Title: Introduction to Quantum Field Theory for Cosmology - Lecture 6

Date: Jan 23, 2014 04:00 PM

URL: http://pirsa.org/14010006

Abstract:

File Edit View Insert Actions Tools Help

Note Title

QFT for Cosmology, Achim Kempf, Winter 2014, Lecture 6

Recall:

- There are two basic mechanisms to increase the amplitudes of oscillators, i.e., also to excite a field's mode oscillators, i.e. to create particles:

  a.) A time-varying driving force $J(t)$

  b.) A time-varying spring "constant" $\omega(t)$

- We are presently considering case a) :

$$\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)$$

1 / 26

1

File Edit View Insert Actions Tools Help

$$\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)$$

with a temporary force: $J(t) = 0$ for all $t \notin [0,T]$

Examples: 1. Temporary emission from antenna, 2. Brief interaction (scattering) of particles.

- We defined a convenient variable $a(t)$,

$$a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)$$

so that: $\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right) - \frac{1}{\sqrt{2\omega}}J(t)\left(a^+(t) + a(t)\right)$

- This meant that we had to solve the simpler problem:

$$* \qquad i\dot{a}(t) = \omega\, a(t) - \frac{1}{\ \ }J(t) \qquad \text{(EOM)}$$

1 / 26

1

File Edit View Insert Actions Tools Help

- We defined a convenient variable $a(t)$,

$$a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)$$

so that: $\hat{H}(t) = \omega\left(a^{+}(t)a(t) + \frac{1}{2}\right) - \frac{1}{\sqrt{2\omega}}\,J(t)\left(a^{+}(t) + a(t)\right)$

- This meant that we had to solve the simpler problem:

* $i\dot{a}(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}}\,J(t)$ (EOM)

* $[a(t), a^{+}(t)] = 1$ for all t (CCR)

- We solved (EOM) with the arbitrary initial condition:

$a(0) = a_{in}$ (an operator on Hilbert space that we still have to choose.)

2 / 26

2

$$a(t) = a_{in} e^{-i\omega t} + 1\frac{i}{\sqrt{2\omega}}\int_0^t J(t') e^{i\omega(t'-t)} dt' \quad (Sol)$$

Proposition:

We can solve (CCR) <u>for all times</u> t by finding an operator a_{in} which obeys:

$$[a_{in}, a_{in}^+] = 1$$

Proof:

Assume $[a_{in}, a_{in}^+] = 1$. Then:

File Edit View Insert Actions Tools Help

finding an operator a_{in} which obeys:

$$[a_{in}, a^+_{in}] = 1$$

Proof:

Assume $[a_{in}, a^+_{in}] = 1$. Then:

$$[a(t), a^+(t)] = [a_{in} e^{-i\omega t} + \underbrace{1 \frac{i}{\sqrt{2\omega}} \int_0^t \ldots dt'}_{\text{number}}, a^+_{in} e^{+i\omega t} - \underbrace{1 \frac{i}{\sqrt{2\omega}} \int_0^t \ldots dt'}_{\text{number}}]$$

$$= \underbrace{[a_{in}, a^+_{in}]}_{=1} e^{-i\omega t} e^{i\omega t}$$

$$= 1 \quad ✓$$

3 / 26

3

File Edit View Insert Actions Tools Help

Structure of the solution

- Recall that the solution (Sol) can also be written as:

$$a(t) = \left(a_{in} + \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega t'} dt' \right) e^{-i\omega t}$$

- Since the force vanishes, $J(t) = 0$, when $t \notin [0,T]$ we noticed that:

$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } 0 \leq t \leq T \\ (a_{in} + J_0)\, e^{-i\omega t} & \text{for } T < t \end{cases}$$

4 / 26

4

File Edit View Insert Actions Tools Help

- Recall that the solution (Sol) can also be written as:

$$a(t) = \left(a_{in} + 1\frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega t'} dt' \right) e^{-i\omega t}$$

- Since the force vanishes, $J(t) = 0$, when $t \notin [0,T]$ we noticed that:

$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t<0 \\ \text{see above} & \text{for } 0 \le t \le T \\ (a_{in} + J_0)\, e^{-i\omega t} & \text{for } T<t \end{cases}$$

with the definition:

$$J_0 := 1\frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$$

File Edit View Insert Actions Tools Help

Strategy:

- We notice that the system is a simple undriven harmonic oscillator in the period before the force acts and again in the period after the force finished acting.
- We focus attention on these two periods.
- We define a_{in}, a_{out}:

$$\boxed{a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t<0 \\ a_{out}\, e^{-i\omega t} & \text{for } t>T \end{cases}}$$

with the definition: $a_{out} := a_{in} + J_0$

5 / 26

5

File Edit View Insert Actions Tools Help

$$a(t) = \begin{cases} a_{in}\, e & \text{for } t<0 \\ a_{out}\, e^{-i\omega t} & \text{for } t>T \end{cases}$$

with the definition: $a_{out} := a_{in} + J_0$

- Exercise: Verify that $[a_{out}, a_{out}^+] = 1$ as well.

  Hint: use that $[a_{in}, a_{in}^+] = 1$

The initial period, $t<0$:

- The dynamical variables:

We have $a(t) = a_{in}\, e^{-i\omega t}$ and therefore we also

5

5 / 26

File Edit View Insert Actions Tools Help

The initial period, $t<0$:

The dynamical variables:

We have $a(t) = a_{in}\, e^{-i\omega t}$ and therefore we also have the dynamics of all other variables, such as:

$$* \quad \hat{q}(t) = \frac{1}{\sqrt{2\omega}}\left(a^{+}_{in}\, e^{i\omega t} + a_{in}\, e^{-i\omega t}\right)$$

$$* \quad \hat{p}(t) = i\sqrt{\frac{\omega}{2}}\left(a^{+}_{in}\, e^{i\omega t} - a_{in}\, e^{-i\omega t}\right)$$

Exercise: verify

$$* \quad \hat{H}(t) = \omega\left(a^{+}(t)\, a(t) + \frac{1}{2}\right)$$

$$\left(a^{+}_{in}\, e^{i\omega t} \ \dots \ e^{-i\omega t} \ \dots\right)$$

6 / 26

6

File Edit View Insert Actions Tools Help

We have $a(t) = a_{in}\, e^{-i\omega t}$ and therefore we also have the dynamics of all other variables, such as:

$$* \quad \hat{q}(t) = \frac{1}{\sqrt{2\omega}}\left(a^+_{in} e^{i\omega t} + a_{in} e^{-i\omega t}\right)$$

$$* \quad \hat{p}(t) = i\sqrt{\frac{\omega}{2}}\left(a^+_{in} e^{i\omega t} - a_{in} e^{-i\omega t}\right)$$

Exercise: verify

$$* \quad \hat{H}(t) = \omega\left(a^+(t)\, a(t) + \frac{1}{2}\right)$$

$$= \omega\left(a^+_{in} e^{i\omega t} a_{in} e^{-i\omega t} + \frac{1}{2}\right)$$

$$= \omega\left(a^+_{in} a_{in} + \frac{1}{2}\right)$$

is constant in time!

6 / 26

6

File Edit View Insert Actions Tools Help

The Hilbert space of states:

* As always, we can write arbitrary Hilbert space vectors as linear combinations of an arbitrary set of basis vectors.

* We could use, for example, the eigenbasis of $\hat{q}(t)$ (or the eigenbasis of $\hat{p}(t)$).

  But: In the Heisenberg picture, this would be inconvenient because $\hat{q}(t)$ has a different eigenbasis for each t.

* However, $\hat{H}$ is time independent (for $t<0$)

7

7 / 26

File Edit View Insert Actions Tools Help

(or the eigenbasis of $\hat{p}(t)$).

But: In the Heisenberg picture, this would be inconvenient because $\hat{q}(t)$ has a different eigenbasis for each t.

