Title: Introduction to Quantum Field Theory for Cosmology - Lecture 2

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Abstract:

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We will study wave phenomena, i.e., oscillating amplitudes.

$\Rightarrow$ Harmonic oscillators will arise in the formalism.

Plan:

1. Recall harmonic oscillators
2. Relativistic fields
3. 2nd quantization
4. The harmonic oscillators of fields & their vacuum fluctuations

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1. Harmonic oscillators

Classical:

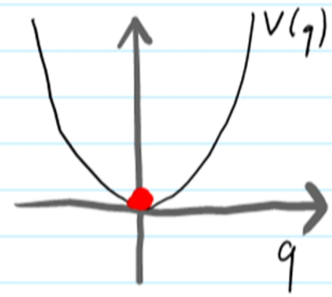
- Hamiltonian: $H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2$

- Eqns of motion: $\dot{p} = -\omega^2 q, \quad \dot{q} = p$

- Lowest energy solution: (later relevant for "vacuum")

$q(t) = 0, \ p(t) = 0$

i.e., $H(t) = 0$ for all t:



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Quantum:

As always when quantizing:

- H and Eqns of motion unchanged.

- But, the canonically conjugate pairs of variables (here, q and p) no longer commute:

- Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2}\hat{q}^2$

- Eqns of motion: $\dot{\hat{p}} = -\omega^2 \hat{q}, \quad \dot{\hat{q}} = \hat{p}$

- And now:

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- Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2}\hat{q}^2$

- Eqns of motion: $\dot{\hat{p}} = -\omega^2 \hat{q}, \quad \dot{\hat{q}} = \hat{p}$

- And now:

$$[\hat{q}(t), \hat{p}(t)] = i\hbar 1$$

- $\Rightarrow$ $\hat{q}(t)$, $\hat{p}(t)$, $\hat{H}$ etc are operator-valued.

- Lowest energy solution now?

The lowest energy state, $|\psi_0\rangle$,

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The lowest energy state, $|\psi_0\rangle$, obeys:

$$\hat{H}|\psi_0\rangle = E_0|\psi_0\rangle$$

with $E_0 = \frac{1}{2}\hbar\omega$

- **We notice:**

Lowest energy is elevated!

(Later for quantum fields $\Rightarrow$ nonzero vacuum energy)

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Lowest energy state $|\psi_0\rangle$?

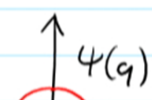
Consider eigenbasis $|q\rangle$ of $\hat{q}$:

$$\hat{q}\,|q\rangle = q\,|q\rangle \quad \text{for } q \in \mathbb{R}$$

$$\langle q|q'\rangle = \delta(q-q')$$

Then, recall:

$$\psi_0(q) = \langle q|\psi_0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\omega}{2\hbar}q^2}$$



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Is oscillator at resting position $q=0$?

In lowest energy state, $|\psi_0\rangle$, we have:

$$\bar{q} = \langle \psi_0 | \hat{q} | \psi_0 \rangle = \int_{-\infty}^{+\infty} \psi_0^*(q)\, q\, \psi_0(q)\, dq = 0$$

i.e. the position expectation vanishes, as in classical mechanics.

But, there are quantum fluctuations!

$$\Delta q = \langle \psi_0 | (\hat{q} - \bar{q})^2 | \psi_0 \rangle^{1/2} = \sqrt{\frac{\hbar}{2m}}$$

i.e. actual measurements yield values spread around $q = 0$.

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Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields
3. 2nd quantization
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

2. Relativistic fields

choose simple case without a potential

- How to make the Schrödinger equation, say

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta \psi(x,t) \qquad (S)$$

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2. Relativistic fields

choose simple case without a potential

▫ How to make the Schrödinger equation, say

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta \psi(x,t) \qquad (S)$$

relativistically covariant?

Laplacian: $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$

▫ Klein & Gordon:

because $= \hat{H} =$ energy

Recall: $p_j = -i\hbar \frac{\partial}{\partial x_j}$ and $E = i\hbar \frac{\partial}{\partial t}$, i.e., the

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$\vec{p}^{\,2} = \sum_{i=1}^{3} p_i^2$

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$\vec{p}^{\,2} = \sum_{i=1}^{3} p_i^2$

$E\psi = \frac{\vec{p}^{\,2}}{2m}\psi$, i.e.:

$$\boxed{E = \frac{\vec{p}^{\,2}}{2m}}$$

i.e. $E = \frac{1}{2} m \dot{x}^2$

But special relativity demands:

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Recall: $p_j = -i\hbar\frac{\partial}{\partial x_j}$ and $E = i\hbar\frac{\partial}{\partial t}$, i.e., the

Schrödinger equation can be written in this form:

