Title: 13/14 PSI - Gravitational Physics Review - Lecture 4

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Abstract:

Lecture 4 Applying Cartan

Spherically symmetric, static

- STATIC : Killing vector $\frac{\partial}{\partial t}$ & $t \leftrightarrow -t$ symm

Lecture 4 Applying Cartan

Spherically symmetric, static

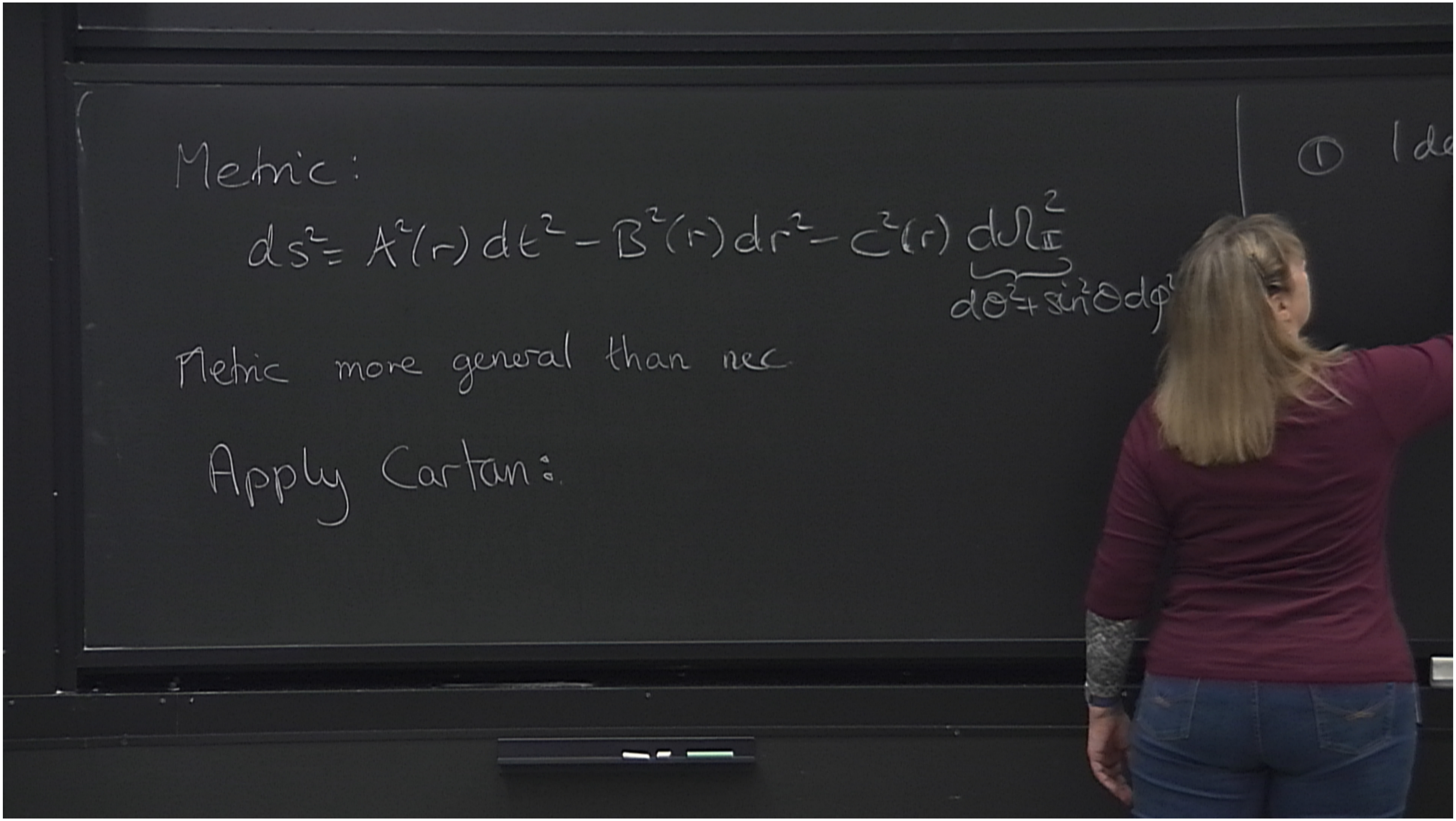
- STATIC: Killing vector $\frac{\partial}{\partial t}$ & $t \leftrightarrow -t$ symm
- SPH SYM: $\exists$ SO(3) algebra of k.v.

$$[\xi_i, \xi_j] = \epsilon_{ijk}\,\xi_k$$

$$\theta, \phi \leftrightarrow S^2$$

Metric:

$$ds^2 = A^2(r)\,dt^2 - B^2(r)\,dr^2 - C^2(r)\,\underbrace{d\Omega_{\mathrm{II}}^2}_{d\theta^2 + \sin^2\theta\, d\varphi^2}$$


Metric:
$ds^2 = A^2(r)\,dt^2 - B^2(r)\,dr^2 - C^2(r)\,\underbrace{d\Omega_{II}^2}_{d\theta^2 + \sin^2\theta\, d\varphi^2}$
Metric more general than nec
Apply Cartan:

$\sin^2\theta\, d\varphi^2$

① Identify o/n basis

$\underline{\omega}^{\hat{t}} = A\,\underline{d}t \qquad \underline{\omega}^{\hat{\theta}} = C\,\underline{d}\theta$

$\underline{\omega}^{\hat{r}} = B\,\underline{d}r \qquad \underline{\omega}^{\hat{\varphi}} = C\sin\theta\,\underline{d}\varphi$

$^2(r)\,dr^2 - C^2(r)\,\underbrace{d\Omega_{II}^2}_{d\theta^2+\sin^2\theta\, d\varphi^2}$

an nec

① Identify o/n basis

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$\underline{\omega}^{\hat{r}} = B\,\underline{d}r \qquad \underline{\omega}^{\hat{\varphi}} = C\sin\theta\,\underline{d}\varphi$

② Diff:

$\underline{d}\underline{\omega}^{t} = A'\,\underline{d}r \wedge \underline{d}t = -\frac{A'}{AB}\,\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}}$

$$\ldots^2(r)\,dr^2 - C^2(r)\,\underbrace{d\Omega_{II}^2}_{d\theta^2 + \sin^2\theta\, d\varphi^2}$$

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① Identify o/n basis

$$\underline{\omega}^{\hat{t}} = A\,\underline{d}t \qquad \underline{\omega}^{\hat{\theta}} = C\,\underline{d}\theta$$

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② Diff:

$$\underline{d}\underline{\omega}^{t} = A'\,\underline{d}r \wedge \underline{d}t = -\frac{A'}{AB}\,\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}}$$

$$\underline{d}\underline{\omega}^{r} = 0$$

ty o/n basis

$dt \qquad \underline{\omega}^{\hat\theta} = C\,\underline{d}\theta$

$\underline{d}r \qquad \underline{\omega}^{\hat\varphi} = C\sin\theta\,\underline{d}\varphi$

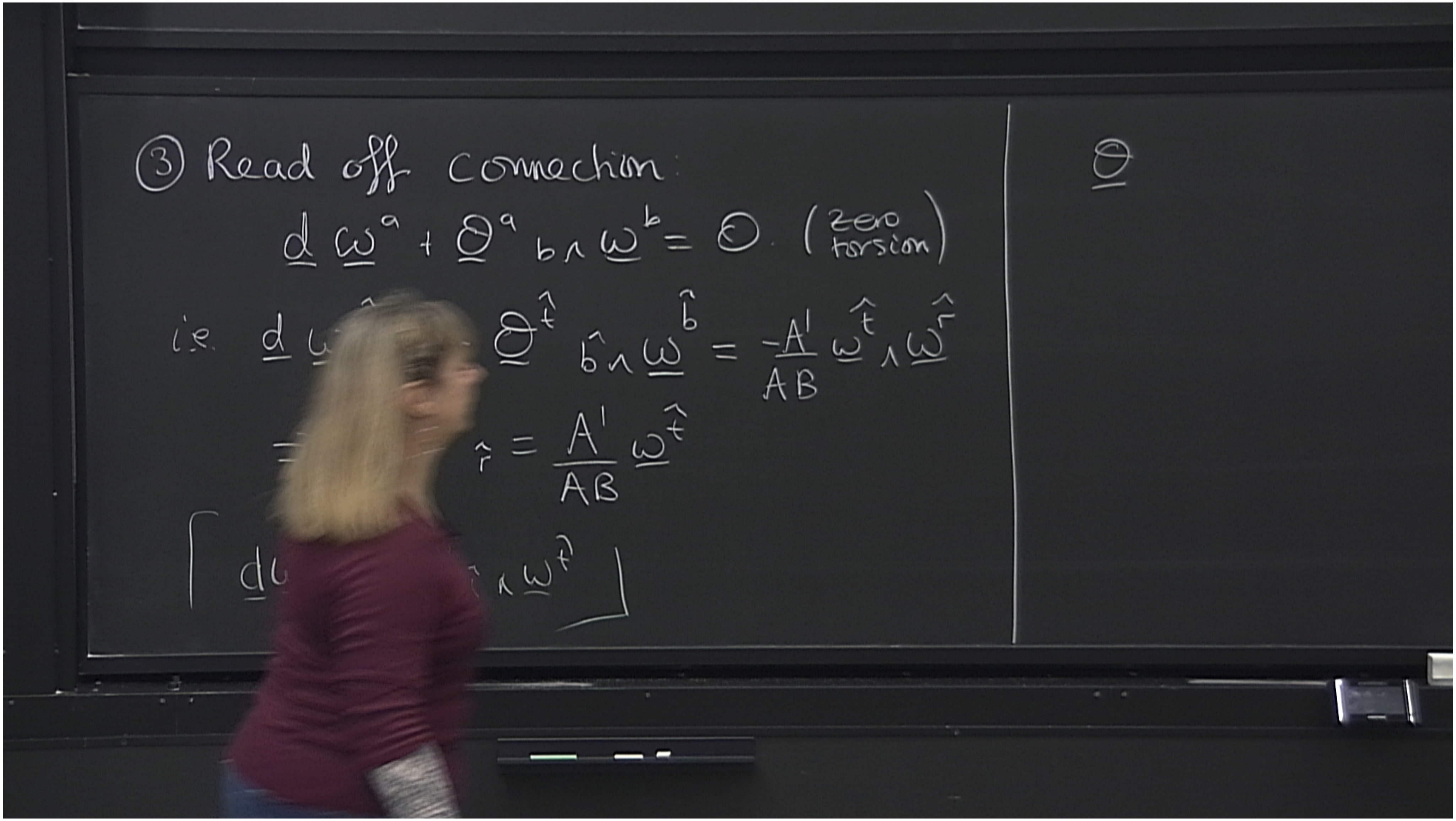
$A'\,\underline{d}r \wedge \underline{d}t = -\frac{A'}{AB}\,\underline{\omega}^{\hat t} \wedge \underline{\omega}^{\hat r}$

