Title: Introduction to Quantum Field Theory for Cosmology - Lecture 4

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Abstract:

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Note Title

QFT for Cosmology, Achim Kempf, Winter 2014, Lecture 4

From the Heisenberg to the Schrödinger picture

Recall Heisenberg picture:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x',t) + \frac{1}{2}\hat{\phi}(x',t)(m^2-\Delta)\hat{\phi}(x',t)\, d^3x'$$

The Heisenberg eqns $i\partial_t \hat{Q}(t) = [\hat{Q}(t), \hat{H}]$ yield:

$\Rightarrow$ K.G. eqn: $\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t)$ and $\dot{\hat{\pi}}(x,t) = (\Delta - m^2)\hat{\phi}(x,t)$

and we have to solve:

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Our results so far:

in box of size $L \times L \times L$

- We obtained explicit solution $\hat{\phi}(x,t)$ in the form

(∓) $$\hat{\phi}(x,t) = L^{-3/2} \sum_k \hat{\phi}_k(t)\, e^{ikx} \quad \text{for } k = \frac{2\pi}{L}(n_1, n_2, n_3)$$

where $\hat{\phi}_k(t)$ are uncoupled complex harmonic oscillators:

$$\ddot{\hat{\phi}}_k(t) = -\omega_k^2 \hat{\phi}_k(t) \quad \text{with} \quad \omega_k = \sqrt{k^2+m^2}$$

- Then,

$$\hat{\phi}_k(t) = \frac{1}{2}\left(\hat{q}_k(t) + \hat{q}_{-k}(t)\right) + \frac{i}{\omega_k\, 2}\left(\hat{p}_k(t) - \hat{p}_{-k}\right)$$

is in terms of ordinary quantum harmonic oscillators $\hat{q}_k, \hat{p}_k$.

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How then to make predictions?

- Assume:

  * $\hat{\phi}(x,t)$, $\hat{\pi}(x,t)$ are known.

  * state $|\Psi\rangle \in \mathcal{H}$ of the K.G. quantum system known.

  ... for example in terms of the $\hat{q}_k(t)$, $\hat{p}_k(t)$ and their action on a Hilbert space.

- Predict, e.g.:

  * expect. value when repeatedly measuring say $\hat{\phi}(x,t)$ at (x,t):

$$\bar{\phi}(x,t) = \langle \Psi | \hat{\phi}(x,t) | \Psi \rangle$$

  * uncertainty in measurement of $\hat{\phi}(x,t)$ at (x,t):

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* uncertainty in measurement of $\hat{\phi}(x,t)$ at (x,t):

(Recall standard deviation: $\Delta A = \sqrt{\overline{(A-\bar{A})^2}}$)

$$\Delta\phi(x,t) = \langle \Psi | \left(\hat{\phi}(x,t) - \bar{\phi}(x,t)\right)^2 | \Psi \rangle^{1/2}$$

(We'll need smearing to make this finite)

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What is prob. amplitude for finding any ϕ_h?

* Choose a state $|\psi\rangle$, e.g., the "Vacuum state":

$|\psi_0\rangle$ = lowest energy state of all $\hat{q}_k, \hat{p}_k$ oscillators

* Recall harm. osc.:

ground state of harm. osc.

$$\psi_0(q) \sim \langle q|\psi_0\rangle \sim e^{-\omega_k q^2/2}$$

* Recall harm. osc.:

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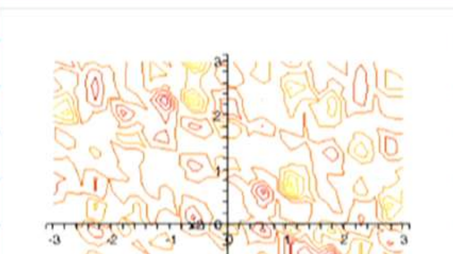
$$\Psi_0(q_k) \sim \langle q_k | \Psi_0 \rangle \sim e^{-\omega_k q_k^2/2}$$

* From this, one can work out (exercise):

$$\text{prob. ampl.}(\phi_k) = \text{const.} \times e^{-\omega_k \phi_k \phi_k^*/2} \qquad (P)$$

Visualization:



1.) Draw ϕ_k values from the prob. distribution (P).

