

Title: 13/14 PSI - Gravitational Physics Review - Lecture 8

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Abstract:

Lecture 8

Geometry of Submanifolds

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- domain wall - approximate by
a $(D-1)$ dim submanifold
- ? $(D-2), (D-3) \dots$ submanifolds?

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- domain wall - approximate by
a $(D-1)$ dim submanifold
- ? $(D-2), (D-3) \dots$ submanifolds?
- Variation of E-H action + Boundary

Gauss - Codazzi

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$\in M$

a submanifold

$n(p+1)$

Gauss-Codazzi



$\Sigma \subset M$
is a submanifold
of dim $(p+1)$
 ↑ ↑
 spatial time

The co-dimension

is $D - (p+1)$

The co-dimension of Σ

is $D - (p+1) = n$, hence $\exists$

n linearly independent

vectors to Σ .

manifold

$(p+1)$
↑
time

ension of Σ

n, hence $\exists$

dependent

to Σ

$= 0$.

s on Σ .

or $\eta_\mu \frac{\partial X^\mu}{\partial \sigma^A} = 0$ where $X^\mu(\sigma^A)$

define



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, hence $\exists$

pendent

Σ

0.

on Σ

or $\eta_{\mu} \frac{\partial X^{\mu}}{\partial \sigma^A} = 0$ where $\frac{X^{\mu}(\sigma^A)}{\text{defines } \Sigma}$

define coord fns of Σ within $\mathcal{M}$

$\mathcal{P}$

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or $\eta_\mu \frac{\partial X^\mu}{\partial \sigma^A} = 0$ where $\frac{X^\mu(\sigma^A)}{\text{defines } \Sigma}$

define the coord fns of Σ within M

The 1st fundamental form

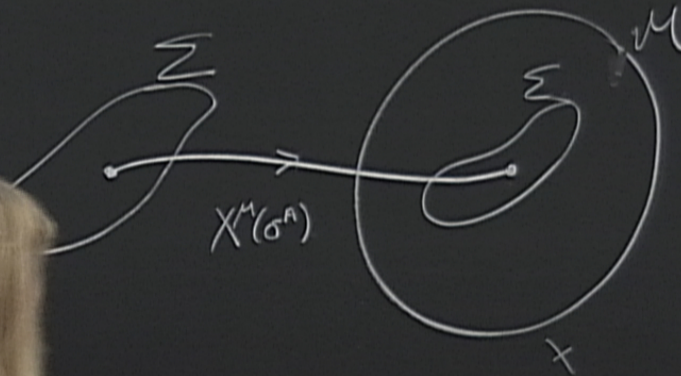
h_{ab} is the projection of the metric onto Σ , and is equivalent to the metric Σ inherits from $\mathcal{M}$, the induced metric

$$\gamma_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^A} \frac{\partial x^\nu}{\partial \sigma^B}$$

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metric onto Σ , and is equivalent
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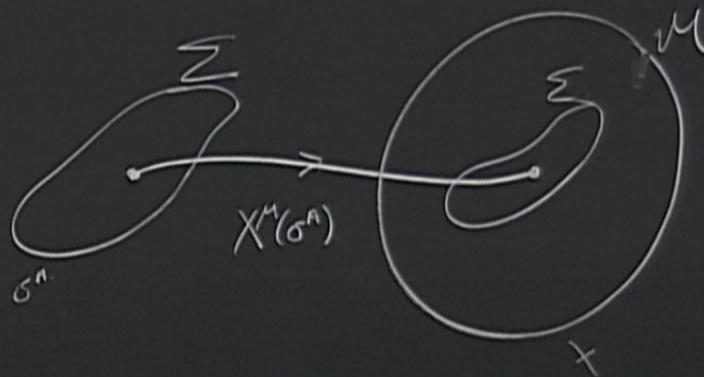
projection of the
 Σ , and is equivalent
 Σ inherits from $\mathcal{M}$,
metric or intrinsic metric

$$g_{AB} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B}$$

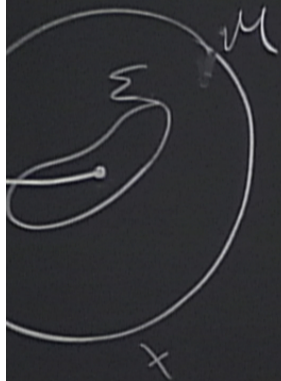


projection of the
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$$g_{AB} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B}$$



