

Title: 13/14 PSI - Gravitational Physics Review - Lecture 6

Date: Jan 13, 2014 10:15 AM

URL: <http://pirsa.org/14010034>

Abstract:

Lecture 6

$$T^{\alpha}_{\beta} \sim \rho, -p \sim T_i$$

The Einstein-Hilbert action (and beyond!)

• Integrating on manifolds

In space, would take

d^4x

manifold $d^4y \rightarrow \det\left(\frac{\partial y}{\partial x}\right) d^4x$

• Integrating on manifolds

In flat space, would take

$$\int d^4x \quad (\text{volume element})$$

On the manifold $d^4y \rightarrow \det\left(\frac{\partial y}{\partial x}\right) d^4x$

But

$$\sqrt{|g|} d^4x \leftrightarrow \sqrt{|g_{\bar{y}}|} d^4y \leftarrow$$

$$\underline{dt} \wedge \underline{dx} \wedge \underline{dy} \wedge \underline{dz}$$

$$dt \left(\frac{\partial y}{\partial x} \right) d^4x$$

$$t \left(\frac{\partial \gamma}{\partial x} \right) d^4x$$

But

$$\sqrt{|g|} d^4x \leftrightarrow \sqrt{|g(y)|} d^4y \leftarrow$$

$$dt \wedge dx \wedge dy \wedge dz$$

To vary $\sqrt{|g|}$ make use of identity:

$$\delta \left(\frac{\partial \mathcal{L}}{\partial x} \right) d^4x$$

But

$$\sqrt{|g|} d^4x \leftrightarrow \sqrt{|g(y)|} d^4y \leftarrow$$

$$dt \wedge dx \wedge dy \wedge dz$$

To vary $\sqrt{|g|}$ make use of
identity: $\det M = \exp \text{tr} \log M$.

$$\delta(\det M) = \delta(\det M)$$

$$\longleftrightarrow \sqrt{|g(y)|} d^4 y \leftarrow$$

$$\begin{aligned} \delta(\det M) &= \delta(\text{tr} \log M) \det M \\ &= \text{tr}(M^{-1} \delta M) \det M \end{aligned}$$

$$\delta \sqrt{-g} = \frac{1}{2} \frac{1}{\sqrt{-g}}$$

$|g|$ make use of

$$\det M = \exp \text{tr} \log$$

$$\leftrightarrow \sqrt{|g_{\mu\nu}|} d^4y \leftarrow$$

make use of
 $\det M = \exp \operatorname{tr} \log M$

$$\begin{aligned} \delta(\det M) &= \delta(\operatorname{tr} \log M) \det M \\ &= \operatorname{tr}(M^{-1} \delta M) \det M \end{aligned}$$

$$\begin{aligned} \Rightarrow \delta \sqrt{-g} &= \frac{1}{2} \frac{1}{\sqrt{-g}} - g^{ab} \delta g_{ab} \\ &= \frac{1}{2} \sqrt{-g} g^{ab} \delta g_{ab} \quad \text{S(metric)} \\ &= -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab} \quad \text{S(inv. met)} \end{aligned}$$

OR

e.g.

$$\partial_a \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{cd} g_{cd,a}$$
$$= \sqrt{-g} \Gamma_a^d{}_d$$

$$= \frac{1}{2} \sqrt{-g} g^{cd} g_{cd,a}$$

$$= \sqrt{-g} \Gamma_a^d{}_d$$

less scalar

$$\frac{1}{2} (\partial\phi)^2$$

$$S_\phi = \int d^4x \sqrt{-g} \frac{1}{2} (\partial\phi)^2 = \frac{1}{2} \int d^4x \sqrt{-g} \partial_a \phi \partial_b \phi g^{ab}$$

$$\delta S_\phi = \int d^4x \sqrt{-g} \left[\partial_a \phi \partial_b \delta\phi g^{ab} + \frac{1}{2} \partial_a \phi \partial_b \phi \delta g^{ab} - \frac{1}{4} g_{ab} (\partial\phi)^2 \delta g^{ab} \right]$$