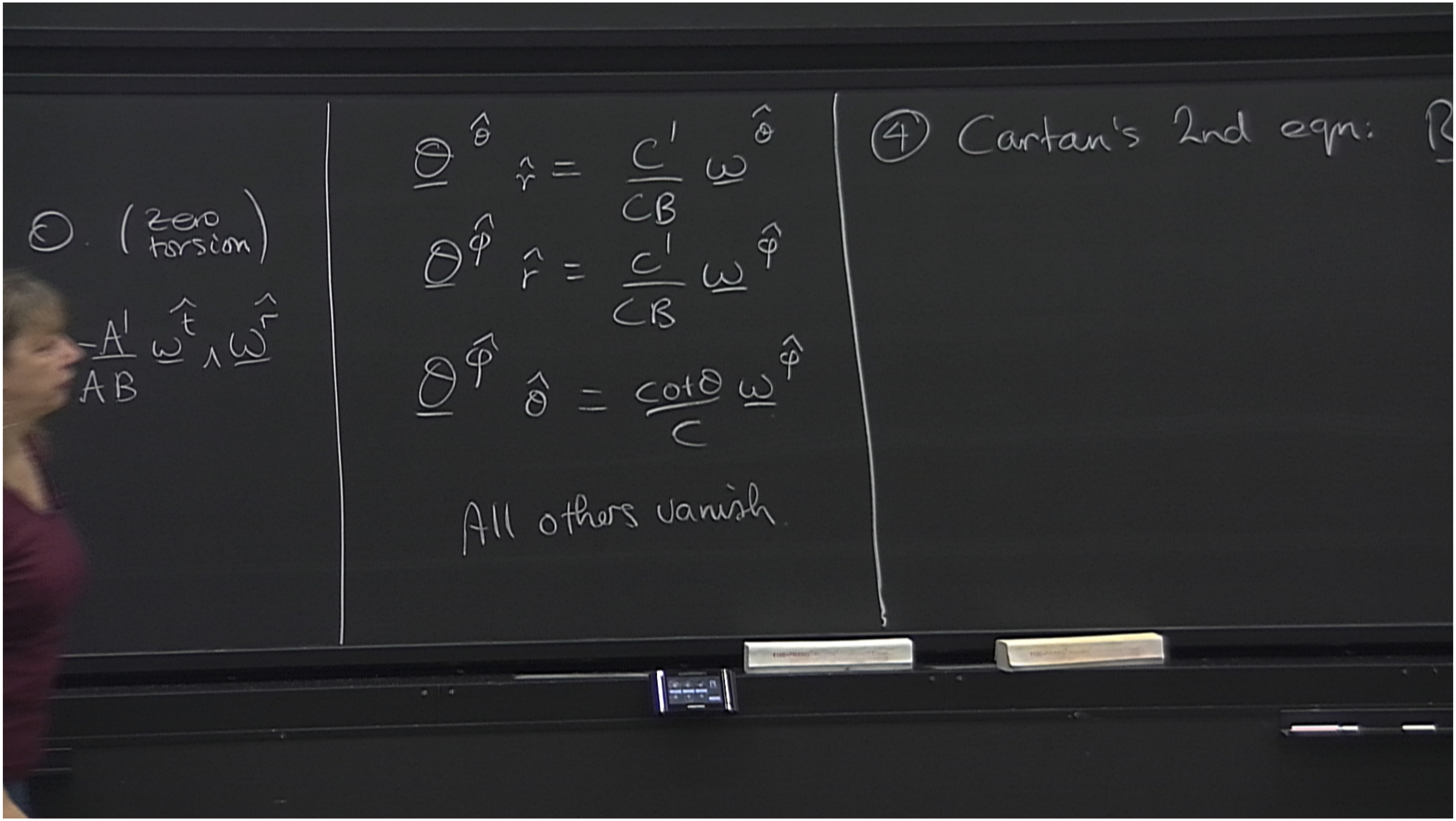
$= 0$

$\underline{d}\,\underline{\omega}^{\hat\theta} = -\frac{C'}{CB}\,\underline{\omega}^{\hat r} \wedge \underline{\omega}^{\hat\theta}$

$$\underline{d}\,\underline{\omega}^{\hat\varphi} = C'\sin\theta\,\underline{d}r \wedge \underline{d}\varphi + C\cos\theta\,\underline{d}\theta \wedge \underline{d}\varphi$$

$$= -\frac{C'}{CB}\,\underline{\omega}^{\hat\varphi} \wedge \underline{\omega}^{\hat r} - \frac{\cot\theta}{C}\,\underline{\omega}^{\hat\varphi} \wedge \underline{\omega}^{\hat\theta}$$


③ Read off connection:
d ω^a + θ^a b ∧ ω^b = 0 (zero torsion)
i.e.
= -A'/AB ω^t ∧ ω^r
= A'/AB ω^t
θ


(zero torsion)
$-\frac{A'}{AB}\,\omega^{\hat{t}} \wedge \omega^{\hat{r}}$
$\Theta^{\hat{\theta}}{}_{\hat{r}} = \frac{C'}{CB}\,\omega^{\hat{\theta}}$
$\Theta^{\hat{\varphi}}{}_{\hat{r}} = \frac{C'}{CB}\,\omega^{\hat{\varphi}}$
$\Theta^{\hat{\varphi}}{}_{\hat{\theta}} = \frac{\cot\theta}{C}\,\omega^{\hat{\varphi}}$
All others vanish.
(4) Cartan's 2nd eqn: R

$$\frac{C'}{CB}\,\underline{\omega}^{\hat\theta}$$

$$\frac{C'}{CB}\,\underline{\omega}^{\hat\varphi}$$

$$\frac{\cot\theta}{C}\,\underline{\omega}^{\hat\varphi}$$

rs vanish

(4) Cartan's 2nd eqn: $\underline{R}^{a}{}_{b} = \underline{d}\,\underline{\theta}^{a}{}_{b} + \underline{\theta}^{a}{}_{c} \wedge \underline{\theta}^{c}{}_{b}$.

$$\underline{R}^{\hat t}{}_{\hat r} = \underline{d}\,\underline{\theta}^{\hat t}{}_{\hat r} + \underbrace{\underline{\theta}^{\hat t}{}_{\hat a}}_{\hat a = \hat r \text{ for nonzero}} \wedge \underbrace{\underline{\theta}^{\hat a}{}_{\hat r}}_{\text{only nonzero for } \hat a = t,\theta,\varphi}$$

$$= \underline{d}\left(\frac{A'}{B}\,\underline{d}t\right)$$

$$= \left(\frac{A'}{B}\right)' \underline{d}r \wedge \underline{d}t = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right) \underline{\omega}^{\hat r} \wedge \underline{\omega}^{\hat t}$$

$$d\underset{\sim}{\omega}^{\hat{r}} = 0 \qquad \frac{\quad}{AB} \qquad -\frac{\cot\theta}{C}\,\underset{\sim}{\omega}^{\hat{\varphi}} \wedge \underset{\sim}{\omega}^{\hat{\theta}}$$

$$\frac{C'}{CB}\,\underset{\sim}{\omega}^{\hat{\theta}} \qquad \frac{C'}{CB}\,\underset{\sim}{\omega}^{\hat{\varphi}} \qquad \frac{\cot\theta}{C}\,\underset{\sim}{\omega}^{\hat{\varphi}}$$

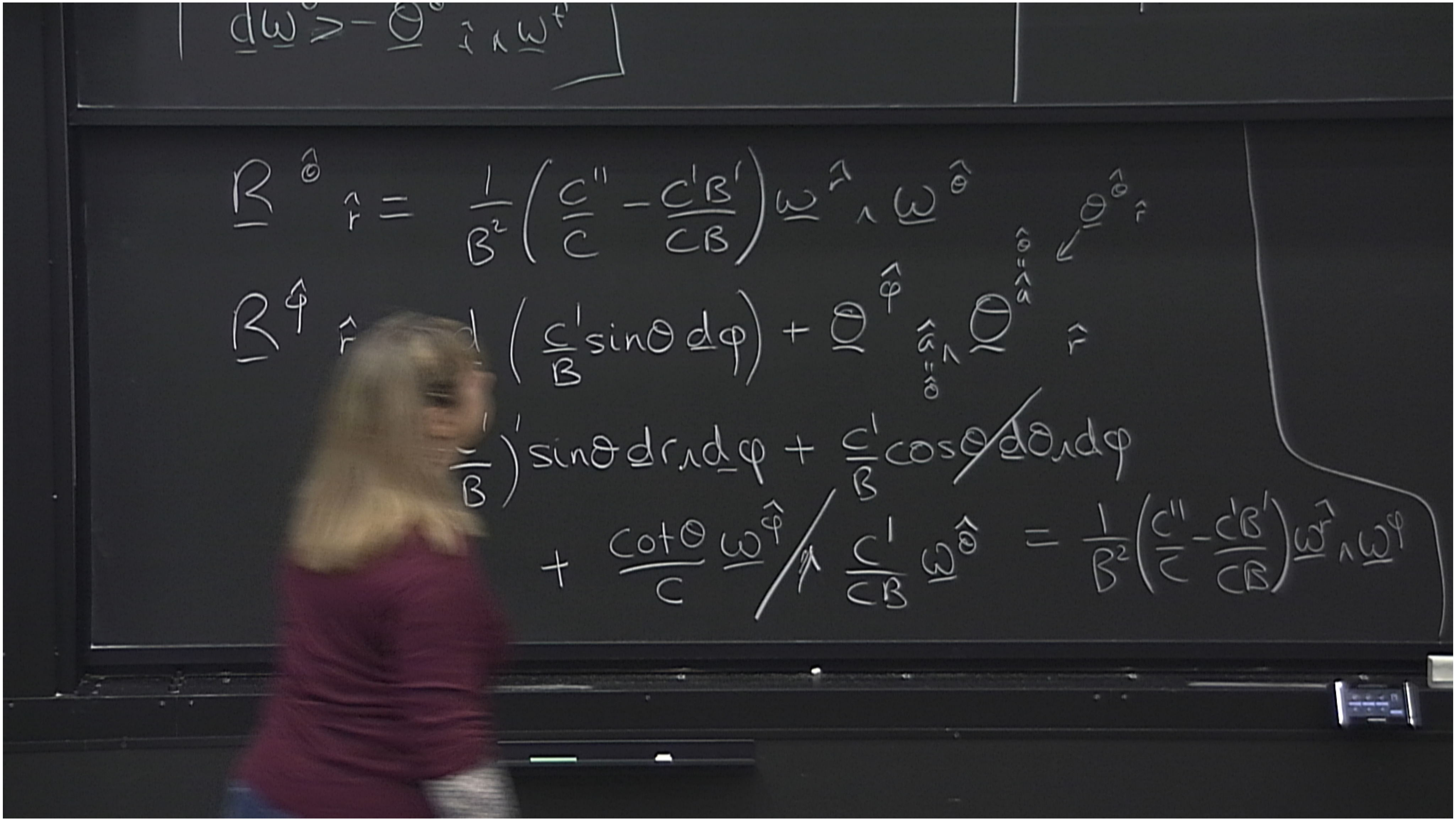
...s vanish

(4) Cartan's 2nd eqn: $\underset{\sim}{R}^{a}{}_{b} = \underset{\sim}{d}\,\underset{\sim}{\theta}^{a}{}_{b} + \underset{\sim}{\theta}^{a}{}_{c} \wedge \underset{\sim}{\theta}^{c}{}_{b}$.

$$\underset{\sim}{R}^{\hat{t}}{}_{\hat{r}} = \underset{\sim}{d}\,\underset{\sim}{\theta}^{\hat{t}}{}_{\hat{r}} + \underbrace{\underset{\sim}{\theta}^{\hat{t}}{}_{\hat{a}}}_{\hat{a}=\hat{r} \text{ for nonzero}} \wedge \underbrace{\underset{\sim}{\theta}^{\hat{a}}{}_{\hat{r}}}_{\text{only nonzero for } \hat{a}=t,\theta,\varphi}$$

$$= \underset{\sim}{d}\left(\frac{A'}{B}\,\underset{\sim}{d}t\right)$$

$$= \left(\frac{A'}{B}\right)' \underset{\sim}{d}r \wedge \underset{\sim}{d}t = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right) \underset{\sim}{\omega}^{\hat{r}} \wedge \underset{\sim}{\omega}^{\hat{t}}$$