$\frac{\partial X^\mu}{\partial \sigma^A}$ is a projection operator
 into Σ $T_p^*(\mathcal{M}) \rightarrow T_p^*(\Sigma)$

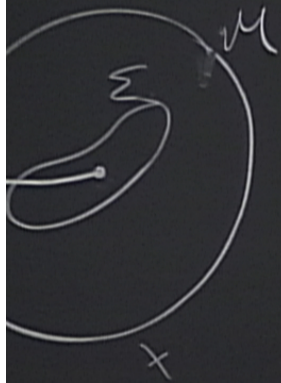


rejection operator.

$$\Sigma \quad T_p^*(M) \rightarrow T_p^*(\Sigma)$$

The 2nd fundamental form



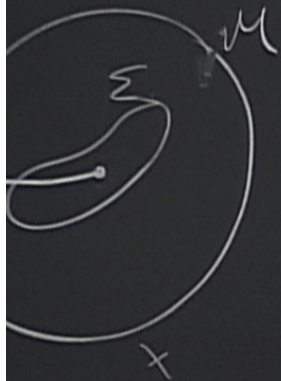


projection operator

$$\Sigma \quad T_p^*(M) \rightarrow T_p^*(\Sigma)$$

The 2nd fundamental form or extrinsic curvature Σ is

$$K_{iab} = \nabla_{\epsilon_i} n^d h_{db}$$

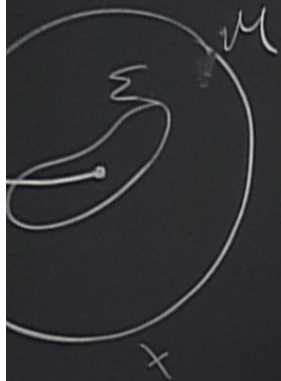


projection operator

$$\Sigma \quad T_p^*(M) \rightarrow T_p^*(\Sigma)$$

The 2nd fundamental form or extrinsic curvature of Σ is

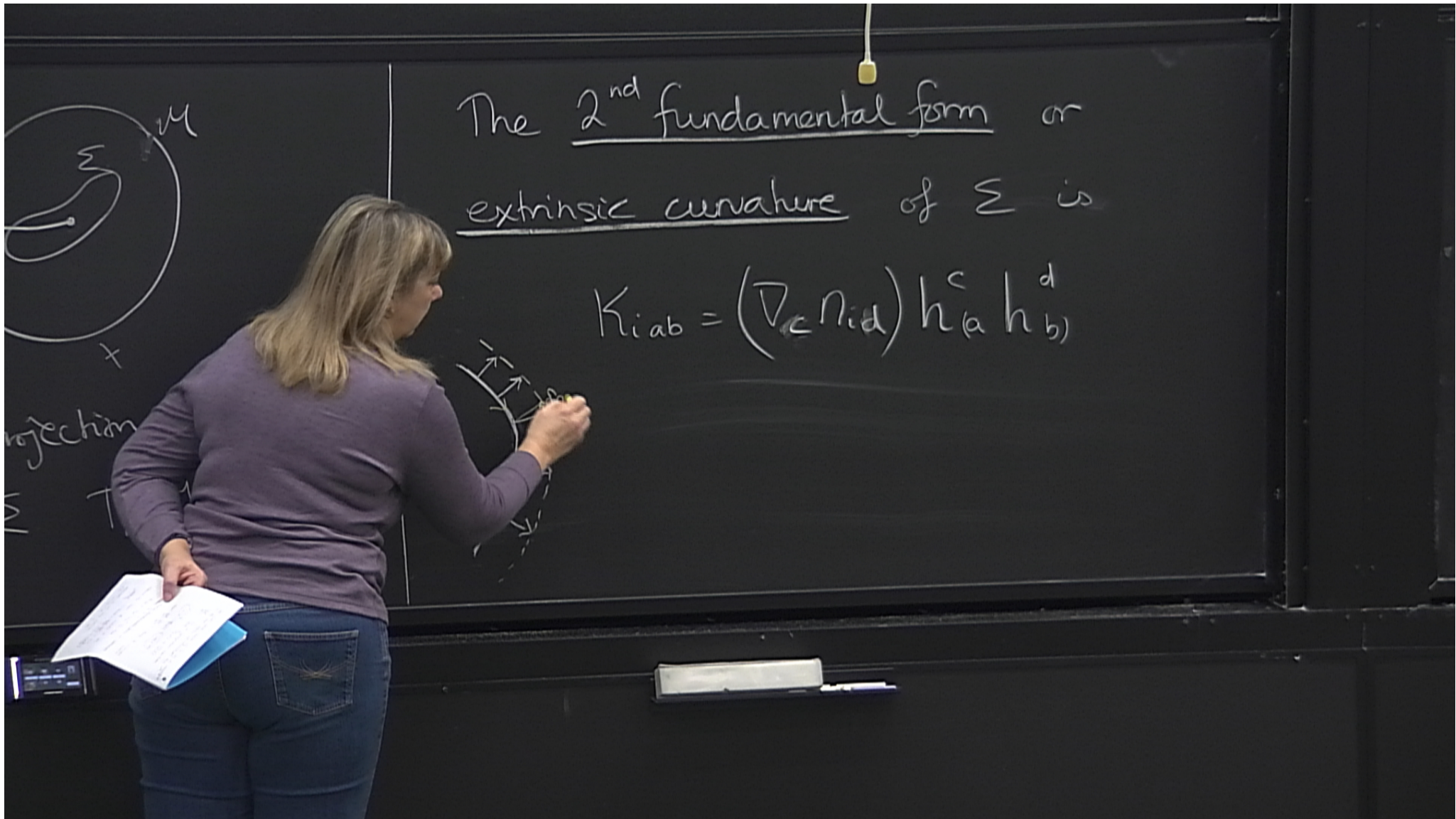
$$K_{iab} = \nabla_c n^c_d h^c_a h^d_b$$



