Title: Rebuilding Mathematics on a Quantum Logical Foundation

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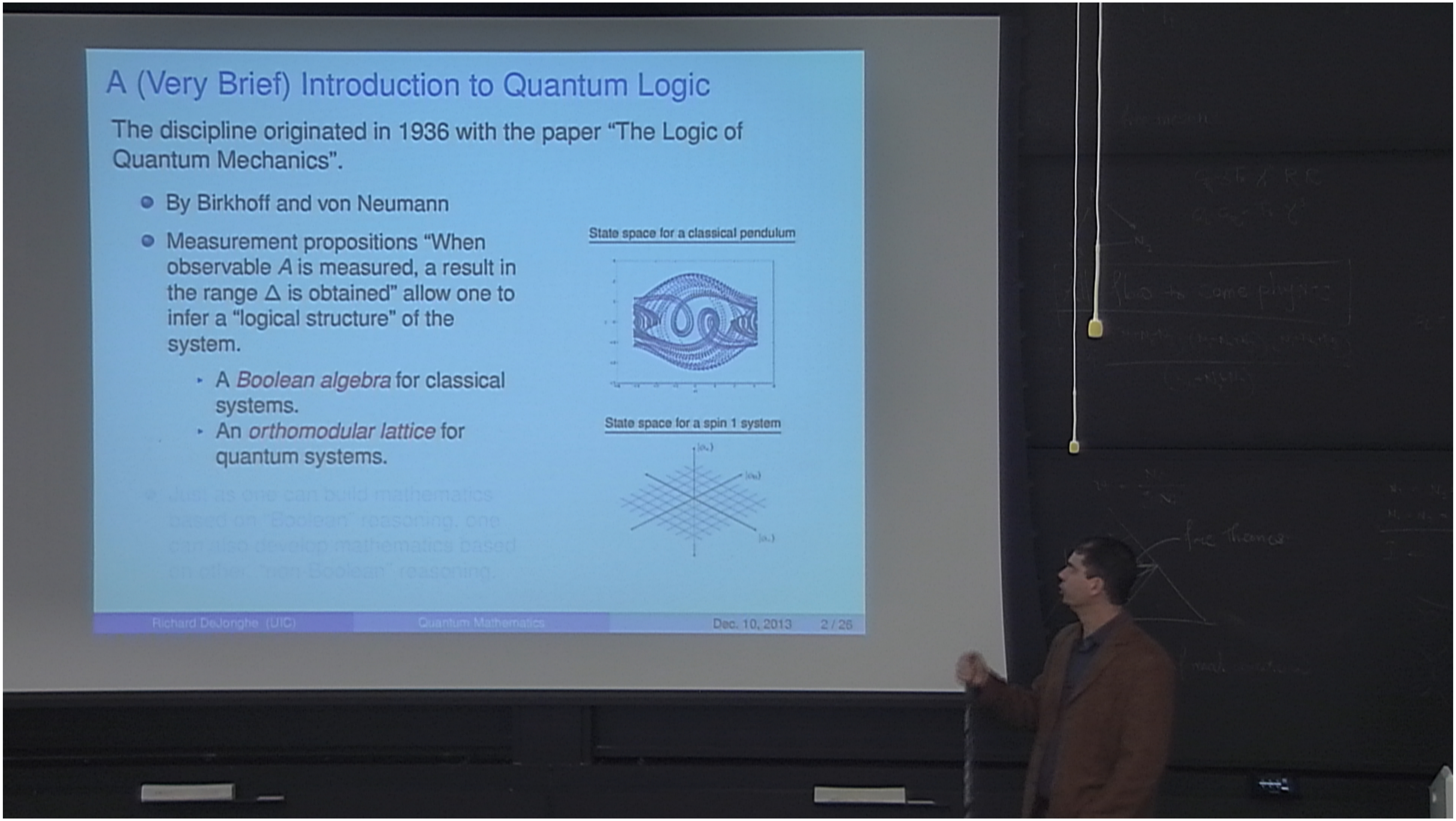
Abstract: It is not unnatural to expect that difficulties lying at the foundations of quantum mechanics can only be resolved by literally going back and rethinking the quantum theory from first principles (namely, the principles of logic). In this talk, I will present a first-order quantum logic which generalizes the propositional quatum logic originated by Birkhoff and von Neumann as well as the standard classical predicate logic used in the development of virtually all of modern mathematics. I will then use this quantum logic to begin to build the foundations of a new ``quantum mathematics" --- in particular a quantum set theory and a quantum arithmetic --- which has the potential to provide a completely new mathematical framework in which to develop the theory of quantum
 mechanics.