$\vec{p}^{\,2} = \sum_{i=1}^{3} p_i^2$

$$E\psi = \frac{\vec{p}^{\,2}}{2m}\psi, \text{ i.e.:} \qquad \boxed{E = \frac{\vec{p}^{\,2}}{2m}} \qquad \text{i.e. } E = \frac{1}{2}m\dot{x}^2$$

But special relativity demands:

$$\frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2 \qquad (\text{Namely: } p_\mu p^\mu = m^2c^4)$$

i.e.: $$\left(-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2\Delta\right)\psi = m^2c^2\psi$$

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i.e.: $\left(-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2\Delta\right)\psi = m^2c^2\psi$

- This "Klein Gordon equation" is usually written as:

$$\boxed{\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\psi = 0}$$

(units chosen so that $c=1$, $\hbar=1$)

Or, also $(\Box + m^2)\psi = 0$ with d'Alembertian $\Box = \partial_t^2 - \Delta$

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- <u>Nonrelativistic limit ok</u>?

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Nonrelativistic limit ok?

Must show that KG eqn reduces

to Schrödinger eqn for small momenta:

Assume K.G. Eqn., i.e.,: $\frac{E^2}{c^2} = m^2c^2 + \vec{p}^{\,2}$

$$\Rightarrow \quad E = \pm\sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$$

Choose positive energy solution:

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Choose positive energy solution:

$$E = \sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$$

Taylor expansion for small $\vec{p}^{\,2}$: (or large c)

$$E = mc^2 + \frac{1}{2}\frac{c^2}{\sqrt{\vec{p}^{\,2}c^2 + m^2c^4}}\Bigg|_{\vec{p}^{\,2}=0} \vec{p}^{\,2} + \mathcal{O}((\vec{p}^{\,2})^2)$$

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$$\Rightarrow \quad E = mc^2 + \frac{\vec{p}^{\,2}}{2m} + \mathcal{O}((\vec{p}^{\,2})^2)$$

$\Rightarrow$ For small momenta the K.G. eqn becomes the Schrödinger eqn:

$$E\psi = \left(\frac{\vec{p}^{\,2}}{2m} + mc^2\right)\psi$$

i.e.: $$i\hbar \frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\Delta + mc^2\right)\psi$$

Note: We obtain an extra term:

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Note: We obtain an extra term:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \underline{mc^2}$$

In QM irrelevant: (use Heisenberg picture)

$$i\hbar \frac{d}{dt}\hat{f} = [\hat{f}, \hat{H} + \text{const}\,1] = [\hat{f}, \hat{H}]$$

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Remarks:

1a) The negative energy solutions spoil the interpretation of the $\psi(x,t)$ as a probability amplitude density!

Namely: Require the negative energy solutions to propagate backwards in time: anti-particles! They look like travelling forward in time with opposite properties.

1b) This problem is deep and led to quantum field theory, where this is solved in terms of anti-particles.

2a) There are many ways to generalize

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Namely:
Require the negative energy solutions to propagate backwards in time: anti-particles! They look like travelling forward in time with opposite properties.

1b) This problem is deep and led to quantum field theory, where this is solved in terms of anti-particles.

2a) There are *many* ways to generalize the Schrödinger equation to obtain a relativistically covariant equation.

2b) E. Wigner (1940s): Complete classification of relativistically covariant wave equations:

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2b) E. Wigner (1940s): Complete classification of relativistically covariant wave equations:

Note: The complete classification allows arbitrarily high spins and distinguishes massive from massless cases. All covariant wave eqns for same spin and mass lead to equivalent QFTs. See, e.g., textbook on QFT by S. Weinberg.

Spin	Standard wave eqn	Examples
0	Klein Gordon eqn.	Higgs, Inflaton, $\pi^0, \pi^\pm$
1/2	Dirac eqn.	e^-, quarks, p^+, n
1	Maxwell YM eqns.	Photons, gluons

Higher spins?

- not observed in truly elementary particles.

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Higher spins?

- not observed in truly elementary particles.
- appear to lead to incurable "divergencies" in QFT.

Note:

- "Graviton" should be a spin 2 particle.

Plan:

1. Recall harmonic oscillators ✓

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Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields ✓
3. 2nd quantization
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

3. 2nd quantization

- We will 2nd quantize only the Klein Gordon equation because:
  - is easiest

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3. 2nd quantization

- We will 2nd quantize only the Klein Gordon equation because:
  - is easiest
  - is only case of cosmological significance that we know of (so far).

- Terminology: We switch from ψ to ϕ and call it a "Field".

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- Terminology: We switch from ψ to ϕ and call it a "Field".