projection operator
 Σ $T_p(M)$

The 2nd fundamental form or extrinsic curvature of Σ is

$$K_{iab} = (\nabla_c n_i a^c) h_a^b h_b^d$$



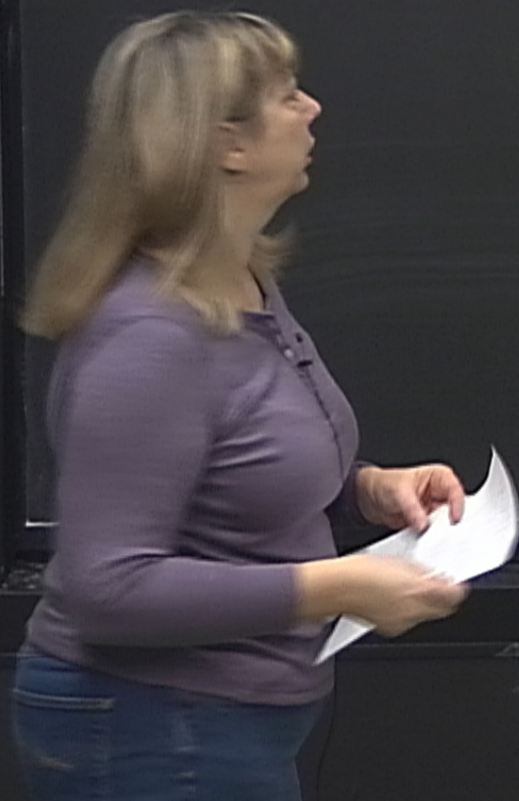
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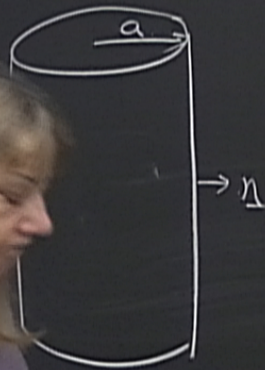


e.g. $x^2 + y^2 = a^2$ C

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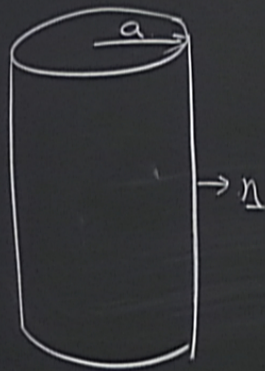


$$n_a = (\cos\theta, \sin\theta, 0)$$

$$h_{ab} = \delta_{ab} - n_a n_b$$

$$= \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

e.g. $x^2 + y^2 = a^2 \subset \mathbb{R}^3$



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$$= \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{CART}} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{Pol.}}$$