2.) Fourier transform to obtain a $\phi(x)$.

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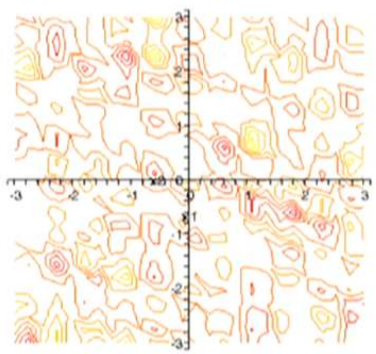

1.) Draw ϕ_k values from the prob. distribution (P).

2.) Fourier transform to obtain a $\phi(x)$.

3.) Plot, e.g., level curves of $\phi(x)$.

Towards the Schrödinger picture (and a derivation of (P) ...)

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Towards the Schrödinger picture (and a derivation of (P) in it)

(First we'll need the analog of Schrödinger wave functions, namely "wave functionals")

- Assume that at a time t all the observables $\hat{\phi}(x,t)$ are simultaneously being measured.

  (We can because $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$)

- At each x we obtain a real-valued measurement outcome, say $f(x)$.

- Thus, the system collapses into a state

$$|f\rangle \in \mathcal{H}$$

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(We can because $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$)

- At each x we obtain a real-valued measurement outcome, say $f(x)$.

- Thus, the system collapses into a state

$$|f\rangle \in \mathcal{H}$$

which is joint eigenstate of all $\hat{\phi}(x,t)$:

$$\hat{\phi}(x,t)\,|f\rangle = f(x)\,|f\rangle$$

Definition: If $f: \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary function, we denote by ..

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Definition: If $f: \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary function, we denote by

$$|f\rangle \in \mathcal{H}$$

the joint eigenvector of all $\hat{\phi}(x,t)$ with eigenvalues $f(x)$:

unique up to a phase

$$\hat{\phi}(x,t)|f\rangle = f(x)|f\rangle \quad \text{for all } x \in \mathbb{R}^3$$

Hilbert basis: The set

$$\{|f\rangle\}$$

of all joint eigenvectors of the $\hat{\phi}(x,t)$ for all $x \in \mathbb{R}^3$ can be used to form a "complete ON basis" of $\mathcal{H}$. (up to functional analytic subtleties).

$\Rightarrow$ For any $|\Psi\rangle \in \mathcal{H}$ we have:

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$\Rightarrow$ For any $|\Psi\rangle \in \mathcal{H}$ we have:

$$|\Psi\rangle = \int_{L^2(\mathbb{R}^3)} |f\rangle\langle f|\Psi\rangle$$

↑ it's more subtle really

analogous to:

$$|\psi\rangle = \int |\vec{x}\rangle \underbrace{\langle\vec{x}|\psi\rangle}_{\psi(\vec{x})} d^3x$$

The "Wave functional"

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The "Wave functional"

Recall QM: ☐ Assume $\{\hat{R}_i\}_{i=1}^{N}$ is compl. set of commuting observables, with joint eigenvectors $|r\rangle$ obeying: $\hat{R}_i|r\rangle = r_i|r\rangle$.

☐ Then the function ψ given by $\psi(r) = \langle r|\psi\rangle$

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▫ Then the function ψ, given by $\psi(r) = \langle r|\psi\rangle$ is called the "**wave function**" of $|\psi\rangle$ in the $\{\hat{R}_i\}$ basis.

Example: $\{\hat{p}_i\}$ yield mom. wave functions $\psi(p) = \langle p|\psi\rangle$

$p = \{p_1, p_2, \dots, p_N\}$

← or, e.g., also the $\{\hat{\pi}(x)\}$.

In QFT: E.g., $\{\hat{\phi}(x)\}_{x\in\mathbb{R}^3}$ is compl. set of com. observables

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with joint eigenvectors $|f\rangle$ obeying $\hat{\phi}(x)|f\rangle = f(x)|f\rangle$.