$d\omega^{\hat\theta} = -\Theta^{\hat\theta}{}_{\hat i}\wedge\omega^{\hat i}$
$R^{\hat\theta}{}_{\hat r} = \frac{1}{B^2}\left(\frac{C''}{C}-\frac{C'B'}{CB}\right)\omega^{\hat r}\wedge\omega^{\hat\theta}$
$\Theta^{\hat\theta}{}_{\hat r}$
$R^{\hat\varphi}{}_{\hat r} = d\left(\frac{C'}{B}\sin\theta\, d\varphi\right) + \Theta^{\hat\varphi}{}_{\hat a}\wedge\Theta^{\hat a}{}_{\hat r}$, $\hat a = \hat\theta$
$\left(\frac{C'}{B}\right)'\sin\theta\, dr\wedge d\varphi + \cancel{\frac{C'}{B}\cos\theta\, d\theta\wedge d\varphi}$
$+ \cancel{\frac{\cot\theta}{C}\omega^{\hat\varphi}\wedge\frac{C'}{CB}\omega^{\hat\theta}} = \frac{1}{B^2}\left(\frac{C''}{C}-\frac{C'B'}{CB}\right)\omega^{\hat r}\wedge\omega^{\hat\varphi}$

$$\underline{\omega}^{\hat r} \wedge \underline{\omega}^{\hat\theta} \qquad \underline{\Theta}^{\hat\theta}{}_{\hat r}$$

$$+\ \underline{\Theta}^{\hat\varphi}{}_{\hat a} \wedge \underline{\Theta}^{\hat a}{}_{\hat r} \qquad \hat a = \hat\theta$$

$$\varphi + \frac{C'}{B}\cancel{\cos\theta\, \underline{d\theta}} \wedge d\varphi$$

$$\underline{\Theta}^{\hat\varphi} \quad \frac{C'}{CB}\,\underline{\omega}^{\hat\theta} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right)\underline{\omega}^{\hat r} \wedge \underline{\omega}^{\varphi}$$

$$\underline{R}^{\hat t}{}_{\hat\theta} = \underline{\Theta}^{\hat t}{}_{\hat a} \wedge \underline{\Theta}^{\hat a}{}_{\hat\theta}$$

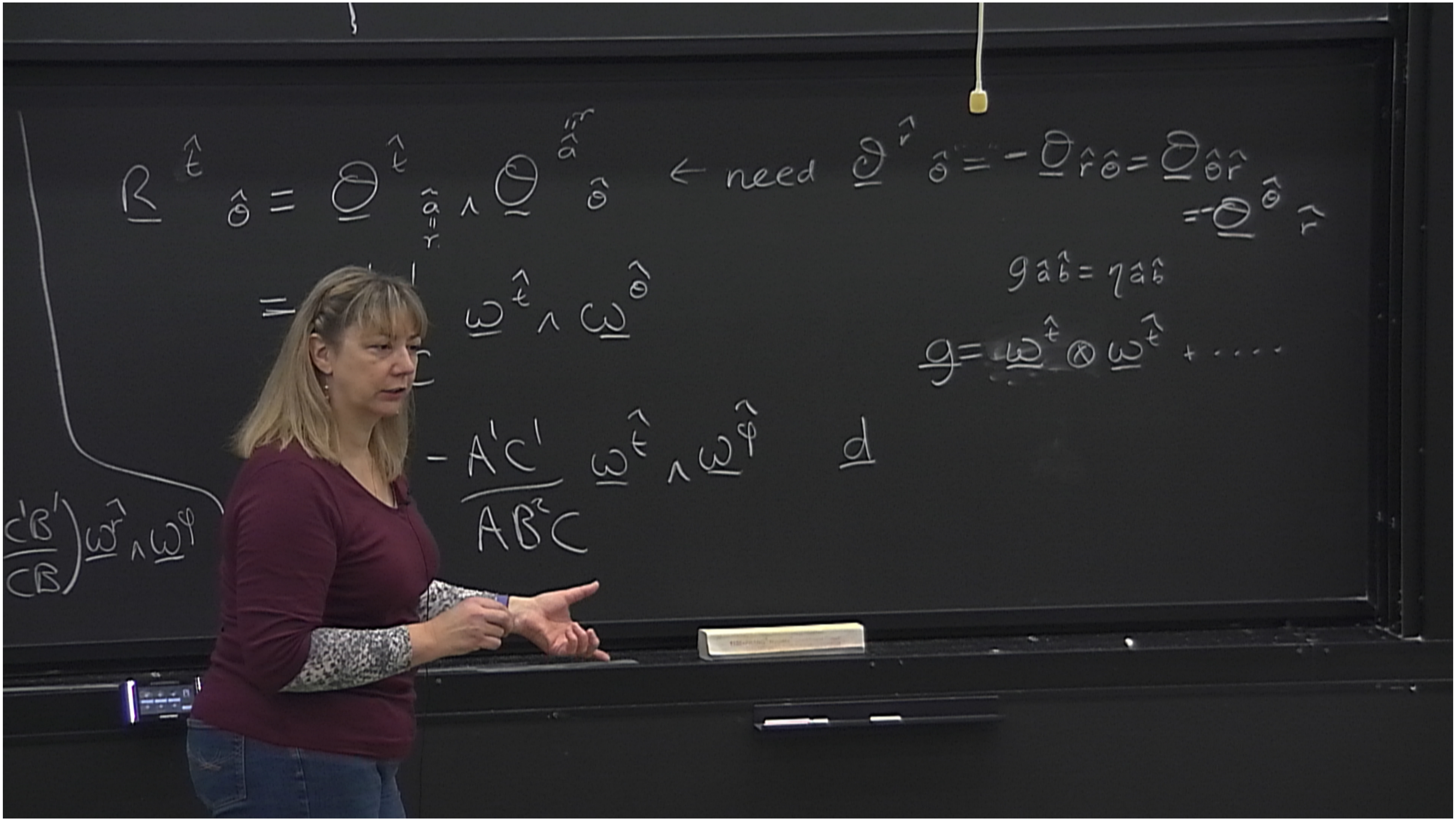
$$\underline{R}^{\hat{t}}{}_{\hat{\theta}} = \underline{\Theta}^{\hat{t}}{}_{\hat{a}} \wedge \underline{\Theta}^{\hat{a}}{}_{\hat{\theta}} \quad \leftarrow \text{need } \underline{\Theta}^{\hat{r}}{}_{\hat{\theta}} = -\underline{\Theta}_{\hat{r}\hat{\theta}} = \underline{\Theta}_{\hat{\theta}\hat{r}} = -\underline{\Theta}^{\hat{\theta}}{}_{\hat{r}}$$

$$\hat{a} = \hat{r}$$

$$g_{\hat{a}\hat{b}} = \eta_{\hat{a}\hat{b}}$$

$$= -\frac{A'C'}{AB^2C}\, \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}}$$

$$\frac{C'B'}{CB}\Big)\, \underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\varphi}$$


← need

$$\underline{R}^{\hat{t}}{}_{\hat{\theta}} = \underline{\Theta}^{\hat{t}}{}_{\hat{a}} \wedge \underline{\Theta}^{\hat{a}}{}_{\hat{\theta}} \quad \leftarrow \text{need } \underline{\Theta}^{\hat{r}}{}_{\hat{\theta}} = -\underline{\Theta}_{\hat{r}\hat{\theta}} = \underline{\Theta}_{\hat{\theta}\hat{r}} = \underline{\Theta}^{\hat{\theta}}{}_{\hat{r}}$$

$$\hat{a} = \hat{r}$$

$$= -\frac{A'C'}{AB^2C}\,\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}}$$

$$g_{\hat{a}\hat{b}} = \eta_{\hat{a}\hat{b}}$$

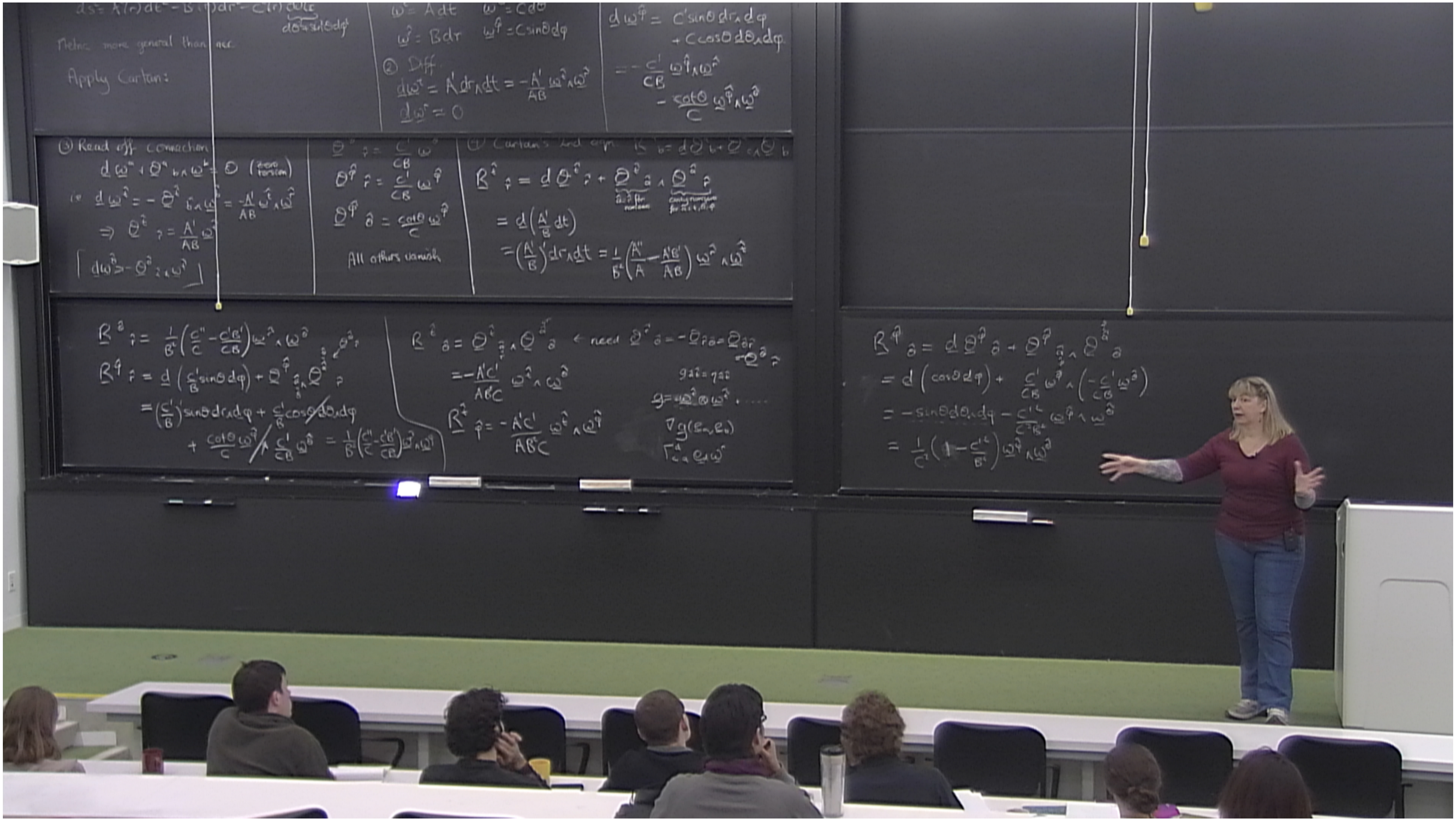
$$\underline{g} = \underline{\omega}^{\hat{t}} \otimes \underline{\omega}^{\hat{t}} + \cdots$$