Take coords $\sigma^A = \{0, z\}$

then $\gamma_{AB} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ Ad.}$$

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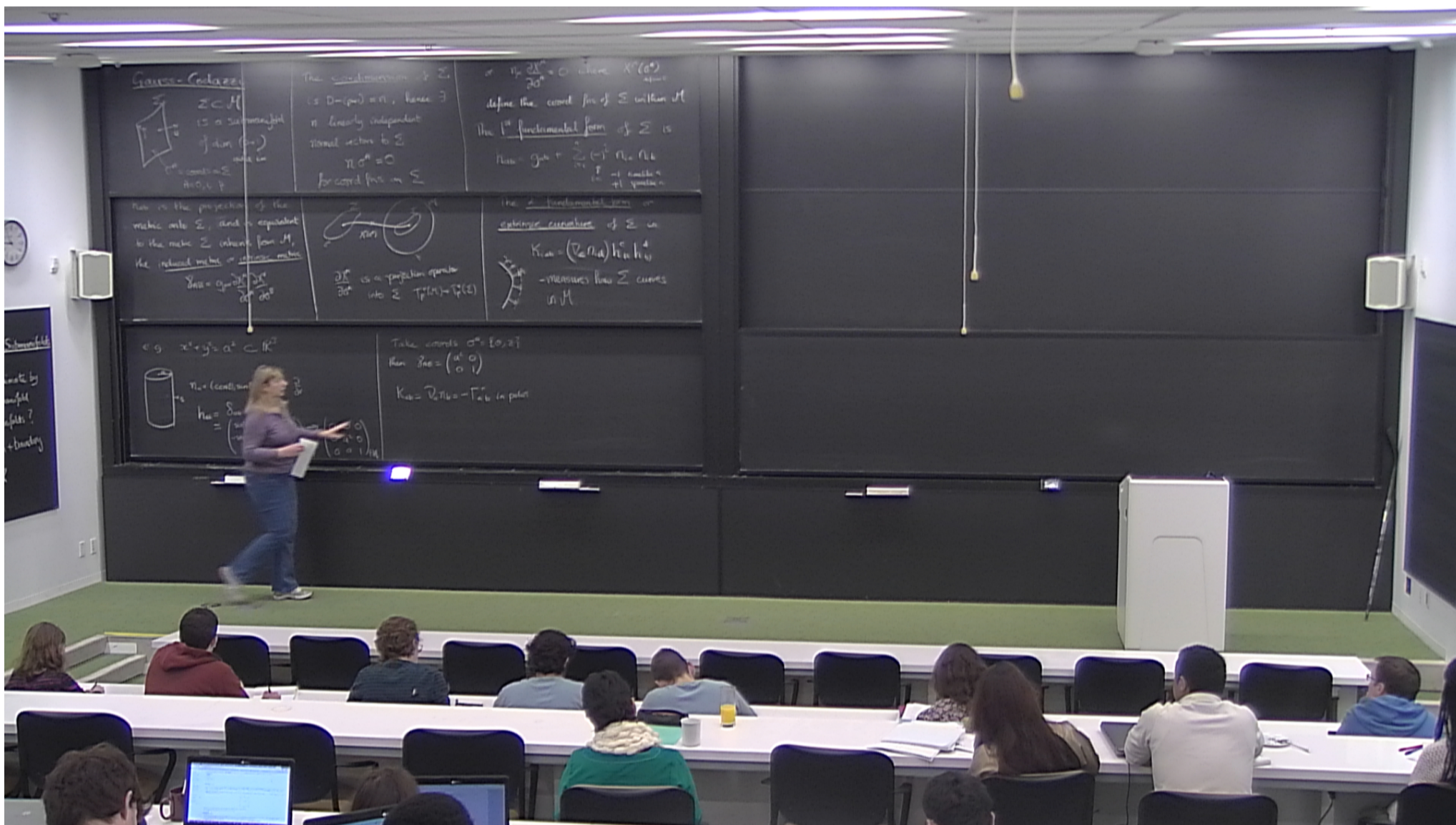
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$K_{ab} = \nabla_a n_b$ in polar