Then, Ψ, given by

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- Then, Ψ, given by

$\{|f\rangle\}$ form field ON eigen basis

(Convention: square bracket because argument is a function)

$\Psi[f] := \langle f|\Psi\rangle$ is called the "wave functional".

(called a "Functional" because argument is a function)

alternatively could use e.g. joint eigenbasis of the $\hat{\pi}(x,t)$.

Interpretation of $\Psi[f]$?

e.g., vacuum $|\Psi_0\rangle$

- Assume the system is in an arbitrary state $|\Psi\rangle \in \mathcal{H}$ at t.

- If measuring now $\hat{\phi}(x,t)$ at all $x \in \mathbb{R}^3$ what is the probability amplitude for finding, say, the values $f(x)$?

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Q: The eqn. of motion for $\Psi[f,t]$?

A: The QFT Schrödinger equation:

- For every quantum theory, we have in the Schrödinger picture of the time evolution:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$$

- Which form does it take for $\Psi[f,t]$?
- Here in QFT:

now independent of time!

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$$\hat{H} = \int \frac{1}{2}\left(\hat{\pi}^2(x) + \hat{\phi}(x)(-\Delta + m^2)\hat{\phi}(x)\right) d^3x$$

- But how do $\hat{\phi}(x)$ and $\hat{\pi}(x)$ act on wave functionals $\Psi[f, t]$?

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(Exercise: check)

- A valid representation of $[\hat{\phi}(x), \hat{\pi}(x')] = i\delta^3(x-x')$ is:

$$\hat{\phi}(x).\ \Psi[f,t] = f(x)\,\Psi[f,t]$$

$$\hat{\pi}(x).\ \Psi[f,t] = -i\frac{\delta}{\delta f(x)}\Psi[f,t]$$

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functional derivative, as in variational principle used to derive Euler Lagrange equations.

- Therefore:

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⌊ inconvenient ⌋

▯ It is more convenient to use infrared-regularized momentum space:

▯ We now need to represent

$$[\hat{\phi}_k, \hat{\pi}_{k'}] = \delta_{k,-k'}$$

on the wave functionals $\Psi[\tilde{f}, t]$.

$\left(\begin{array}{l}\tilde{f}_k \text{ is Fourier} \\ \text{transform of } f(x)\end{array}\right)$

▯ As is easy to verify, this works:

$$\hat{\phi}_k \cdot \Psi[\tilde{f}, t] = \tilde{f}_k \Psi[\tilde{f}, t]$$

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Note: Ordinary derivatives here because set of variables $\{\tilde{f}_k\}$ is discrete, since $k = \frac{2\pi}{L}(n_1, n_2, n_3)$, $\vec{n} \in \mathbb{Z}^3$.

Schrödinger equation: $i\partial_t |\psi\rangle = \hat{H}|\psi\rangle$ becomes:

$$i\partial_t \Psi[\tilde{f}, t] = \sum_k \frac{1}{2}\left(-\frac{\partial}{\partial\tilde{f}_k}\frac{\partial}{\partial\tilde{f}_{-k}} + (k^2+m^2)\tilde{f}_k\tilde{f}_{-k}\right)\Psi[\tilde{f}, t]$$

Recall: For QM harm. osc., ground state Schrödinger wave function is:

$$\psi(x,t) = N\, e^{-\frac{1}{2}\omega x^2 - i\omega_k t}$$

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Schrödinger equation:

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Recall: For QM harm. osc., ground state Schrödinger wave function is:

$$\Psi(x,t) = N\, e^{-\frac{1}{2}\omega x^2 - i\omega_0 t}$$

Exercise: check it. Can you solve for excited states?

Ground state solution in QFT reads, similarly:

$= (\vec{k}^2 + m^2)^{1/2}$

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$$\Psi[\tilde{f},t] = N\, e^{-\sum_k\left(\frac{1}{2}\omega_k \tilde{f}_k\tilde{f}_{-k} - i\omega_k t\right)}$$

Exercise: verify

... which we had already claimed before.

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Generic wave functionals

- Assume the system is in a state, $|\alpha\rangle$, other than $|\Psi_0\rangle$.

  $\Rightarrow$ Not for all modes' oscillators is $|\alpha\rangle$ the ground state.