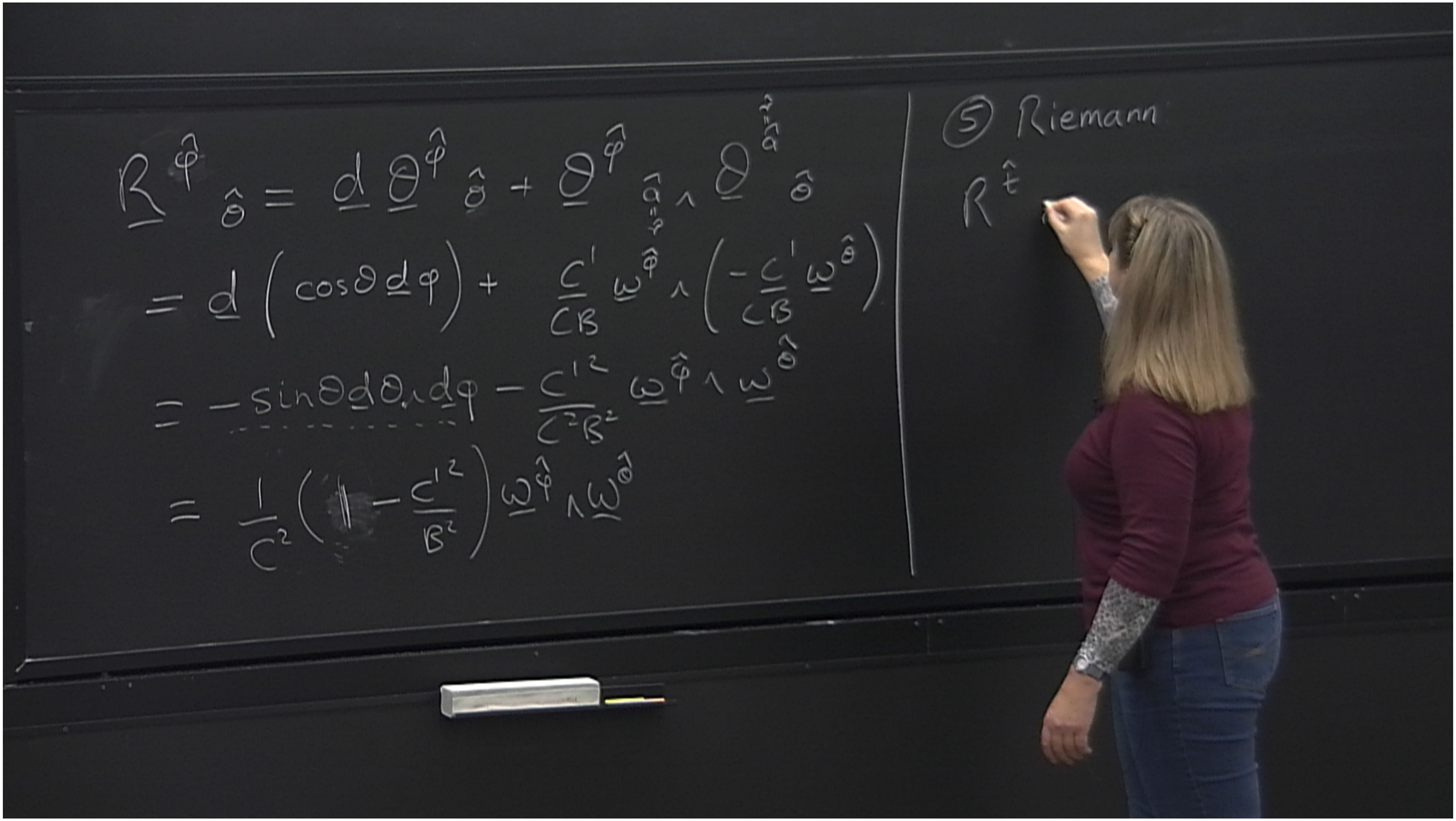
$$\underline{R}^{\hat{t}}{}_{\hat{\varphi}} = -\frac{A'C'}{AB^2C}\,\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\varphi}}$$

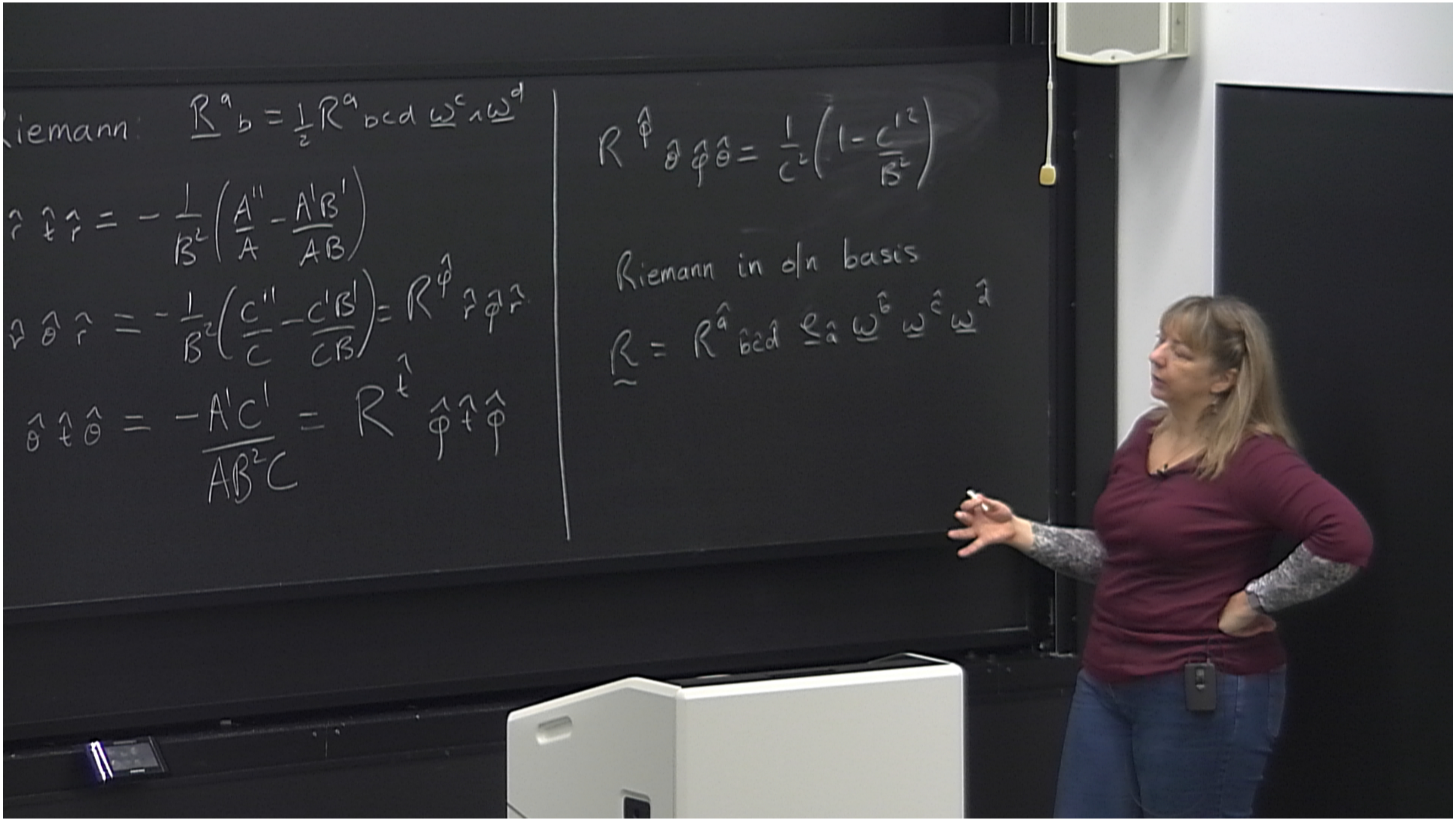
$$\nabla \underline{g}(\underline{e}_a, \underline{e}_b)$$

$$\Gamma^d{}_{ca}\,\underline{e}_d\,\underline{\omega}^c$$

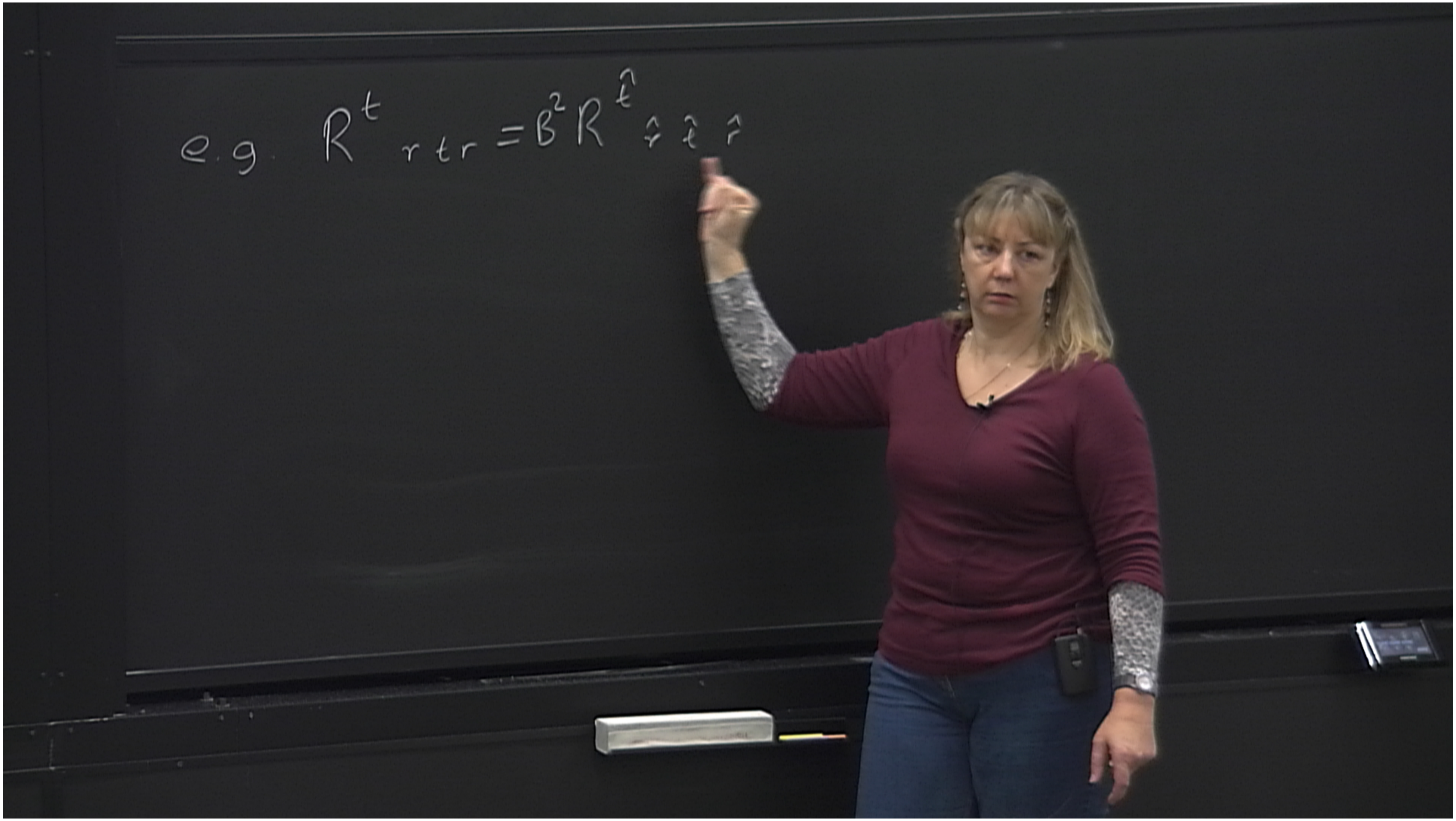
$$\left(\frac{C'B'}{CB}\right)\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\varphi}$$

