Title: Introduction to Quantum Field Theory for Cosmology - Lecture 5

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Abstract:

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QFT for Cosmology, Achim Kempf, Winter 2014, Lecture 5

Note Title

Recall: The QFT problem

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\hat{\phi}(x,t) = 0 \quad \text{and} \quad [\hat{\phi}(x,t), \dot{\hat{\phi}}(x',t)] = i\,\delta(x-x')$$

when Fourier-expanded into k modes (in a box of size $L \times L \times L$)

$$\hat{\phi}(x,t) = L^{-3/2} \sum_k \hat{\phi}_k(t)\, e^{ikx}$$

$\leftarrow k = \frac{2\pi}{L}(n_1, n_2, n_3),\ n_i \in \mathbb{Z}$

becomes a harmonic oscillator for each mode:

$$\ddot{\hat{\phi}}_k(t) = -(k^2+m^2)\hat{\phi}_k(t) \quad \text{and} \quad [\hat{\phi}_k, \dot{\hat{\phi}}_{k'}] = i\,\delta_{k,-k'}$$

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- **Found interpretation:** If QFT state $|\alpha\rangle$ is such that, e.g., the k mode oscillator is in n'th energy eigenstate then n particles of momentum k are present.

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Special case: States in QFT which describe a single particle.

- Recall:

$$\hat{H} = \sum_k \hat{H}_k \text{, with } \hat{H}_k = \frac{1}{2}\hat{\pi}_k^+\hat{\pi}_k + \frac{\omega_k^2}{2}\hat{\phi}_k^+\hat{\phi}_k$$

- The state |one particle of momentum k⟩ is defined by:

$$\hat{H}_k |\text{one particle of momentum } k\rangle = \hbar\omega_k\left(\tfrac{1}{2}+1\right)|\text{one particle of momentum } k\rangle$$

and for $k' \neq k$:

$$\hat{H}_{k'} |\text{one particle of momentum } k\rangle = \hbar\omega_{k'}\left(\tfrac{1}{2}+0\right)|\text{one particle of momentum } k\rangle$$

Recall:

$$\psi(k) = \int d^3x\, \psi(x) e^{ixk}$$

- By linear combination, the state

$$|1 \text{ particle w. prob. density } \psi(x)\rangle$$

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Fourier transform of $\psi(x)$

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Fourier transform of $\psi(x)$

$$|1\text{ particle w. prob. density } \psi(x)\rangle = \frac{1}{L^{3/2}}\sum_k \tilde{\psi}_k\,|\text{one particle of momentum } k\rangle$$

$$= \frac{1}{(2\pi L)^{3/2}}\sum_k \int e^{-ixk}\,\psi(x)\,|\text{one particle of momentum } k\rangle\, d^3x$$

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But: Particle interpretation has limited applicability!

Why? ▫ Because the particle interpretation relies upon:

a.) Fourier decomposition (not covariant in general relativity)

b.) Harmonic oscillators have discrete spectrum (not if wavelength larger than "horizon")

c.) Vacuum state is lowest energy state (if an oscillator's parameters time dependent: what is, e.g., its momentary frequency?)

▫ Each, a, b, c, have problems in presence of gravity, as we will see later!

▫ Note: while the wave interpretation of $\hat{\phi}(x,t)$ always applies,

the field $\hat{\phi}(x,t)$ is usually only indirectly measurable.

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Mechanisms for mode excitation / particle creation?

- Equivalent question:

  What are mechanisms for exciting harmonic oscillators?

- 2 types of mechanism: (here, $\hat{q}(t)$ stands for $\hat{\phi}_k(t)$)

we'll begin with this effect → **a.)** A "driving force" shakes the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2 \hat{q}(t) + \hat{J}(t)$$

e.g.: 

b.) A time dependence of ω affects the oscillator: 

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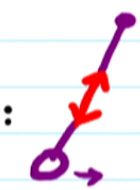
e.g.: 

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Both occur in QFT:

a.) Multiple fields enter into the Hamiltonian and into the eqns of motion. Thus, fields provide each other with J terms, e.g.:

$$H(\hat{\phi},\hat{\psi}) = \hat{H}_1(\hat{\phi}) + \hat{H}_2(\hat{\psi}) + \int_{\mathbb{R}^3} \lambda\, \hat{\phi}(x,t)\, \hat{\psi}(x,t)\, d^3x$$

- Wave interpretation: Nontrivial interaction of waves of different types of fields
- Particle interpretation: The collision of particles happens when their mode oscillators drive another.
  → Collissions can create and annihilate particles.
- Strongest effects?