$$= \frac{1}{2} \sqrt{-g} g^{cd} g_{cd,a}$$

$$= \sqrt{-g} \Gamma_a^d{}_d$$

less scalar

$$\frac{1}{2} (\partial\phi)^2$$

$$S_\phi = \int d^4x \sqrt{-g} \frac{1}{2} (\partial\phi)^2 = \frac{1}{2} \int d^4x \sqrt{-g} \partial_a \phi \partial_b \phi g^{ab}$$

$$\delta S_\phi = \int d^4x \sqrt{-g} \left[\partial_a \phi \partial_b \delta\phi g^{ab} + \frac{1}{2} \partial_a \phi \partial_b \phi \delta g^{ab} - \frac{1}{4} g_{ab} (\partial\phi)^2 \delta g^{ab} \right]$$

$$= \int d^4x \sqrt{-g} \left[\frac{\delta S}{\delta \phi} \delta\phi + \frac{\delta S}{\delta g^{ab}} \delta g^{ab} \right]$$

$$(\partial\phi)^2 = \frac{1}{2} \int d^4x \sqrt{-g} \partial_a \phi \partial_b \phi g^{ab}$$

$$\partial_b \delta\phi g^{ab} + \frac{1}{2} \partial_a \phi \partial_b \phi \delta g^{ab}$$

$$\frac{1}{4} g_{ab} (\partial\phi)^2 \delta g^{ab}]$$

$$\phi + \frac{\delta S}{\delta g^{ab}} \delta g^{ab}]$$

$$\frac{\delta S}{\delta \phi} = - \frac{1}{\sqrt{-g}} [\partial_b \sqrt{-g} g^{ab} \partial_a \phi]$$

$$= - \partial_b \nabla^b \phi - \frac{1}{\sqrt{-g}} (\partial_b \sqrt{-g}) \nabla^b \phi$$

$$= - [\partial_b \nabla^b \phi + \frac{1}{\sqrt{-g}} \partial_b \sqrt{-g} \nabla^b \phi]$$

$$= - \square \phi$$

Note $\frac{\delta S_\phi}{\delta g_{ab}} = \frac{1}{2} \left[\partial_a \phi \partial_b \phi - \frac{1}{2} (\partial \phi)^2 g_{ab} \right]$
 $= \frac{1}{2} T_{\phi ab}$

Now construct a gravitational lagrangian

$$-\frac{1}{2}(\partial\phi)^2 g_{ab}]$$

ional Lagrangian.
as candidate

Need δR

$$\delta(R_{ab} g^{ab}) = \delta R_{ab} g^{ab} + \underbrace{R_{ab}}_{\substack{\rightarrow \text{Einstein tensor} \\ \downarrow}} \delta g^{ab}$$

Variation of R_{ab} given by Palatini's Lemma:

$$\delta R_{ab} = \nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ca}$$

$$g^{ab}) = \delta R_{ab} g^{ab} + \underbrace{R_{ab}}_{\substack{\rightarrow \text{Einstein tensor} \\ \downarrow}} \delta g^{ab}$$

R_{ab} given by Palatini's Lemma:

$$R_{ab} = \nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ca}$$

- Simple proof using normal co-ordinates.

Normal coords are ones in which Γ vanishes "locally".

In N.C

$$\delta \Gamma_{ab,c}^c = \partial_c \left[\frac{1}{2} g^{cd} (\delta g_{da,b} + \delta g_{db,a} - \delta g_{ab,d}) \right]$$

In N.C

$$\delta \Gamma_{ab,c}^c = \partial_c \left[\frac{1}{2} g^{cd} (\delta g_{da,b} + \delta g_{db,a} - \delta g_{ab,d}) \right]$$
$$= \frac{1}{2} g^{cd} [\delta g_{da,bc} + \delta g_{db,ac} - \delta g_{ab,dc}]$$

