Title: 13/14 PSI - Gravitational Physics Review - Lecture 3

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URL: http://pirsa.org/14010029

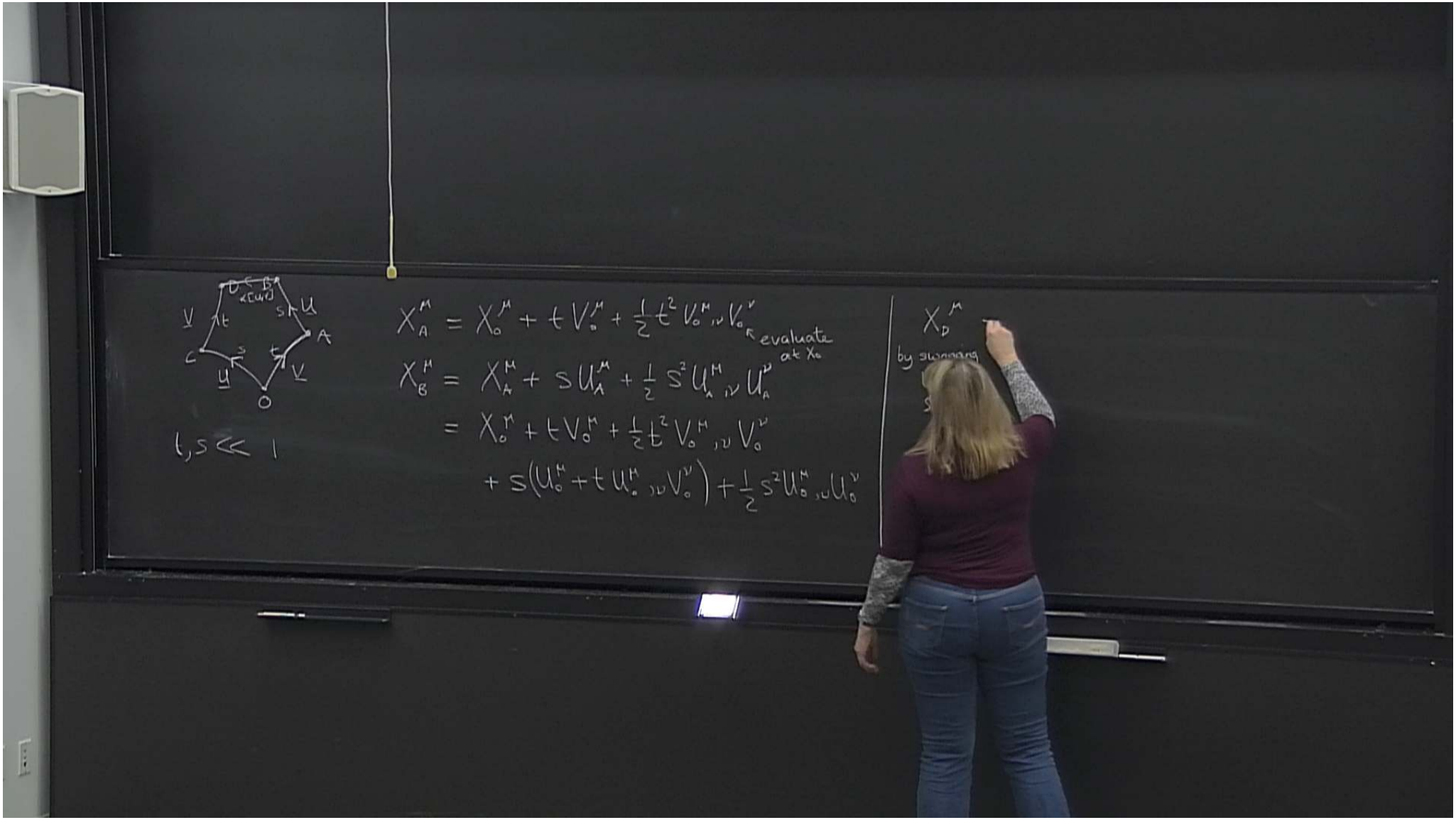
Abstract:

Lecture 3 : Connections & Curvature

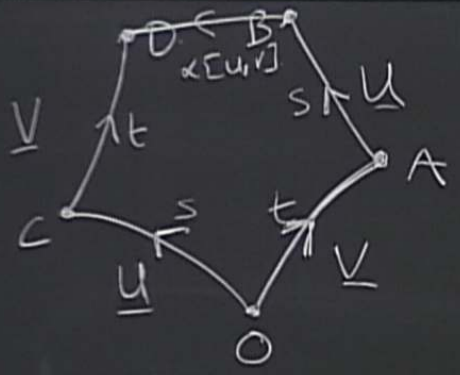
Recap $(\mathcal{L}_V U)^\mu = U^\mu{}_{,\nu} V^\nu - V^\mu{}_{,\nu} U^\nu$

$\mathcal{L}_V U = [V, U]$

$(VU - UV) f$



$X_A^\mu = X_0^\mu + t V_0^\mu + \frac{1}{2} t^2 V_0^\mu{}_{,\nu} V_0^\nu$
evaluate at X_0
$X_B^\mu = X_A^\mu + s U_A^\mu + \frac{1}{2} s^2 U_A^\mu{}_{,\nu} U_A^\nu$
$= X_0^\mu + t V_0^\mu + \frac{1}{2} t^2 V_0^\mu{}_{,\nu} V_0^\nu$
$+ s(U_0^\mu + t U_0^\mu{}_{,\nu} V_0^\nu) + \frac{1}{2} s^2 U_0^\mu{}_{,\nu} U_0^\nu$
$t, s \ll 1$
X_D^μ
by swapping



$t, s \ll 1$

$$X_A^\mu = X_0^\mu + t V_0^\mu + \frac{1}{2} t^2 V_0^\mu{}_{,\nu} V_0^\nu \quad \text{evaluate at } X_0$$

$$X_B^\mu = X_A^\mu + s U_A^\mu + \frac{1}{2} s^2 U_A^\mu{}_{,\nu} U_A^\nu$$

$$= X_0^\mu + t V_0^\mu + \frac{1}{2} t^2 V_0^\mu{}_{,\nu} V_0^\nu$$

$$+ s\left(U_0^\mu + t U_0^\mu{}_{,\nu} V_0^\nu\right) + \frac{1}{2} s^2 U_0^\mu{}_{,\nu} U_0^\nu$$

$\frac{1}{2} V_0^\mu{}_{,\nu} V_0^\nu$ ← evaluate at X_0

$S^2 U_A^\mu{}_{,\nu} U_A^\nu$

$\frac{1}{2} V_0^\mu{}_{,\nu} V_0^\nu$

$V_0^\nu) + \frac{1}{2} S^2 U_0^\mu{}_{,\nu} U_0^\nu$

$X_D^\mu = X_0^\mu + S U_0^\mu + \frac{1}{2} S^2 U_0^\mu{}_{,\nu} U_0^\nu$

$+ t(V_0^\mu + S V_0^\mu{}_{,\nu} U_0^\nu) + \frac{1}{2} t^2 V_0^\mu{}_{,\nu} V_0^\nu$

by swapping
$U \leftrightarrow V$
$S \leftrightarrow t$

$\tfrac{1}{2} V_0^\mu{}_{,\nu} V_0^\nu$ ← evaluate at X_0

$s^2 U_A^\mu{}_{,\nu} U_A^\nu$

$\tfrac{1}{2} V_0^\mu{}_{,\nu} V_0^\nu$

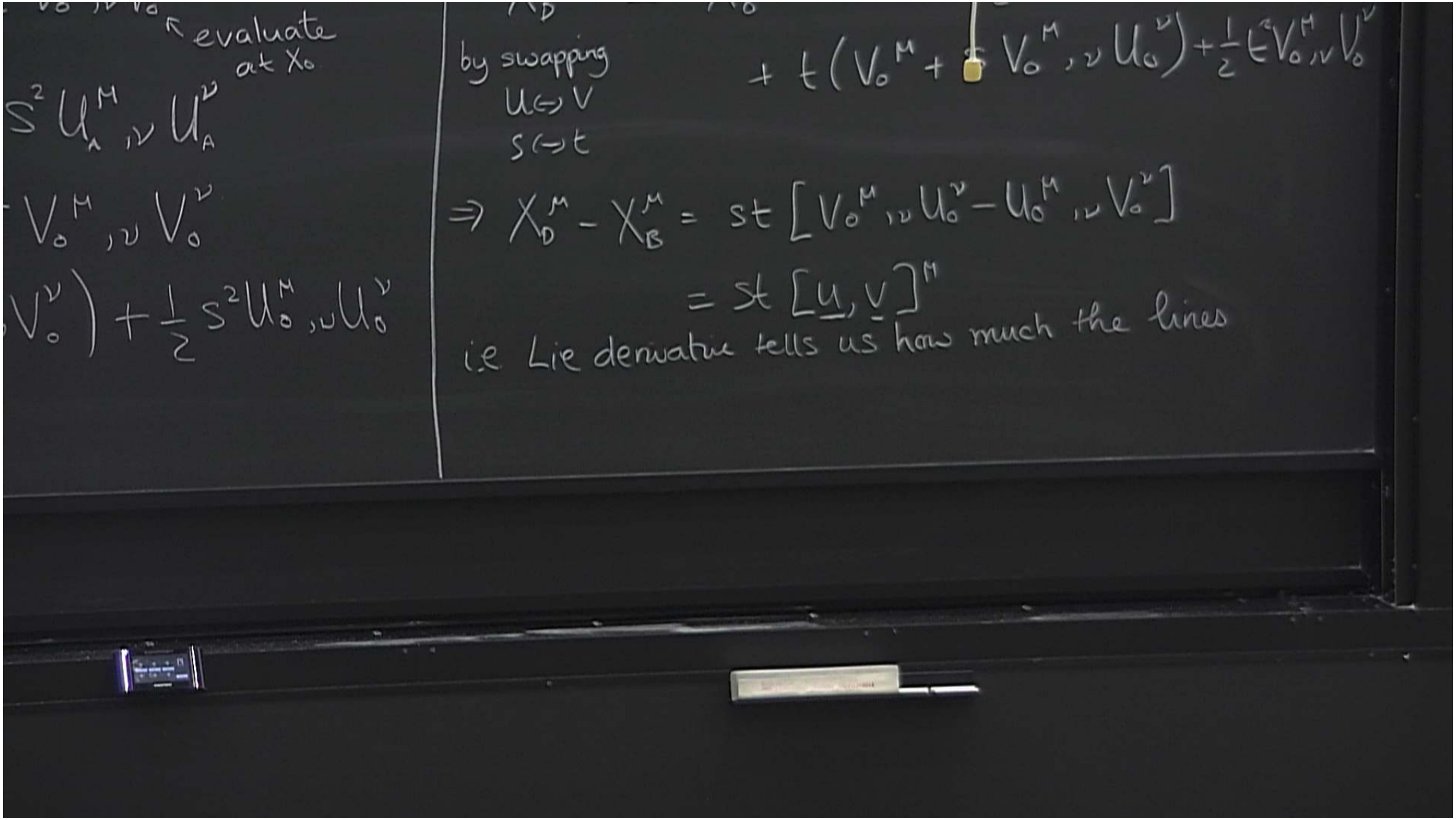
$V_0^\nu) + \tfrac{1}{2} s^2 U_0^\mu{}_{,\nu} U_0^\nu$

$$X_D^\mu = X_0^\mu + s U_0^\mu + \tfrac{1}{2} s^2 U_0^\mu{}_{,\nu} U_0^\nu + t(V_0^\mu + s V_0^\mu{}_{,\nu} U_0^\nu) + \tfrac{1}{2} t^2 V_0^\mu{}_{,\nu} V_0^\nu$$

by swapping
$U \leftrightarrow V$
$s \leftrightarrow t$

$$\Rightarrow X_D^\mu - X_B^\mu = st\left[V_0^\mu{}_{,\nu} U_0^\nu - U_0^\mu{}_{,\nu} V_0^\nu\right]$$

$$= st\,[\underline{U}, \underline{V}]^\mu$$


evaluate at X_0
$s^2 U^\mu_{,\nu} U^\nu_A$
$V^\mu_{0,\nu} V^\nu_0$
$V^\nu_0) + \frac{1}{2} s^2 U^\mu_{0,\nu} U^\nu_0$
by swapping
$U \to V$
$s \to t$
$+ t(V_0^\mu + s V_{0,\nu}^\mu U_0^\nu) + \frac{1}{2} t^2 V_{0,\nu}^\mu V_0^\nu$
$\Rightarrow X_D^\mu - X_B^\mu = st\,[V_{0,\nu}^\mu U_0^\nu - U_{0,\nu}^\mu V_0^\nu]$
$= st\,[\underline{U}, \underline{V}]^\mu$
i.e Lie derivative tells us how much the lines

$X_D = X_0 + sU_0 + \frac{1}{2}s\,U_{0,\nu}U_0$

by swapping $U \leftrightarrow V$, $s \leftrightarrow t$

$+ t(V_0^\mu + s\,V_{0,\nu}^\mu U_0^\nu) + \frac{1}{2}t^2 V_{0,\nu}^\mu V_0^\nu$

$\Rightarrow X_D^\mu - X_B^\mu = st\left[V_{0,\nu}^\mu U_0^\nu - U_{0,\nu}^\mu V_0^\nu\right]$

$= st\,[\underline{U}, \underline{V}]^\mu$

i.e. Lie derivative tells us how much the lines or integral curves of U & V fail to close.

A Killing vector is a vector field along which the metric is Lie invariant

$$\mathcal{L}_V g = 0$$

} tells us about symmetries of metric.

e.g. $V = \frac{\partial}{\partial \phi}$ AXIAL SYMMETRY.

Reln: $\underline{d}$ & $\mathcal{L}_{\underline{v}}$:

$$\langle \underline{d}\underline{\omega} \mid \underline{u}, \underline{v} \rangle = \underline{u}(\langle \underline{\omega} \mid \underline{v} \rangle) - \underline{v}(\langle \underline{\omega} \mid \underline{u} \rangle) - \langle \underline{\omega} \mid [\underline{u}, \underline{v}] \rangle$$

Reln: $\underline{d} \propto \mathcal{L}$:

$$\langle \underline{d\omega} \mid \underline{u}, \underline{v} \rangle = \underline{u}(\langle \underline{\omega} \mid \underline{v} \rangle) - \underline{v}(\langle \underline{\omega} \mid \underline{u} \rangle) - \langle \underline{\omega} \mid [\underline{u}, \underline{v}] \rangle$$

Connections - links tangent spaces.

$$\nabla \underline{e}_b = \Gamma_{cb}^{a} \, \underline{e}_a \otimes \underline{\omega}^c$$

Reln: $\underline{d} \propto \mathcal{L}$:

$$\langle \underline{d}\omega \,|\, \underline{u}, \underline{v} \rangle = \underline{u}(\langle \omega | \underline{v} \rangle) - \underline{v}(\langle \omega | \underline{u} \rangle) - \langle \omega | [\underline{u}, \underline{v}] \rangle$$

Connections - links tangent spaces

$$\underline{\nabla} \underline{e}_b = \Gamma^{a}_{cb}\, \underline{e}_a \otimes \underline{\omega}^c \quad \text{or} \quad \Gamma^{a}_{bc} = \langle \underline{\omega}^a | \underbrace{\nabla_b}_{\text{"}\underline{e}_b \cdot \underline{\nabla}\text{"}} \underline{e}_c \rangle$$

or integral curves of u & v fail to close.

Reln: $\underline{d}$ & $\mathcal{L}$:

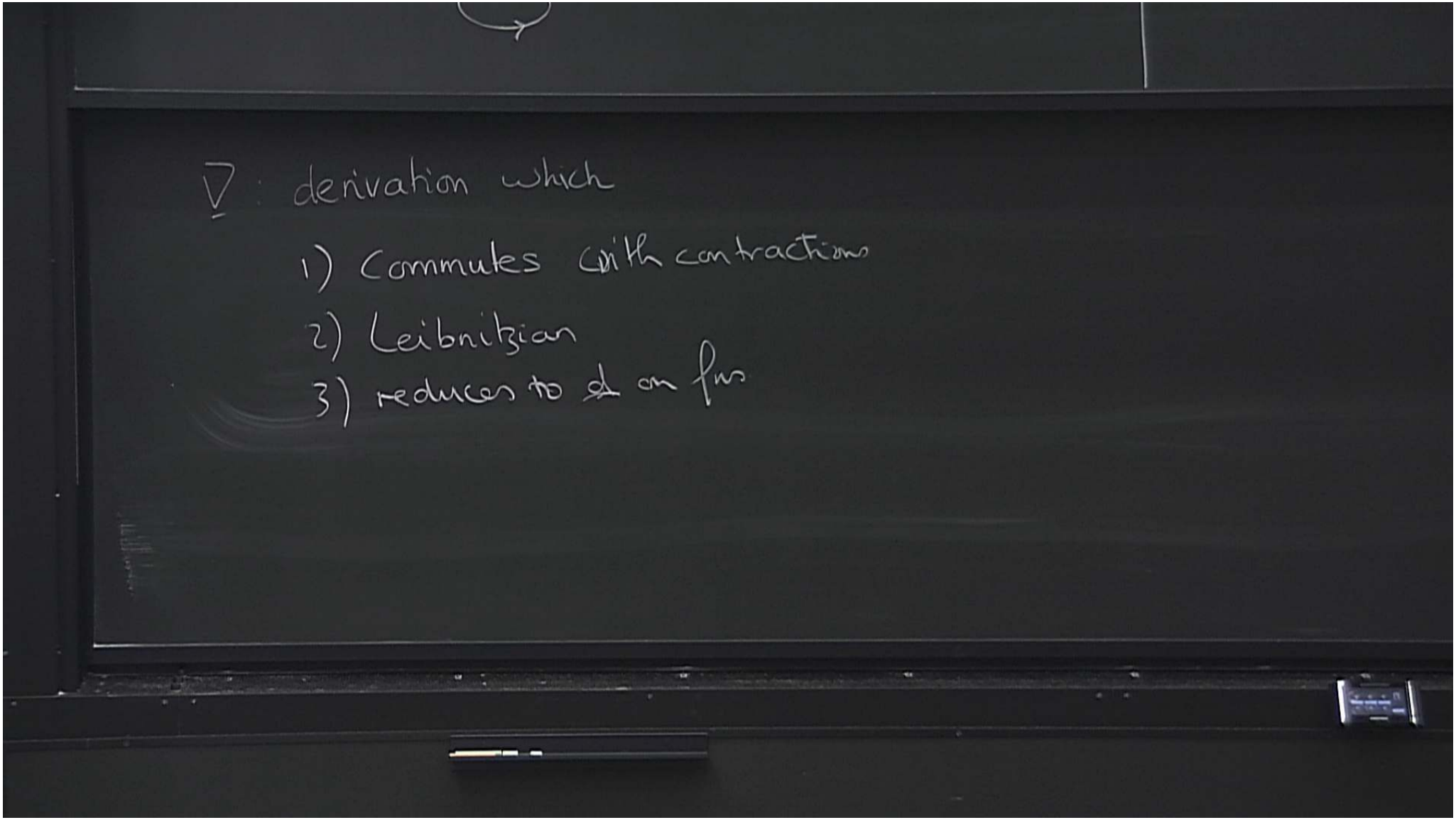
$$\langle \underline{d}\omega \,|\, u, v\rangle = \underline{u}(\langle \omega | \underline{v}\rangle) - \underline{v}(\langle \underline{\omega} | \underline{u}\rangle) - \langle \underline{\omega} | [u, v]\rangle$$