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- Strongest effects?

When oscillator "resonates" with driving force

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b.) The presence of gravity can effectively influence the $\omega_k(t)$.

- **Wave interpretation:** * E.g. cosmic expansion stretches the wavelength

  $\Rightarrow$ expect $\omega = \omega(t)$ decreases. True, and also:

  * if wavelength > horizon then $\omega^2(t) < 0$!

  $\Rightarrow$ runaway harmonic mode oscillators

  

  (then: field amplification but no particle interpretation)

- **Particle interpretation:**

  Gravity can excite mode oscillators, i.e. it can create particles from the vacuum.

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Particle interpretation:

Gravity can excite mode oscillators, i.e. it can create particles from the vacuum.

Strongest effects? When oscillator resonates with $\omega(t)$. This effect is called parametric resonance.

Case a: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:

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Case a: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:

- We model the electromagnetic field as a Klein Gordon field.

  (The fact that EM fields have polarization and have $m=0$ is not important here)

- Consider an arbitrary mode of the electromagnetic field:

  $\hat{\phi}_k(t)$

  should really be quantized too

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▫ In a rough simplification, the EM **k** mode obeys:

$$\hat{H}_k = \frac{1}{2}\hat{\pi}_k^+(t)\,\hat{\pi}_k(t) + \frac{1}{2}\omega_k^2\,\hat{\phi}_k^+(t)\,\hat{\phi}_k(t) + \hat{\phi}_k(t)\,j_k(t)$$

⇒ If the current $j(t)$ varies in time it can excite the mode oscillators, thus creating photons.

⇒ **Need to study the quantized driven harmonic oscillator!**

for $\hat{H}_k(t)$ — for $\hat{\pi}_k(t)$ — stands for a field mode $\hat{\phi}_k(t)$

$$\boxed{\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)}$$

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stands for a mode $J_k(t)$ of another classical (or

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I Preparation:

- Recall that for all observables $\hat{f}$:

state at initial time

$$\bar{f}(t) = \langle \psi_0 | \hat{U}^+(t) \hat{f}_0 \hat{U}(t) | \psi_0 \rangle$$

operator at initial time

with the time-evolution operator obeying:

$$\hat{U}(t_0) = 1, \qquad i \frac{d}{dt} \hat{U}(t) = \hat{U}(t) \hat{H}(t)$$

the original Hamiltonian

"Heisenberg Hamiltonian"

- Schrödinger picture? We write equivalently:

Exercise: check!

$$\bar{f}(t) = \left(\langle \psi_0 | \hat{U}^+(t)\right) \hat{f}_0 \underbrace{\left(\hat{U}(t) | \psi_0 \rangle\right)}^{= |\psi(t)\rangle}$$

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$$\bar{f}(t) = \left(\langle\psi_0|\hat{U}^+(t)\right)\hat{f}_0\left(\overbrace{\hat{U}(t)|\psi_0\rangle}^{=|\psi(t)\rangle}\right)$$

$$= \langle\psi(t)|\hat{f}_0|\psi(t)\rangle$$

Recall: $\hat{H}_s(t) = \hat{H}(t)$ only if $\frac{d}{dt}\hat{H}(t) = 0$

The dynamics is $i\frac{d}{dt}|\psi(t)\rangle = \hat{H}_s(t)|\psi(t)\rangle$

with Schrödinger Hamiltonian: $\hat{H}_s(t) = \hat{U}(t)\hat{H}(t)\hat{U}^+(t)$

Exercise: check

- We will use, equivalently, the Heisenberg picture:

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We will use, equivalently, the Heisenberg picture:

$$\bar{f}(t) = \langle \psi_0 | \underbrace{\left(\hat{U}^+(t)\, \hat{f}_0\, \hat{U}(t) \right)}_{=} | \psi_0 \rangle$$

$$= \langle \psi_0 | \; \hat{f}(t) \; | \psi_0 \rangle$$

with dynamics:

$$i \frac{d}{dt} \hat{f}(t) = [\hat{f}(t), \hat{H}(t)]$$

II Aspects of the Heisenberg picture:

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II Aspects of the Heisenberg picture:

- The state of the quantum system stays the same Hilbert space vector, say $|\psi\rangle \in \mathcal{H}$ (from measurement to measurement).

