$y \; G(x-y) \operatorname{Log}\left(1 - q Q(y) e^{-\varphi(y)}\right) \qquad \varphi = \sum_n q^n \varphi_n$$

$$\cup x_i = \{1,\dots,k\}$$

$$Z = \sum_{q=0}^{\infty} \frac{q^k}{k!} \int \dots \underbrace{\prod_{I<J} e^{-V(\Phi_{IJ})}} = e^{\Delta}$$

$$= e^{\Delta\left(\frac{\delta}{\delta\varphi}\right)} \prod_{I=1}^{k} e^{-\varphi(\Phi_I)}\Big|_{\varphi=0}$$

$$\Delta\left(\frac{\delta}{\delta\varphi}\right) = -\frac{1}{2}\int dx\, V(x-y) \frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)}$$


$= e^{\Delta(\frac{\delta}{\delta\varphi})} \left(\sum_{n=0}^{\infty} \frac{q^n}{n!} \int \prod \frac{d\Phi_I}{} \frac{Q(\Phi_I) e^{-\varphi(\Phi_I)}}{\epsilon_2} \prod_{I<J} \frac{\Phi_{IJ}^2}{\Phi_{IJ}^2 - \epsilon_2^2} \right)$
$\exp(\frac{1}{\epsilon_2} \int d\Phi_0 \, Li_2(q Q e^{-\varphi}))$
$e^{\Delta(\frac{\delta}{\delta\varphi})} F[\varphi] \Big|_{\varphi=0} = \int [D\rho D\varphi] \, e^{\frac{1}{\hbar}(-\hbar\Delta(\frac{\rho}{\hbar}) + \int dx \, \rho(x)\varphi(x))} F[\varphi]$
$e^{\frac{1}{\epsilon_2}[Y[\varphi,\varphi] + \dots]}$