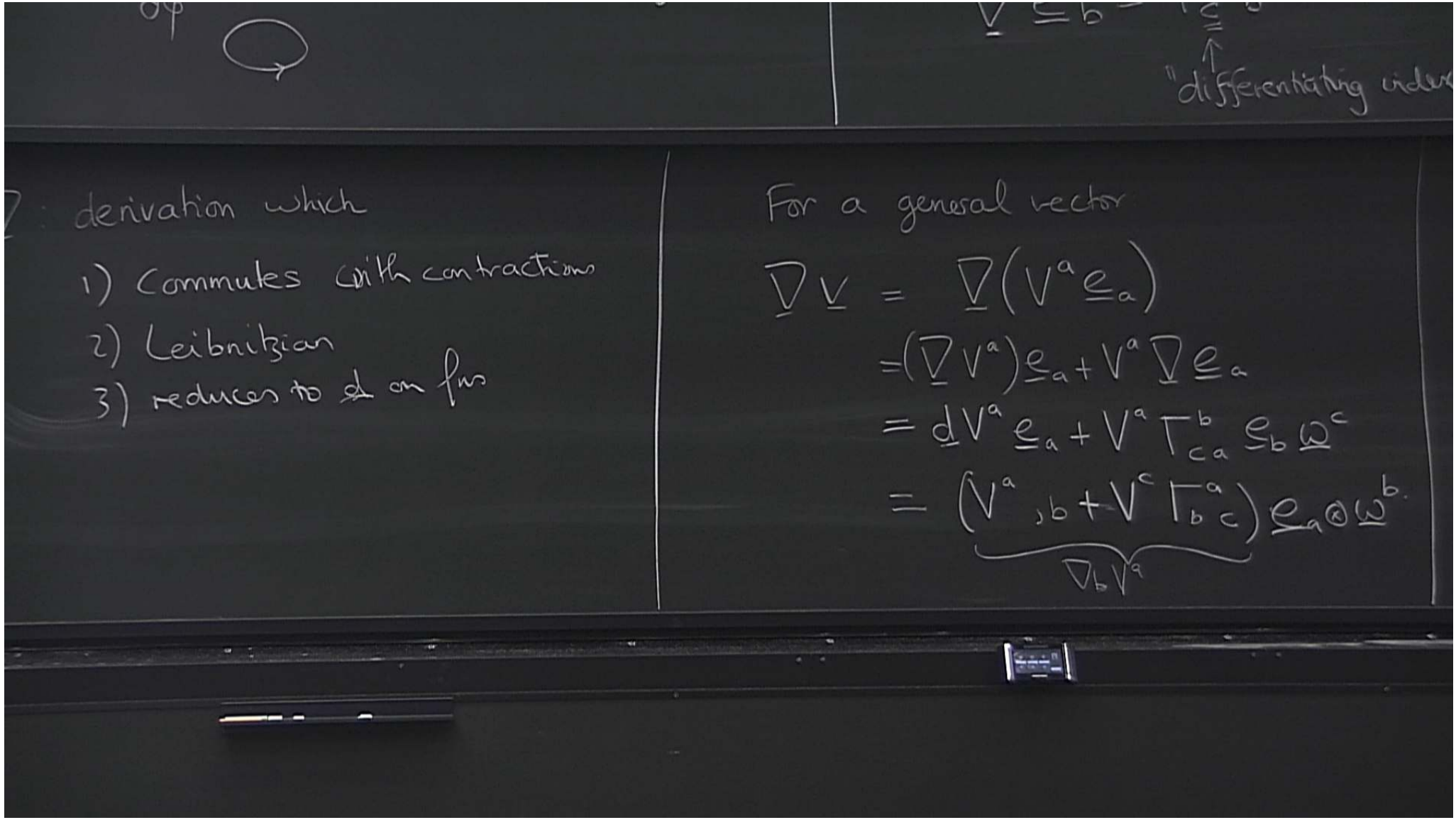
Connections - links tangent spaces

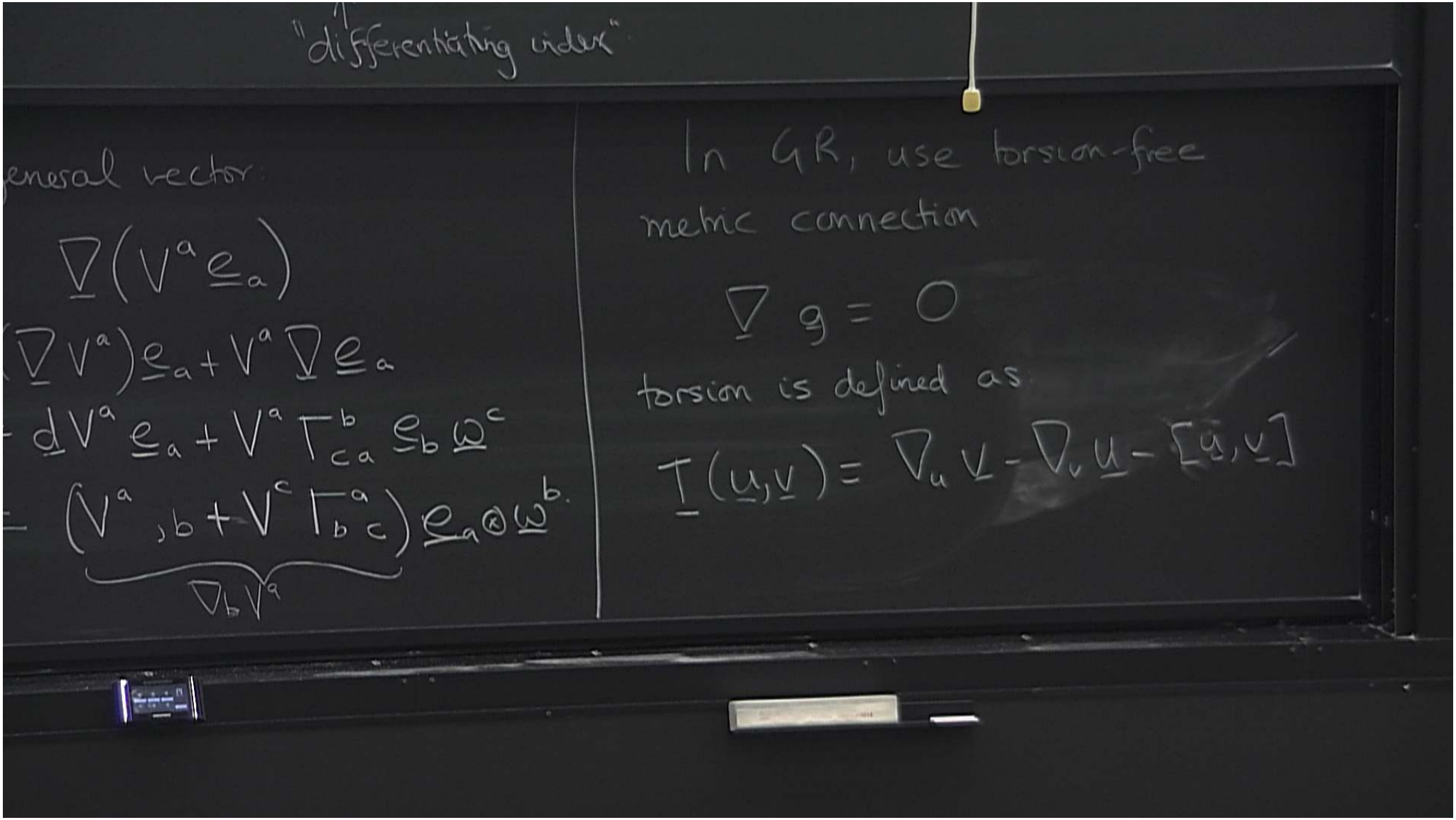
$$\nabla \underline{e}_b = \Gamma^{a}_{cb}\, \underline{e}_a \otimes \underline{\omega}^c \quad \text{or} \quad \Gamma^{a}_{bc} = \langle \underline{\omega}^a | \nabla_b \underline{e}_c \rangle$$

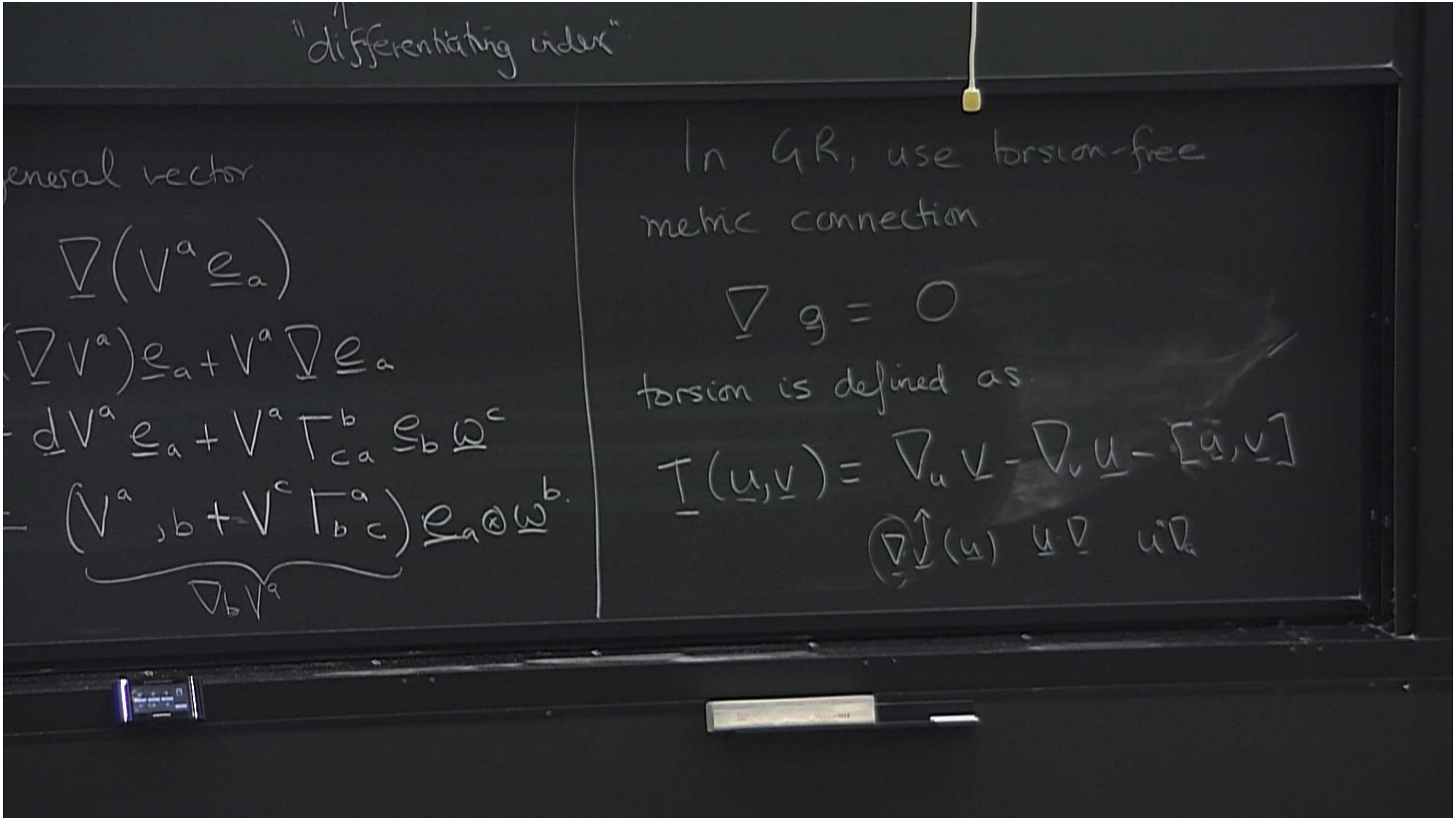
"differentiating index"

"$\underline{e}_b \cdot \nabla$"


∇: derivation which
1) Commutes with contractions
2) Leibnitzian
3) reduces to ∂ on $f \in$


"differentiating index
derivation which
1) Commutes with contractions
2) Leibnitzian
3) reduces to d on fns
For a general vector
$\nabla \underline{V} = \nabla(V^a \underline{e}_a)$
$= (\nabla V^a)\underline{e}_a + V^a \nabla \underline{e}_a$
$= dV^a \underline{e}_a + V^a \Gamma^b_{ca} \underline{e}_b \underline{\omega}^c$
$= (V^a{}_{,b} + V^c \Gamma^a_{bc}) \underline{e}_a \otimes \underline{\omega}^b$
$\nabla_b V^a$


"differentiating index"
general vector
$\nabla(V^a \underline{e}_a)$
$(\nabla V^a)\underline{e}_a + V^a \nabla \underline{e}_a$
$= dV^a \underline{e}_a + V^a \Gamma^b_{ca} \underline{e}_b \underline{\omega}^c$
$= (V^a{}_{,b} + V^c \Gamma^a_{bc}) \underline{e}_a \otimes \underline{\omega}^b$
$\nabla_b V^a$
In GR, use torsion-free metric connection
$\nabla g = 0$
torsion is defined as
$\underline{T}(\underline{u},\underline{v}) = \nabla_u \underline{v} - \nabla_v \underline{u} - [\underline{u},\underline{v}]$

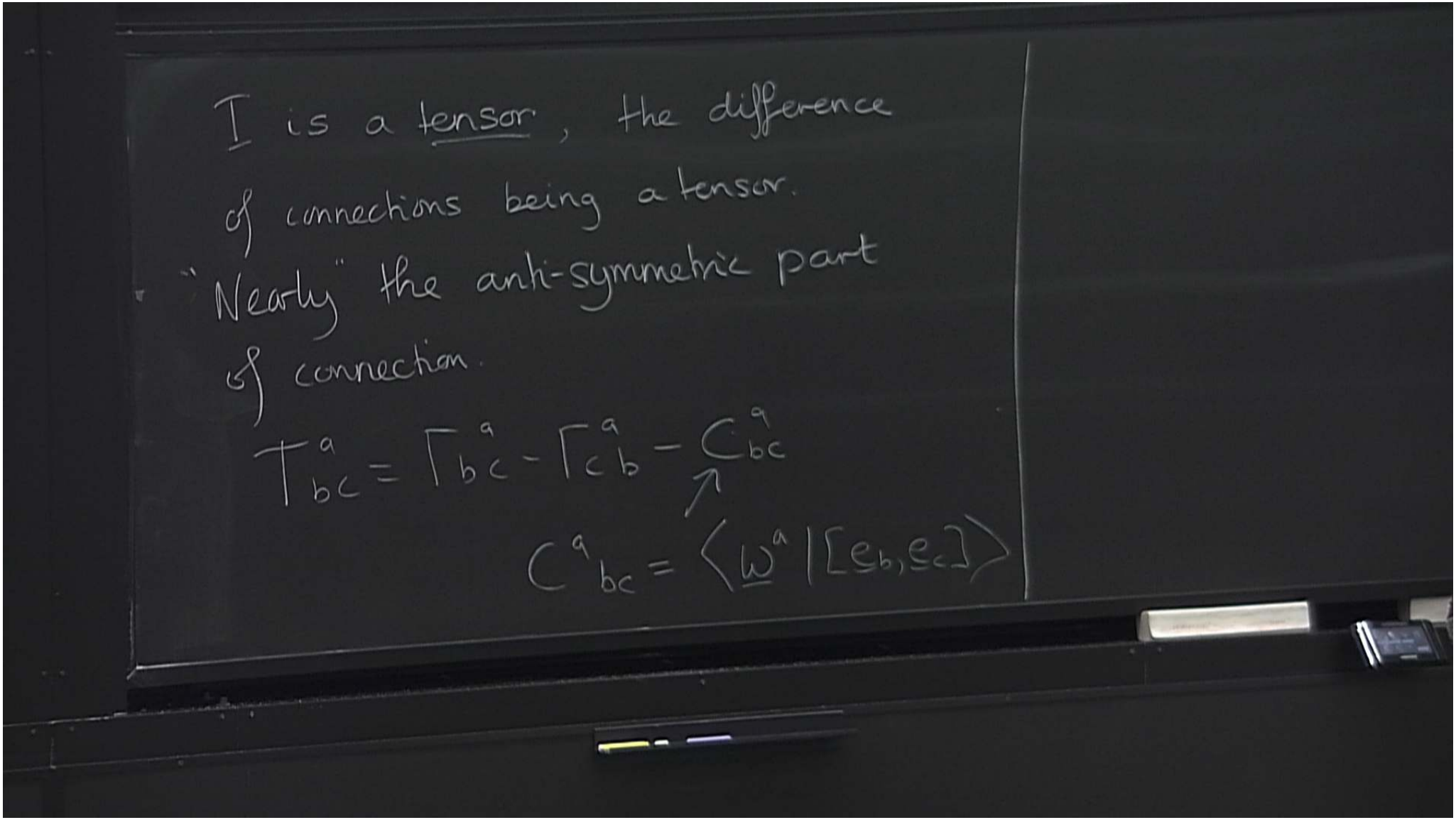

"differentiating index"
general vector
$\nabla(V^a e_a)$
$(\nabla V^a) e_a + V^a \nabla e_a$
$dV^a e_a + V^a \Gamma^b_{ca} e_b \omega^c$
$(V^a{}_{,b} + V^c \Gamma^a_{bc}) e_a \otimes \omega^b$
$\nabla_b V^a$
In GR, use torsion-free metric connection
$\nabla g = 0$
torsion is defined as
$T(u,v) = \nabla_u v - \nabla_v u - [u,v]$
$(\nabla v)(u)$ u v

$\underline{T}$ is a tensor, the difference of connections being a tensor.

"Nearly" the anti-symmetric part of connection.