- Definition:

we will do the general definition later

The canonically conjugate field $\pi(x,t)$ to $\phi(x,t)$

is defined as: $\pi(x,t) = \dot{\phi}(x,t)$ (analogous to $p_i = \dot{q}_i$)

- Klein Gordon equation can now be written in the form:

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The canonically conjugate field $\pi(x,t)$ to $\phi(x,t)$ is defined as: $\pi(x,t) = \dot{\phi}(x,t)$ (analogous to $p_i = \dot{q}_i$)

- Klein Gordon equation can now be written in the form:

$$\dot{\pi}(x,t) - \Delta\phi(x,t) + m^2\phi(x,t) = 0$$

Notice:

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Notice:

The K.G. equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\phi = 0 \qquad (\hbar = 1 = c)$$

does not couple $Re(\phi)$ to $Im(\phi)$:
each separately fulfills the K.G. eqn.

$\Rightarrow$ It suffices to study real-valued ϕ.

Making ϕ complex is then straightforward.

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$\Rightarrow$ It suffices to study real-valued ϕ.

Making ϕ complex is then straightforward.

- Quantization conditions:

analogous to:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\, \delta^3(x-x') \qquad [\hat{q}_a(t), \hat{p}_{a'}(t)] = i\hbar\, \delta_{aa'}$$

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

$$[\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0 \qquad [\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$$

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$$\left[\hat{\phi}(x,t),\ \hat{\phi}(x',t)\right] = 0$$

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$$\left[\hat{\pi}(x,t),\ \hat{\pi}(x',t)\right] = 0$$

$$[\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$$

- We keep the equations of motion:

(E1) $$\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t)$$

$$\dot{\hat{q}}_a(t) = \hat{p}_a(t)$$

(E2) $$\dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\,\hat{\phi}(x,t)$$

$$\dot{\hat{p}}_a(t) = -K_a\hat{q}_a(t)$$

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$$[\hat\phi(x,t), \hat\pi(x',t)] = i\hbar\,\delta(x-x')$$ $[\hat q_a(t), \hat p_{a'}(t)] = i\hbar\,\delta_{aa'}$

$$[\hat\phi(x,t), \hat\phi(x',t)] = 0$$ $[\hat q_a(t), \hat q_{a'}(t)] = 0$

$$[\hat\pi(x,t), \hat\pi(x',t)] = 0$$ $[\hat p_a(t), \hat p_{a'}(t)] = 0$

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(E2) $$\dot{\hat\pi}(x,t) = -(-\Delta + m^2)\hat\phi(x,t)$$ $\dot{\hat p}_a(t) = -K_a \hat q_a(t)$

- **Note**: $\phi^*(x,t) = \phi(x,t)$ now implies hermiticity: $\hat\phi^+(x,t) = \hat\phi(x,t)$

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Is there a Hamiltonian for 2nd quantization? Yes!

analogous to:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x,t) + \frac{1}{2}\hat{\phi}(x,t)\left(m^2 - \Delta\right)\hat{\phi}(x,t)\, d^3x$$

$$\hat{H} = \sum_a{}' \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2}{2}\hat{q}_a^2$$

Proposition:

With this definition of $\hat{H}$, the Heisenberg equations $i\hbar\dot{\hat{f}} = [\hat{f}, \hat{H}]$

$$i\hbar\,\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}]$$

$$i\hbar\,\dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$$

$$i\hbar\,\dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}] \quad (*)$$

$$i\hbar\,\dot{\hat{p}}_a(t) = [\hat{p}_a(t), \ldots$$

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analogous to:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x,t) + \frac{1}{2}\hat{\phi}(x,t)(m^2-\Delta)\hat{\phi}(x,t)\, d^3x$$

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Proposition:

With this definition of $\hat{H}$, the Heisenberg equations $i\hbar \dot{\hat{f}} = [\hat{f}, \hat{H}]$

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$$i\hbar \dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}] \qquad (*)$$

$$i\hbar \dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$$

$$i\hbar \dot{\hat{p}}_a(t) = [\hat{p}_a(t), \hat{H}]$$

yield the proper eqns of motion: E1, E2.

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Indeed, e.g.:

$$i\hbar\, \dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), H] = \left[\hat{\phi}(x,t), \int_{\mathbb{R}^3} \frac{1}{2}\, \hat{\pi}^2(x',t) \; + \text{something}(\hat{\phi})\, d^3x'\right]$$

$$= \frac{1}{2}\int [\hat{\phi}(x,t), \hat{\pi}(x',t)]\,\hat{\pi}(x',t) + \hat{\pi}(x',t)\,[\hat{\phi}(x,t), \hat{\pi}(x',t)]\, d^3x'$$

$$= \frac{i\hbar}{2}\int \delta^3(x-x')\,\hat{\pi}(x',t) + \hat{\pi}(x',t)\,\delta^3(x-x')\, d^3x' = \hat{\pi}(x,t)\, i\hbar \checkmark$$

Exercise: Prove (*)

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