* However, $\hat{H}$ is time independent (for $t<0$).
~> Let us construct and use its eigenbasis:

The eigenbasis of $\hat{H}$ for $t<0$:

* We have

7 / 26

7

File Edit View Insert Actions Tools Help

The eigenbasis of $\hat{H}$ for $t<0$:

* We have

$$\hat{H}_{t<0} = \omega\left(a^+_{in} a_{in} + \frac{1}{2}\right)$$

with:

$$[a_{in}, a^+_{in}] = 1 \qquad \text{(CCR)}$$

* Assume there exists a vector, denoted say $|0_{in}\rangle$, which obeys:

$$a_{in}|0_{in}\rangle = 0$$

the Hilbert space vector with zero length

Recall: the energy eigenvalues of any harmonic oscillator is $E_n = \hbar\omega(n+\frac{1}{2})$ i.e. we have here $E_0 = \hbar\omega\frac{1}{2}$ (with $\hbar = 1$).

* Then it is eigenvector of $H_{t<0}$:

$$\hat{H}_{t<0}|0_{in}\rangle = \omega\left(a^+_{in}a_{in} + \frac{1}{2}\right)|0_{in}\rangle = \frac{1}{2}\omega|0_{in}\rangle$$

8

8 / 26

File Edit View Insert Actions Tools Help

$$\hat{H}_{t<0} = \omega\left(a^+_{in} a_{in} + \frac{1}{2}\right)$$

with:

$$[a_{in}, a^+_{in}] = 1 \qquad \text{(CCR)}$$

* Assume there exists a vector, denoted say $|0_{in}\rangle$, which obeys:

$$a_{in}|0_{in}\rangle = 0$$

← the Hilbert space vector with zero length

* Then it is eigenvector of $H_{t<0}$:

Recall: the energy eigenvalues of any harmonic oscillator is $E_n = \hbar\omega(n+\frac{1}{2})$ i.e. we have here $E_0 = \hbar\omega\frac{1}{2}$ (with $\hbar = 1$).

$$\hat{H}_{t<0}|0_{in}\rangle = \omega(a^+_{in}a_{in} + \tfrac{1}{2})|0_{in}\rangle = \tfrac{1}{2}\omega|0_{in}\rangle$$

⇒ We recognize $|0\rangle$: it is the lowest energy eigenvector of $\hat{H}$ (and thus it indeed exists)

8 / 26

8

File Edit View Insert Actions Tools Help

* Consider now the state $|1_{in}\rangle := a^+_{in}\,|0_{in}\rangle$:

$$\hat{H}_{t<0}|1_{in}\rangle = \hat{H}_{t<0}\,a^+_{in}\,|0_{in}\rangle = \omega\left(a^+_{in}a_{in} + \tfrac{1}{2}\right)a^+_{in}\,|0_{in}\rangle$$

$$= \left(\omega\, a^+_{in}\left(a^+_{in}\cancel{a_{in}} + 1\right) + \frac{\omega}{2}\,a^+_{in}\right)|0_{in}\rangle$$

$$= \omega\,\frac{3}{2}\,a^+|0_{in}\rangle$$

$$= \frac{3}{2}\,\omega\,|1_{in}\rangle$$

$\Rightarrow$ The state $|1_{in}\rangle$ is eigenstate of $\hat{H}$ with eigenvalue $\frac{3}{2}\omega$. So it must be the 1st excited state.

* Is the vector $|1_{in}\rangle$ normalized?

9

9 / 26

File Edit View Insert Actions Tools Help

* Is the vector $|1_{in}\rangle$ normalized?

$$\langle 1_{in}|1_{in}\rangle = \langle 0_{in}|a_{in}\, a^+_{in}|0_{in}\rangle = \langle 0_{in}|a^+_{in}\, a_{in} + 1|0_{in}\rangle = \langle 0_{in}|0_{in}\rangle = 1$$

✓

* **Proposition:**

The set of vectors $\{|n_{in}\rangle\}_{n=0}^{\infty}$ defined through

$$|n_{in}\rangle := \frac{1}{\sqrt{n!}}\left(a^+_{in}\right)^n |0_{in}\rangle$$

is orthonormal, i.e., $\langle n|n'\rangle = \delta_{n,n'}$. Exercise: verify

File Edit View Insert Actions Tools Help

* Proposition:

The set of vectors $\{|n_{:}\rangle\}_{n=0}^{\infty}$ defined through

$$|n_{:}\rangle := \frac{1}{\sqrt{n!}} (a^{+}_{:})^{n} |0_{:}\rangle$$

is orthonormal, i.e., $\langle n|n'\rangle = \delta_{n,n'}$. Exercise: verify

* Proposition:

The vectors $|n_{:}\rangle$ are eigenvectors of $\hat{H}_{t<0}$:

$$\hat{H}_{t<0} |n_{:}\rangle = E_n |n_{:}\rangle$$

with Exercise: verify

10 / 26

10

File Edit View Insert Actions Tools Help

* Proposition:

The vectors $|n_{in}\rangle$ are eigenvectors of $\hat{H}_{t<0}$:

$$\hat{H}_{t<0}|n_{in}\rangle = E_n |n_{in}\rangle$$

with

$$E_n = \omega\left(n+\frac{1}{2}\right)$$

Exercise: verify

* Proposition: $\{|n_{in}\rangle\}$ is complete eigenbasis of $\hat{H}$.

⇝ Summary re choice of basis for $t<0$:

o The Hamiltonian $\hat{H}(t)$ is constant for $t<0$.

10 / 26

10

File Edit View Insert Actions Tools Help

⇝ Summary re choice of basis for $t<0$:

- The Hamiltonian $\hat{H}(t)$ is constant for $t<0$.
- Thus it has one eigenbasis for all $t<0$, namely $\{|n_{in}\rangle\}$.
- We may expand every arbitrary vector $|\psi\rangle$ of the Hilbert space, $\mathcal{H}$, in this basis:

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n_{in}\rangle$$

- E.g., the state of our quantum system could be:

$$|\psi\rangle = |5_{in}\rangle$$

11 / 26

11

File Edit View Insert Actions Tools Help

Hilbert space, $\mathcal{H}$, in this basis:

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n_{in}\rangle$$

- E.g., the state of our quantum system could be:

$$|\psi\rangle = |5_{in}\rangle$$

- The system always stays in state $|\psi\rangle = |5_{in}\rangle$.

Recall: - But $|\psi\rangle = |5_{in}\rangle$ generally ceases to be eigenvector of $\hat{H}(t)$ for $t>0$!

The period $t > T$: (after the force ceased to act)

- Once the driving force acts, $\hat{H}(t)$ starts to change.

11 / 26

11

File Edit View Insert Actions Tools Help

The period $t > T$: (after the force ceased to act)

- Once the driving force acts, $\hat{H}(t)$ starts to change.
- **But:** After the force finished, $t > T$, the Hamiltonian simply reads

$$\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \tfrac{1}{2}\right) - \frac{a^+(t)+a(t)}{\sqrt{2\omega}}\,\cancel{J(t)} \quad \text{for } t > T$$

$$\cancel{J(t)} = 0$$

and from above, therefore:

$$\hat{H}(t) = \omega\left(a^+_{out}\,\cancel{e^{i\omega t}}\,a_{out}\,\cancel{e^{-i\omega t}} + \tfrac{1}{2}\right) \quad \text{with} \quad a_{out} = a_{in} + J_0$$

12 / 26

12

File Edit View Insert Actions Tools Help

$$H(t) = \omega\left(a^+(t)\,a(t) + \tfrac{1}{2}\right) - \frac{a^+(t)+a(t)}{\sqrt{2\omega}}\,\underset{\substack{\shortparallel\\0}}{\cancel{J(t)}} \quad \text{for } t>T$$

and from above, therefore:

$$\hat{H}(t) = \omega\left(a^+_{out}\,\cancel{e^{i\omega t}}\,a_{out}\,\cancel{e^{-i\omega t}} + \tfrac{1}{2}\right) \quad \text{with } a_{out} = a_{in} + J_0$$

$$\Rightarrow \quad \hat{H}_{t>T} = \omega\left(a^+_{out}\,a_{out} + \tfrac{1}{2}\right) \Rightarrow \hat{H} \text{ is then constant again!}$$

Note: we can construct a basis from $a_{out}|0_{out}\rangle = 0$ etc.