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}$$

$$C^a_{bc} = \langle \underline{\omega}^a | [e_b, e_c] \rangle$$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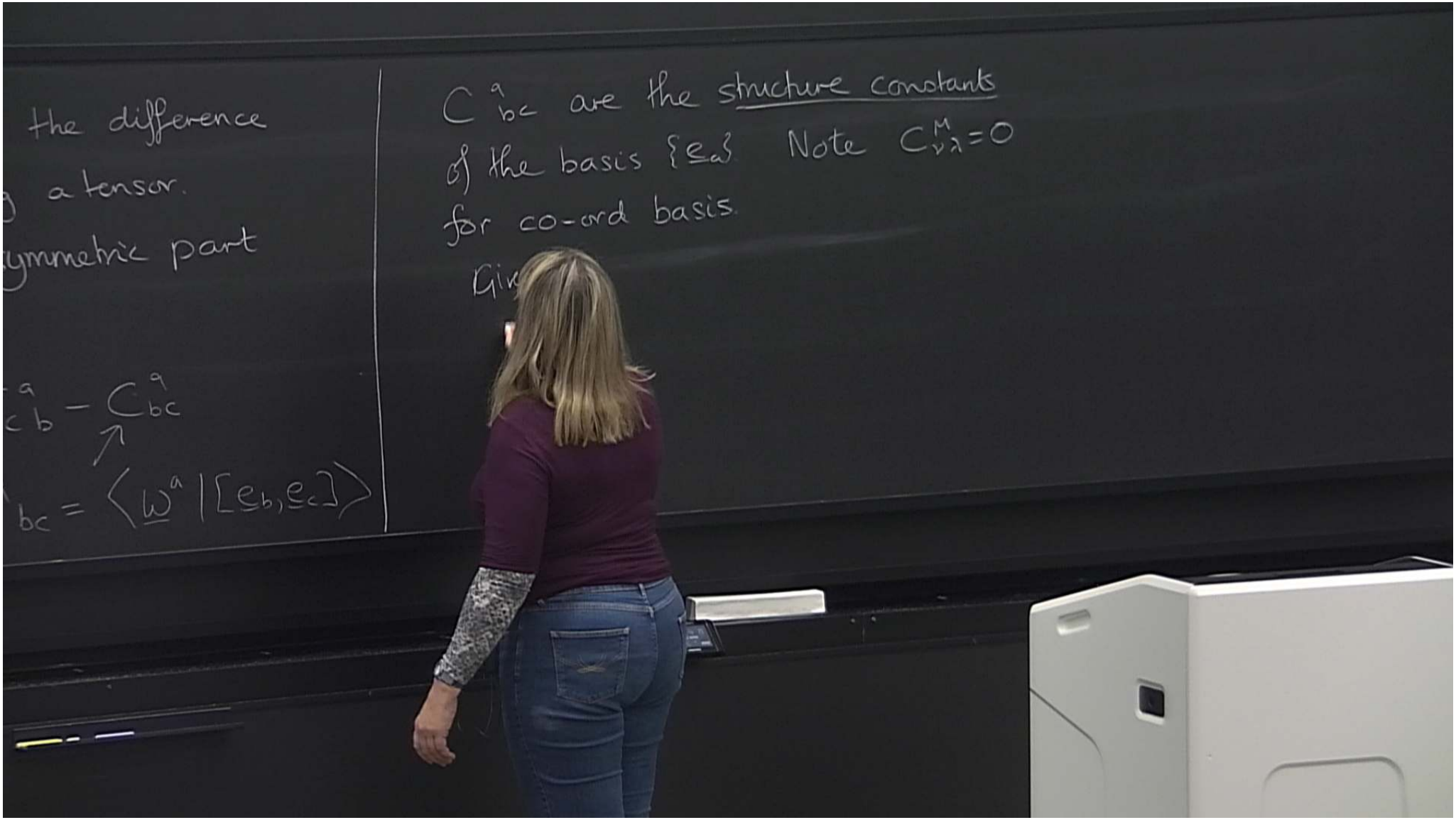
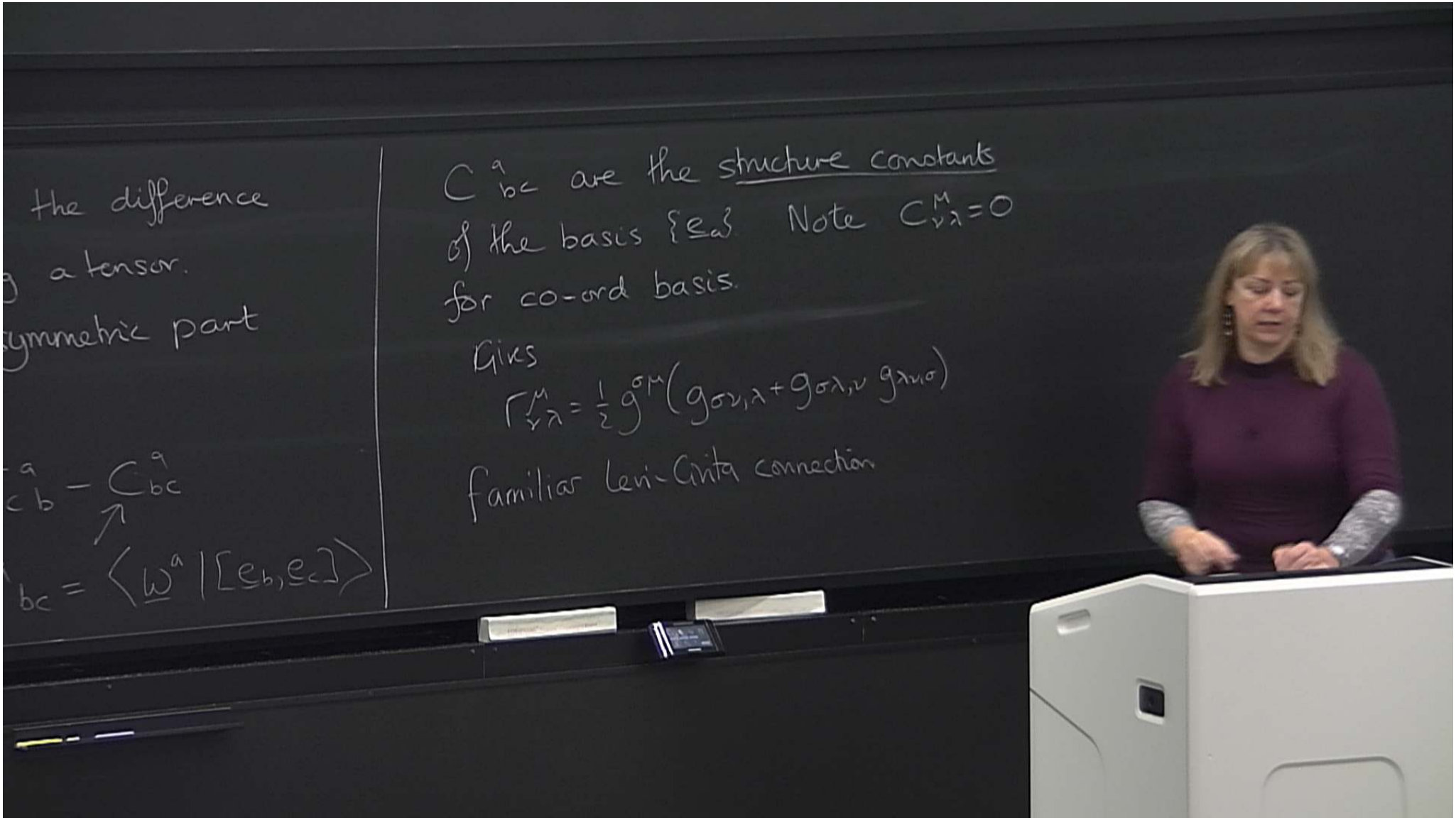

the difference
a tensor.
ymmetric part
C^a_{bc} are the structure constants
of the basis $\{e_a\}$ Note $C^M_{\nu\lambda}=0$
for co-ord basis.
$C^a_{cb} - C^a_{bc}$
$bc = \langle \omega^a | [e_b, e_c] \rangle$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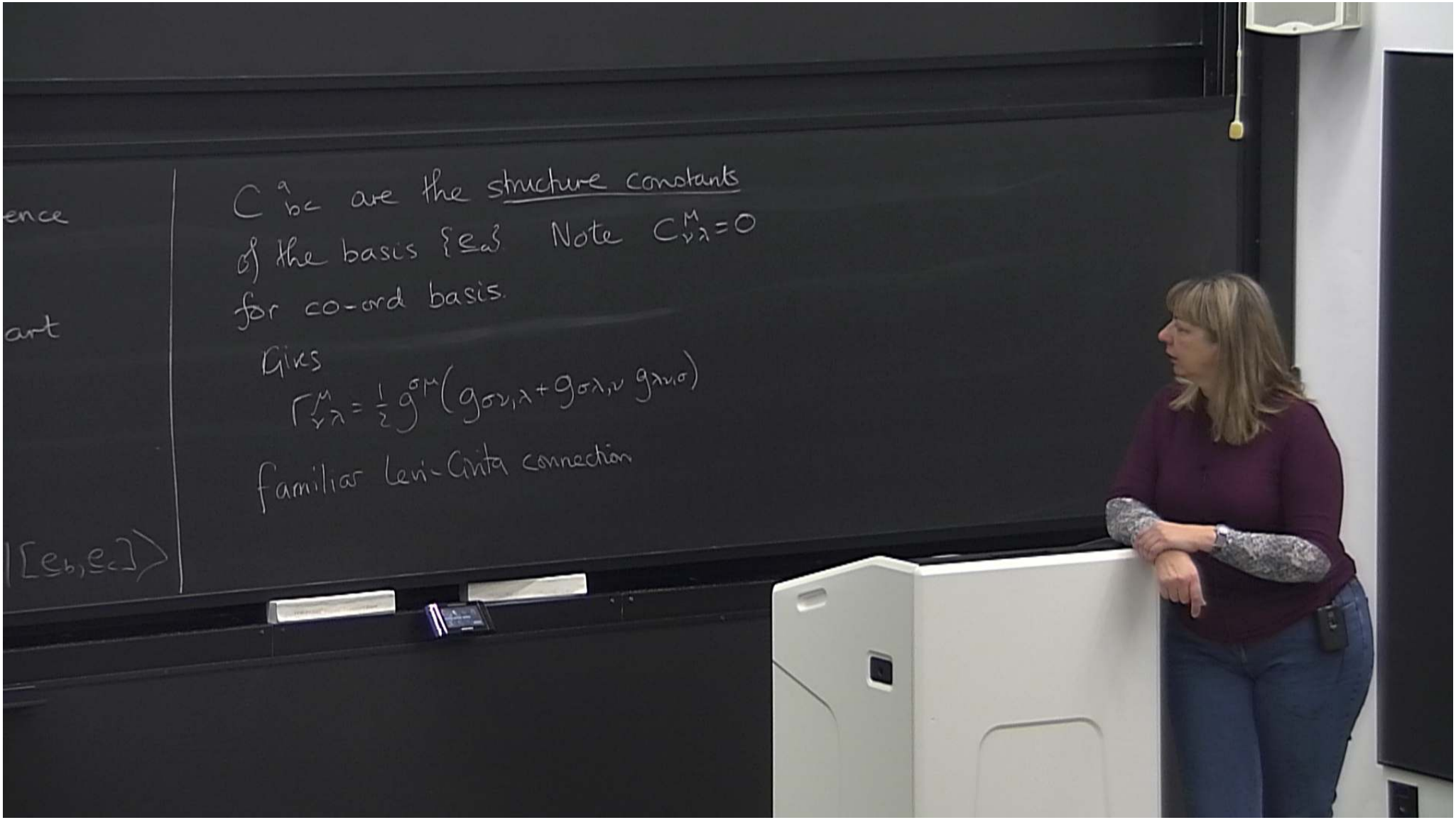
Gives
$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\sigma\mu}(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} g_{\lambda\mu,\sigma})$
familiar Levi-Civita connection
$C^a_{cb} - C^a_{bc}$
$C^a_{bc} = \langle \omega^a | [e_b, e_c] \rangle$


ence
art
$[e_b, e_c]$
C^a_{bc} are the structure constants
of the basis $\{e_a\}$ Note $C^\mu_{\nu\lambda} = 0$
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Gives
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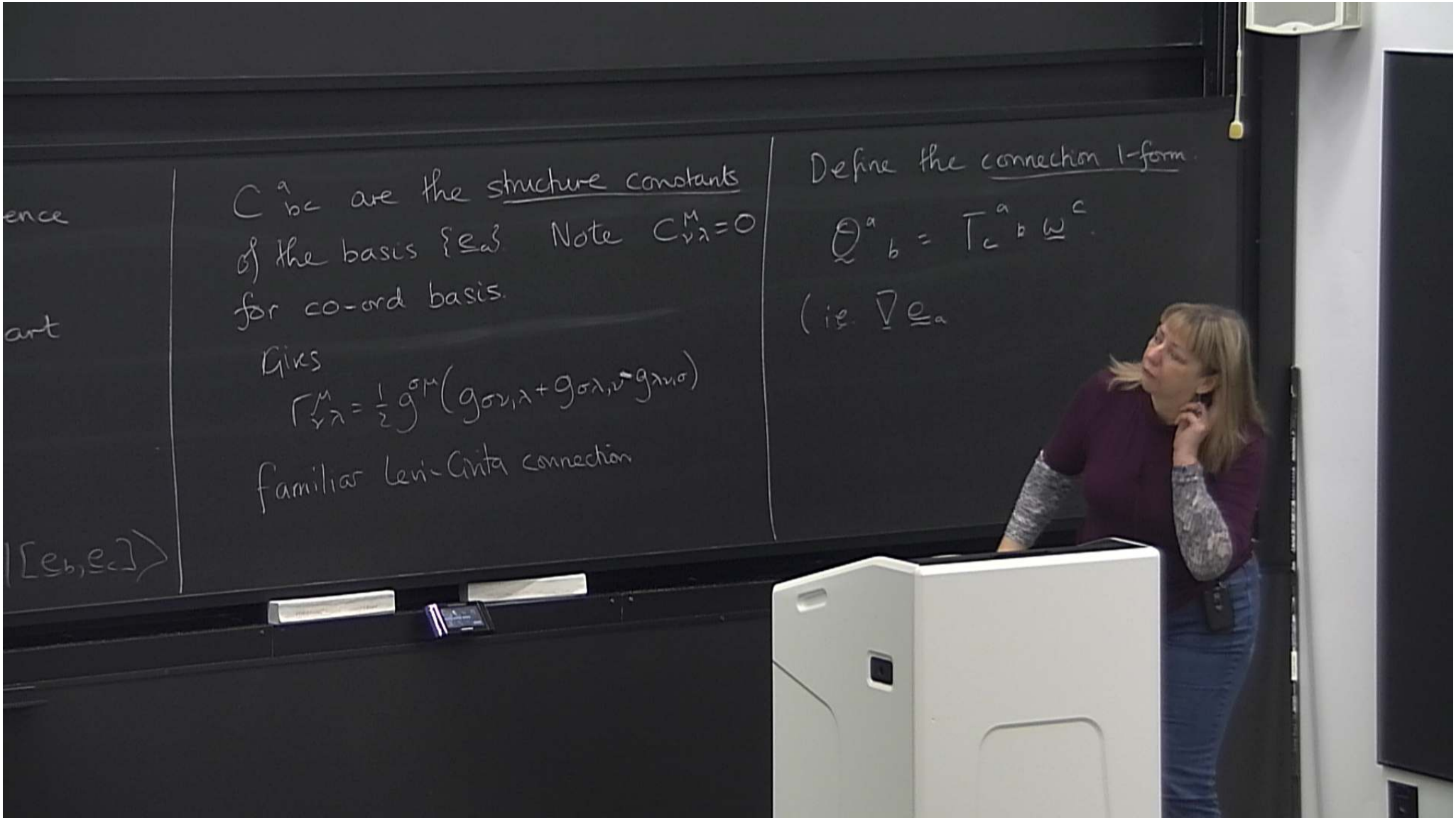
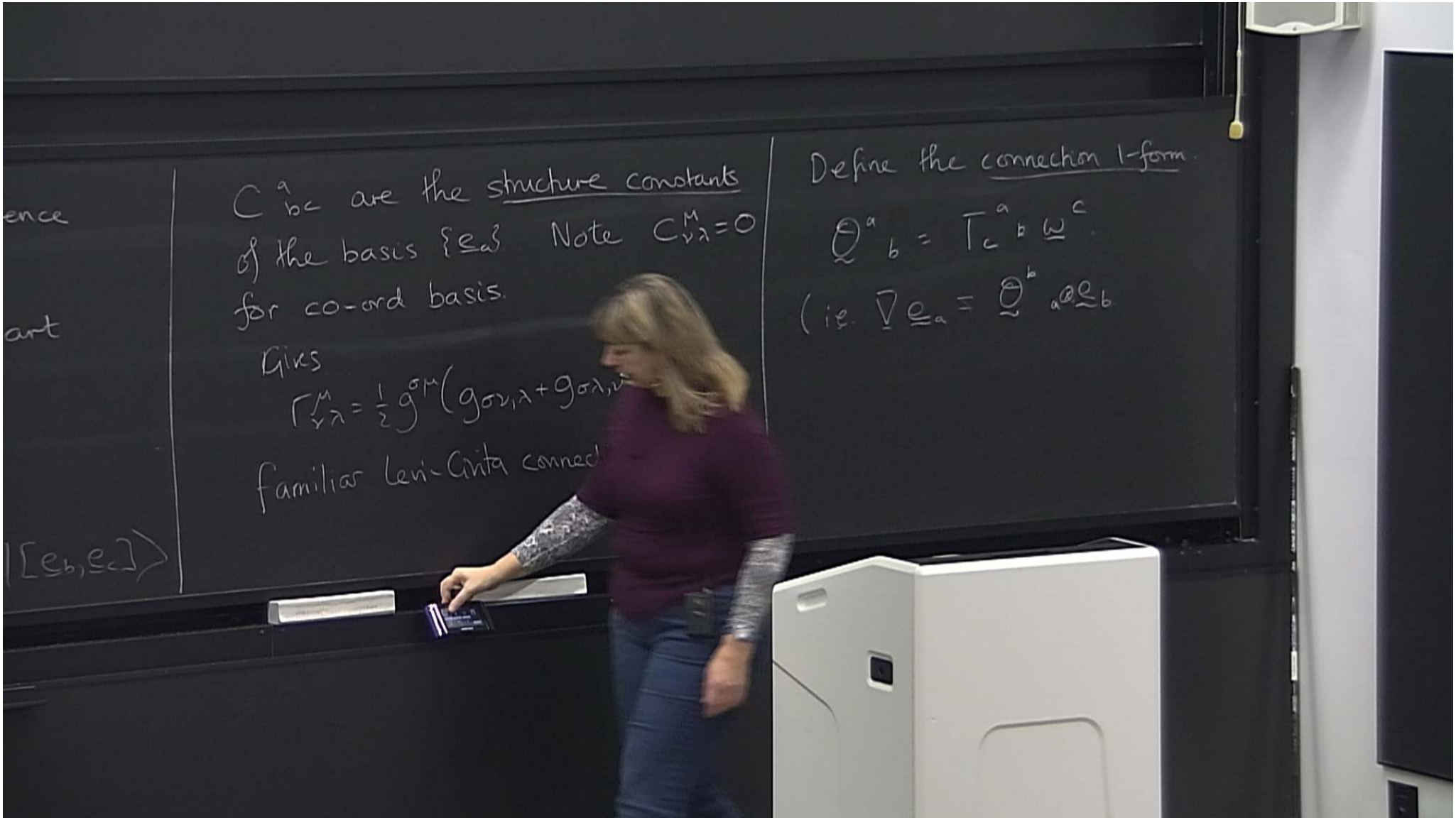
familiar Levi-Civita connection


ence
art
$[e_b, e_c]$
C^a_{bc} are the structure constants of the basis $\{e_a\}$ Note $C^\mu_{\nu\lambda} = 0$ for co-ord basis.
Gives
$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\sigma\mu}(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\lambda\nu,\sigma})$
familiar Levi-Civita connection
Define the connection 1-form
$\underline{\theta}^a{}_b = \Gamma^a_{cb}\,\underline{\omega}^c$
(ie. $\underline{\nabla} \underline{e}_a$