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ Pol.}$



Take coords $\sigma^A = \{t, r\}$

then $\gamma_{AB} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$

$$K_{ab} = \nabla_a n_b = -\Gamma_{ab}^r \text{ in polars}$$

Take coords $\sigma^A = \{0, z\}$

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$K_{ab} = \Gamma_{ab}^r - \Gamma_{ab}^r$ in polars

$\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$

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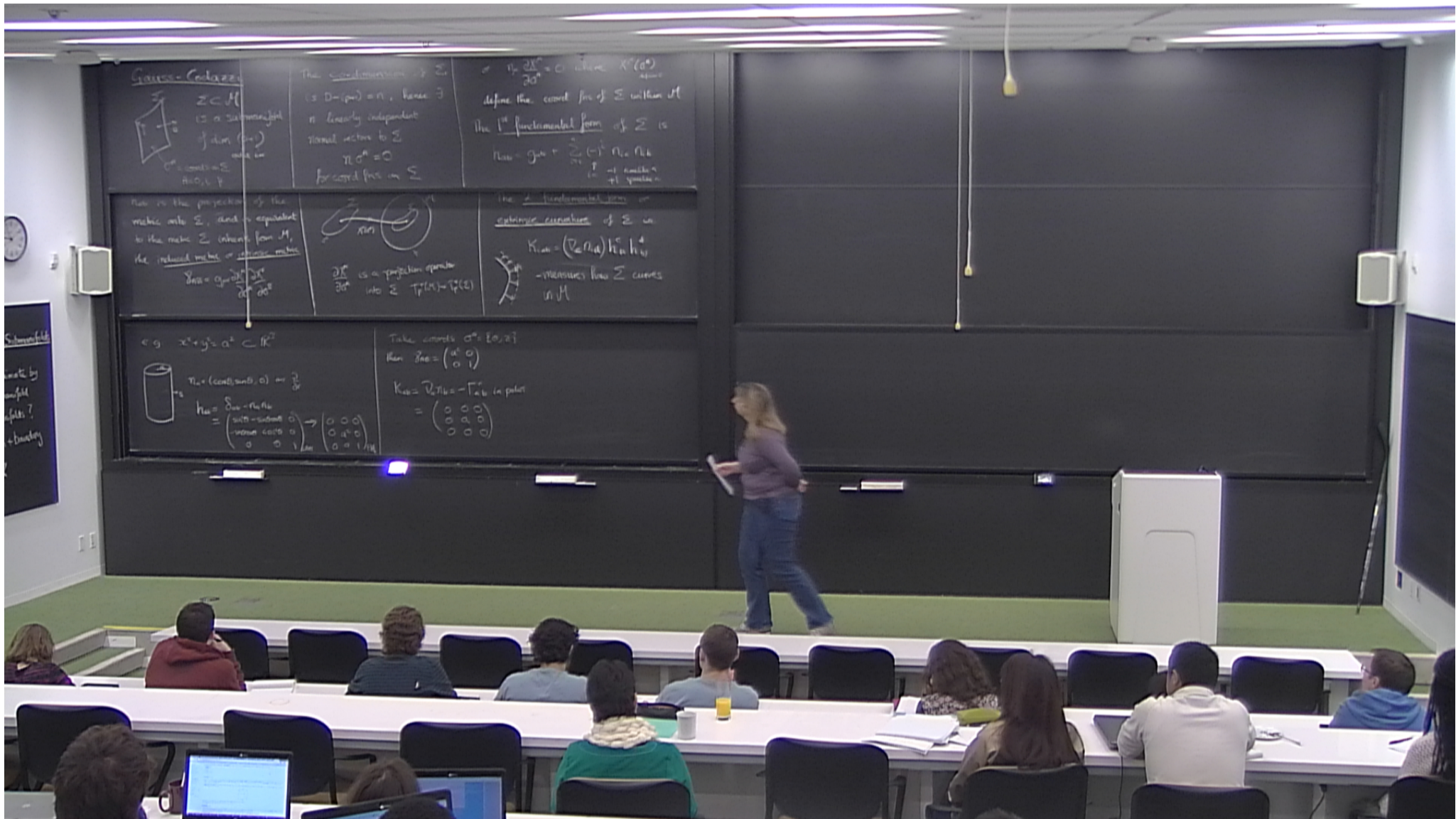
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e.g. $x^2 + y^2 = a^2 \subset \mathbb{R}^3$