- The observables, say $\hat{H}(t)$, $\hat{f}(t)$, etc, are time-dependent operators in Hilbert space.

- Important implication:

The eigenbases and the eigenvalues of observables, such as $\hat{H}(t)$ and any $\hat{f}(t)$ depend on time!

$$\hat{f}(t)\,|f_n(t)\rangle = f_n(t)\,|f_n(t)\rangle$$

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Example: * Assume the driven harmonic oscillator starts out at time t_1 in n'th energy state, say $|\psi\rangle = |E_n(t_1)\rangle$:

$$\hat{H}(t_1)|E_n(t_1)\rangle = E_n(t_1)|E_n(t_1)\rangle$$

* State vector of the system stays $|\psi\rangle$ for $t > t_1$.

* But at later times, say $t > t_1$ the Hamiltonian and its eigenvectors and eigenvalues are

$$\hat{H}(t)|E_m(t)\rangle = E_m(t)|E_m(t)\rangle$$

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$\Rightarrow$ At time t_2 system is still in state $|\chi\rangle$ and still

$$|\chi\rangle = |E_n(t_1)\rangle$$

but $|\chi\rangle$ is generally no longer n'th (or any other) energy eigenstate!

In particular:

* Assume system starts out at t_1 in lowest energy state (i.e. in vacuum): $|\chi\rangle = |E_0(t_1)\rangle$

* Then if $|\chi\rangle = |E_0(t_1)\rangle \neq |E_0(t_2)\rangle$

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In particular:

* Assume system starts out at t_1 in lowest energy state (i.e. in vacuum): $|\psi\rangle = |E_0(t_1)\rangle$

* Then if $|\psi\rangle = |E_0(t_1)\rangle \neq |E_0(t_2)\rangle$

$\Rightarrow$ At t_2 the system's state $|\psi\rangle$ is not the ground state i.e. not the vacuum state, i.e. particles (e.g. photons) exist at time t_2.

III Strategy for solving quantized driven harmonic oscillator

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III Strategy for solving quantized driven harmonic oscillator

Problem:

* CCR: $[\hat{q}(t), \hat{p}(t)] = i\mathbb{1}$

* Hermiticity: $\hat{q}^+(t) = \hat{q}(t), \quad \hat{p}^+(t) = \hat{p}(t)$

* Hamiltonian: $\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\hat{q}(t)$

* Heisenberg eqns $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ yield:

$$\dot{\hat{q}}(t) = \hat{p}(t)$$

$$\dot{\hat{p}}(t) = -\omega^2\hat{q}(t) + J(t)$$

This is a good strategy with and without a driving force

is operator even though no "hat".

Strategy: * Combine $a(t) := \alpha\,\hat{q}(t) + i\beta\,\hat{p}(t)$ (analogous to "real" & "imaginary" parts)

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IV Determine α and β:

- Notice first that once we have $a(t)$ we immediately obtain $\hat{q}(t), \hat{p}(t)$: Use of $a^+(t) = \alpha\,\hat{q}(t) - i\beta\,\hat{p}(t)$ yields:

$$\hat{q}(t) = \frac{1}{2\alpha}\left(a^+(t) + a(t)\right)$$

$$\hat{p}(t) = \frac{i}{2\beta}\left(a^+(t) - a(t)\right)$$

- Use this to express $[\hat{q},\hat{p}] = i$ in terms of new variable $a(t)$:

$$\Rightarrow \quad [a(t), a^+(t)] = 2\alpha\beta$$

For simplicity, we choose $\beta = \frac{1}{2\alpha}$ so that:

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- Now express $\hat{H}(t)$ in terms of new variable $a(t)$:

$$\hat{H}(t) = -\frac{1}{2}\alpha^2 \left(a^+(t) - a(t)\right)^2 + \frac{\omega^2}{2}\frac{1}{4\alpha^2}\left(a^+(t) + a(t)\right)^2$$

$$- J(t)\frac{1}{2\alpha}\left(a^+(t) + a(t)\right)$$

We notice that the terms $\sim a^+(t)^2$ and $\sim a(t)^2$ drop out if we choose:

$$-\frac{1}{2}\alpha^2 + \frac{\omega^2}{2}\frac{1}{4\alpha^2} = 0$$

Thus, we choose: $\alpha = \sqrt{\frac{\omega}{2}}$ and therefore $\beta = \frac{1}{\sqrt{2\omega}}$