C^a_{bc} are the structure constants
of the basis $\{e_a\}$ Note $C^\mu_{\nu\lambda}=0$
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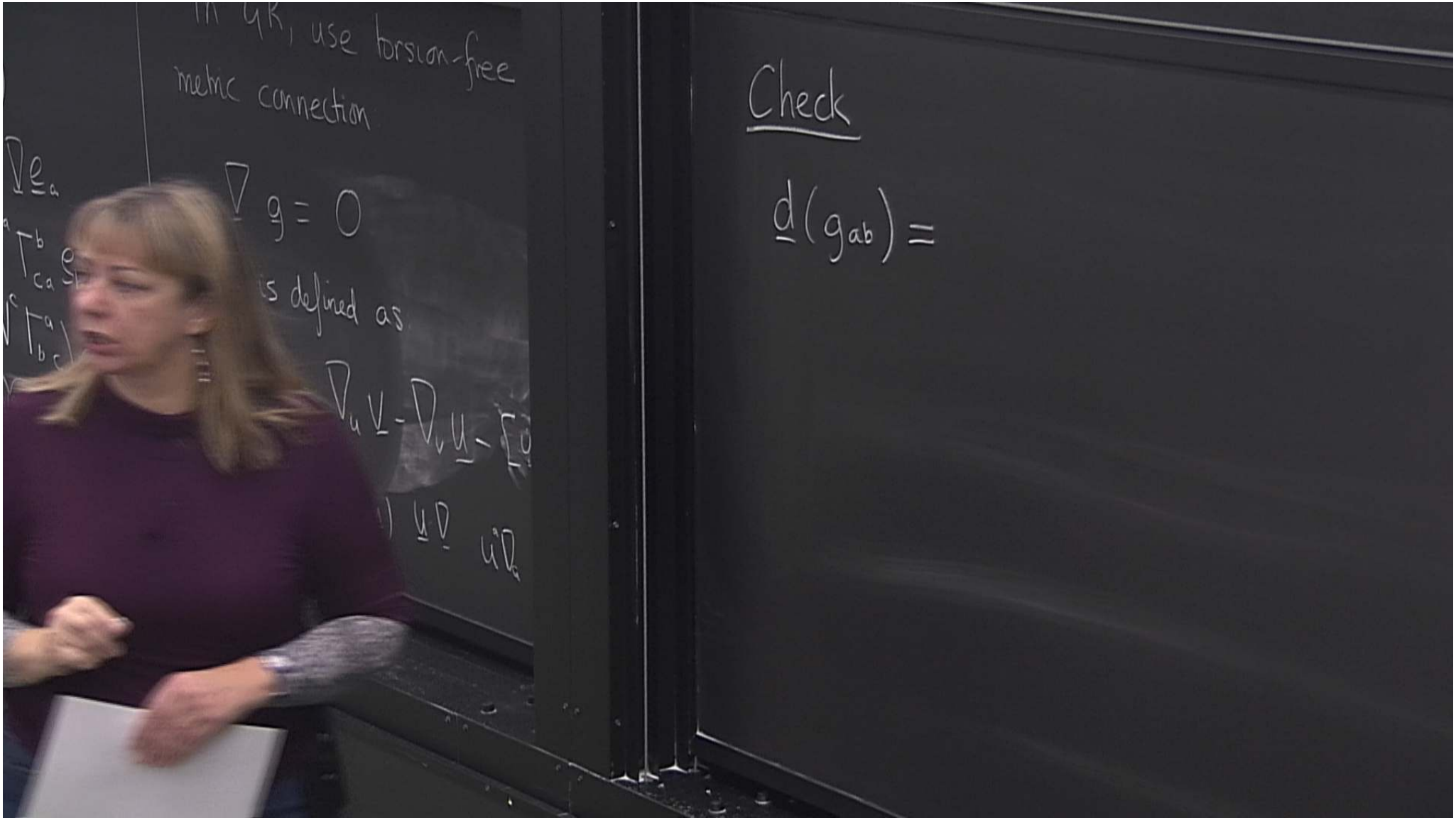
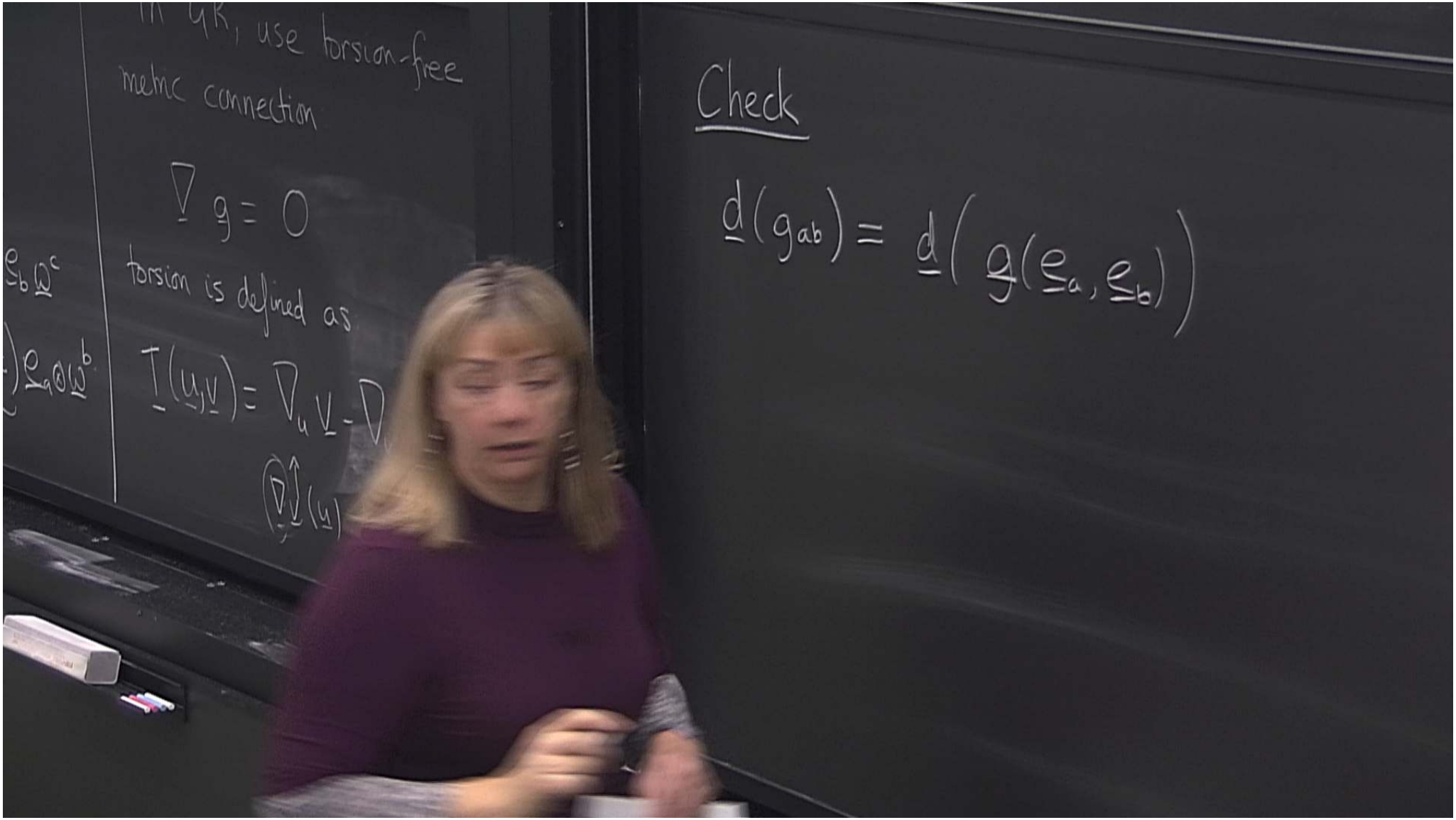
familiar Levi-Civita connect
Define the connection 1-form
$\underset{\sim}{\theta}^a{}_b = \Gamma^a_{cb}\,\underset{\sim}{\omega}^c$
(i.e. $\nabla \underset{\sim}{e}_a = \underset{\sim}{\theta}^b{}_a \otimes \underset{\sim}{e}_b$
$[e_b, e_c]$


use torsion-free
metric connection
$\nabla g = 0$
is defined as
Check
$\underline{d}(g_{ab}) =$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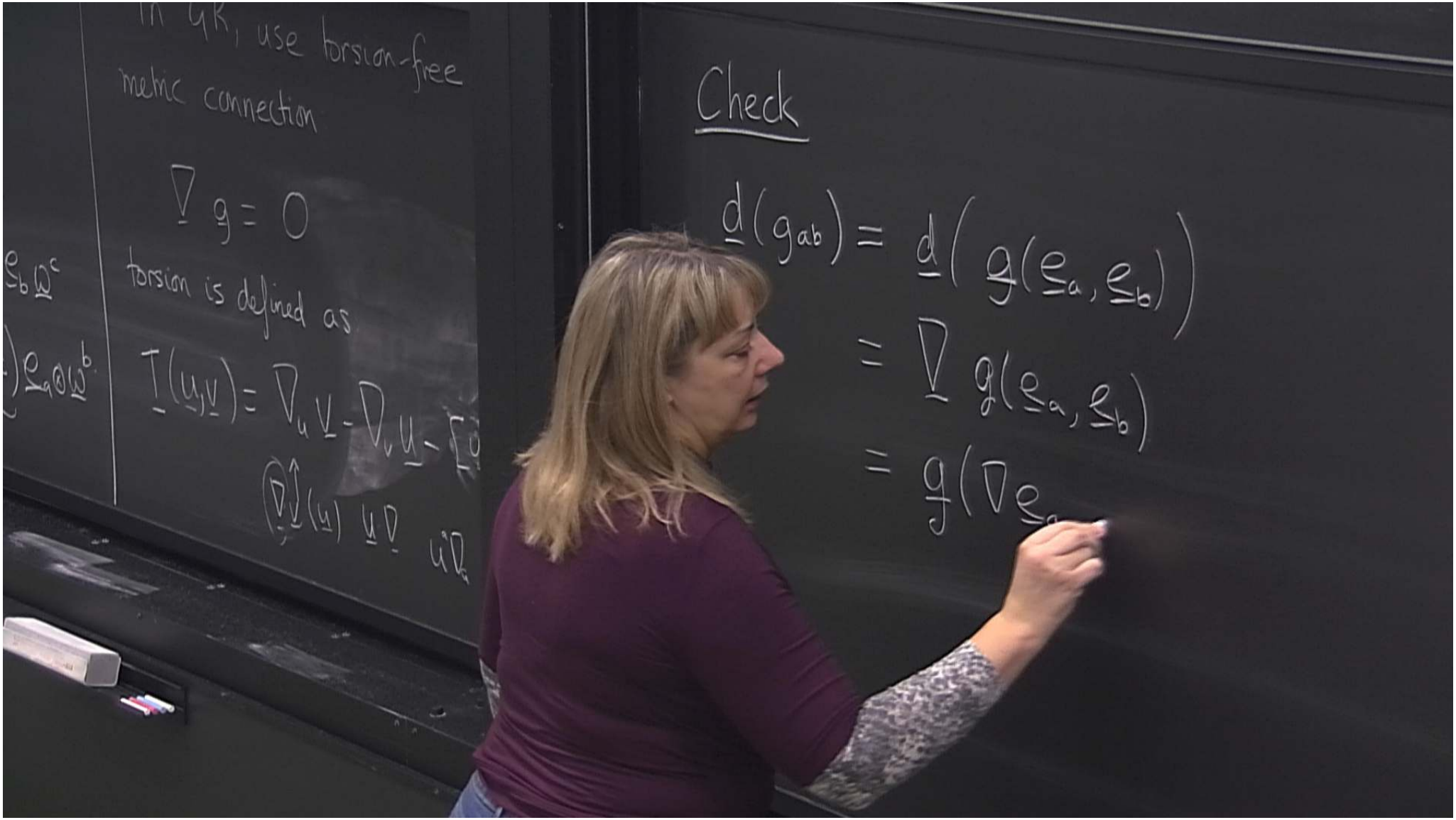
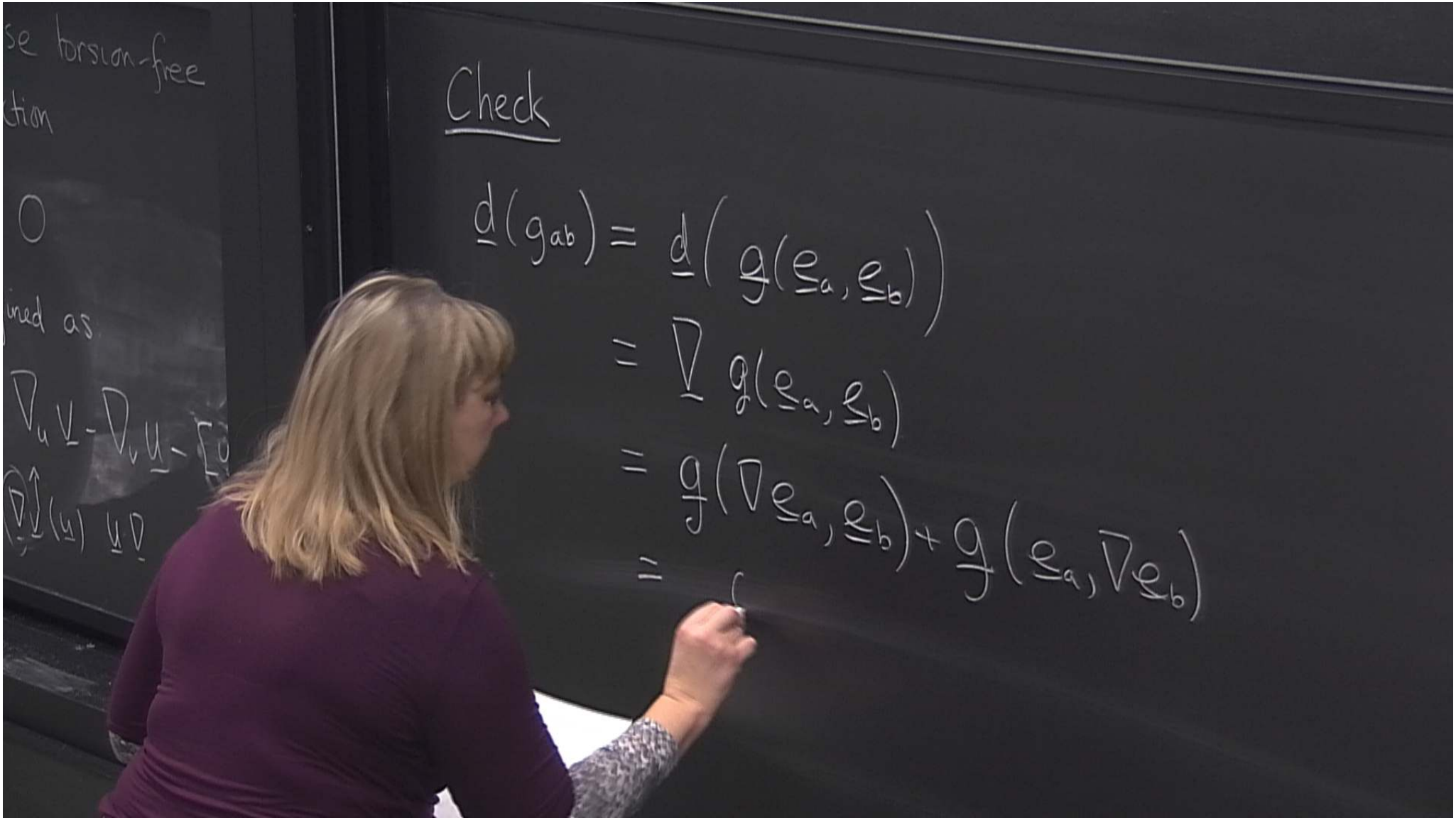
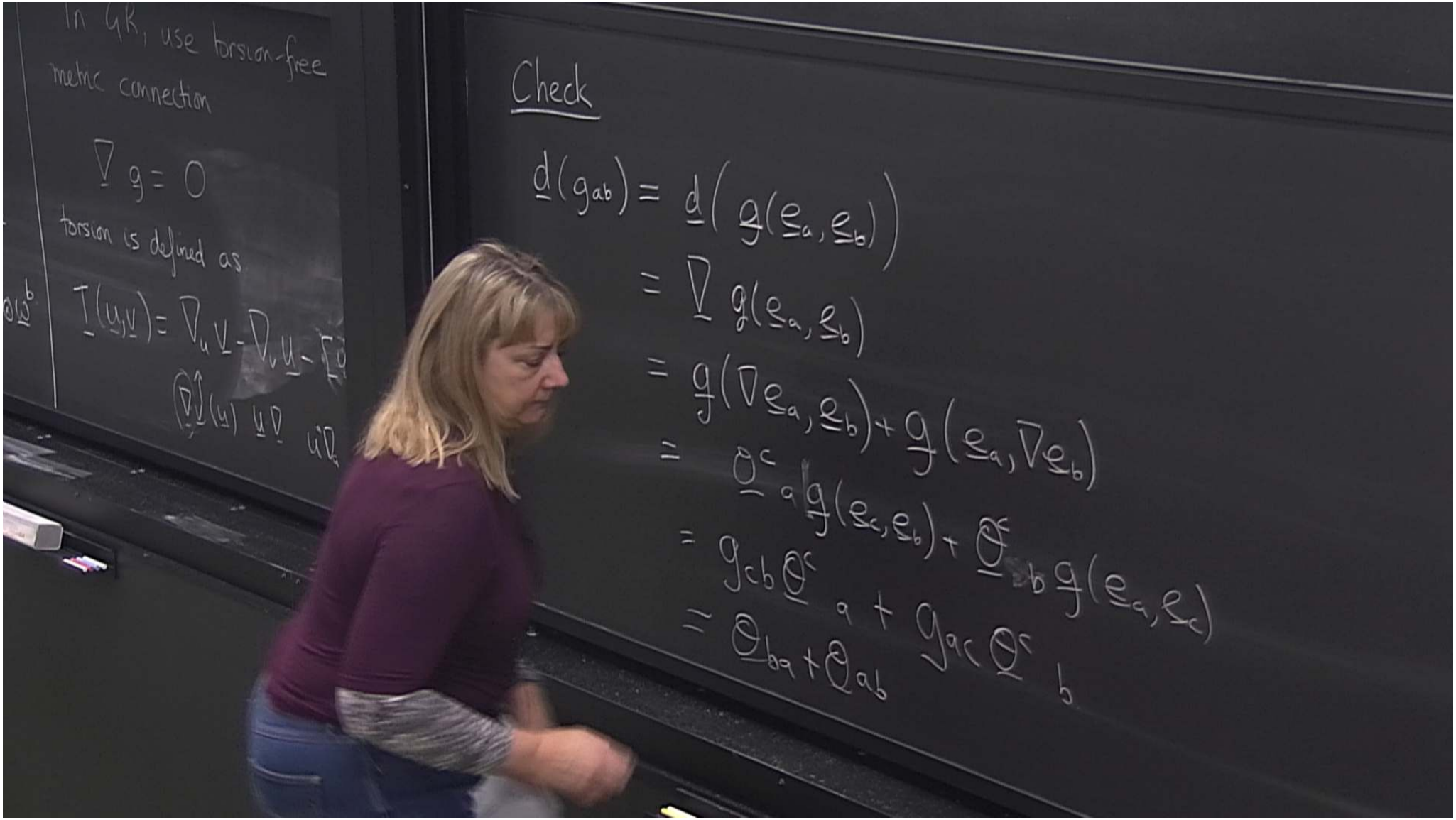
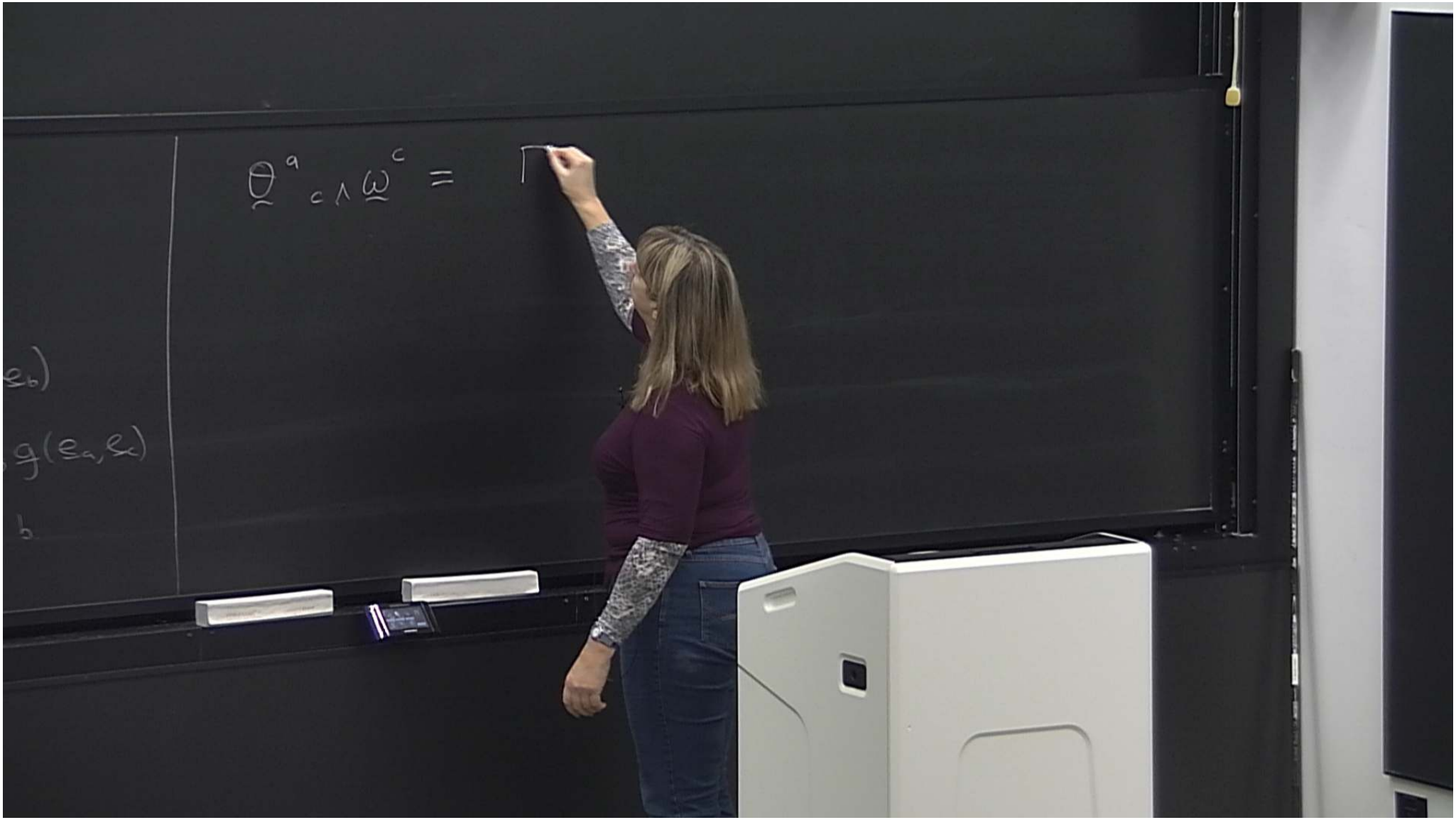
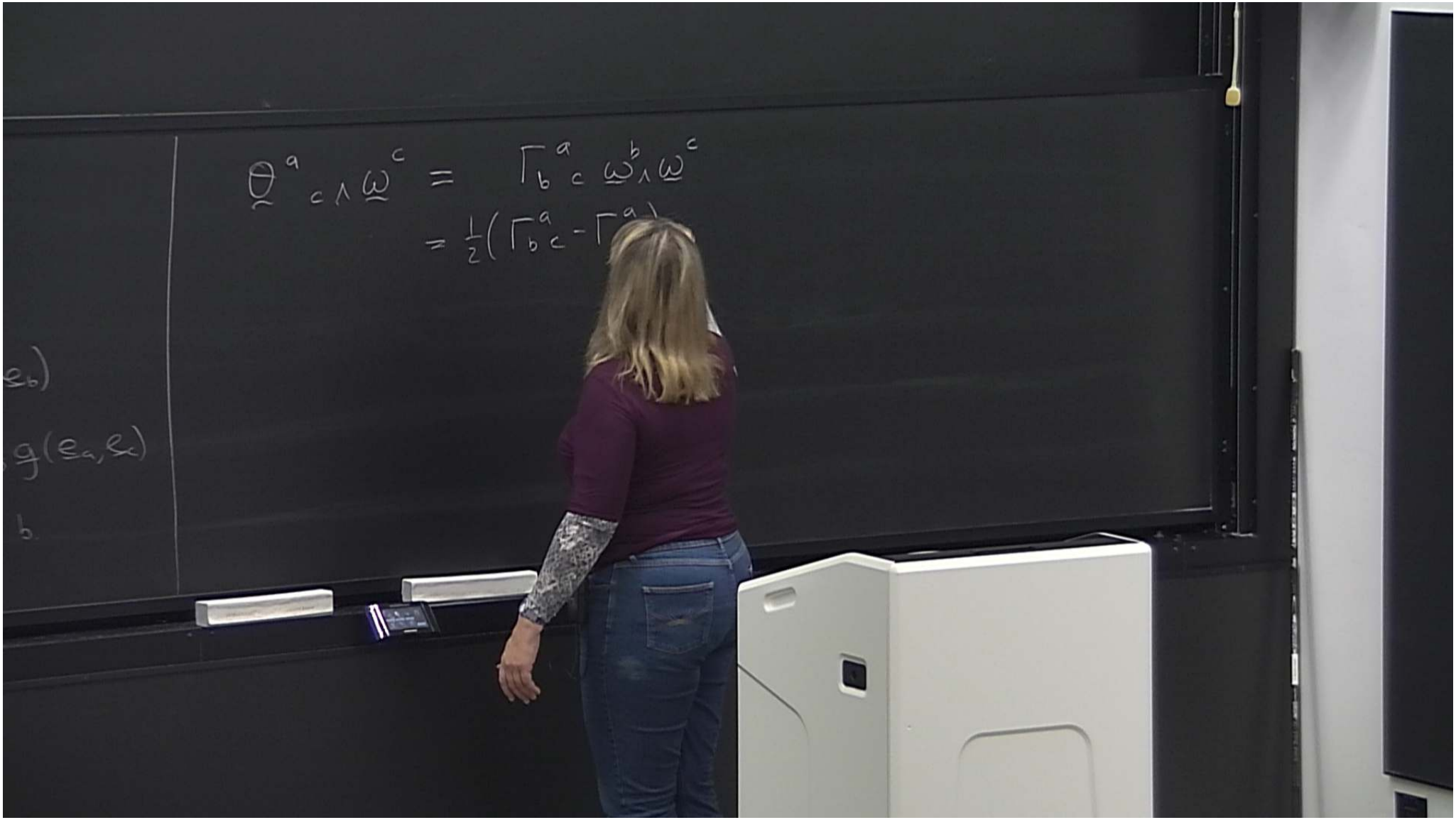