$$dx^2 + dy^2 + dz^2$$

$$dr^2 + r^2 d\theta^2 + dz^2$$

$$n_a = (\cos\theta, \sin\theta, 0) \Leftrightarrow \frac{\partial}{\partial r}$$

$$h_{ab} = \delta_{ab} - n_a n_b$$

$$= \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{CART}} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{Pol.}}$$

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$\sin^2\theta = y^2/a^2$
 $\cos^2\theta = 1 - y^2/a^2$

For $\text{codim} > 1$, the normals can vary, or "twist", throughout the submanifold.



The normal fundamental forms are defined as

$$\beta_{\mu ij} = \eta_{\nu\mu} \nabla_{\mu} \eta_j^{\nu}.$$

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The normal fundamental forms
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$$\beta_{\mu ij} = n_i \nabla_\mu n_j^\nu$$

this is the connection on
the normal bundle of Σ .

(Will a

normals can
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fundamental forms

as

$$\nabla_\mu n_j^\nu$$

this is the connection on
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(Will assume n_i^μ extend
geodesically off Σ into M .)



this is the connection on
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(Will assume n_i^M extend
geodesically off Σ into M
& $\nabla_{n_i} n_j = 0$)

We can now relate the curvature

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We can now relate the curvatures
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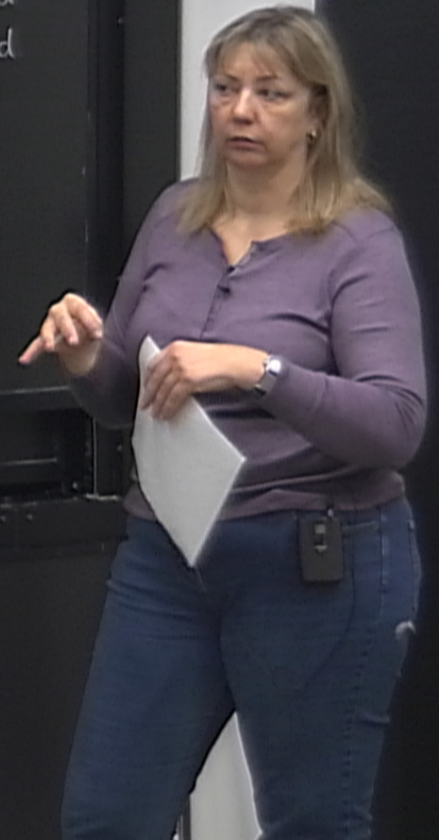


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$${}^{(p+)}R^a{}_{bcd} = {}^{(D)}R^{a'}{}_{b'c'd'} h^a{}_{a'} h^b{}_{b'} h^c{}_{c'} h^d{}_{d'}$$



this is the connection on
the normal bundle of Σ .

(Will assume n, M extend
geodesically off Σ into M
& ∇_n, ∇_{n^*})

We can now relate the curvatures
of M and Σ via the Gauss eqn.

$${}^{(p)}R^a{}_{bcd} = {}^{(D)}R^{a'}{}_{b'c'd'} h^a{}_{a'} h^b{}_{b'} h^c{}_{c'} h^d{}_{d'} \\ - \sum_{i=1}^n (-)^i [K^a{}_c K_{i b d} - K_{i d} K^a{}_{b c}]$$

Proof for $n=1$ Use

Proof for $n=1$ Use Riemann Identity.

$$2 \nabla_c^{(p_{11})} \nabla_d^{(p_{11})} V^a$$

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$$2 \nabla_c^{(p+1)} \nabla_d^{(p+1)} V^a = {}^{(p+1)}R^a_{bcd} V^b \quad (V^b h_b^a = V^a)$$

Proof for $n=1$ Use Riemann Identity.

$$\overset{(p+1)}{\nabla_d} V^a = {}^{(p+1)}R^a_{bcd} V^b \quad (V^b h_b^a = V^a)$$