In N.C

$$\begin{aligned}
 \delta \Gamma_{ab,c}^c &= \partial_c \left[\frac{1}{2} g^{cd} (\delta g_{da,b} + \delta g_{db,a} - \delta g_{ab,d}) \right] \\
 &= \frac{1}{2} g^{cd} [\delta g_{da,bc} + \delta g_{db,ac} - \delta g_{ab,dc}] \\
 &= -\frac{1}{2} (\nabla_c \nabla_b \delta g^{ce}) g_{ea} - \frac{1}{2} (\nabla_c \nabla_a \delta g^{ce}) g_{be} \\
 &\quad + \frac{1}{2} g_{ac} g_{bd} \square \delta g^{cd}
 \end{aligned}$$

$$R_{ab} = R^c{}_{acb} = \Gamma^c{}_{ab;c} - \Gamma^c{}_{ac;b} \quad (NC)$$

$$\Rightarrow \delta R_{ab} = \delta \Gamma^c{}_{ab;c} - \delta \Gamma^c{}_{ac;b}$$

Hence

$$g^{ab} \delta R_{ab} = - \nabla_a \nabla_b \delta g^{ab} + g_{ab} \square \delta g^{ab}$$

$$R_{ab} = R^c{}_{acb} = \Gamma^c{}_{ab;c} - \Gamma^c{}_{ac;b} \quad (NC)$$

$$\Rightarrow \delta R_{ab} = \delta \Gamma^c{}_{ab;c} - \delta \Gamma^c{}_{ac;b}$$

Hence

$$g^{ab} \delta R_{ab} = \underbrace{- \nabla_a \nabla_b \delta g^{ab} + g_{ab} \square \delta g^{ab}}_{\text{total derivative}}$$

$$\text{Try } S_g = C \int d^4x \sqrt{-g} R$$

$$\delta S_g = C \int d^4x \sqrt{-g} \left[R_{ab} \delta g^{ab} - \frac{1}{2} R g_{ab} \delta g^{ab} - \nabla_a \nabla_b \delta g^{ab} + g_{ab} \square \delta g^{ab} \right]$$

$$\frac{1}{\sqrt{-g}} \overset{\uparrow}{\partial_a (\sqrt{-g} \nabla_b \delta g^{ab})} - \frac{1}{\sqrt{-g}} \partial_c \sqrt{-g} \delta^c [g_{ab} \delta g^{ab}]$$

$$= \int d^4x \sqrt{-g} \left(C_{ab} \delta g^{ab} \right.$$

$$\left. + \int_{\partial M} d^3x \sqrt{g_3} \left[-\nabla_b \delta g^{ab} + \nabla^a g_{cd} \delta g^{cd} \right] n_a \right)$$

variation of normal derivatives
to boundary

$$= \int d^4x \sqrt{g} C G_{ab} \delta g^{ab}$$

normal
to ∂M

$$+ C \int_{\partial M} d^3x \sqrt{g_3} \left[-\nabla_b \delta g^{ab} + \nabla^a g_{cd} \delta g^{cd} \right] n_a$$

Variation of normal derivatives
to boundary

$$\rightarrow C = -\frac{1}{16\pi G}$$

i.e. $S_{E-H} = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R$

$$= \int d^4x \sqrt{-g} C G_{ab} \delta g^{ab}$$

normal
to ∂M

$$+ C \int_{\partial M} d^3x \sqrt{g_3} \left[-\nabla_b \delta g^{ab} + \nabla^a g_{cd} \delta g^{cd} \right] n_a$$

Variation of normal derivatives
to boundary

$$\rightarrow C = -\frac{1}{16\pi G}$$

i.e.
$$S_{E-H} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R$$

For now ignore boundary

"winkle"

Can go beyond Einstein-Hilbert:

e.g. $\mathcal{L} = \Phi R$

$\wedge$ replacing $1/16\pi G$

(Jordan)
(Brans-
Dicke)

Can go beyond Einstein-Hilbert:

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Scalar-tensor theories of gravity

$$g^{ab} \Phi \delta R_{ab} = -\Phi \nabla_a \nabla_b \delta g^{ab} + \Phi g_{ab} \square \delta g^{ab}$$

$$\stackrel{\text{"by parts"}}{=} -\delta g^{ab} \nabla_a \nabla_b \Phi + \delta g^{ab} \square \Phi g_{ab}$$

Einstein tensor becomes

$$\Phi G_{ab} - \nabla_a \nabla_b \Phi + g_{ab} \square \Phi$$

In Brans-Dicke

$$\mathcal{L} = -\Phi R + \omega \frac{(\partial\Phi)^2}{\Phi} + 16\pi \mathcal{L}_m$$

eqns of motion

$$\Phi G_{ab} = 8\pi T_{ab} + \nabla_a \nabla_b \Phi - g_{ab} \Box \Phi + \frac{\omega}{\Phi} (\partial_a \Phi \partial_b \Phi - \frac{1}{2} g_{ab} (\partial\Phi)^2)$$

$$R + 2\omega \frac{\Box \Phi}{\Phi} = \omega \frac{(\nabla\Phi)^2}{\Phi^2}$$

Combine $\rightarrow$

$$\frac{\Box \Phi}{\Phi} = \frac{8\pi T}{(3+2\omega)}$$

Note matter couples minimally to g

e.g. $(\partial\phi)^2 = g^{ab} \partial_a \phi \partial_b \phi.$

but grav action is not

E-M & grav eqns more complicated

Often change frame to make grav eqns like Einstein,
but matter nonminimally coupled.

Conformal transformations in GR

$$\hat{g}_{ab} = \Omega^2(x) g_{ab} \quad \Omega(x) \text{ a local rescaling}$$

does not change angles

$$\hat{\Gamma}_{bc}^a = \Gamma_{bc}^a + \Omega^{-1} \left(\Omega_{,c} \delta_b^a + \Omega_{,b} \delta_c^a - g^{ae} \Omega_{,e} g_{bc} \right)$$

(ex) $\rightarrow$
 $\hat{R}_{bd} =$

(ex)
→

$$\hat{R}_{bd} = R_{bd} + (2-D)\Omega^{-1}\nabla_b\nabla_d\Omega$$

$$- g_{bd}\Omega^{-1}\square\Omega + 2(D-2)\Omega^{-2}\nabla_b\Omega\nabla_d\Omega$$

$$- (D-3)g_{bd}\Omega^{-2}(\nabla\Omega)^2$$

$$D = \dim M$$

(ex)
→

$$\hat{R}_{bd} = R_{bd} + (2-D)\Omega^{-1}\nabla_b\nabla_d\Omega$$

$$- g_{bd}\Omega^{-1}\square\Omega + 2(D-2)\Omega^{-2}\nabla_b\Omega\nabla_d\Omega$$

$$- (D-3)g_{bd}\Omega^{-2}(\nabla\Omega)^2$$

$$D = \dim M$$

$$\hat{R} = \Omega^{-2}R$$

$$\sqrt{\hat{g}} = \Omega^D\sqrt{g}$$

$$\rightarrow \hat{R}\sqrt{\hat{g}} = \Omega^2 R\sqrt{g}$$

(ex)
→

$$\hat{R}_{bd} = R_{bd} + (2-D)\Omega^{-1}\nabla_b\nabla_d\Omega$$

$$- g_{bd}\Omega^{-1}\Box\Omega + 2(D-2)\Omega^{-2}\nabla_b\Omega\nabla_d\Omega$$

$$- (D-3)g_{bd}\Omega^{-2}(\nabla\Omega)^2$$

$$D = \dim M$$

$$\hat{R} = \Omega^{-2}R + \dots$$

$$\sqrt{\hat{g}} = \Omega^D\sqrt{g}$$

$$\rightarrow \hat{R}\sqrt{\hat{g}} = \Omega^2 R\sqrt{g}$$

Einstein frame