Compare $t<0$ to $t>T$:

QFT:

File Edit View Insert Actions Tools Help

Compare $t<0$ to $t>T$:

QFT:

A. Motion: $\bar{q}(t)$ (large $\bar{q}$ means large $\bar{\phi}_k$ means large waves)

B. Resonance: best $J(t)$? (consider e.g. antenna)

C. Energy expectation: $\bar{E}(t)$ (large $\bar{E}$ means large $\bar{E}_k$ means energy in mode k)

D. Energy eigenstates: $\{|E_n(t)\rangle\}$ (particle creation)

We will consider the example where the system starts out in the lowest energy state (the vacuum).

13 / 26

13

File Edit View Insert Actions Tools Help

We will consider the example where the system starts out in the lowest energy state (the vacuum):

$$|\psi\rangle = |0_{in}\rangle$$

A. Motion $\bar{q}(t)$:

$$\bar{q}(t) = \langle\psi|\hat{q}(t)|\psi\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right)|0_{in}\rangle$$

13 / 26

13

File Edit View Insert Actions Tools Help

A. Motion $\bar{q}(t)$:

$$\bar{q}(t) = \langle \psi \,|\, \hat{q}(t) \,|\, \psi \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+(t) + a(t) \right) | 0_{in} \rangle$$

* For $t < 0$ we obtain:

$$\bar{q}(t) = \frac{1}{\sqrt{2\omega}} \langle 0_{in} | a^+_{in} e^{i\omega t} + a_{in} e^{-i\omega t} | 0_{in} \rangle$$

$$= 0$$

14

14 / 26

File Edit View Insert Actions Tools Help

* For $t > T$ we obtain:

$a_{out} = a_{in} + J_0$

$$\bar{q}(t) = \langle \psi | \hat{q}(t) | \psi \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+(t) + a(t) \right) | 0_{in} \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+_{out} e^{i\omega t} + a_{out} e^{-i\omega t} \right) | 0_{in} \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left((\cancel{a^+_{in}} + J_0^*) e^{i\omega t} + (\cancel{a_{in}} + J_0) e^{-i\omega t} \right) | 0_{in} \rangle$$

$$= \frac{1}{\sqrt{2\omega}} \left(J_0^* e^{i\omega t} + J_0 e^{-i\omega t} \right) \qquad (*)$$

$$= \int^T \frac{\sin((t-t')\omega)}{\quad} J(t')\, dt'$$

(Remark: same as classical $q(t)$ due

15

15 / 26

File Edit View Insert Actions Tools Help

* For $t>T$ we obtain:

$a_{out} = a_{in} + J_0$

$$\bar{q}(t) = \langle\gamma|\, \hat{q}(t) \,|\gamma\rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+(t) + a(t)\right) | 0_{in} \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+_{out} e^{i\omega t} + a_{out} e^{-i\omega t}\right) | 0_{in} \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left((\cancel{a^+_{in}} + J_0^*) e^{i\omega t} + (\cancel{a_{in}} + J_0) e^{-i\omega t}\right) | 0_{in} \rangle$$

$$= \frac{1}{\sqrt{2\omega}} \left(J_0^* e^{i\omega t} + J_0 e^{-i\omega t}\right) \qquad (*)$$

$$= \int^T \frac{\sin((t-t')\omega)}{\ldots} J(t')\, dt'$$

(Remark: same as classical $q(t)$ due

15

15 / 26

File Edit View Insert Actions Tools Help

B. Resonance:

* The amplitude of the excited motion of the oscillator is determined by J_0, as equation (*) shows.

* We expect that the driving force $J(t)$ is most efficient at creating a large J_0 if it oscillates at roughly the oscillator's natural frequency ω.