Proof for $n=1$ Use Riemann Identity.

$$2 \nabla_c \nabla^c V^a = {}^{(p+1)}R^a_{bcd} V^b \quad (V^b h_b^a = V^a)$$

$${}^{(p+1)}R^a_{bcd}$$

Proof for $n=1$ Use Riemann Identity.

$$2^{(p+1)} \nabla_c^{(p+1)} \nabla_d^{(p+1)} V^a = {}^{(p+1)} R^a{}^b (V^b h_b^a = V^a)$$

$${}^{(p+1)} R^a{}_{bcd} V^b = h_c^a h_c^f f_d^f \nabla_d V^a$$

Proof for $n=1$ Use Riemann Identity.

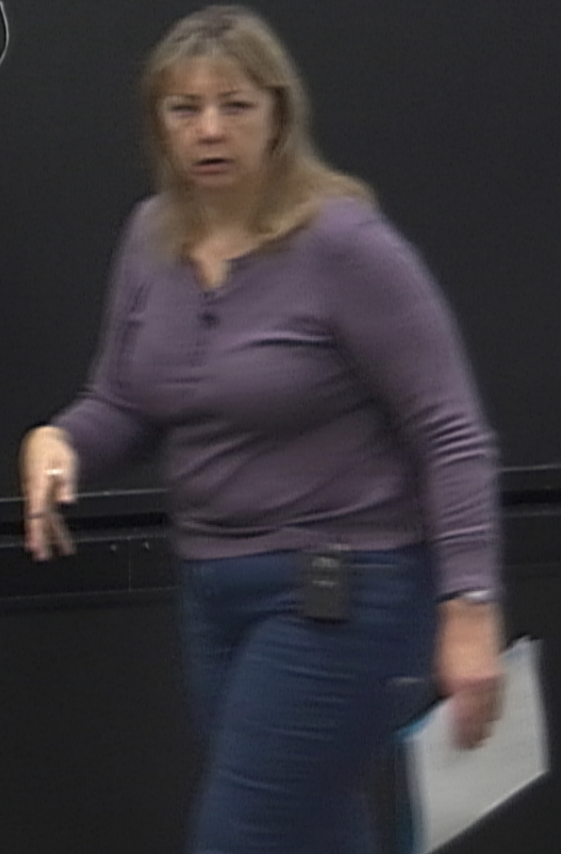
$$2 \nabla_c^{(p+1)} \nabla_d^{(p+1)} V^a = {}^{(p+1)}R^a_{bcd} V^b \quad (h^a_b V^b = V^a)$$

$${}^{(p+1)}R^a_{bcd} V^b = h^a_c h^f_d \nabla_f^{(p+1)} \nabla_c V^e$$

Proof for $n=1$ Use Riemann Identity.

$$2 \nabla_c^{(p+1)} \nabla_d^{(p+1)} V^a = {}^{(p+1)}R^a_{bcd} V^b \quad (V^b h_b^a = V^a)$$

$${}^{(p+1)}R^a_{bcd} V^b = h_c^a h_e^f h_d^p \nabla_f^{(p+1)} \nabla_p V^e - c c d$$



Proof for $n=1$ Use Riemann Identity.

$$2 \nabla_c^{(p+1)} \nabla_d^{(p+1)} V^a = {}^{(p+1)}R^a_{b} V^b \quad (V^b h_b^a = V^a)$$

$$\begin{aligned} {}^{(p+1)}R^a_{bcd} V^b &= h_c^a h_c^f h_d^p \nabla_f \nabla_p V^b - (c \leftrightarrow d) \\ &= h_c^a h_c^f h_d^p \nabla_f V^b \end{aligned}$$

Proof for $n=1$ Use Riemann Identity.

$$2 \nabla_{[e}^{(p+1)} \nabla_{d]}^{(p+1)} V^a = {}^{(p+1)}R^a_{bcd} V^b \quad (V^b h_b^a = V^a)$$

$$\begin{aligned} {}^{(p+1)}R^a_{bcd} V^b &= h_e^a h_c^f h_d^p \nabla_f^{(p+1)} \nabla_p V^e - c c_{cd} \\ &= h_e^a h_c^f h_d^p \nabla_f (h_p^q h_r^e \nabla_q V^r) - c c_{cd} \end{aligned}$$

$$= h_e^a h_c^f h_d^q h_r^e \nabla$$

Proof for $n=1$ Use Riemann Identity.