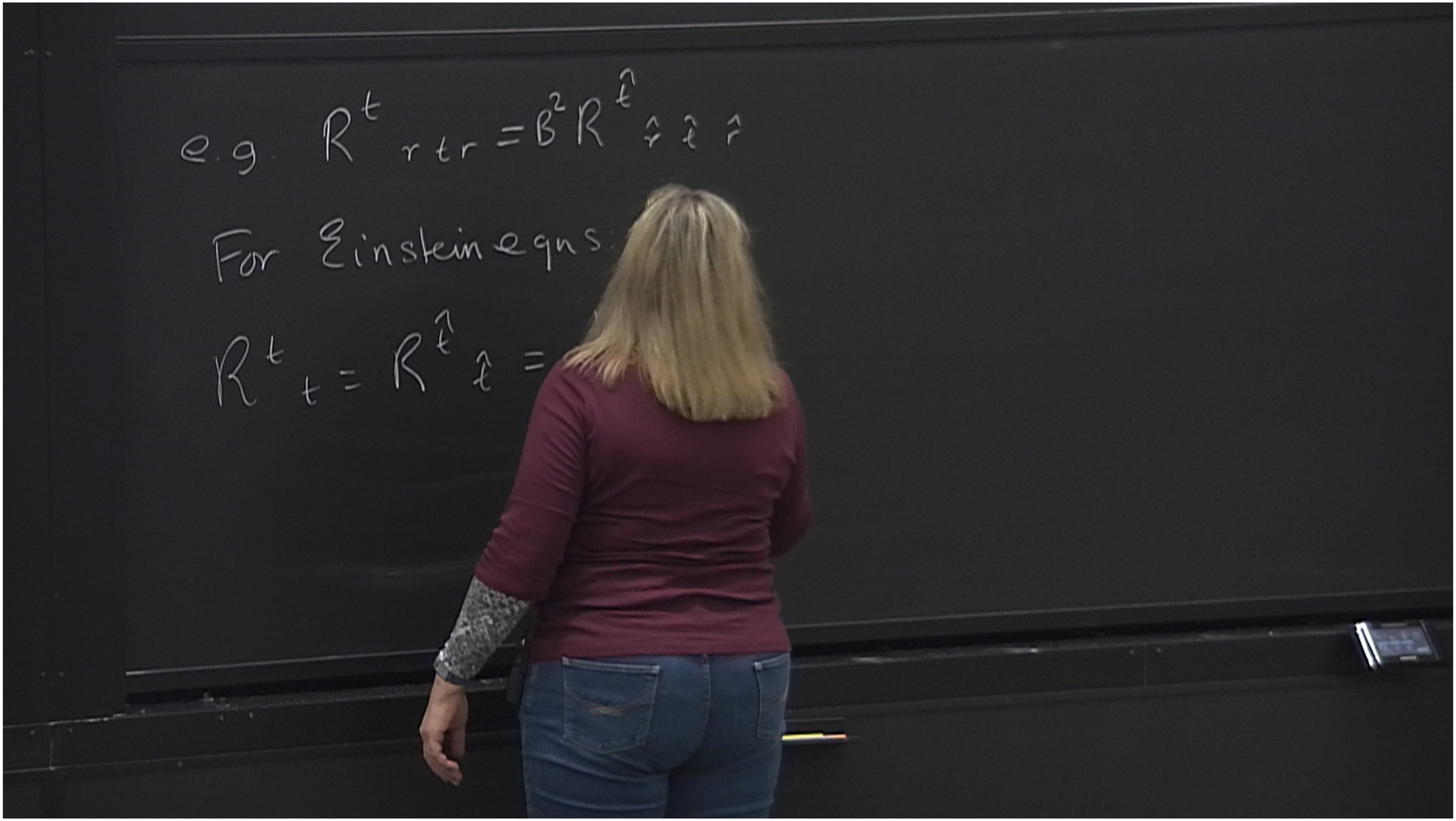



$R^{\hat\varphi}{}_{\hat\theta} = d\theta^{\hat\varphi}{}_{\hat\theta} + \theta^{\hat\varphi}{}_{\hat\alpha} \wedge \theta^{\hat\alpha}{}_{\hat\theta}$
$\hat\alpha = \hat\varphi$
$= d(\cos\theta\, d\varphi) + \frac{C'}{CB}\omega^{\hat\varphi} \wedge \left(-\frac{C'}{CB}\omega^{\hat\theta}\right)$
$= -\sin\theta\, d\theta \wedge d\varphi - \frac{C'^2}{C^2B^2}\omega^{\hat\varphi} \wedge \omega^{\hat\theta}$
$= \frac{1}{C^2}\left(1 - \frac{C'^2}{B^2}\right)\omega^{\hat\varphi} \wedge \omega^{\hat\theta}$
⑤ Riemann:
$R^{\hat t}$



iemann: $\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$
$\hat{r}\hat{t}\hat{r} = -\frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)$
$\hat{\theta}\hat{\theta}\hat{r} = -\frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right) = R^{\hat{\varphi}}{}_{\hat{r}\hat{\varphi}\hat{r}}$
$\hat{\theta}\hat{t}\hat{\theta} = -\frac{A'C'}{AB^2C} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}}$
$R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = \frac{1}{C^2}\left(1 - \frac{C'^2}{B^2}\right)$
Riemann in o/n basis
$\underset{\sim}{R} = R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\, \underline{e}_{\hat{a}}\, \underline{\omega}^{\hat{b}}\, \underline{\omega}^{\hat{c}}\, \underline{\omega}^{\hat{d}}$

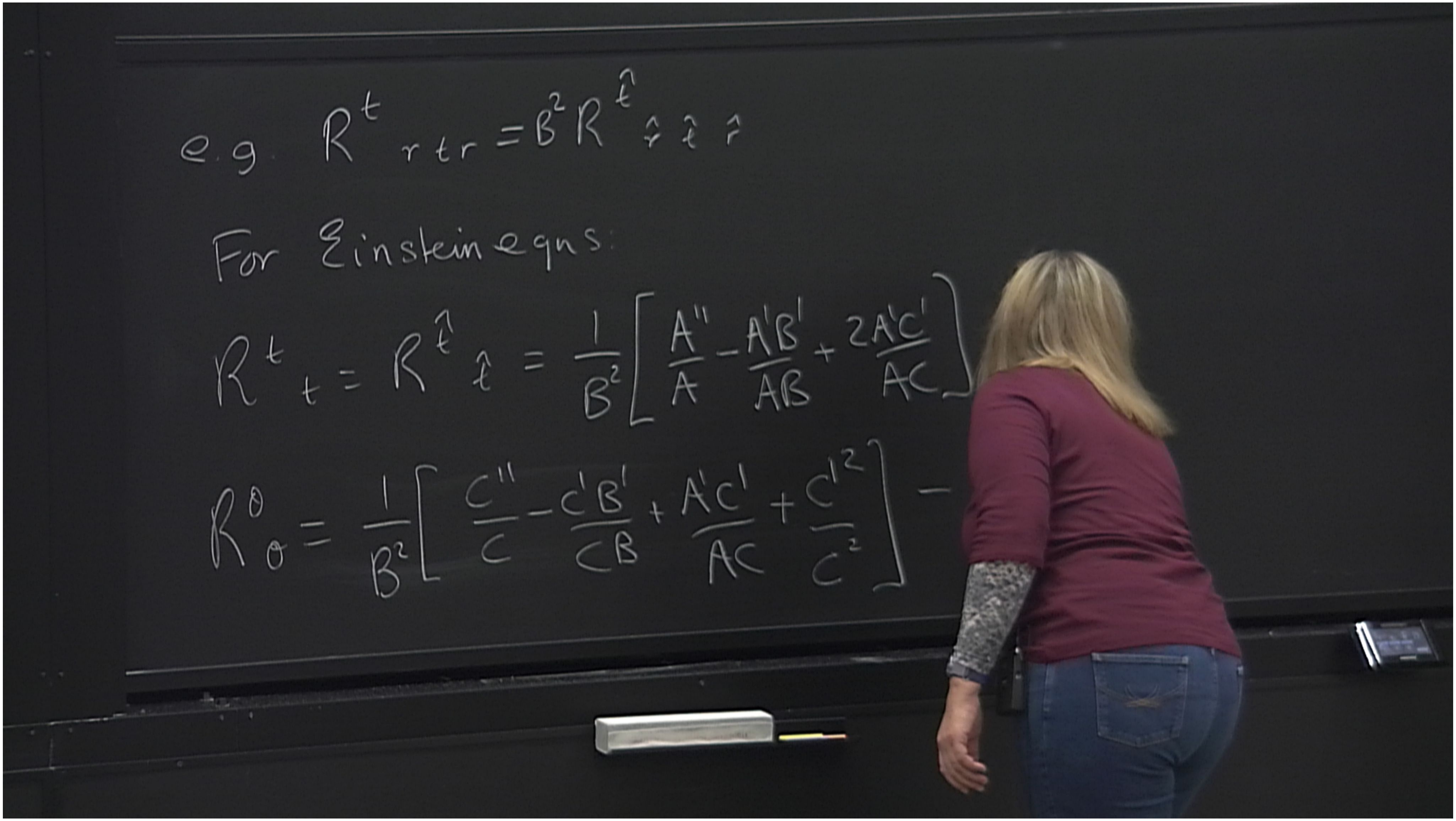

e.g. $R^{t}{}_{rtr} = B^2 R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}}$

