Title: Introduction to Quantum Field Theory for Cosmology - Lecture 3

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Abstract:

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QFT for Cosmology, Achim Kempf, Winter 2014, Lecture 3

Note Title

- Quantization conditions:

$$\left[\hat{\phi}(x,t),\ \hat{\pi}(x',t)\right] = i\hbar\,\delta^3(x-x')$$

analogous to:

$$\left[\hat{q}_a(t),\ \hat{p}_{a'}(t)\right] = i\hbar\,\delta_{aa'}$$

$$\left[\hat{\phi}(x,t),\ \hat{\phi}(x',t)\right] = 0$$

$$\left[\hat{q}_a(t),\ \hat{q}_{a'}(t)\right] = 0$$

$$\left[\hat{\pi}(x,t),\ \hat{\pi}(x',t)\right] = 0$$

$$\left[\hat{p}_a(t),\ \hat{p}_{a'}(t)\right] = 0$$

- We keep the equations of motion:

$$\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t)$$

(EI)

$$\dot{\hat{q}}_a(t) = \hat{p}_a(t)$$

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$$\left[\hat{\pi}(x,t),\hat{\pi}(x',t)\right]=0 \qquad \left[\hat{p}_a(t),\hat{p}_{a'}(t)\right]=0$$

- We keep the equations of motion:

$$\dot{\hat{\phi}}(x,t)=\hat{\pi}(x,t) \qquad \text{(E1)} \qquad \dot{\hat{q}}_a(t)=\hat{p}_a(t)$$

$$\dot{\hat{\pi}}(x,t)=-\left(-\Delta+m^2\right)\hat{\phi}(x,t) \qquad \text{(E2)} \qquad \dot{\hat{p}}_a(t)=-K_a\hat{q}_a(t)$$

- Note: $\phi^*(x,t)=\phi(x,t)$ now implies hermiticity: $\hat{\phi}^\dagger(x,t)=\hat{\phi}(x,t)$

- Proposition:

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Proposition:

E1, E2 follow from the Heisenberg eqns

$$i\hbar\,\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}]$$

$$i\hbar\,\dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}]$$

analogous to:

$$i\hbar\,\dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$$

$$i\hbar\,\dot{\hat{p}}_a(t) = [\hat{p}_a(t), \hat{H}]$$

with this QFT Hamiltonian:

$$\hat{H} = \int \frac{1}{2}\hat{\pi}^2(x',t) + \frac{1}{2}\hat{\phi}(x',t)\,(m^2 - \Delta)\,\hat{\phi}(x',t)\,d^3x'$$

$$\hat{H} = \sum_a{}' \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2}{2}\hat{q}_a^2$$

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Proposition:

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$$\hat{H} = \int \frac{1}{2}\hat{\pi}^2(x',t) + \frac{1}{2}\hat{\phi}(x',t)(m^2 - \Delta)\hat{\phi}(x',t)\,d^3x'$$

$$\hat{H} = \sum_a \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2}{2}\hat{q}_a^2$$

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Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields ✓
3. 2nd quantization ✓
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

4. Harmonic oscillators in quantum fields

- From the above, we need to solve 2 equations:

a.) The K.G. eqn: $\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right) \hat{\phi}(x,t) = 0$, with $x = (x_1, x_2, x_3)$

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4. Harmonic oscillators in quantum fields

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a.) The K.G. eqn: $\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right) \hat{\phi}(x,t) = 0$, with $x = (x_1, x_2, x_3)$

b.) The commutation rels: $[\hat{\phi}(x,t), \hat{\dot{\phi}}(x',t)] = i\hbar\,\delta(x-x')$

- Q: How to solve these eqns?

A: Use similarity to harmonic oscillator problem

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- **Q**: How to solve these eqns?

A: Use similarity to harmonic oscillator problem after overcoming a few technical difficulties:

1st Difficulty: (in reducing the QFT problem to harmonic oscillators)

- In the K.G. equation,

$$\dot{\hat{\pi}}(x,t) = -\left(-\Delta + m^2\right)\hat{\phi}(x,t) \quad \text{Analogy} \quad \dot{\hat{p}}_a(t) = -K_a\,\hat{q}_a(t)$$

we notice that $(-\Delta + m^2)$, unlike K_a, is not a number!