* Indeed: J_0 is the Fourier component of $J(t)$ for the frequency ω on the interval $[0,T]$:

$$J_0 := \frac{i}{\sqrt{2\omega}} \int^T J(t') e^{i\omega t'} dt'$$

16 / 26

16

File Edit View Insert Actions Tools Help

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left((\cancel{a^+_{in}} + J_0^*) e^{i\omega t} + (\cancel{a_{in}} + J_0) e^{-i\omega t} \right) | 0_{in} \rangle$$

$$= \frac{1}{\sqrt{2\omega}} \left(J_0^* e^{i\omega t} + J_0 e^{-i\omega t} \right) \qquad (*)$$

Exercise: verify ↗

$$= \int_0^T \frac{\sin((t-t')\omega)}{\omega} J(t')\, dt'$$

(Remark: same as classical $q(t)$ due to Ehrenfest theorem)

$\Rightarrow$ $\bar{q}$ oscillates with frequency ω, as expected.

B. Resonance:

* The amplitude of the excited motion of the

15 / 26

File Edit View Insert Actions Tools Help

oscillator is determined by J_0, as equation (*) shows.

* We expect that the driving force $J(t)$ is most efficient at creating a large J_0 if it oscillates at roughly the oscillator's natural frequency ω.

* Indeed: J_0 is the Fourier component of $J(t)$ for the frequency ω on the interval $[0,T]$:

$$J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$$

Thus, indeed, the more of the frequency ω is contained in $J(t)$, the larger is $|J_0|$.

16 / 26

16

File Edit View Insert Actions Tools Help

C. Energy expectation

* For $t<0$ we have:

$$\bar{H}(t) = \langle\psi|\hat{H}(t)|\psi\rangle \quad \text{(always)}$$

$$= \langle 0_{in}|\,\omega\left(a^{+}_{in}a_{in}+\tfrac{1}{2}\right)|0_{in}\rangle \quad (\text{for } t<0)$$

$$= \frac{\omega}{2}$$

i.e., the energy of the ground state of the Hamiltonian $\hat{H}_{t<0}$.

* For $t>T$ we have:

$$\bar{H}(t) = \langle\psi|\hat{H}(t)|\psi\rangle \quad \text{(always)}$$

17 / 26

17

File Edit View Insert Actions Tools Help

$$= \langle 0_{in} | \, \omega \left(a^+_{in} \cancel{a}_{in} + \tfrac{1}{2} \right) | 0_{in} \rangle \qquad (\text{for } t<0)$$

$$= \frac{\omega}{2}$$

i.e., the energy of the ground state of the Hamiltonian $\hat{H}_{t<0}$.

* For $t>T$ we have:

$$\bar{H}(t) = \langle \psi | \hat{H}(t) | \psi \rangle \qquad (\text{always})$$

$$= \langle 0_{in} | \, \omega \left(a^+_{out} a_{out} + \tfrac{1}{2} \right) | 0_{in} \rangle \quad (\text{for } t>T)$$

$$= \omega \langle 0_{in} | \left(\cancel{a^+_{in}} + J_0^* \right) \left(\cancel{a_{in}} + J_0 \right) + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \langle 0_{in} | \, J_0^* J_0 + \tfrac{1}{2} \, | 0_{in} \rangle$$

17 / 26

17

File Edit View Insert Actions Tools Help

* For $t>T$ we have:

$$\bar{H}(t) = \langle\psi|\hat{H}(t)|\psi\rangle \qquad \text{(always)}$$

$$= \langle 0_{in}|\,\omega\left(a^{+}_{out}a_{out}+\tfrac{1}{2}\right)|0_{in}\rangle \qquad (\text{for } t>T)$$

$$= \omega\,\langle 0_{in}|\left(\cancel{a}^{+}_{in}+J_0^{*}\right)\left(\cancel{a}_{in}+J_0\right)+\tfrac{1}{2}\,|0_{in}\rangle$$

$$= \omega\,\langle 0_{in}|\,J_0^{*}J_0+\tfrac{1}{2}\,|0_{in}\rangle$$

$$= \omega\left(\tfrac{1}{2}+|J_0|^2\right) \qquad \text{which is elevated!}$$

Remark: We notice that the oscillator's energy

17 / 26

17

File Edit View Insert Actions Tools Help

$$= \omega \langle 0_{in} | (\cancel{a}^+_{in} + J_0^*)(\cancel{a}_{in} + J_0) + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \langle 0_{in} | J_0^* J_0 + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \left(\tfrac{1}{2} + |J_0|^2\right)$$ which is elevated!