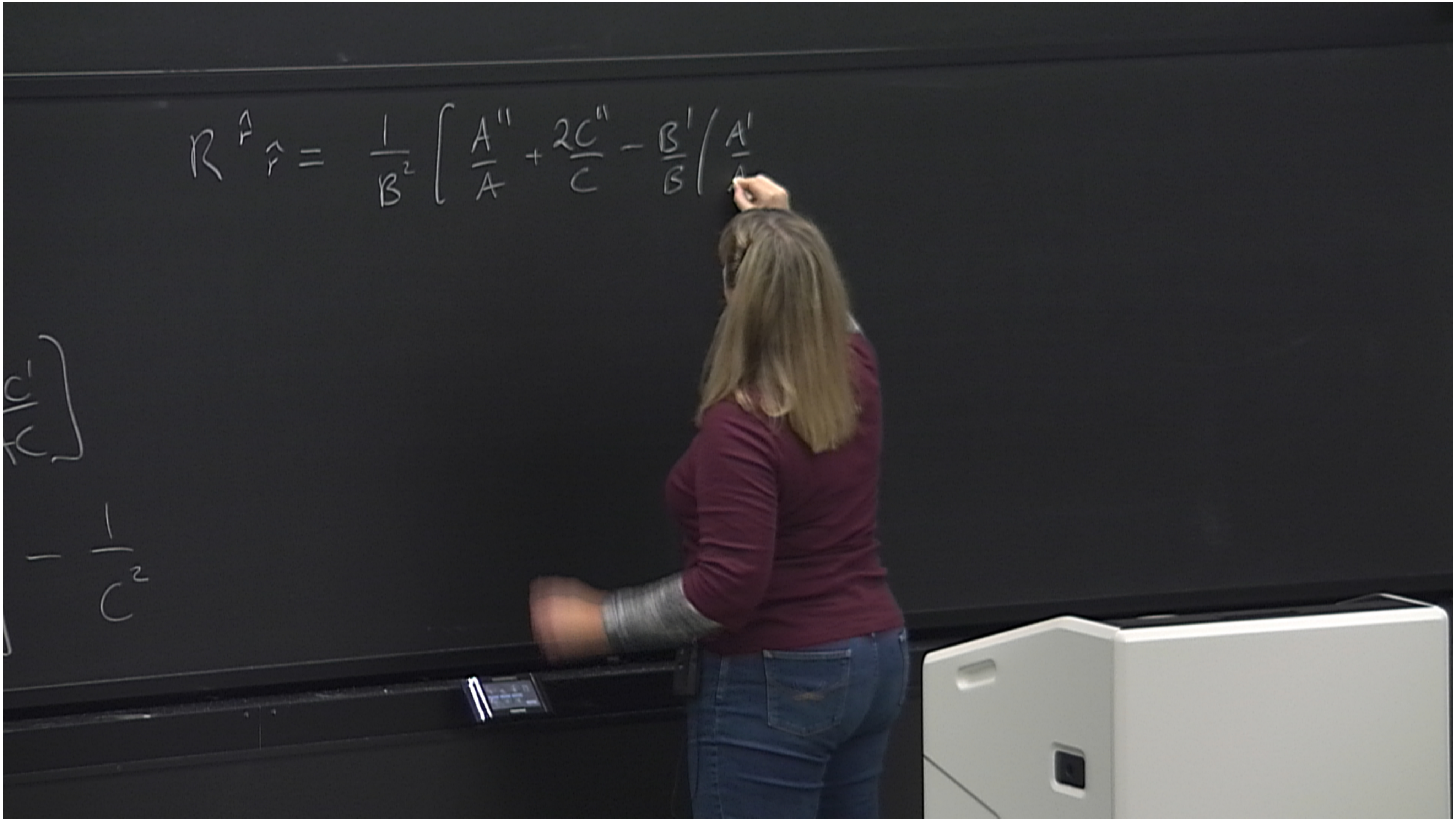

e.g. $R^{t}{}_{rtr} = B^{2} R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}}$
For Einstein eqns:
$R^{t}{}_{t} = R^{\hat{t}}{}_{\hat{t}} =$

e.g. $R^{t}{}_{rtr} = B^2 R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}}$

For Einstein eqns:

$$R^{t}{}_{t} = R^{\hat{t}}{}_{\hat{t}} = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + \frac{2A'C'}{AC}\right]$$


e.g. $R^t{}_{rtr} = B^2 R^{\hat t}{}_{\hat r \hat t \hat r}$
For Einstein eqns:
$R^t{}_t = R^{\hat t}{}_{\hat t} = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + 2\frac{A'C'}{AC}\right]$
$R^\theta{}_\theta = \frac{1}{B^2}\left[\frac{C''}{C} - \frac{C'B'}{CB} + \frac{A'C'}{AC} + \frac{C'^2}{C^2}\right] -$


$R^{\hat{r}}{}_{\hat{r}} = \frac{1}{B^2}\Big[\frac{A''}{A} + \frac{2C''}{C} - \frac{B'}{B}\Big(\frac{A'}{A}$
$-\frac{1}{C^2}$

$$\frac{2A'C'}{AC}\Big]$$

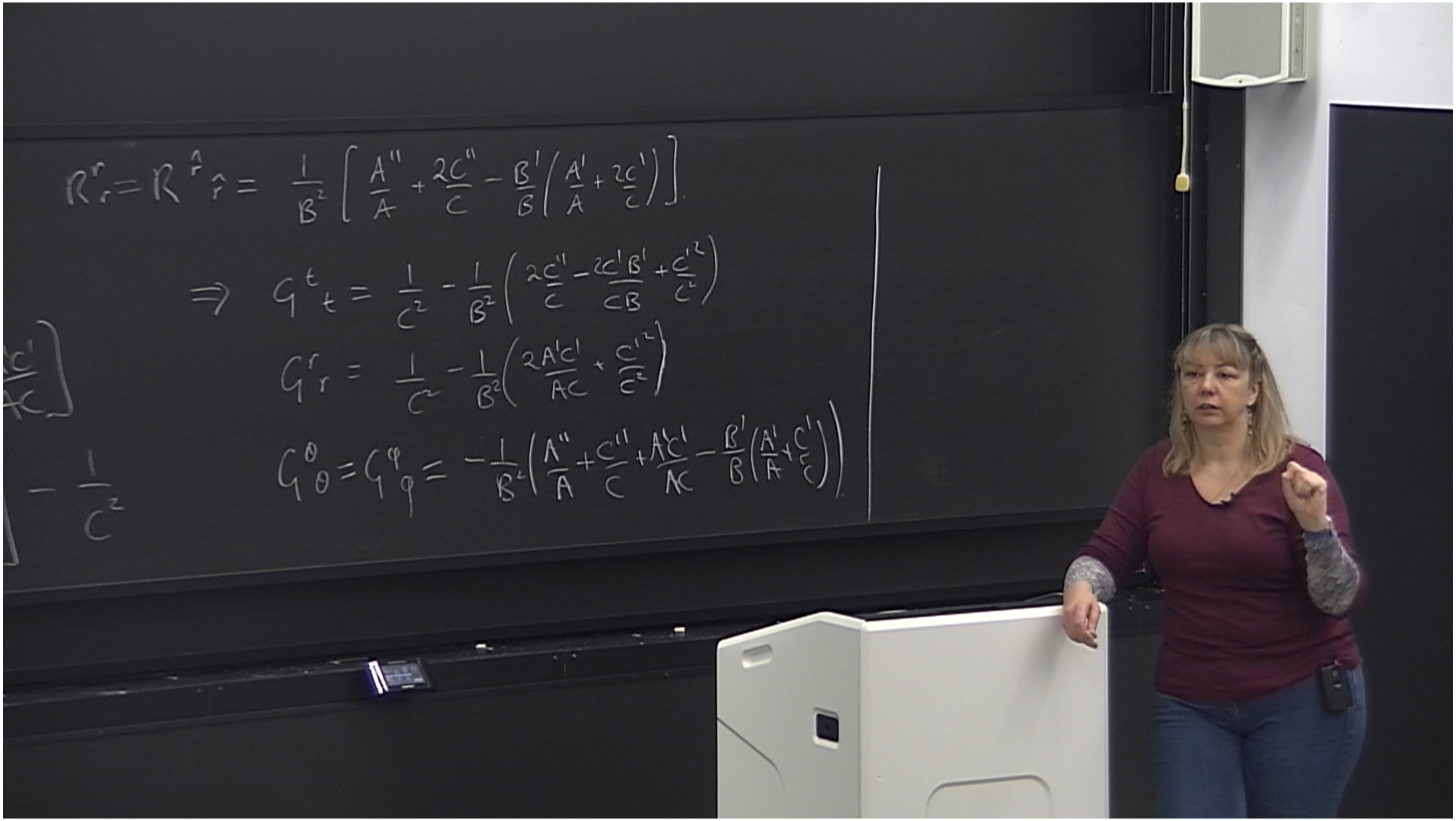
$$\Big] - \frac{1}{C^2}$$

$$R^r{}_r = R^{\hat r}{}_{\hat r} = \frac{1}{B^2}\left[\frac{A''}{A} + \frac{2C''}{C} - \frac{B'}{B}\left(\frac{A'}{A} + \frac{2C'}{C}\right)\right].$$

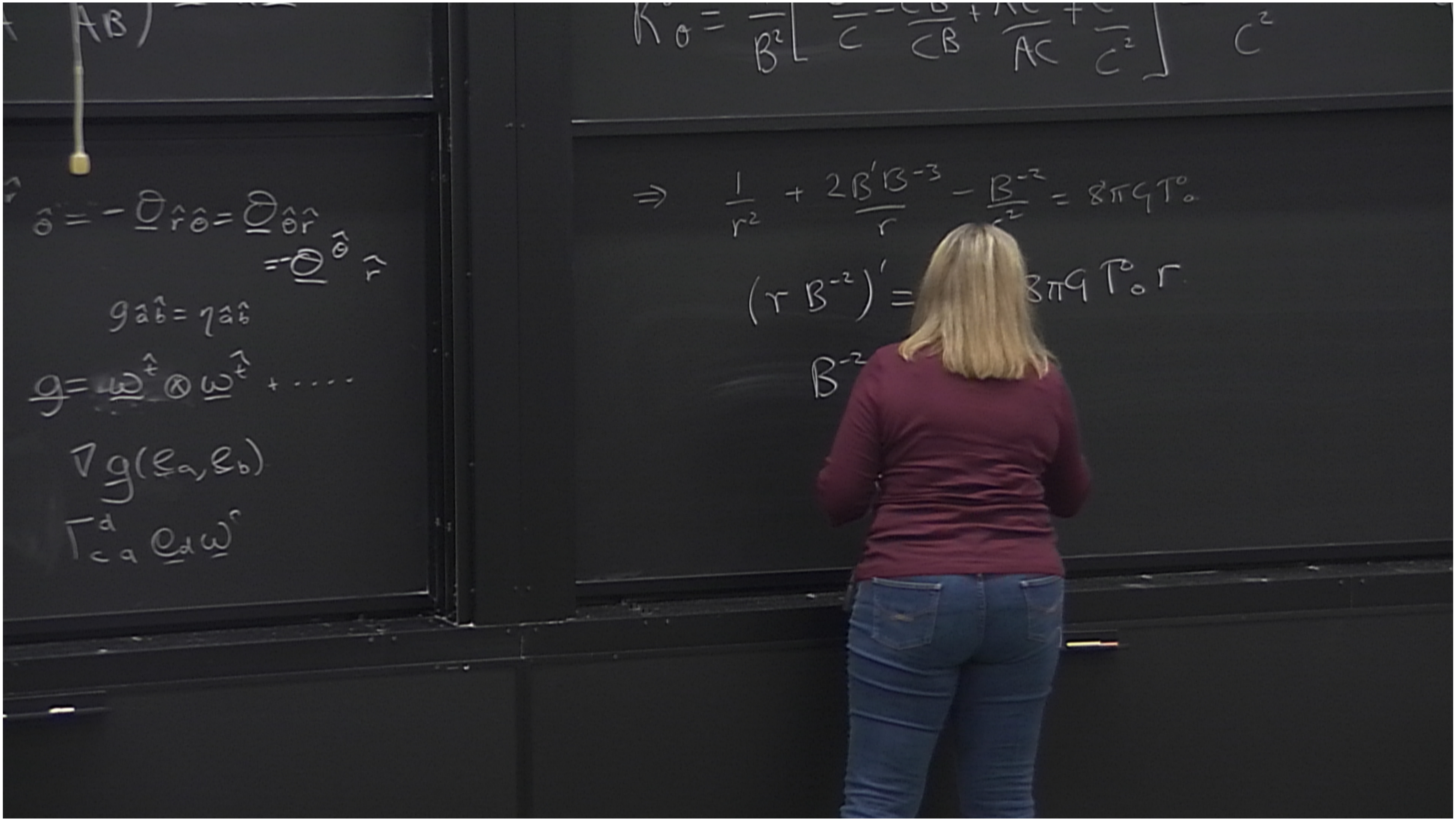
$$\Rightarrow \quad G^t{}_t = \frac{1}{C^2} - \frac{1}{B^2}\left(\frac{2C''}{C} - \frac{2C'B'}{CB} + \frac{C'^2}{C^2}\right)$$