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1st Difficulty: (in reducing the QFT problem to harmonic oscillators)

- In the K.G. equation,

$$\dot{\hat{\pi}}(x,t) = -\left(-\Delta + m^2\right)\hat{\phi}(x,t) \quad \text{Analogy} \quad \dot{\hat{p}}_a(t) = -K_a\,\hat{q}_a(t)$$

we notice that $(-\Delta + m^2)$, unlike K_a, is not a number!

- Q: Can we "transform" $(-\Delta + m^2)$ into a number?

A: Yes: Fourier transform turns derivatives into numbers!

Fourier transform of the spatial variables x_i:

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Fourier transform of the spatial variables x_i:

Definition:

$$\hat{\phi}(k,t) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix\cdot k}\, \hat{\phi}(x,t)\, d^3x$$

$x \cdot k = \sum_{i=1}^{3} x_i k_i \quad ; \quad k = (k_1, k_2, k_3)$

Traditional notation: $\hat{\phi}_k(t) := \hat{\phi}(k,t)$

Traditional terminology: $\hat{\phi}_k(t)$ is called the field's **k-mode**.

Inverse Fourier transform:

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Proposition: (Exercise: show this)

Analogous to:

a.) $\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}_k^+(t)\hat{\pi}_k(t) + \frac{1}{2}\hat{\phi}_k^+(t)\left(k^2+m^2\right)\hat{\phi}_k(t)\, d^3k$, $k^2 = \sum_{i=1}^{3} k_i^2$

$\hat{H} = \sum_a{}' \frac{1}{2}\hat{p}_a\hat{p}_a + \frac{1}{2}\omega_a \hat{q}_a\hat{q}_a$

b.) $[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\,\delta^3(k+k')$

$[\hat{q}_a, \hat{p}_{a'}] = i\hbar\,\delta_{a,a'}$

$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0$

$[\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$

$[\hat{\pi}_k(t), \hat{\pi}_{k'}(t)] = 0$

$[\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$

c.) $\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t)$

$\dot{\hat{q}}_a(t) = \hat{p}_a(t)$

$\dot{\hat{\pi}}_k(t) = -(k^2+m^2)\hat{\phi}_k(t)$

$\dot{\hat{p}}_a(t) = -\omega_a^2\hat{q}_a(t)$

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$$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0 \qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

$$[\hat{\pi}_k(t), \hat{\pi}_{k'}(t)] = 0 \qquad [\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$$

c) $\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t)$ $\qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$

$\dot{\hat{\pi}}_k(t) = -(k^2 + m^2)\,\hat{\phi}_k(t)$ $\qquad \dot{\hat{p}}_a(t) = -\omega_a^2\,\hat{q}_a(t)$

$\Rightarrow$ For each mode $\vec{k}$ we seem to have a harmonic oscillator with $\omega_k = \sqrt{k^2 + m^2}$.

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Exercise:

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Exercise:

Show that a) + b) + Heisenberg eqn $\dot{\hat{f}}(t) = \frac{1}{i\hbar}[\hat{f}(t), \hat{H}]$ yields c.)

($\hat{f}$ is arbitrary. E.g. $\hat{f} = \hat{\phi}_k$ or $\hat{f} = \hat{\pi}_k$)

2nd Difficulty: (in reducing the QFT problem to harmonic oscillators)

We notice that the commutation relations

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\,\delta^3(k+k') \quad \text{and} \quad [\hat{q}_a, \hat{p}_{a'}] = i\hbar\,\delta_{a,a'}$$

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2nd Difficulty: (in reducing the QFT problem to harmonic oscillators)

- We notice that the commutation relations

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\,\delta^3(k+k') \quad \text{and} \quad [\hat{q}_a, \hat{p}_{a'}] = i\hbar\,\delta_{a,a'}$$

do not match, because the Kronecker δ is only either 0 or 1, unlike the Dirac δ!