$$2^{(p+2)} \nabla_{[e}^{(p+1)} \nabla_{d]}^{(p+1)} V^a = {}^{(p+1)} R^a_{bcd} V^b \quad (V^b h_b^a = V^a)$$

$${}^{(p+1)} R^a_{bcd} V^b = h_e^a h_c^f h_d^p \nabla_f^{(p)} \nabla_p V^e - c \leftrightarrow d$$

$$= h_e^a h_c^f h_d^p \nabla_f (h_p^q h_r^e \nabla_q V^r) - c \leftrightarrow d$$

$$\downarrow \delta_p^q (-)^i n_p n^q \text{ \& remember } n_p h_d^p = 0$$

$$= h_e^a h_c^f h_d^p (-)^i h_e^a$$

$$= h_e^a h_c^f h_d^q h_r^e \nabla_f \nabla_q V^r - \text{c.c.}$$

$$(-)^i h_e^a h_c^f h_d^p h_r^e n^q (\nabla_f n_p) \nabla_q V^r - \text{c.c.}$$

$$h_b^a = V^a)$$

$$- \text{c.c.}$$

$$\text{but } n_p h_d^p = 0$$

$$= h_e^a \underbrace{h_c^f h_d^q h_r^e}_{\text{c} \leftrightarrow \text{d}} \nabla_f \nabla_q V^r - \text{c} \leftrightarrow \text{d} \quad \checkmark$$

$$(-)^i h_e^a h_c^f h_d^p h_r^e \overbrace{n^q (\nabla_f n_p) \nabla_q V^r}^{\text{c} \leftrightarrow \text{d}} - \text{c} \leftrightarrow \text{d} \quad (\nabla_n V = 0)$$

$$(-)^j h_e^a h_c^f h_d^q \overbrace{n_r (\nabla_f n^e) \nabla_q V^r}^{\text{c} \leftrightarrow \text{d}} - \text{c} \leftrightarrow \text{d}$$

$$(n_r \nabla_q V^r = -V^r \nabla_q n_r)$$

$$= h$$

$$(-) h_e h_c h_d h_r n^r (V_f n^p) V_q V = c \leftrightarrow d.$$

$$(-)^i h_e^a h_c^f h_d^g h_r^e \overbrace{n^r (V_f n^e) V_q} V^r = c \leftrightarrow d. \quad (n_r \nabla_q V^r = -V^r \nabla_q n_r)$$

$$= h_e^a h_c^f h_d^g h_r^e R^r_{p f q} V^p$$

$$- (-)^i K^a_c V^r K_{dr} + (-)^i K^a_d V^r K_{cr}$$

$\delta_p^v (-)^p \Pi_p \Pi^p$ & remember $\Pi_p \Pi^p = 0$

$${}^{(p+1)}R^a{}_{bcd} V^b = h^a{}_{a'} h^{c'}_c h^{d'}_d R^{a'}{}_{b' c' d'} V^{b'}$$

$$(-)^i [K^a{}_d K_{bc} - K^a{}_c K_{bd}] V^b.$$

$\delta_p^v (-)^{p+1} \Pi_p \Pi^v$ & remember $\Pi_p \Pi^v = 0$

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$\delta_p^{p'} (-)^p \Pi_p \Pi^{p'}$ & remember $\Pi_p \Pi^{p'} = 0$

$${}^{(p+1)}R^a{}_{bcd} V^b = h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d R^{a'}{}_{b c' d'} V^b$$

$$(-)^i [K^a{}_d K_{bc} - K^a{}_c K_{bd}] V^b$$

→ gives $R^a{}_{bcd} \equiv 0$ for cylinder.

$\delta_p^v(-) \Pi_p \Pi^v$ & remember $\Pi_p \Pi^v = 0$

$${}^{(p+1)}R^a{}_{bcd} V^b = h^a{}_{a'} h^{c'}_c h^{d'}_d R^{a'}{}_{b' c' d'} V^{b'}$$

$$= i [K^a{}_d K_{bc} - K^a{}_c K_{bd}] V^b$$

→ gives $R^a{}_{bcd} \equiv$ ω tensor.