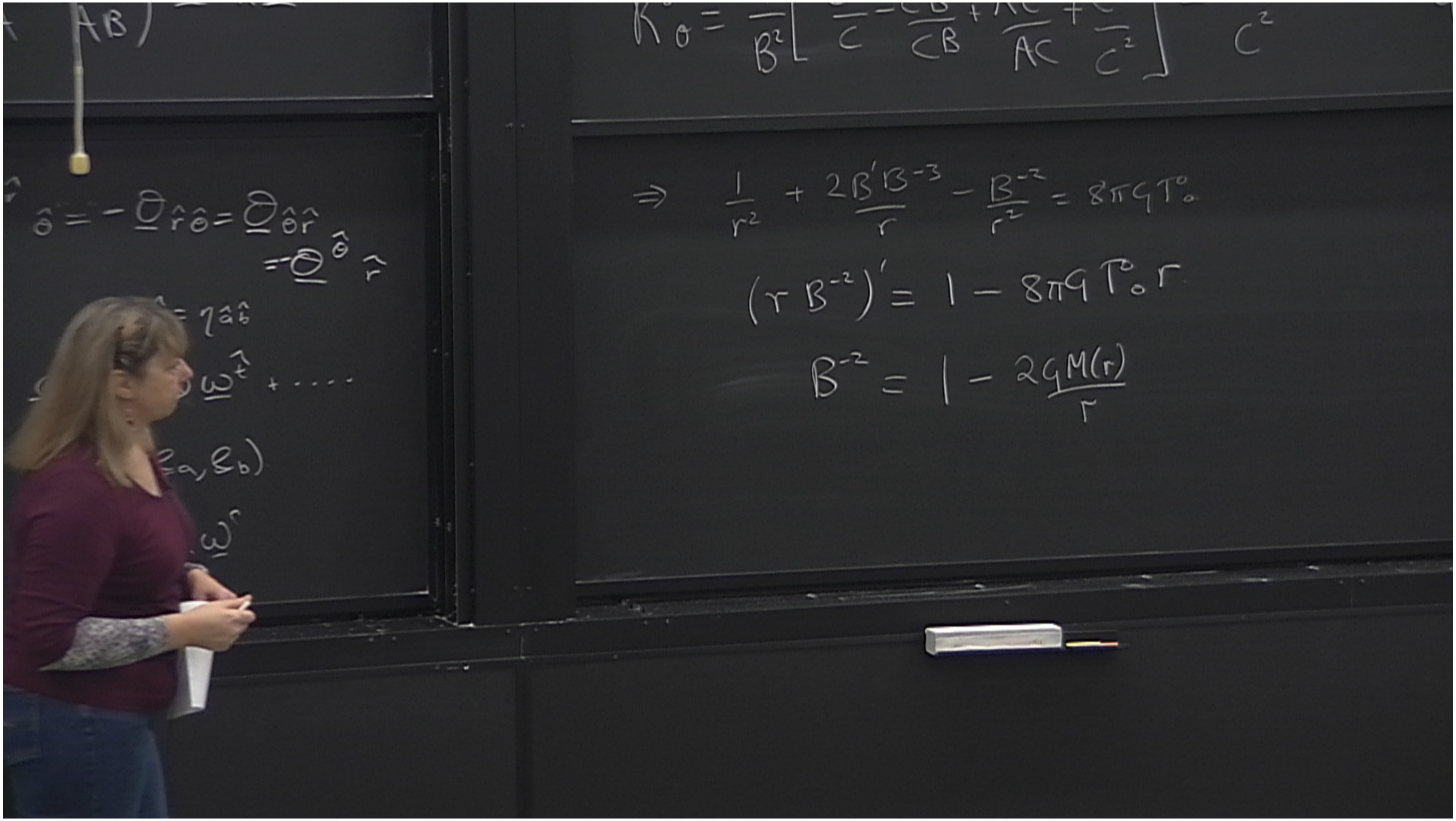
$$G^r{}_r = \frac{1}{C^2} - \frac{1}{B^2}\left(\frac{2A'C'}{AC} + \frac{C'^2}{C^2}\right)$$

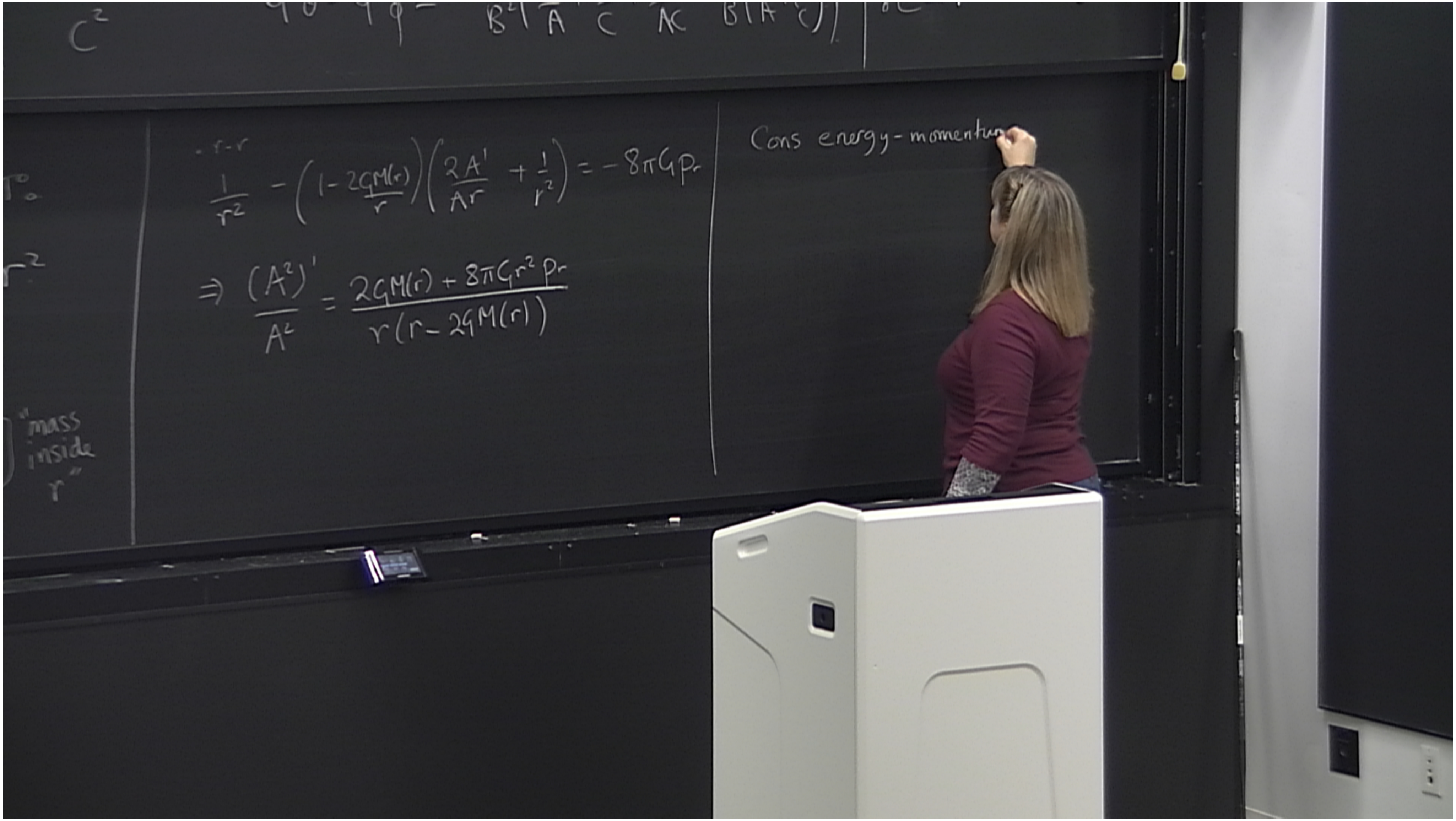
$$G^\theta{}_\theta = G^\varphi{}_\varphi = -\frac{1}{B^2}\left(\frac{A''}{A} + \frac{C''}{C} + \frac{A'C'}{AC} - \frac{B'}{B}\left(\frac{A'}{A} + \frac{C'}{C}\right)\right).$$



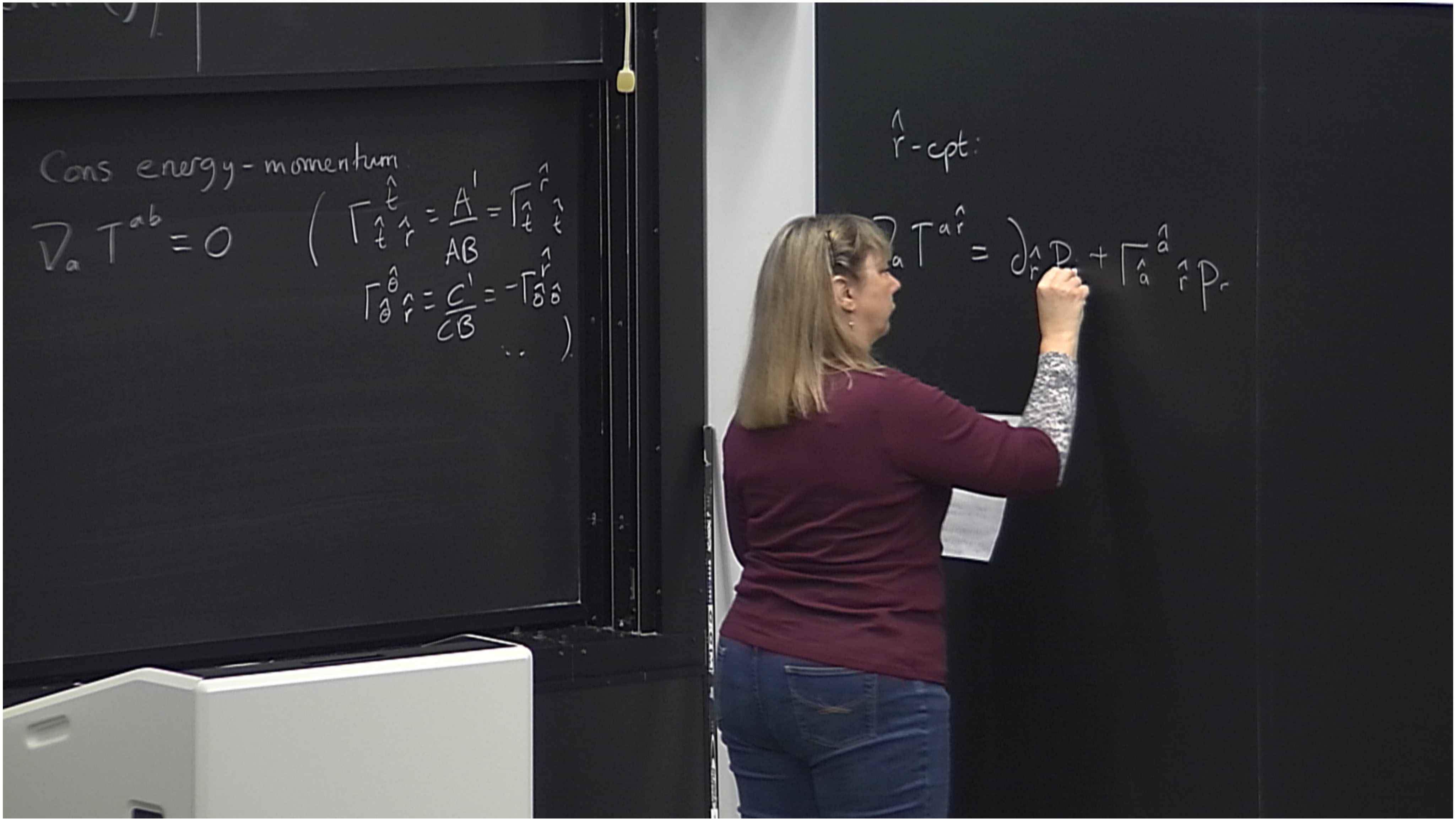
$R^r{}_r = R^{\hat r}{}_{\hat r} = \frac{1}{B^2}\left[\frac{A''}{A} + \frac{2C''}{C} - \frac{B'}{B}\left(\frac{A'}{A} + \frac{2C'}{C}\right)\right]$
$\Rightarrow G^t{}_t = \frac{1}{C^2} - \frac{1}{B^2}\left(\frac{2C''}{C} - \frac{2C'B'}{CB} + \frac{C'^2}{C^2}\right)$
$G^r{}_r = \frac{1}{C^2} - \frac{1}{B^2}\left(\frac{2A'C'}{AC} + \frac{C'^2}{C^2}\right)$
$G^\theta{}_\theta = G^\varphi{}_\varphi = -\frac{1}{B^2}\left(\frac{A''}{A} + \frac{C''}{C} + \frac{A'C'}{AC} - \frac{B'}{B}\left(\frac{A'}{A} + \frac{C'}{C}\right)\right)$