- But if an oscillator is excited, then its wave function spreads out - classically its amplitude of oscillation would increase.

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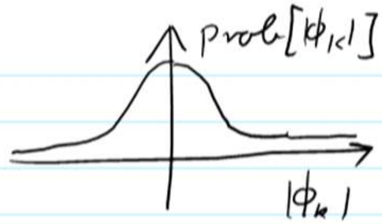
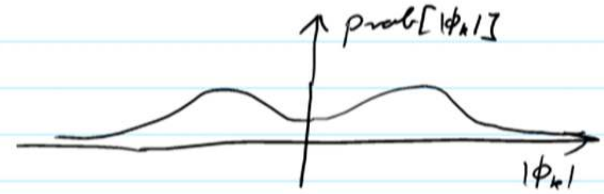


ground state



example of excited state

□ The more a mode k is excited, the more likely is a measurement of $\hat{\phi}_k$ to yield a $f_k = \phi_k$ with a large modulus $|\phi_k|$.

Can you produce a ...

⇒ If, e.g., a mode k is very highly excited then $|\phi_k|$ is likely very large, i.e., a measurement

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The particle interpretation

- General states, i.e., states $|\alpha\rangle$ other than the vacuum state $|\Psi_0\rangle$ are states "with particles". Why?

- Recall:

$$\hat{H} = \sum_k \left(\frac{1}{2}\hat{\pi}_k^+(t)\hat{\pi}_k(t) + \frac{1}{2}\hat{\phi}_k^+(t)\left(k^2+m^2\right)\hat{\phi}_k(t)\right)$$

commuting

$$= \sum_k \hat{H}_k \qquad \text{with } \hat{H}_k = \frac{1}{2}\hat{\pi}_k^+\hat{\pi}_k + \frac{1}{2}\hat{\phi}_k^+\left(k^2+m^2\right)\hat{\phi}_k$$

$\Rightarrow$ Any energy eigenstate of the QFT is also

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$\Rightarrow$ Any energy eigenstate of the QFT is also eigenstate to each $\hat{H}_k$ – whose spectrum is discrete!

$E_k(n) = \hbar \omega_k (\frac{1}{2} + n_k)$

$\Rightarrow$ Any energy eigenstate $|E\rangle \in \mathcal{H}$ of the QFT can be specified by listing to which energy level n_k each mode k is excited:

$$|E\rangle = |\{n_k\}_{\text{all } k}\rangle$$

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Example: $|E\rangle = |n_{k_1}=3, n_{k_2}=7, \text{all other } n_k = 0\rangle$

$\ast$ $|E\rangle$ is the 3rd and 7th excited state for $\hat{H}_{k_1}$ and $\hat{H}_{k_2}$, respecti...

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□ Example: $|E\rangle = |n_{k_a}=3,\ n_{k_b}=7,\ \text{all other } n_k=0\rangle$

* $|E\rangle$ is the 3rd and 7th excited state for $\hat{H}_{k_a}$ and $\hat{H}_{k_b}$ respectively
* $|E\rangle$ is the ground state for all other $\hat{H}_k$.

□ Energy: using $E_{n_k} = \overset{=1}{\hbar}\omega_k(n_k+\frac{1}{2})$

$$\hat{H}|E\rangle = \left(3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k}\frac{1}{2}\omega_k\right)|E\rangle$$

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$$\downarrow$$

$$\hat{H}|E\rangle = \left(3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k} \frac{1}{2}\omega_k\right)|E\rangle$$

□ Crucial observation:

* If we increase the n_k of a mode k by 1

$\rightsquigarrow$ total energy increases by $\omega_k = \sqrt{k^2 + m^2}$!

* But recall from special relativity: $E^2 - p^2 = m^2$

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* But recall from special relativity: $E^2 - p^2 = m^2$.

$\Rightarrow$ $E_{particle} = \sqrt{k^2_{particle} + m^2_{particle}} = \omega_k$

⇝ Interpretation (which works at least in Minkowski space:) 18 / 19

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$\rightsquigarrow$ Interpretation (which works at least in Minkowski space:)

$$\boxed{\text{Mode excitation} = \text{particle creation}}$$

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