- Idea: If use Fourier series instead, should have discrete values of k, thus Kronecker δ for CCR!

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- Idea: If use Fourier series instead, should have discrete values of k, thus Kronecker δ for CCR!

- Strategy:

1. Put system into a large box $[-L/2, L/2]^{\times 3}$
2. Assume (for example) periodic boundary conditions.
   (If box large enough it should not matter here what happens at the boundary of the box)
3. Instead of Fourier transform, we can now use Fourier series.

Terminology: Putting a system in a box is called

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Strategy:

1. Put system into a large box $[-L/2, L/2]^{\times 3}$
2. Assume (for example) periodic boundary conditions.
   (If box large enough it should not matter here what happens at the boundary of the box)
3. Instead of Fourier transform, we can now use Fourier series.

Terminology: Putting a system in a box is called "Infrared regularization".
Because "long" wavelengths are removed.

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Infrared regularization:

* $(k_1, k_2, k_3) = \frac{2\pi}{L}(n_1, n_2, n_3)$ with $n_1, n_2, n_3 \in \mathbb{Z}$

* $V = L^3$ (Volume of box)

* Fourier series expansion coefficients:

$$\hat{\phi}_k(t) = V^{-1/2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \hat{\phi}(x,t)\, e^{-ixk}\, d^3x$$

* The inverse is the Fourier series:

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The QFT problem in the box:

a.) $\hat{H} = \sum_k \frac{1}{2} \hat{\pi}_k^+ \hat{\pi}_k + \frac{1}{2} \omega_k^2 \hat{\phi}_k^+ \hat{\phi}_k$

$\omega_k^2 = k^2 + m^2$

analogous to

$\hat{H} = \sum_a \frac{1}{2} \hat{p}_a \hat{p}_a + \frac{1}{2} \omega_a^2 \hat{q}_a \hat{q}_a$

b.) $[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar \delta_{k,-k'}$ ← Kronecker δ

$[\hat{q}_a, \hat{p}_{a'}] = i\hbar \delta_{a,a'}$

$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0$

$[\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$

$[\hat{\pi}_k(t), \hat{\pi}_{k'}(t)] = 0$

$[\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$

c.) $\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t)$

$\dot{\hat{q}}_a(t) = \hat{p}_a(t)$

$\dot{\hat{\pi}}_k(t) = -(k^2 + m^2)\hat{\phi}_k(t) + \ldots$

$\dot{\hat{p}}_a(t) = \ldots$

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The QFT problem in the box:

a) $\hat{H} = \sum_k \frac{1}{2} \hat{\pi}_k^+ \hat{\pi}_k + \frac{1}{2} \omega_k^2 \hat{\phi}_k^+ \hat{\phi}_k$

$\omega_k^2 = k^2 + m^2$

analogous to

$\hat{H} = \sum_a \frac{1}{2} \hat{p}_a \hat{p}_a + \frac{1}{2} \omega_a^2 \hat{q}_a \hat{q}_a$

b) $[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar \delta_{k,-k'}$ (Kronecker δ)

$[\hat{q}_a, \hat{p}_{a'}] = i\hbar \delta_{a,a'}$

$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0$

$[\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$

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c) $\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t)$

$\dot{\hat{q}}_a(t) = \hat{p}_a(t)$

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3rd Difficulty: (in reducing the QFT problem to harmonic oscillators)

Hermiticity:

We notice that $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$, $\hat{\pi}^+(x,t) = \hat{\pi}(x,t)$ implies

$$\hat{\phi}_k^+(t) = \hat{\phi}_{-k}(t), \quad \hat{\pi}_k^+(t) = \hat{\pi}_{-k}(t) \qquad (H)$$

(Indeed:

$$\hat{\phi}_k^+(t) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\,\hat{\phi}^+(x,t)\,d^3x = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\,\hat{\phi}(x,t)\,d^3x = \hat{\phi}_{-k}(t)$$

)