use torsion-free
metric connection
$\nabla g = 0$
torsion is defined as
Check
$d(g_{ab}) = d(g(e_a, e_b))$
$= \nabla g(e_a, e_b)$
$= g(\nabla e_$

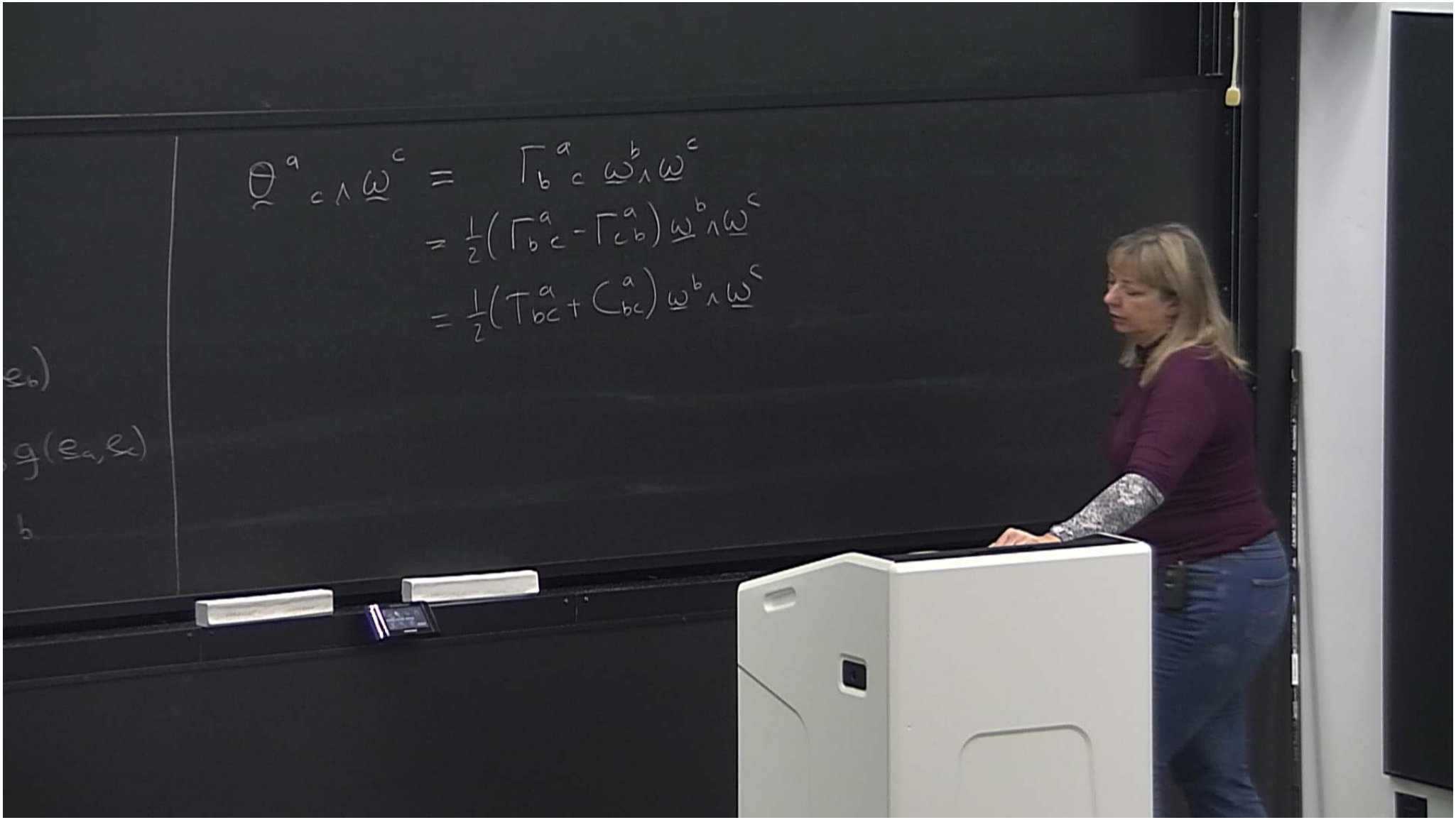

Check
$\underline{d}(g_{ab}) = \underline{d}(g(e_a, e_b))$
$= \underline{\nabla}\, g(e_a, e_b)$
$= g(\nabla e_a, e_b) + g(e_a, \nabla e_b)$

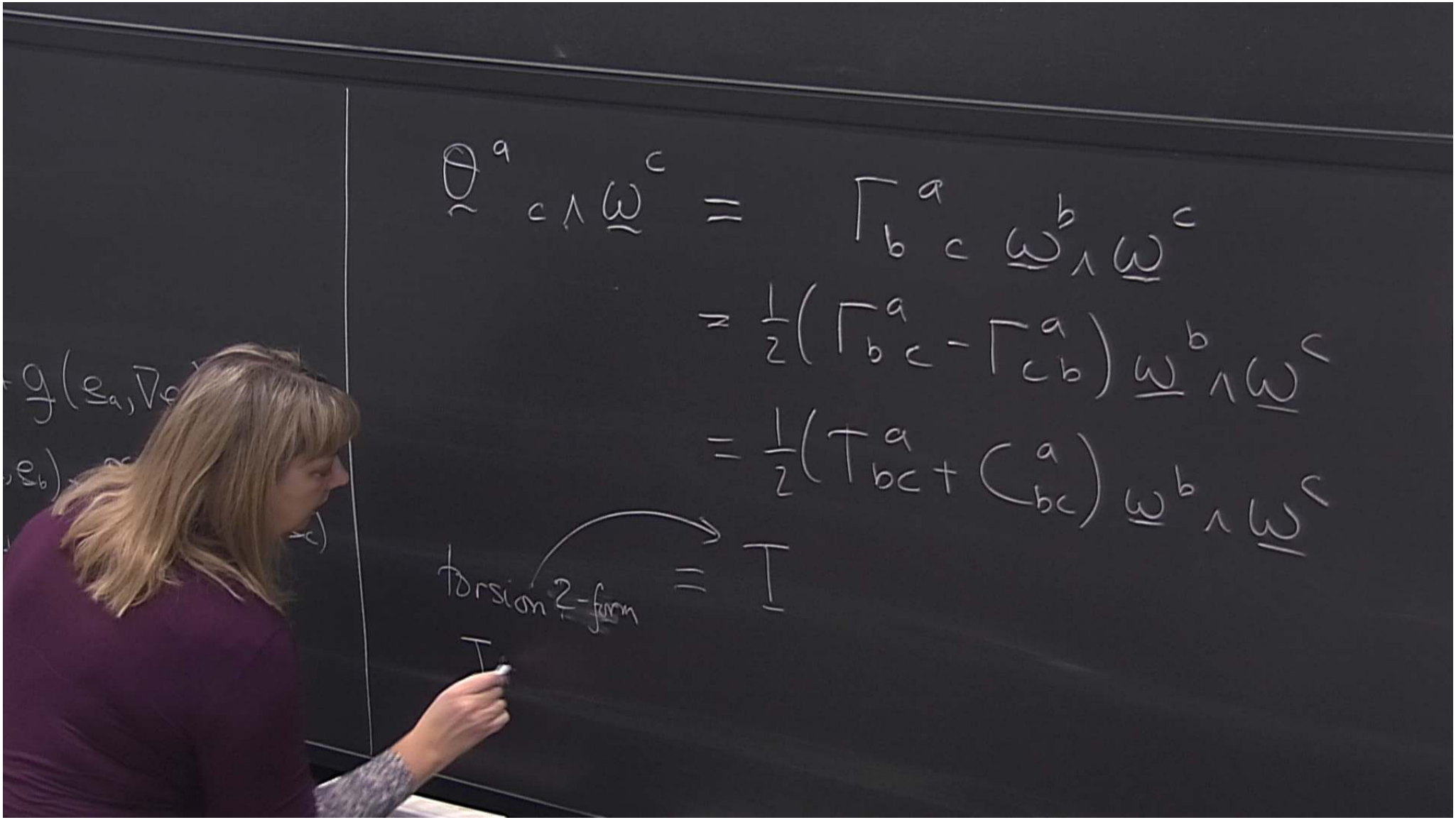

In GR, use torsion-free metric connection
$\nabla g = 0$
torsion is defined as
Check
$d(g_{ab}) = d(g(e_a, e_b))$
$= \nabla g(e_a, e_b)$
$= g(\nabla e_a, e_b) + g(e_a, \nabla e_b)$
$= \omega^c{}_a g(e_c, e_b) + \omega^c{}_b g(e_a, e_c)$
$= g_{cb}\omega^c{}_a + g_{ac}\omega^c{}_b$
$= \omega_{ba} + \omega_{ab}$