A (Very Brief) Introduction to Quantum Logic
The discipline originated in 1936 with the paper "The Logic of Quantum Mechanics".
By Birkhoff and von Neumann
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A Boolean algebra for classical systems.
An orthomodular lattice for quantum systems.
Just as one can build mathematics based on "Boolean" reasoning, one can also develop mathematics based on other "non-Boolean" reasoning.
State space for a classical pendulum
State space for a spin 1 system
Richard DeJonghe (UIC)
Quantum Mathematics
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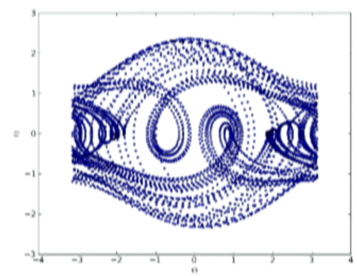

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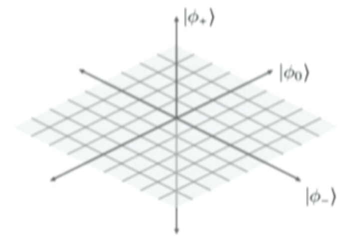

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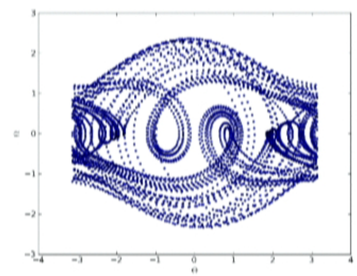

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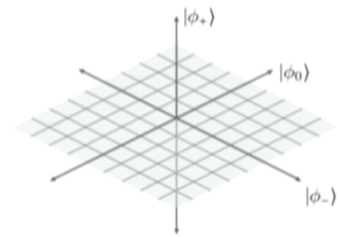

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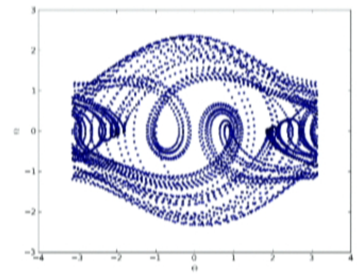

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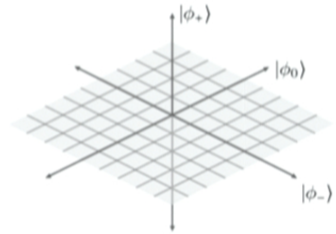

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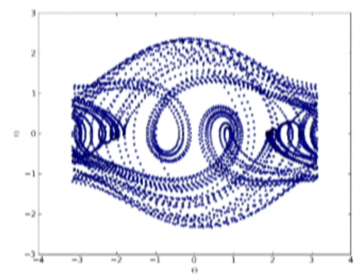

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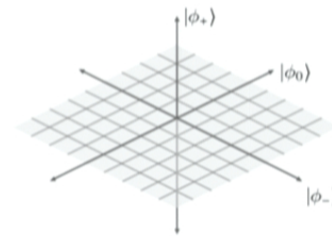