But eqns (H) do not match:

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(Indeed:

$$\hat{\phi}_k^+(t) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\,\hat{\phi}^+(x,t)\,d^3x = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\,\hat{\phi}(x,t)\,d^3x = \hat{\phi}_{-k}(t)$$

)

But eqns (H) do not match:

$$\hat{q}_a^+(t) = \hat{q}_a(t) \qquad \hat{p}_a^+(t) = \hat{p}_a(t)$$

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Proper treatment:

- Define new variables $\hat{q}_k, \hat{p}_k$, which are proper oscillators:

  Eqns of motion: $\dot{\hat{p}}_k = \dot{\hat{q}}_k$, $\dot{\hat{p}}_k = -\omega_k^2 q_k$

  Canon. com. rels: $[\hat{q}_k, \hat{p}_{k'}] = i\delta_{k,k'}$

  Hermiticity: $\hat{q}_k^+ = \hat{q}_k$ $\hat{p}_k^+ = \hat{p}_k$

- Then, try ansatz:

$$\hat{\phi}_k = \frac{1}{2}\left(\hat{q}_k + \hat{q}_{-k}\right) + \frac{i}{\omega_k 2}\left(\hat{p}_k - \hat{p}_{-k}\right) \qquad (A)$$

Remark: With $a_k := \sqrt{\omega_k}\,\hat{x}_k + \frac{i}{\sqrt{\omega_k}}\hat{p}_k$ this ansatz reads:

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Proper treatment:

- Define new variables $\hat{q}_k, \hat{p}_k$, which are proper oscillators:

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- Then, try ansatz:

$$\hat{\phi}_k = \frac{1}{2}\left(\hat{q}_k + \hat{q}_{-k}\right) + \frac{i}{\omega_k 2}\left(\hat{p}_k - \hat{p}_{-k}\right) \qquad (A)$$

Remark: With $a_k := \sqrt{\omega_k}\,\hat{x}_k + \frac{i}{\sqrt{\omega_k}}\hat{p}_k$ this ansatz reads:

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- Now, show that ansatz (A) succeeds, i.e., that indeed:

Hamiltonian $\hat{H} = \sum_k \frac{1}{2}\hat{p}_k^2 + \frac{1}{2}\omega_k^2 \hat{q}_k^2$ (H)

Eqns of motion: $\dot{\hat{q}}_k = \hat{p}_k$, $\dot{\hat{p}}_k = -\omega_k^2 \hat{q}_k$

Canon. com. rels: $[\hat{q}_k, \hat{p}_{k'}] = i\,\delta_{k,-k'}$

Hermiticity cond.: $\hat{q}_k^\dagger = \hat{q}_{-k}$, $\hat{p}_k^\dagger = \hat{p}_{-k}$

- Finally, via inverse Fourier series, show that:

$$\hat{\phi}(x) = \sqrt{\frac{2}{V}} \sum_k \left\{ \cos(xk)\,\hat{q}_k - \frac{1}{\omega_k} \sin(xk)\,\hat{p}_k \right\} \qquad (B)$$

Remark: *Is this ansatz unique?

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For now: We solved, using (A), the QFT eqns of the K.G. field.

Namely, we have now solved:

Eqns of motion:
$$\begin{cases} \dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t) \\ \dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\,\hat{\phi}(x,t) \end{cases}$$

Hermiticity: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$, $\hat{\pi}^+(x,t) = \hat{\pi}(x,t)$

Can. comm. rels: $[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x')$

Example: How to calculate quant. fluct. of K.G. field?

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Hermiticity cond.: $\varphi_k = \varphi_{-k}$, $\pi_k = \pi_{-k}$

Finally, via inverse Fourier series, show that:

$$\hat{\phi}(x) = \sqrt{\frac{2}{V}} \sum_k \left\{ \cos(xk)\, \hat{q}_k - \frac{1}{\omega_k} \sin(xk)\, \hat{p}_k \right\} \qquad (B)$$

Remark: * Is this ansatz unique?

* No: x's and p's could be more mixed, and (H) need not hold.