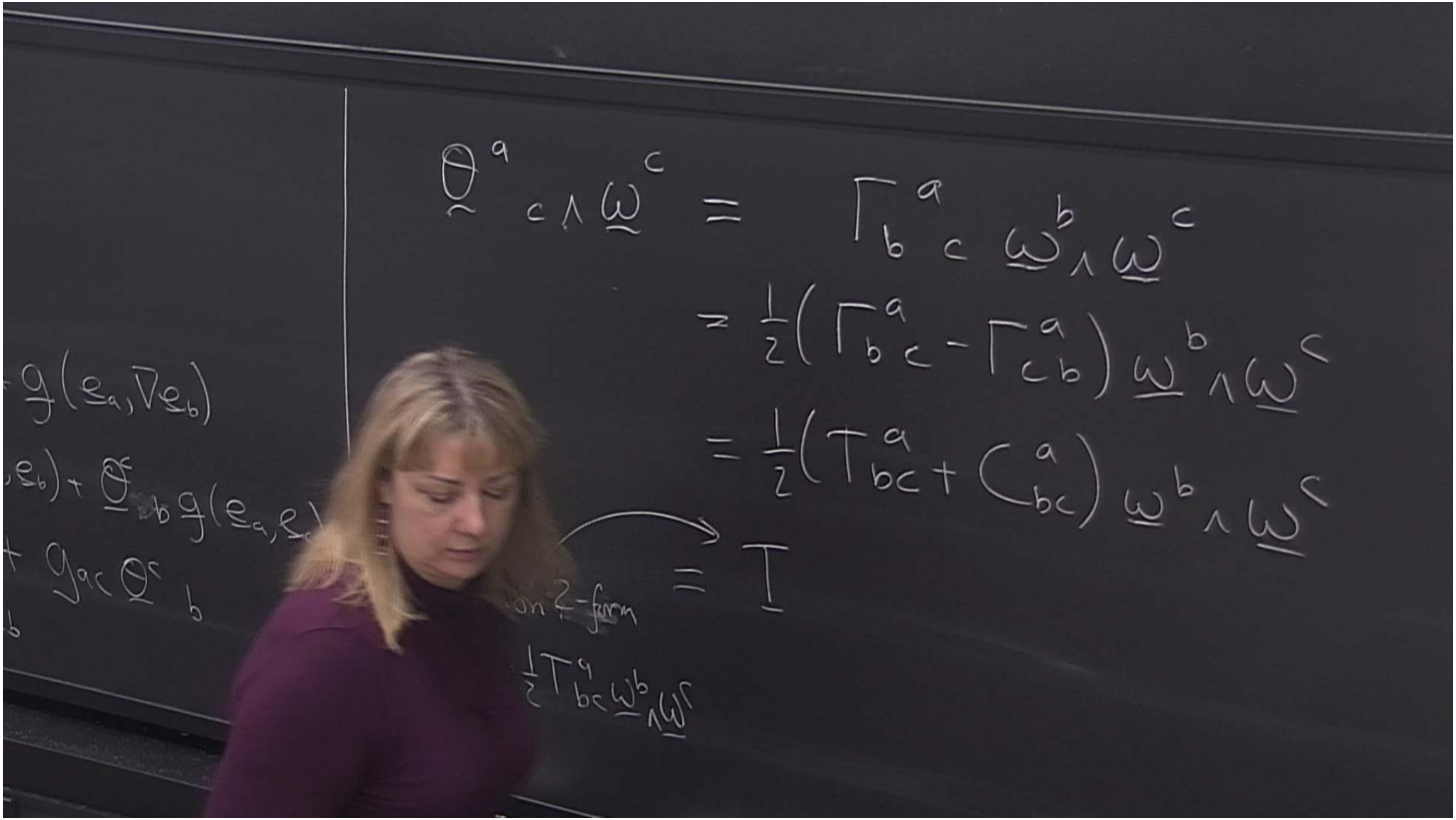

$\Theta^a{}_c \wedge \omega^c =$
$e_b)$
$g(e_a, e_c)$
b

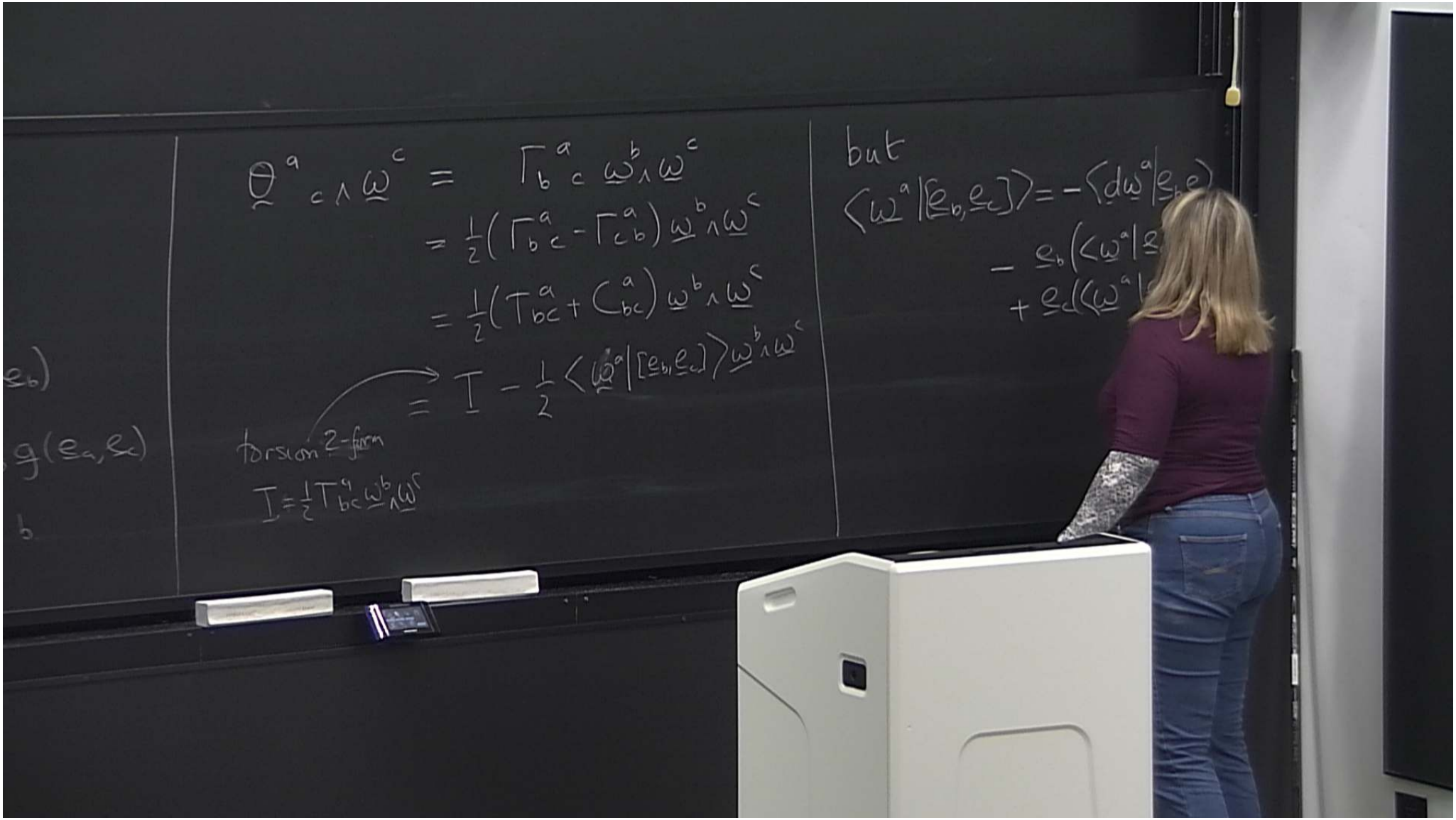
$\Theta^a{}_c \wedge \omega^c = \Gamma^a_{bc}\, \omega^b \wedge \omega^c$
$e_b)$
$g(e_a, e_c)$
b

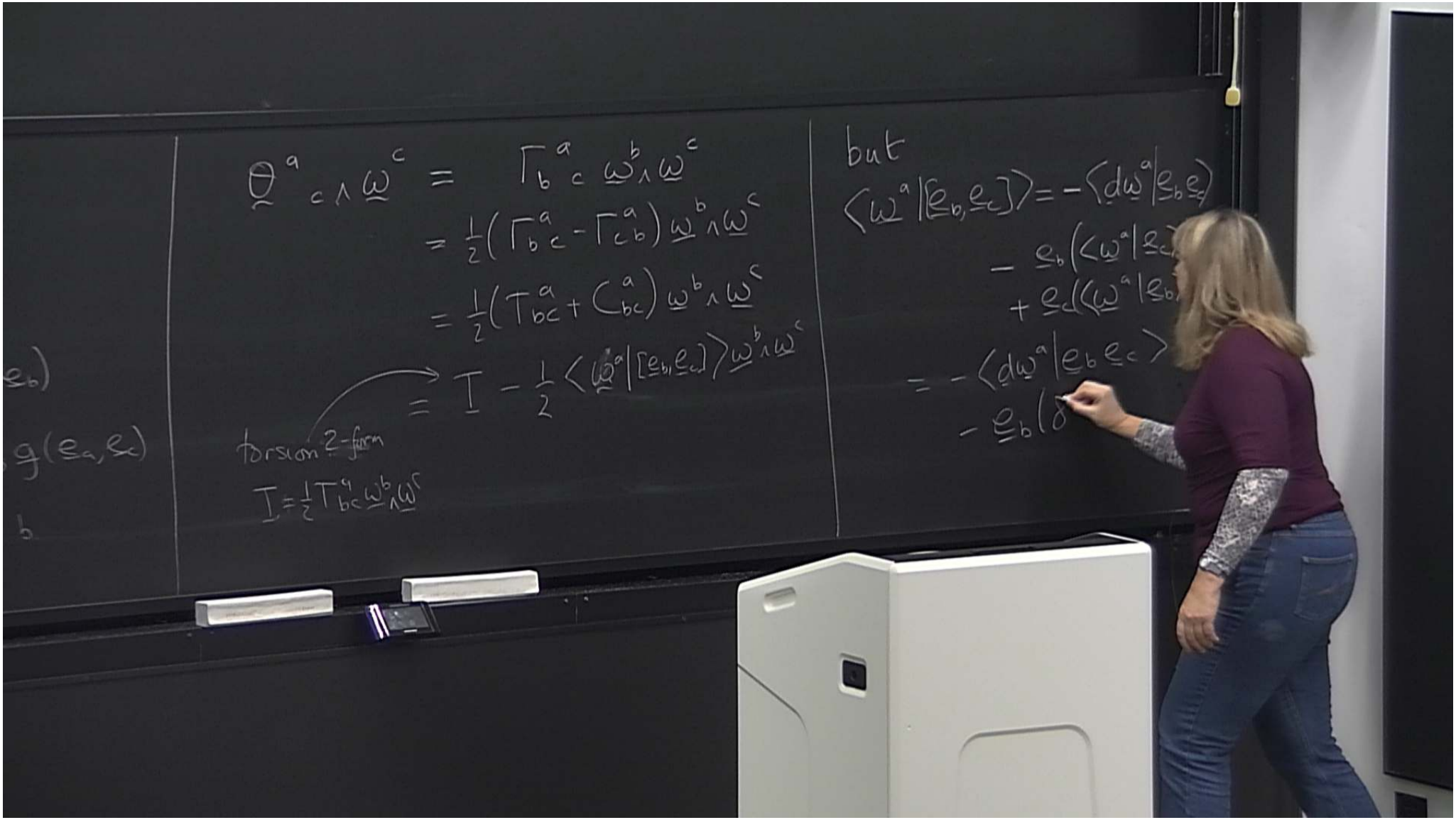

$\Theta^a{}_c \wedge \omega^c = \Gamma^a_{bc}\, \omega^b \wedge \omega^c$
$= \frac{1}{2}(\Gamma^a_{bc} - \Gamma^a$
$e_b)$
$g(e_a, e_c)$
b.


$\Theta^a{}_c \wedge \omega^c = \Gamma^a_{bc}\, \omega^b \wedge \omega^c$
$= \frac{1}{2}\left(\Gamma^a_{bc} - \Gamma^a_{cb}\right) \omega^b \wedge \omega^c$
$= \frac{1}{2}\left(T^a_{bc} + C^a_{bc}\right) \omega^b \wedge \omega^c$
$e_b)$
$g(e_a, e_c)$
b


$\Theta^a = \Gamma^a_{bc}\,\omega^b \wedge \omega^c$
$= \frac{1}{2}\left(\Gamma^a_{bc} - \Gamma^a_{cb}\right)\omega^b \wedge \omega^c$
$= \frac{1}{2}\left(T^a_{bc} + C^a_{bc}\right)\omega^b \wedge \omega^c$
torsion 2-form $= T$


$\Theta^a{}_c \wedge \omega^c = \Gamma^a_{bc}\, \omega^b \wedge \omega^c$
$= \frac{1}{2}\left(\Gamma^a_{bc} - \Gamma^a_{cb}\right) \omega^b \wedge \omega^c$
$= \frac{1}{2}\left(T^a_{bc} + C^a_{bc}\right) \omega^b \wedge \omega^c$
$= T$
$\frac{1}{2} T^a_{bc}\, \omega^b \wedge \omega^c$


$\Theta^a{}_c \wedge \omega^c = \Gamma^a_{bc}\, \omega^b \wedge \omega^c$
$= \frac{1}{2}(\Gamma^a_{bc} - \Gamma^a_{cb})\, \omega^b \wedge \omega^c$
$= \frac{1}{2}(T^a_{bc} + C^a_{bc})\, \omega^b \wedge \omega^c$
$= T - \frac{1}{2} \langle \omega^a | [e_b, e_c] \rangle \omega^b \wedge \omega^c$
torsion 2-form
$T = \frac{1}{2} T^a_{bc}\, \omega^b \wedge \omega^c$
but
$\langle \omega^a | [e_b, e_c] \rangle = -\langle d\omega^a | e_b, e_c \rangle$
$- e_b(\langle \omega^a |$
$+ e_c(\langle \omega^a |$
$e_b)$
$g(e_a, e_c)$
b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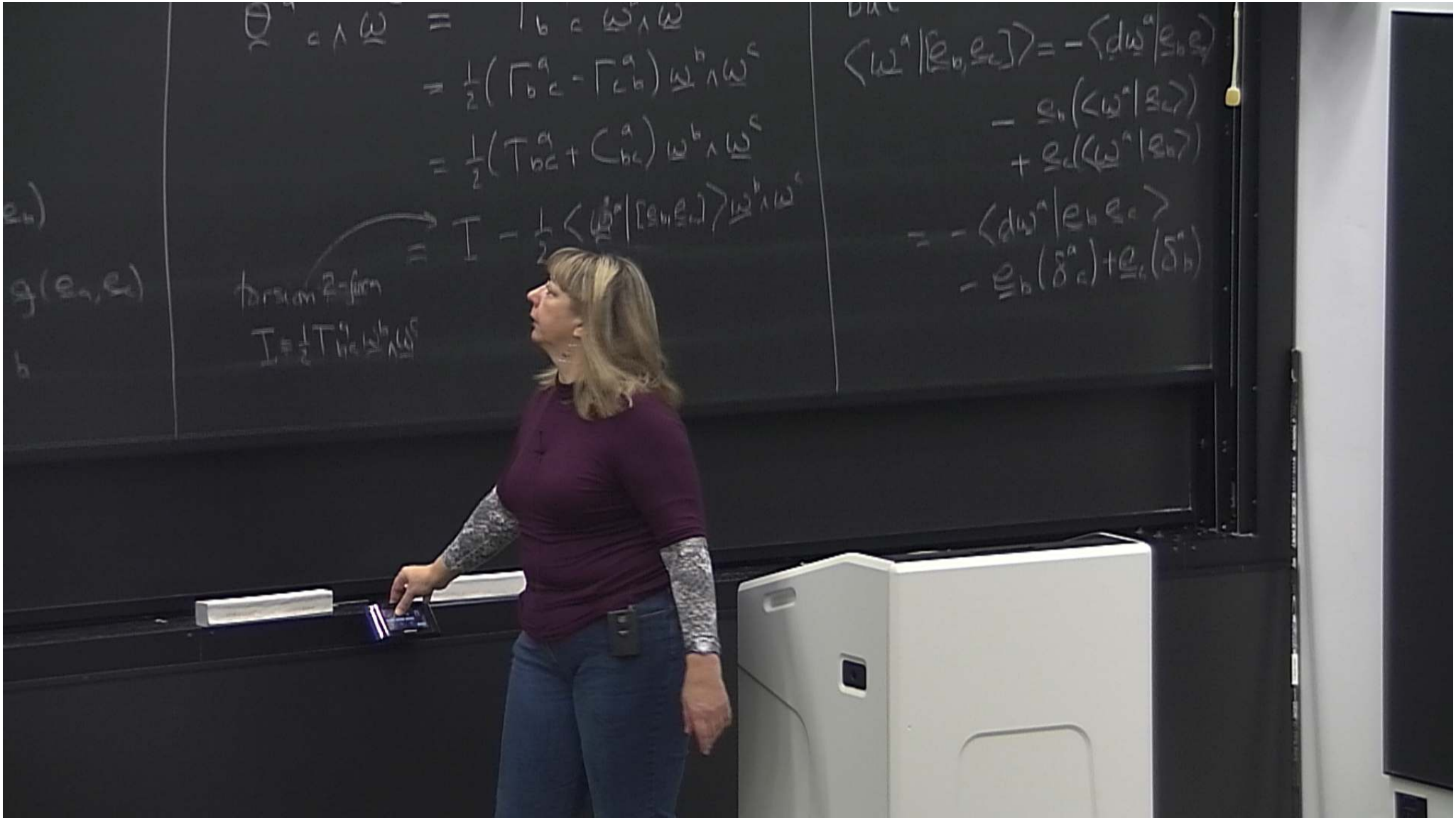
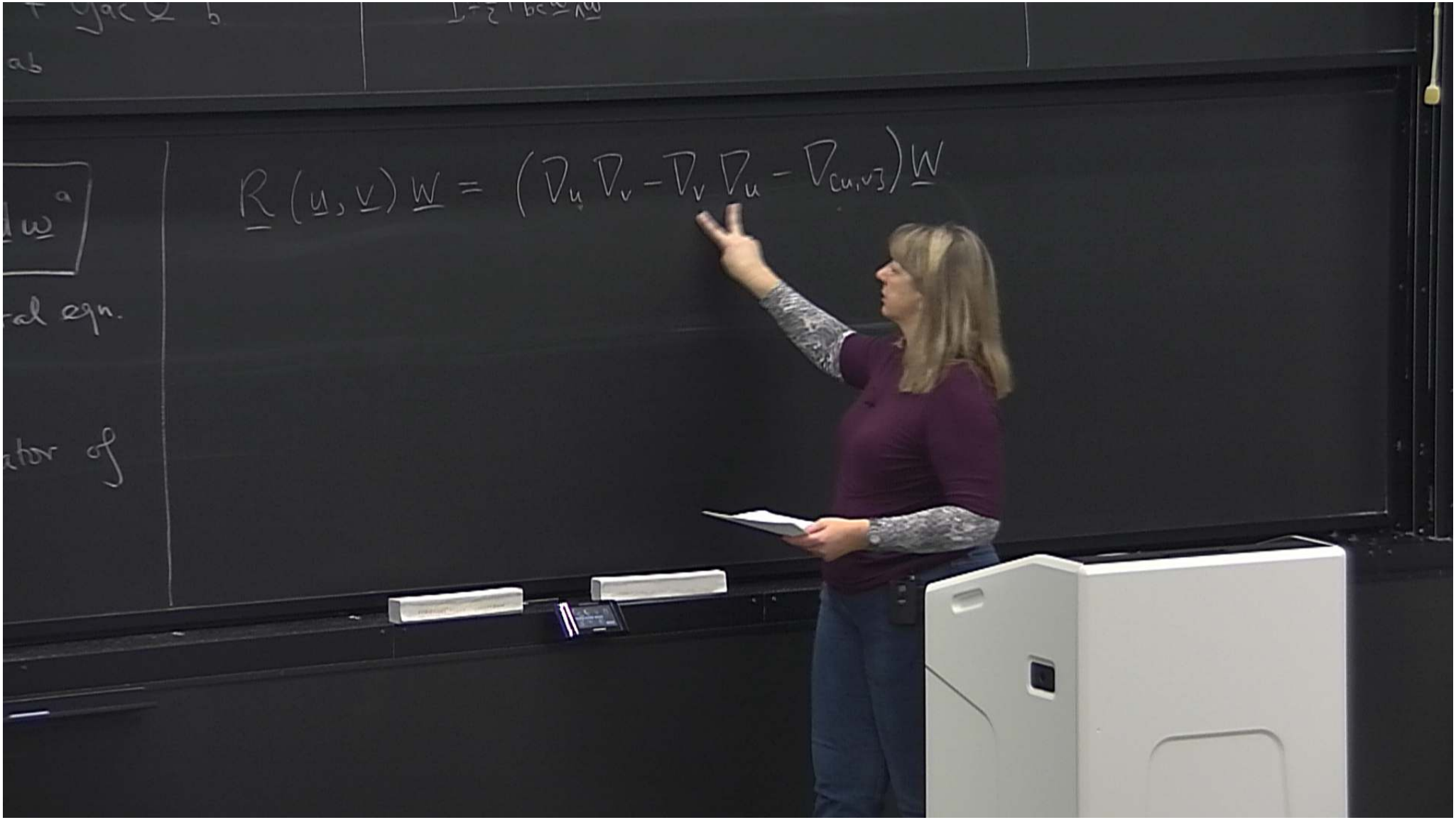
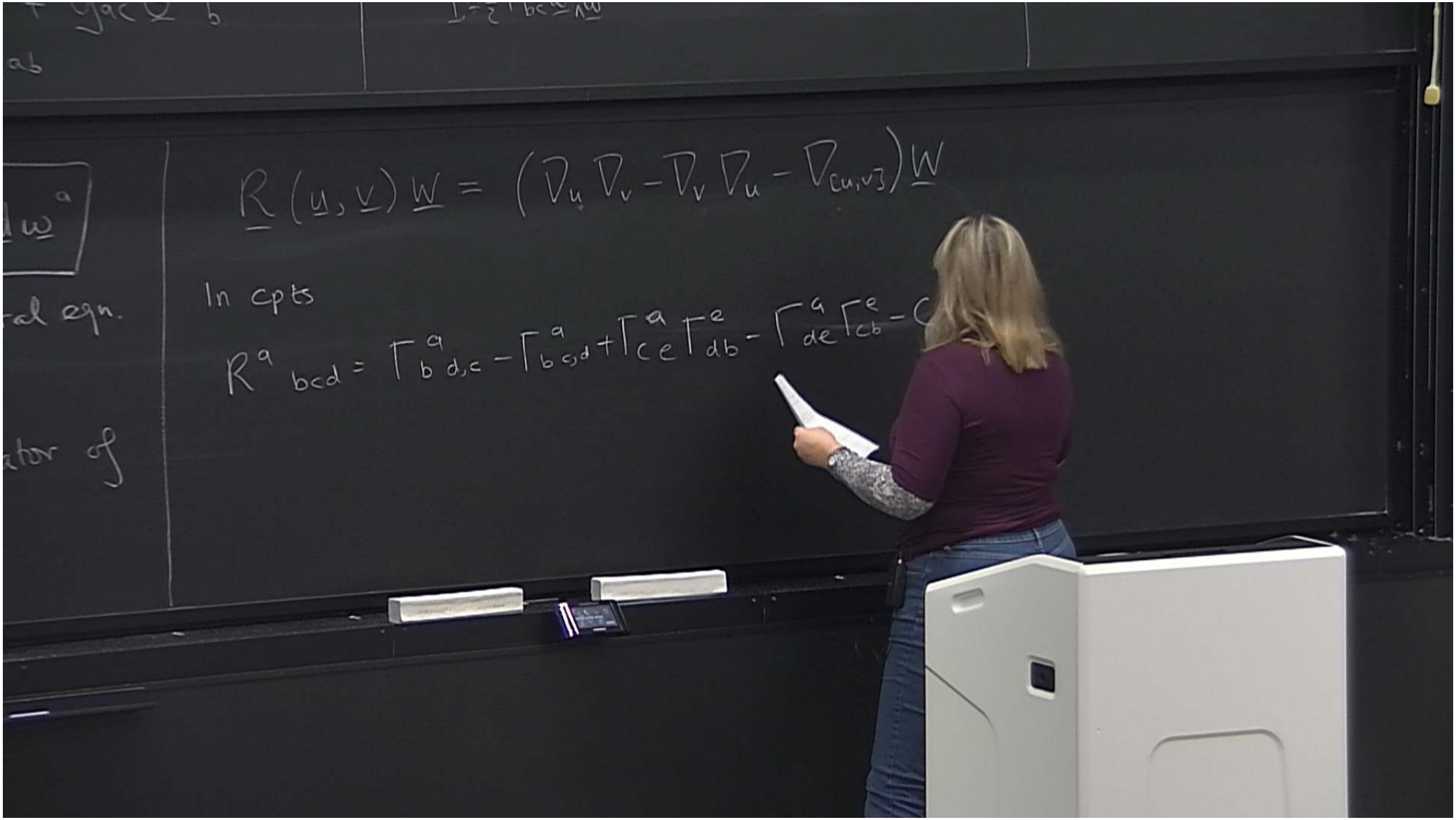