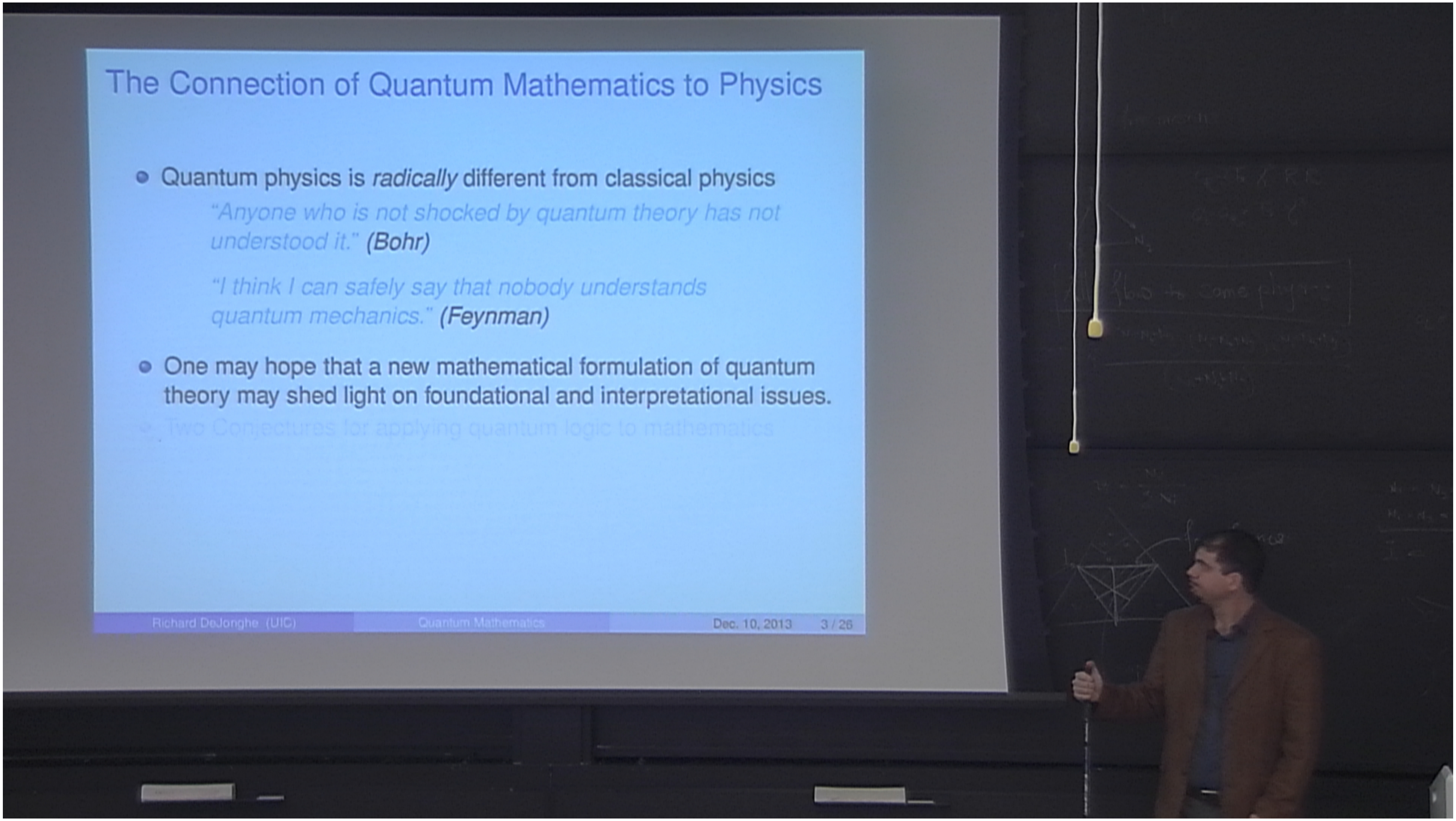
The Connection of Quantum Mathematics to Physics
Quantum physics is radically different from classical physics
"Anyone who is not shocked by quantum theory has not understood it." (Bohr)
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One may hope that a new mathematical formulation of quantum theory may shed light on foundational and interpretational issues.
Two Conjectures for applying quantum logic to mathematics
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- One may hope that a new mathematical formulation of quantum theory may shed light on foundational and interpretational issues.
- Two Conjectures for applying quantum logic to mathematics

  Strong Version: Quantum logic is *the right logic*.

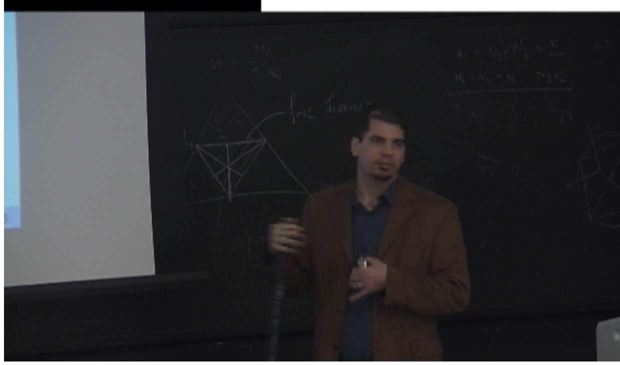
  Version: Quantum logic is *a useful logic* for developing

  matics to describe the microscopic world.

Previous Forays Into Quantum Mathematics

So far not much has been done in the field of quantum mathematics, but there have been a couple of notable developments.

- Dunn (1980): Proved that the usual axiomatization of Peano arithmetic is “inherently classical”, i.e. all the usual classical theorems of these axioms are also theorems under quantum logic.
- Takeuti (1981): Developed a quantum set theory that, while having a rich structure, is a bit unwieldy. In his own words, . . .