Remark: We notice that the oscillator's energy increases the more the larger $|J_0|$, i.e., from B, the closer the driving force is to the oscillator's natural frequency ω.

Remark: In QFT, say when electrical current drives electromagnetic field modes, the

18 / 26

18

File Edit View Insert Actions Tools Help

Implications:

- At late times energy is elevated

  $\Rightarrow$ State of system, $|\psi\rangle = |0_{in}\rangle$ can no longer be the ground state.

- Indeed, ground state $|0_{out}\rangle$ of

$$H_{t>T} = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right)$$
$$= \omega\left(a^+_{out}\,a_{out} + \frac{1}{2}\right)$$

would have to have eigenvalue $\omega/2$, i.e., $a_{out}|0_{out}\rangle = 0$.

And: $\quad a_{out}|\psi\rangle = a_{out}|0_{in}\rangle = (a_{in} + J_0)|0_{in}\rangle$

i.e.: $\quad a_{out}|\psi\rangle = J_0|\psi\rangle \neq 0$

19 / 26

19

File Edit View Insert Actions Tools Help

$$= \omega\left(a^{+}_{out}\, a_{out} + \tfrac{1}{2}\right)$$

would have to have eigenvalue $\omega/2$, i.e., $a_{out}|0_{out}\rangle = 0$.

And: $a_{out}|\chi\rangle = a_{out}|0_{in}\rangle = (a_{in} + J_0)|0_{in}\rangle$

i.e.: $a_{out}|\chi\rangle = J_0|\chi\rangle \neq 0$

→ Q: So what kind of excited state is $|\chi\rangle$ at late times?

A: Since $|\chi\rangle$ is eigenstate of a lowering operator,

namely a_{out}, $|\chi\rangle$ is what is called a Coherent State.

Recall:

19 / 26

19

File Edit View Insert Actions Tools Help

A: Since $|\chi\rangle$ is eigenstate of a lowering operator, namely a_{out}, $|\chi\rangle$ is what is called a Coherent State.

Recall:

Coherent states saturate the uncertainty relation:

If $|\psi\rangle$ is a coherent state, then

$$\Delta q_{|\psi\rangle} \, \Delta p_{|\psi\rangle} = \frac{\hbar}{2}$$

→ These are the states which come closest to having definite values for both q and p, i.e., they are as close as possible to obeying:

20 / 26

20

File Edit View Insert Actions Tools Help

$$\Delta q_{|\psi\rangle} \, \Delta p_{|\psi\rangle} = \frac{\hbar}{2}$$

→ These are the states which come closest to having definite values for both q and p, i.e., they are as close as possible to obeying:

$$\hat{q}|\psi\rangle = \langle\hat{q}\rangle|\psi\rangle \text{ and } \hat{p}|\psi\rangle = \langle\hat{p}\rangle|\psi\rangle$$

Q: Significance of driven harmonic oscillators always ending up in a coherent state for QFT?

A: Consider example of classical currents and charges

20 / 26

20

File Edit View Insert Actions Tools Help

Q: Significance of driven harmonic oscillators always ending up in a coherent state for QFT?

A: Consider example of classical currents and charges driving the mode oscillators of the electromagnetic QFT:

- The charges and currents drive the EM oscillators into a coherent state. ← (In Heisenberg picture: State stays constant but its meaning relative to the then time-dependent operators changes)

- ⇒ The $\hat{q}_k$, $\hat{p}_k$ of the EM field (essentially the $\hat{E}_i$ and $\hat{B}_i$ fields) will be as sharp as possible, i.e., the EM

21 / 26

21

File Edit View Insert Actions Tools Help

Remarks:

- It is also often said, when teaching quantum mechanics, that an electron's wave function will obey the Schrödinger equation (and thus evolve unitarily) only until it next interacts with some other system.