torsion 2-form

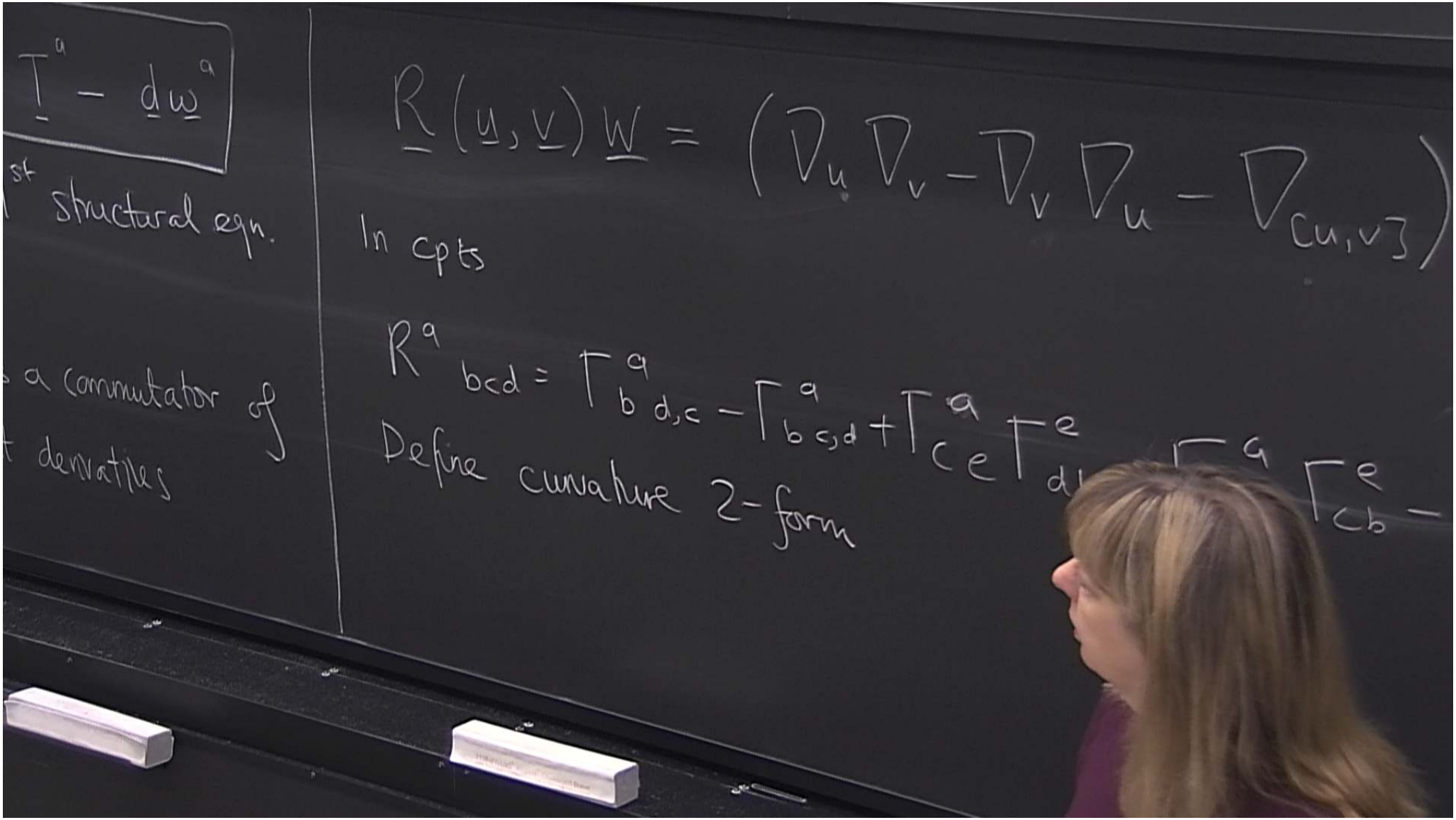

$\underline{R}(\underline{u},\underline{v})\underline{w} = (\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]})\underline{w}$

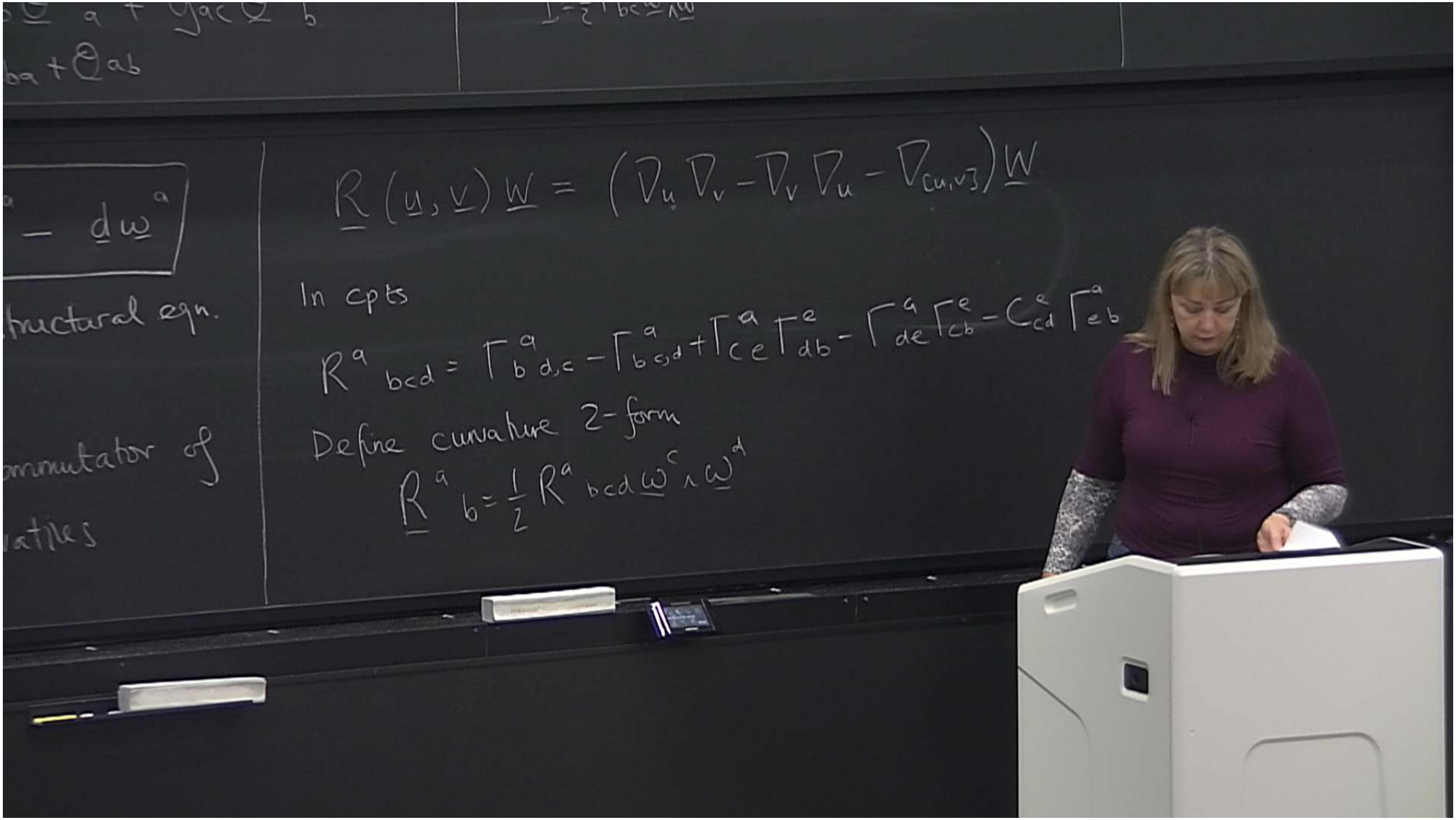

$\underline{R}(\underline{u},\underline{v})\underline{W} = (\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]})\underline{W}$
In cpts
$R^a{}_{bcd} = \Gamma^a_{b\,d,c} - \Gamma^a_{b\,c,d} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C$

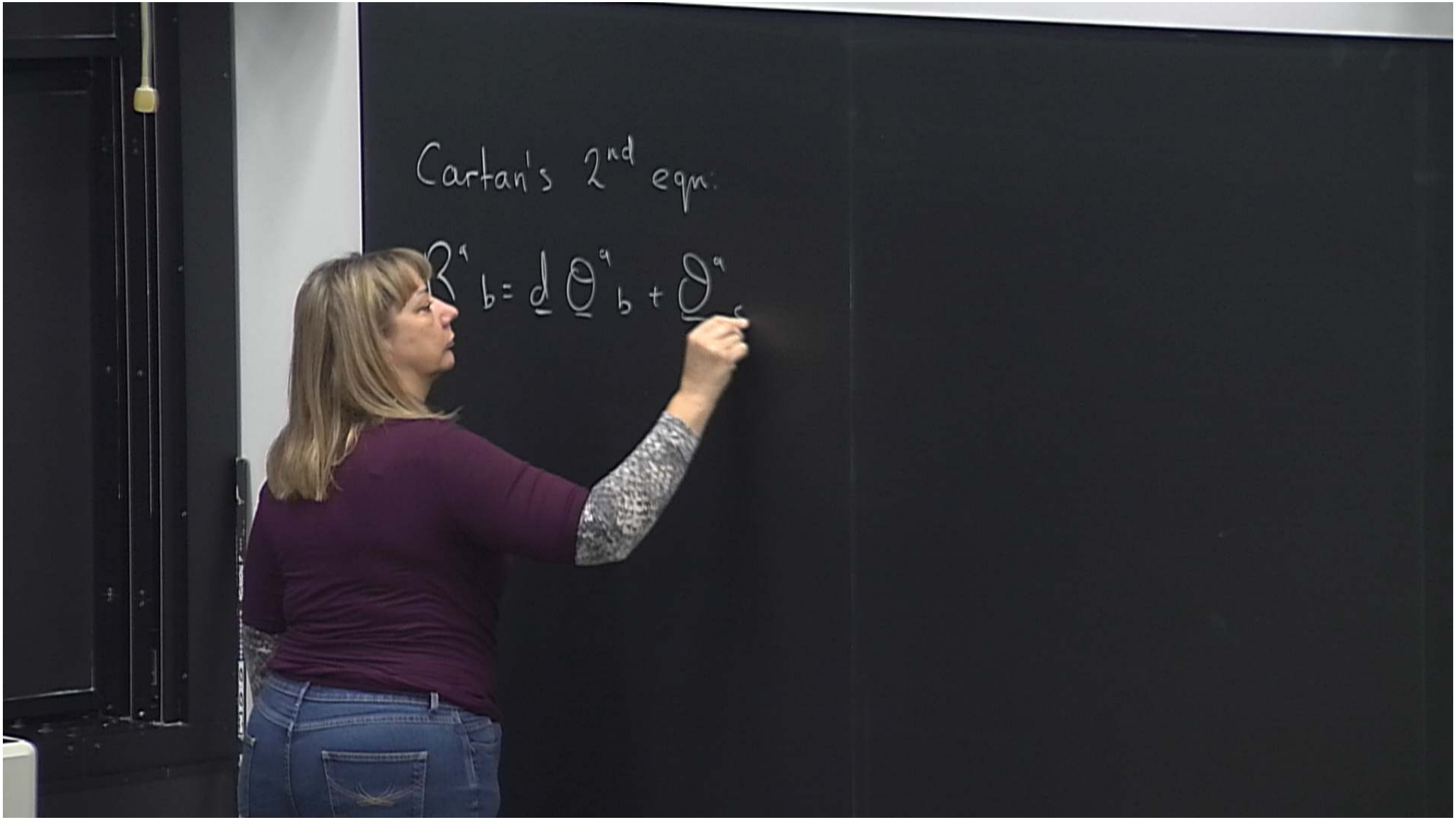
$$\underline{R}(\underline{u},\underline{v})\underline{W} = \left(\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}\right)\underline{W}$$

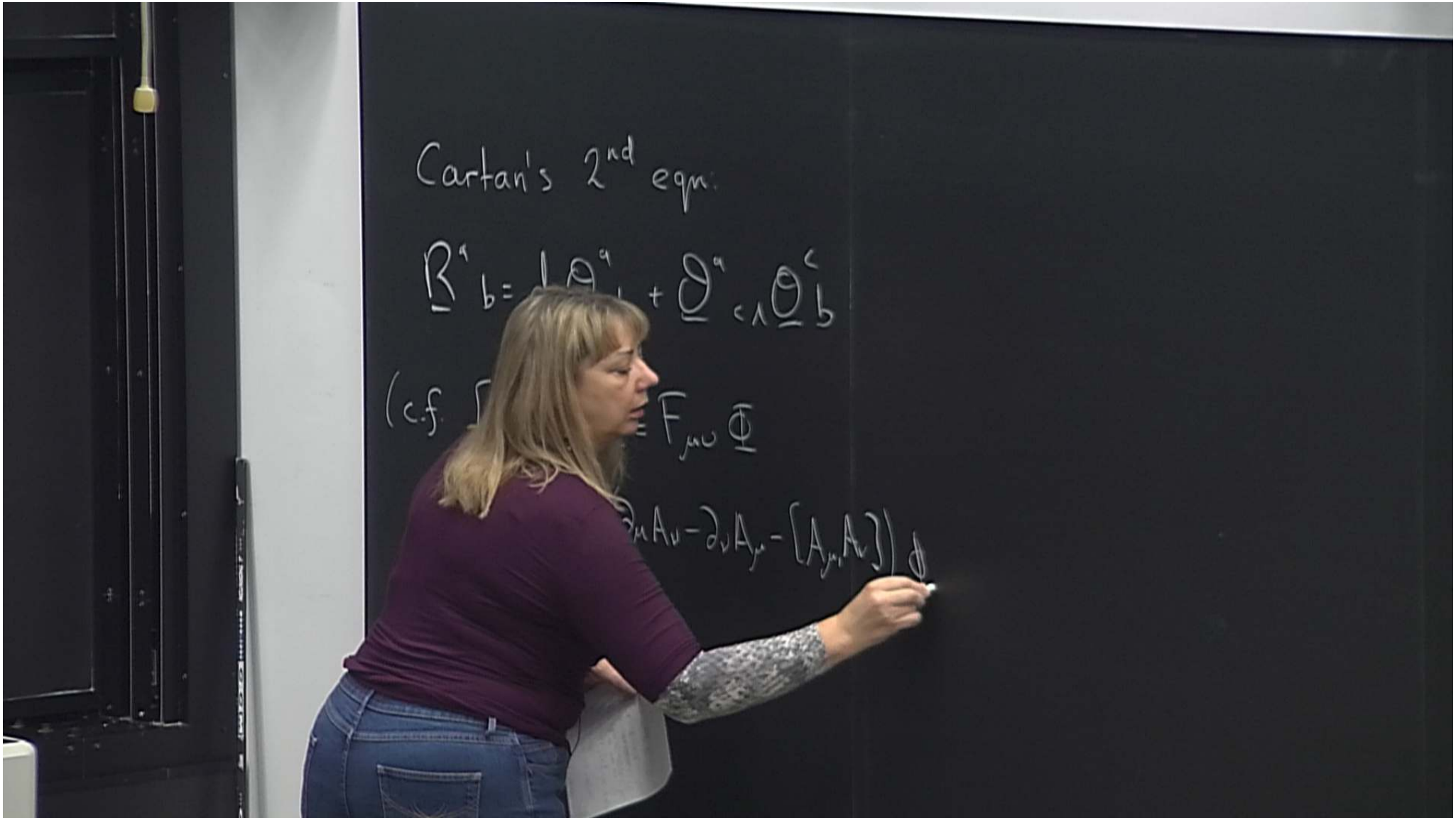
In cpts

$$R^a{}_{bcd} = \Gamma^a_{b\,d,c} - \Gamma^a_{b\,c,d} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}$$