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  "A development of mathematics with quantum logic is not impossible. However I now feel that it is not very worthwhile because of its extreme difficulty." (Takeuti)

Outline

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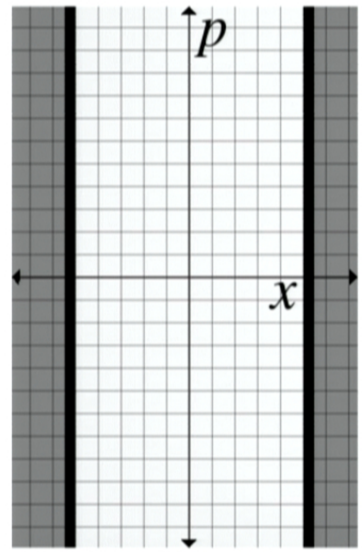
Logic of Physical Systems Encoded in State Spaces

- Classical Physics —
  Classical logic embodied in "measurement propositions"
- Classical measurement propositions equivalent to algebra of subsets of phase space

Phase space of a particle in a box:



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Example

$P: -3 \leq p \leq 3$

$\leq x \leq 3$

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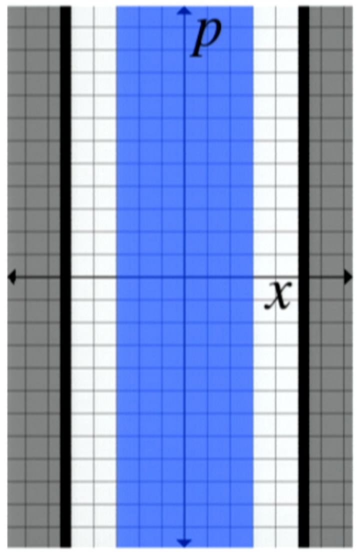

Q in blue

Logic of Physical Systems Encoded in State Spaces

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Example

Trivial propositions —
 T: whole phase space
 F: $\varnothing$

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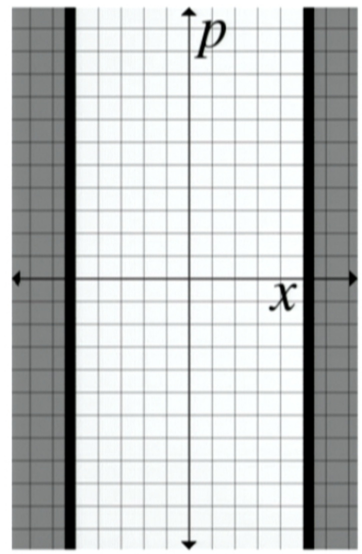


Algebra of Subsets of Phase Space

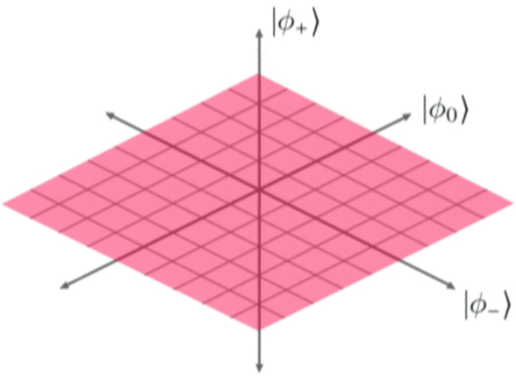
Form a *Boolean Algebra*.

Logic of a Quantum System Encoded in its State Space

- Quantum logic embodied in "quantum measurement propositions", of the form "outcome of measuring observable $\hat{A}$ is in range Δ".
- Quantum measurement propositions are equivalent to subspaces/projection operators by spectral theorem

Hilbert Space of a spin 1 particle:

o.n.b. $\{|\phi_-\rangle, |\phi_0\rangle, |\phi_+\rangle\}$



Example

$\hat{A} = -|\phi_-\rangle\langle\phi_-| + |\phi_+\rangle\langle\phi_+|$

P: outcome is ≤ 0,
xy-plane / $|\phi_-\rangle\langle\phi_-| + |\phi_0\rangle\langle\phi_0|$

'not *P* ': outcome > 0
xy-plane$^\perp$ = z-axis / $|\phi_+\rangle\langle\phi_+|$