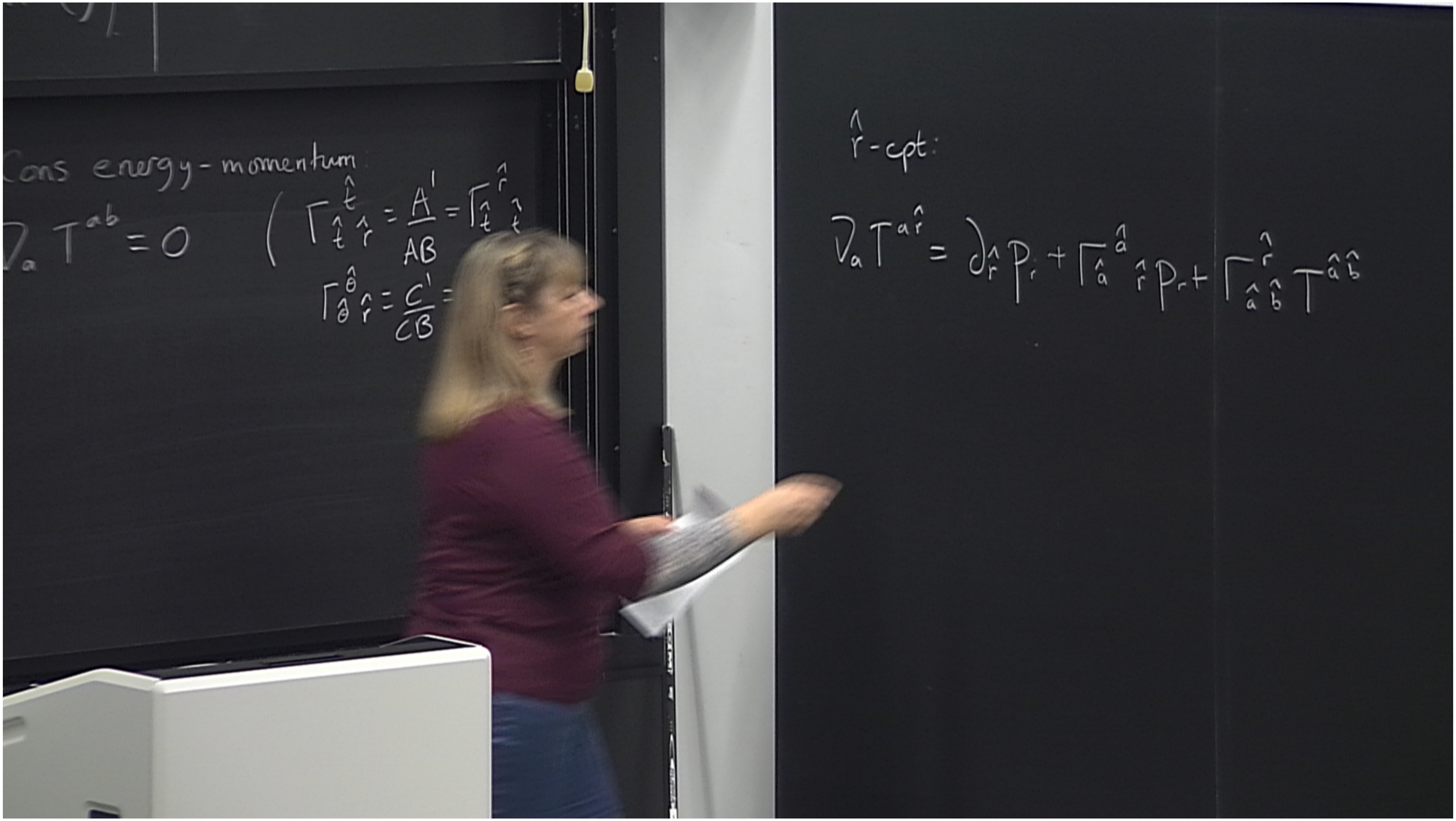

(r B^-2)' = 8πG T°o r

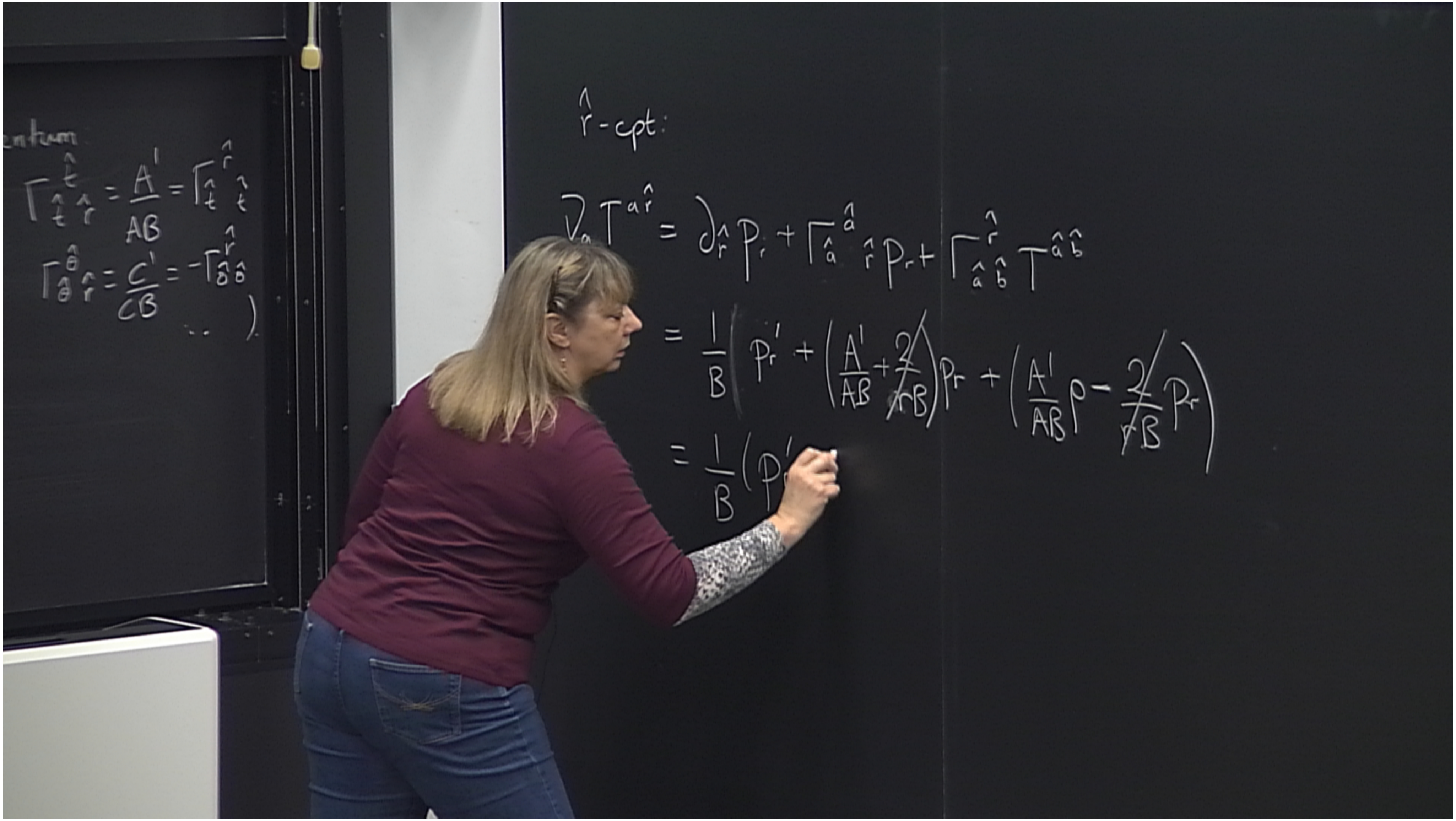

$\Rightarrow \frac{1}{r^2} + \frac{2B'B^{-3}}{r} - \frac{B^{-2}}{r^2} = 8\pi G T^0{}_0$
$(r B^{-2})' = 1 - 8\pi G T^0{}_0 r$
$B^{-2} = 1 - \frac{2GM(r)}{r}$



1/r² − (1 − 2GM(r)/r)(2A'/Ar + 1/r²) = −8πGpᵣ
⇒ (A²)'/A² = (2GM(r) + 8πGr² pᵣ) / (r(r − 2GM(r)))
"mass inside r"
Cons energy-momentum


Cons energy-momentum
$\nabla_a T^{ab} = 0$
$\left(\Gamma^{\hat{t}}_{\hat{t}\hat{r}} = \frac{A'}{AB} = \Gamma^{\hat{r}}_{\hat{t}\hat{t}} \right.$
$\Gamma^{\hat{\theta}}_{\hat{\theta}\hat{r}} = \frac{C'}{CB} = -\Gamma^{\hat{r}}_{\hat{\theta}\hat{\theta}}$
$\left. \dots \right)$
$\hat{r}$-cpt:
$\nabla_a T^{a\hat{r}} = \partial_{\hat{r}} P + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} P_r$


Cons energy-momentum
$\nabla_a T^{ab} = 0$
$\Gamma^{\hat{t}}{}_{\hat{t}\hat{r}} = \frac{A'}{AB} = \Gamma^{\hat{r}}{}_{\hat{t}\hat{t}}$
$\Gamma^{\hat{\theta}}{}_{\hat{\theta}\hat{r}} = \frac{C'}{CB}$
$\hat{r}$-cpt:
$\nabla_a T^{a\hat{r}} = \partial_{\hat{r}} p_r + \Gamma^{\hat{a}}{}_{\hat{a}\hat{r}} p_r + \Gamma^{\hat{r}}{}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$


$\hat{r}$ - cpt:
$\nabla_a T^{a\hat{r}} = \partial_{\hat{r}} p_r + \Gamma_{\hat{a}}{}^{\hat{a}}{}_{\hat{r}} p_r + \Gamma_{\hat{a}}{}^{\hat{r}}{}_{\hat{b}} T^{\hat{a}\hat{b}}$
$= \frac{1}{B}\left(p_r' + \left(\frac{A'}{AB} + \cancel{\frac{2}{rB}}\right) p_r + \left(\frac{A'}{AB}\rho - \cancel{\frac{2}{rB}} p_r\right)\right)$
$= \frac{1}{B}\left(p_r' \right.$
$\Gamma_{\hat{t}}{}^{\hat{t}}{}_{\hat{r}} = \frac{A'}{AB} = \Gamma_{\hat{t}}{}^{\hat{r}}{}_{\hat{t}}$
$\Gamma_{\hat{\theta}}{}^{\hat{\theta}}{}_{\hat{r}} = \frac{C'}{CB} = -\Gamma_{\hat{\theta}}{}^{\hat{r}}{}_{\hat{\theta}}$