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- Thus, by definition:

$$a(t) = \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)$$

- The Hamiltonian simplifies to become:

$$\hat{H}(t) = \omega\left(a^+(t)a(t) + \frac{1}{2}\right) - J(t)\frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right)$$

V Solve for $a(t)$:

- The Heisenberg equation $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ reads for $a(t)$:

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V Solve for $a(t)$:

- The Heisenberg equation $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ reads for $a(t)$:

$$i\,\dot{a}(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}} J(t)$$

- Let us give $a(t=0)$ a name: $a_{in} = a(0)$. Then:

Exercise: verify.

$$\boxed{a(t) = a_{in}\, e^{-i\omega t} + 1\, \frac{i}{\sqrt{2\omega}} \int_0^t J(t')\, e^{i\omega(t'-t)}\, dt'}$$

VI Case of force of finite duration

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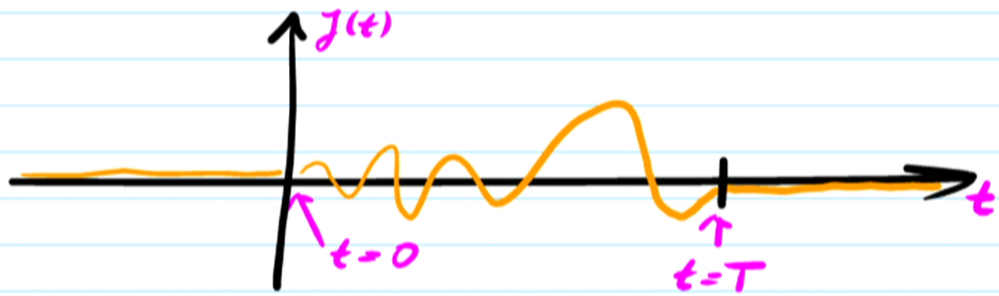
VI Case of force of finite duration

- Assume $J(t) = 0$ for all $t \notin [0, T]$



- Define $$J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t')\, e^{i\omega t'}\, dt'$$

- Then: $C_0 \cdot e^{-i\omega t}$ for $t < 0$

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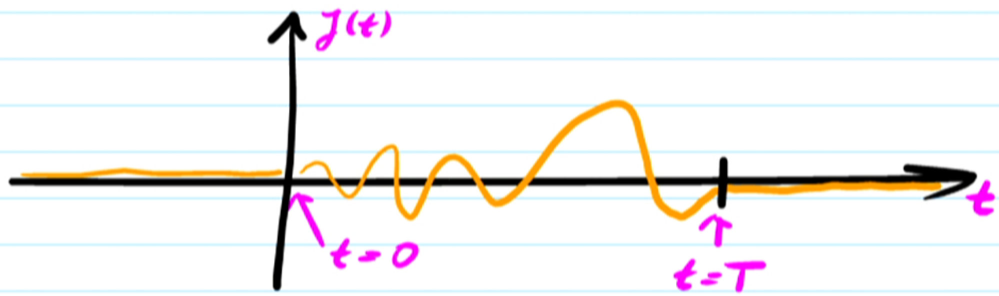
□ Assume $J(t) = 0$ for all $t \notin [0, T]$



□ Define
$$J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t')\, e^{i\omega t'}\, dt'$$

□ Then:
$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0, T] \\ (a_{in} + J_0)\, e^{-i\omega t} & \text{for } t > T \end{cases}$$

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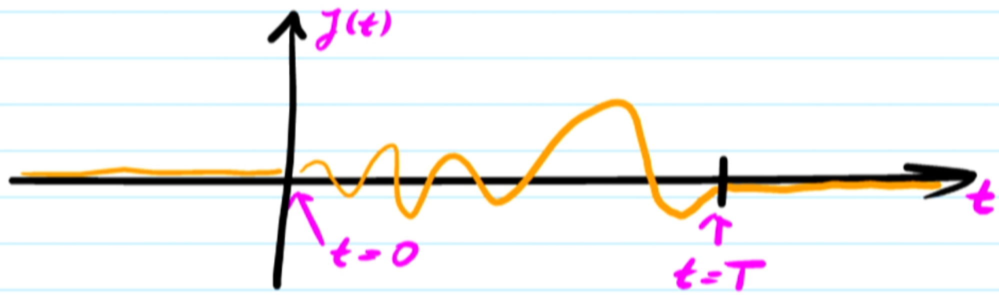
- Assume $J(t) = 0$ for all $t \notin [0,T]$



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