$\underline{T}^a - \underline{d}\underline{\omega}^a$
st structural eqn.
a commutator of
derivatives
$\underline{R}(\underline{u},\underline{v})\underline{w} = (\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]})$
In cpts
$R^a{}_{bcd} = \Gamma^a_{b\,d,c} - \Gamma^a_{b\,c,d} + \Gamma^a_{ce}\Gamma^e_{d}$
Define curvature 2-form


$\underline{R}(\underline{u},\underline{v})\underline{W} = (\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]})\underline{W}$
In cpts
$R^a{}_{bcd} = \Gamma^a_{b\,d,c} - \Gamma^a_{b\,c,d} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}$
Define curvature 2-form
$\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\,\underline{\omega}^c \wedge \underline{\omega}^d$


Cartan's 2nd eqn: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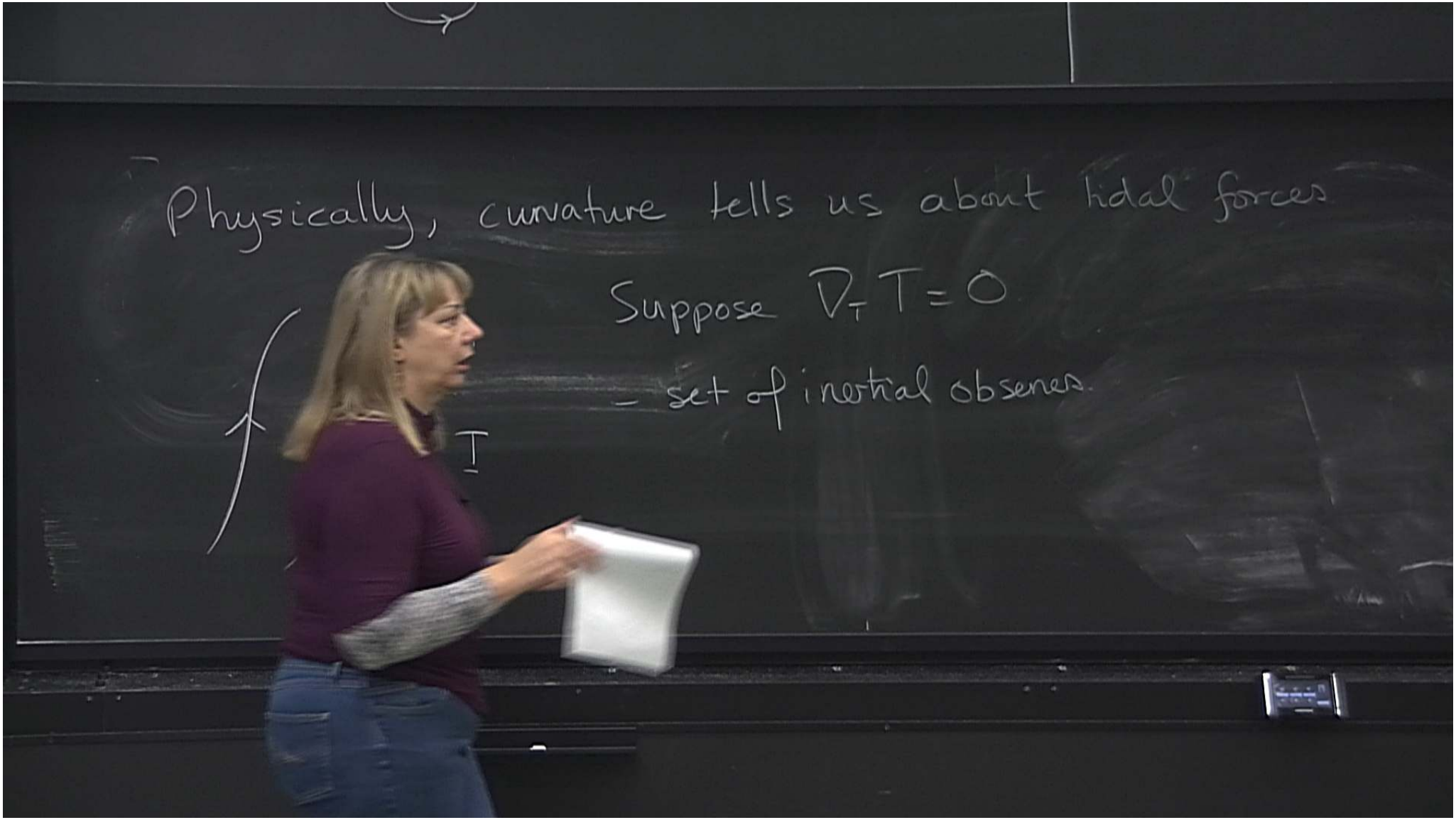
Physically, curvature tells us about tidal forces
Suppose $D_T T = 0$
– set of inertial observers.
T

Physically, curvature tells us about tidal forces

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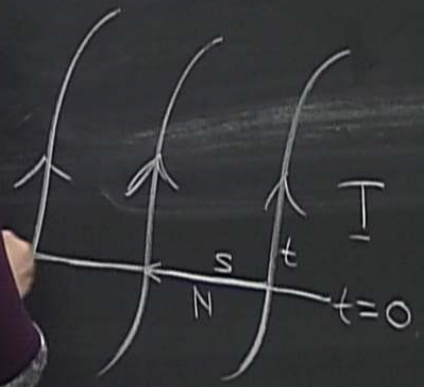

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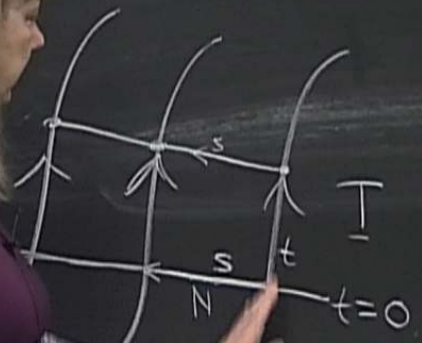
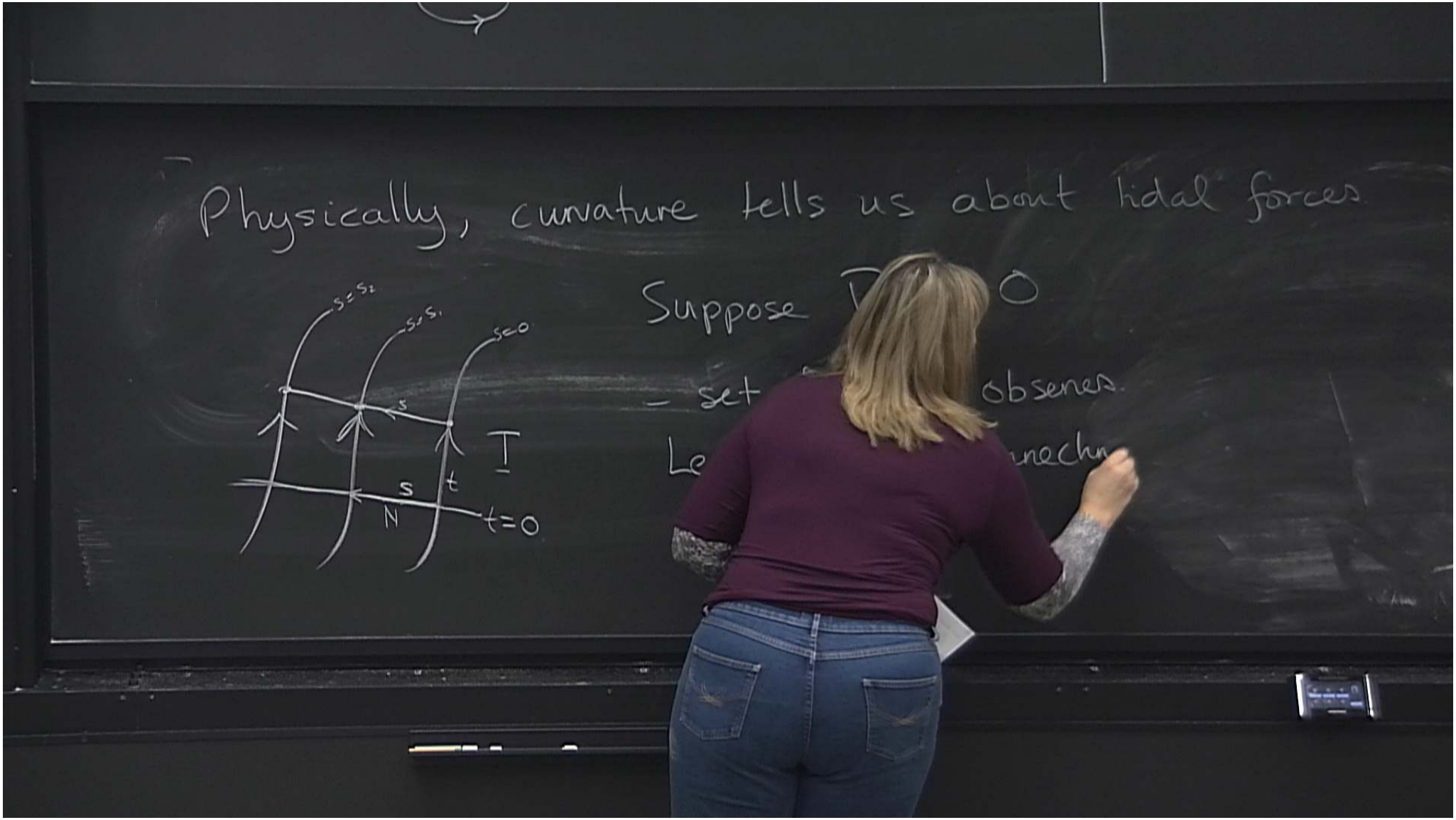
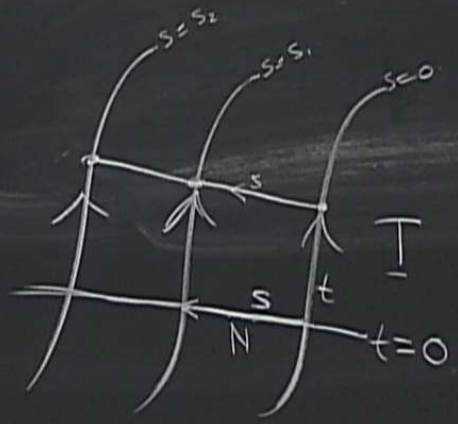



Physically, curvature tells us about tidal forces
Suppose
obsenes.
s=s₂
s=s₁
s=0
s
t
I
s
N
t=0

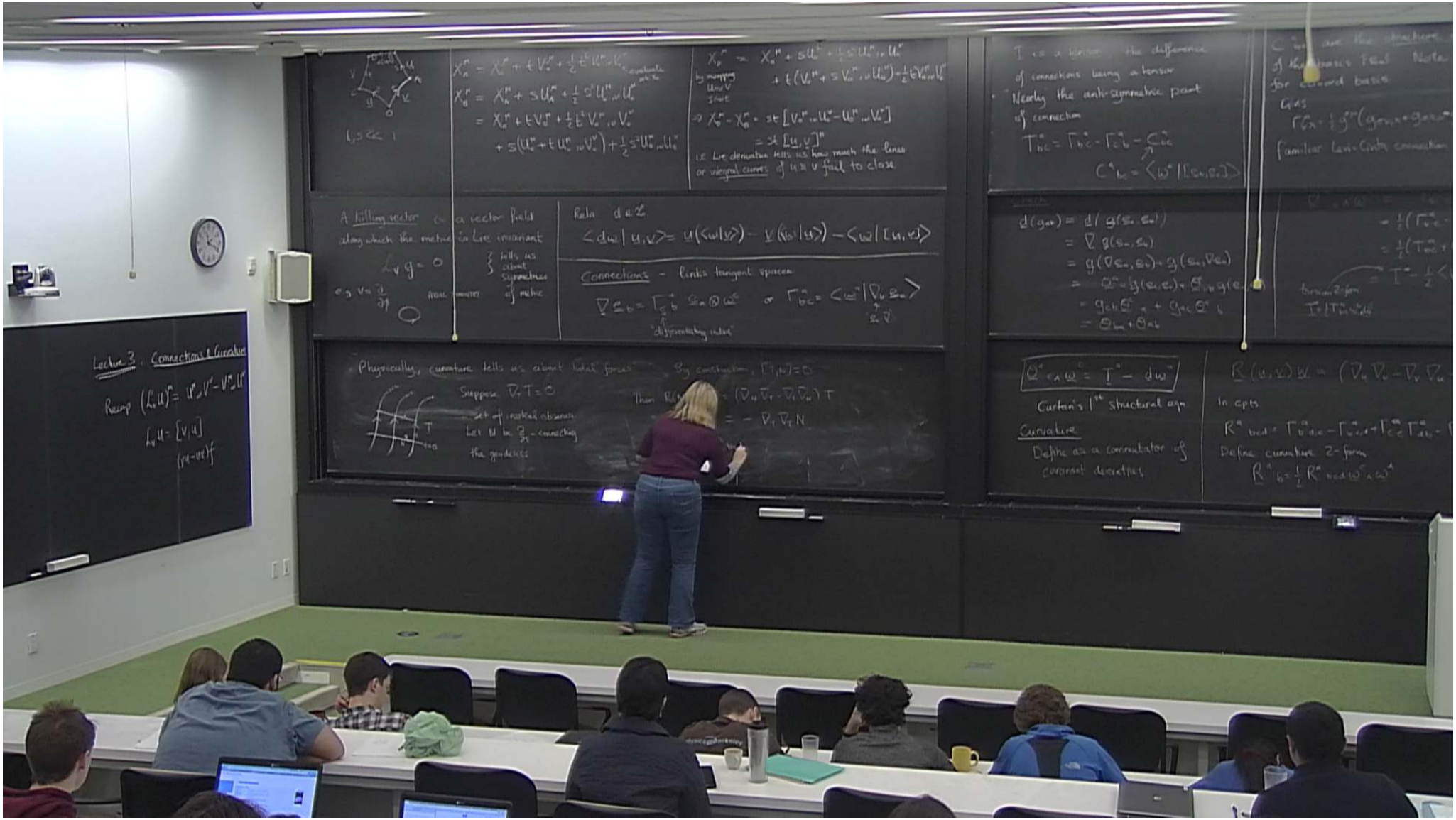
Physically, curvature tells us about tidal forces

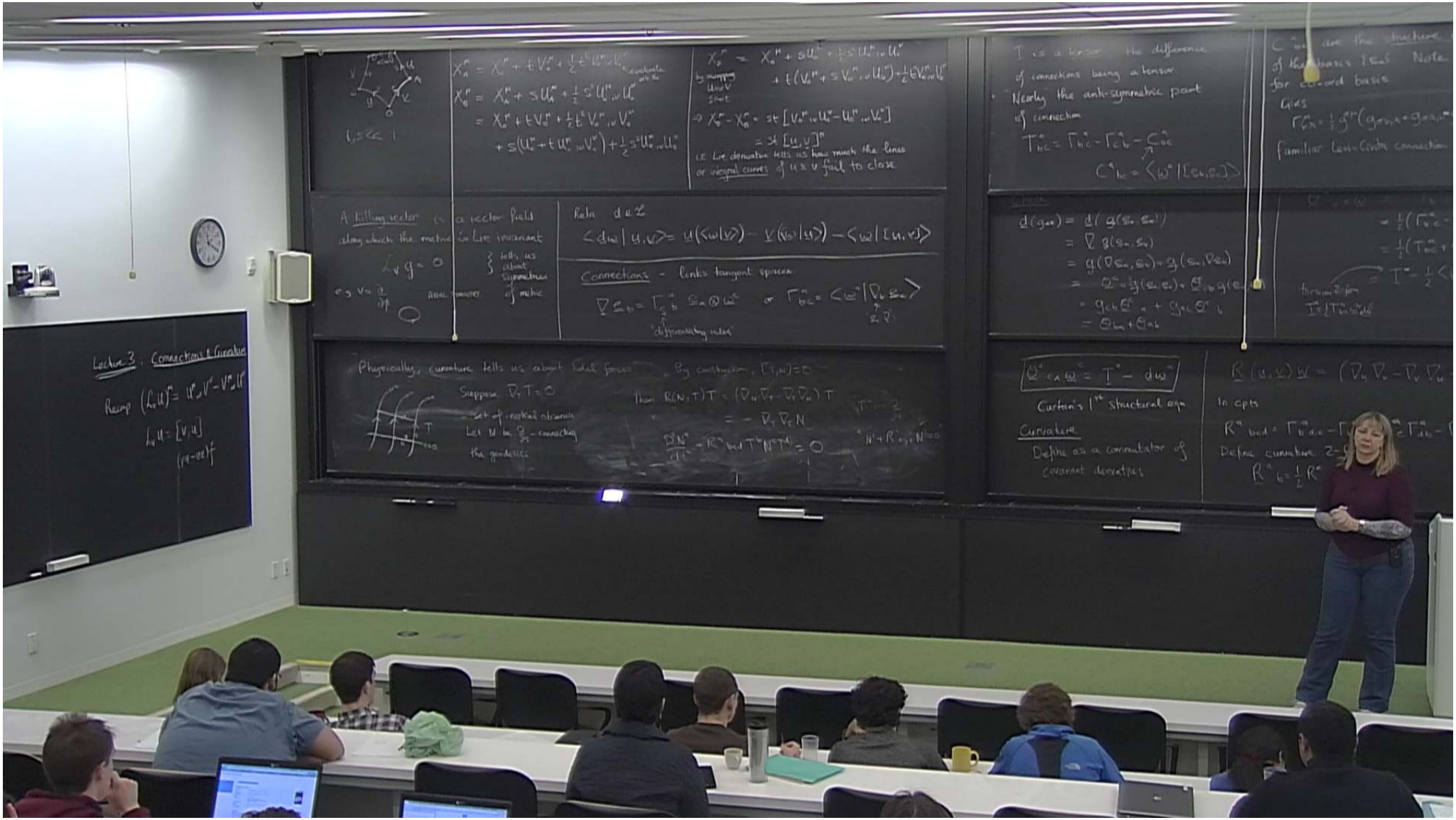


Suppose $D_T T = 0$

- set of inertial observers.

Let N be $\frac{\partial}{\partial s}$ - connecting the geodesics


Lecture 3 : Connections & Curvature
Connections - links tangent spaces
Physically, curvature tells us about tidal forces
Curvature
Define as a commutator of covariant derivatives


Lecture 3: Connections & Curvature
A Killing vector is a vector field along which the metric is Lie invariant
Connections - links tangent spaces
Physically, curvature tells us about tidal forces
Curvature
Define as a commutator of covariant derivatives
Cartan's 1st structural eqn
In cpts
Familiar Levi-Civita connection