Significance of non-uniqueness?

"No particles state"

* Ground state of q_k, p_k oscillators $\Rightarrow$ Vacuum

* This need not be lowest energy state of the QFT Hamiltonian

→ Problem of vacuum identification on curved space.

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Example: How to calculate quant. fluct. of K.G. field?

1. Solve the system of ∞ many quantum harmonic oscillator degrees of freedom

$$\hat{q}_k, \hat{p}_k$$

with $\hat{H} = \sum_k \frac{1}{2}\hat{p}_k^2 + \frac{\omega_k^2}{2}\hat{q}_k^2, \qquad \omega_k = \sqrt{k^2+m^2}$

for all $k = (k_1, k_2, k_3) = \frac{2\pi}{L}(n_1, n_2, n_3)$ where $n_1, n_2, n_3 \in \mathbb{Z}$

2. Choose a state $|\Psi\rangle$ of that quantum system.

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Example: The oscillators could all be in their lowest energy state.

We preliminarily call this $|4\rangle$ the vacuum state $|4_0\rangle$.

3. Given a state $|4\rangle$, we can calculate the probability (amplitude density) for finding arbitrary values $q_a(t)$, $P_a(t)$.

Example:

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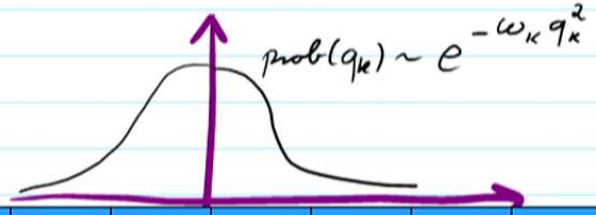
the vacuum state $|0\rangle$.

3. Given a state $|\psi\rangle$, we can calculate the probability (amplitude density) for finding arbitrary values $q_k(t)$, $p_k(t)$.

Example:

- In vacuum state, we know that the probability distribution of the $\hat{q}_k$ (and $\hat{p}_k$ as well) is gaussian:



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4. Given $|\Psi\rangle$, calculate the probability distribution of the Fourier coefficients:

$$\hat{\phi}_k, \hat{\pi}_k$$

Can do because they are simply linear combinations of the harmonic oscillator variables $\hat{q}_k, \hat{p}_k$. (Exercise: calculate)

Example:

- For $|\Psi_0\rangle$, since q_k, p_k are gaussian distributed, also the ϕ_k, π_k are gaussian distributed:

$$-\omega_k \phi_k^* \phi_k$$ (straight lines)

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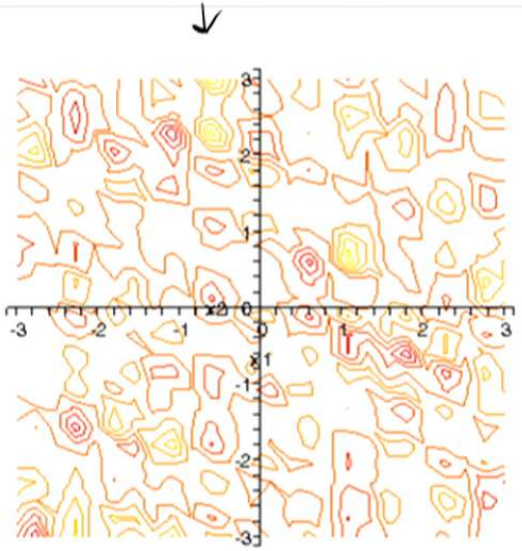
... to obtain prob. distribution of $\phi(x)$:

Example:

- Consider $|\Psi_0\rangle$.
- Draw a field $\phi(x)$ from the above calculated probability distribution for fields $\phi(x)$.

Contour lines of a typical $\phi(x,t)$ drawn from the vacuum's probability distribution for ϕ's.



← Actual draw from that distribution.

The fluctuations trace back to the Fourier coefficients and to the $\hat{q}_k$, $\hat{p}_k$ which fluctuate even in lowest energy state.

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