- That other system would gain information about our electron during the interaction, and this should make the electron's wave function collapse. We may not measure the other system to learn what it learned, but in any case, if the e^- started in a pure state, its interaction with the

23 / 26

23

A: Not every interaction constitutes a measurement:

- Consider the quantum systems of the e^- and the EM QFT:

$$\hat{H}^{(tot)} = \hat{H}^{(EM)} + \hat{H}^{(e^-)} + \hat{H}^{(int)}$$

Here, $\hat{H}^{int} = \hat{p}\,\hat{B}$, for example.

- Now assume the initial state of the total system, $|\psi\rangle$, is unentangled: $|\psi\rangle = |\varphi_0^{e^-}\rangle \otimes |\phi_0^{EM}\rangle$.

- In general, the time evolution operator

File Edit View Insert Actions Tools Help

- In general, the time evolution operator

$$\hat{U}(t) = e^{-it\hat{H}^{tot}}$$

will (e.g., in the Schrödinger picture) make

$$|\psi(t)\rangle = e^{-it\left(\hat{H}^{EM} + \hat{H}^{e^-} + \hat{p}\hat{B}\right)} |\varphi_0^{e^-}\rangle \otimes |\phi_0^{EM}\rangle$$

an entangled state, because the interaction term contains operators $(\hat{p}, \hat{B})$ from both quantum systems.

- This means that when we trace over the EM field, the electron's state is seen to evolve from pure to mixed.

- However, it is a good approximation

25 / 26

25

File Edit View Insert Actions Tools Help

will (e.g., in the Schrödinger picture) make

$$|\psi(t)\rangle = e^{-it\left(\hat{H}^{EM}+\hat{H}^{e^-}+\hat{p}\hat{B}\right)}|\varphi_0^{e^-}\rangle\otimes|\phi_0^{EM}\rangle$$

an entangled state, because the interaction term contains operators $(\hat{p},\hat{B})$ from both quantum systems.

- This means that when we trace over the EM field, the electron's state is seen to evolve from pure to mixed.

- However, if in good approximation,

$$\hat{B}_i(x)|\phi^{EM}\rangle \approx \underbrace{\overbrace{\langle\hat{B}_i(x)\rangle}^{\text{number-valued!}}}_{=:B_i(x)}|\phi^{EM}\rangle$$

25 / 26

25

File Edit View Insert Actions Tools Help

$$|\psi(t)\rangle = e^{-it\left(\hat{H}^{EM}+\hat{H}^{e^-}+\hat{p}\hat{B}\right)}|\varphi_0^{e^-}\rangle\otimes|\phi_0^{EM}\rangle$$

an entangled state, because the interaction term contains operators $(\hat{p},\hat{B})$ from both quantum systems.

- This means that when we trace over the EM field, the electron's state is seen to evolve from pure to mixed.

- However, if in good approximation,

$$\hat{B}_i(x)|\phi^{EM}\rangle \approx \overbrace{\underbrace{\langle\hat{B}_i(x)\rangle}_{=:\,B_i(x)}}^{\text{number-valued!}}|\phi^{EM}\rangle$$

25 / 26

25

File Edit View Insert Actions Tools Help

then $\hat{H}^{tot} \approx \underset{\substack{\text{acts only}\\ \text{on H.space}\\ \text{of EM field}}}{\hat{H}^{EM}} + \underbrace{\hat{H}^{e^-} + \hat{p}B}_{\substack{\text{act only on Hilbert}\\ \text{space of } e^-}}$.

$\Rightarrow$ $\hat{U}(t)$ is, in good approximation, not entangling.

i.e. the electron's state evolves from pure to pure, in spite of its interaction with the EM field (because it was essentially not a measurement).

Next:

Study the coherent state that a driven harmonic

26 / 26

26