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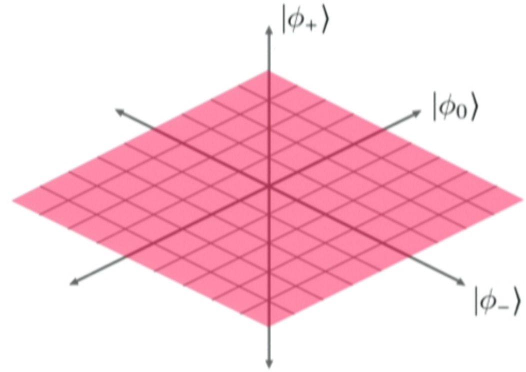

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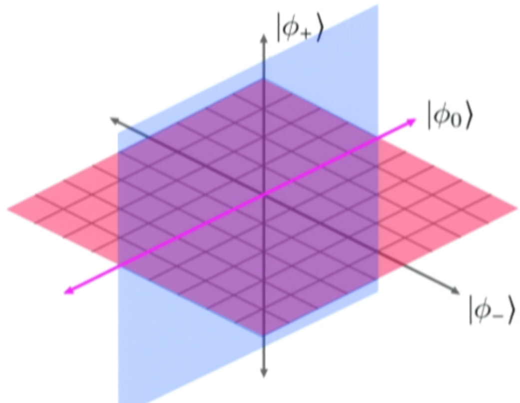

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'P and Q' :

xy-plane $\cap$ yz-plane

$=$ y-axis / $|\phi_0\rangle\langle\phi_0|$

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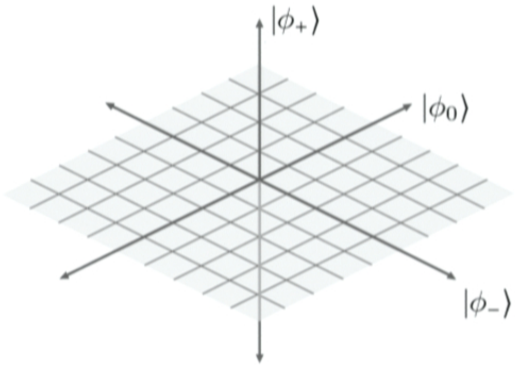

Example

Trivial propositions —

T: $\mathcal{H}$ / I

F: $\{|0\rangle\}$ / 0

Projection Operators on Hilbert Space

Form an *Orthomodular Lattice* (called the *Projection Lattice*).

Boolean Algebras vs. Orthomodular Lattices

Standardize Notation —

$$\text{and: } \cap \Leftrightarrow \wedge \,; \quad \text{or: } \cup, \text{ span} \Leftrightarrow \vee \,; \quad \text{not: } {}^{c}, \ {}^{\perp} \Leftrightarrow \neg$$

$$\text{T}: \text{ phase space}, \ \mathcal{H} \Leftrightarrow 1; \quad \text{F}: \ \varnothing, \{|0\rangle\} \Leftrightarrow 0$$

A Boolean algebra / orthomodular lattice (OML) is defined to be an abstract algebra — i.e. set L along with operations $(\wedge, \vee, \neg, 1, 0)$ satisfying certain algebraic identities.

- Many algebraic propreties in common —

$$P \vee \neg P = 1, \quad P \wedge \neg P = 0, \quad P \vee P = P, \quad P \wedge P = P \ldots$$

- Characterizing the difference — distributivity in Boolean algebras

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R),$$

but only the (weaker) orthomodularity in OMLs

$$P \vee (\neg P \vee (P \wedge Q)) = P \wedge Q$$

- There is a "quintessential" Boolean algebra — $\{0, 1\}$.

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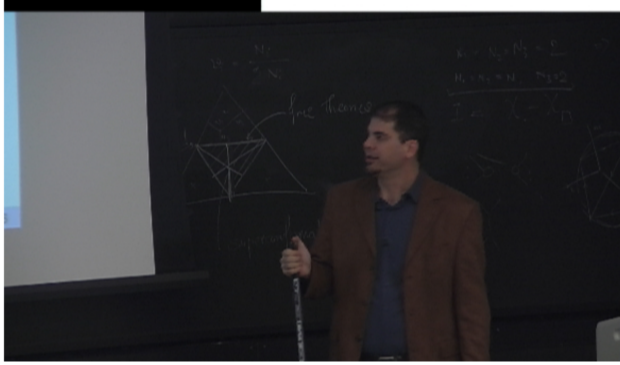

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Syntax — Formalization of Mathematical Statements

We will only be working with two different "languages", one for set theory ($\mathcal{L}_{set}$) and another for arithmetic ($\mathcal{L}_A$). This syntax is the same for both classical and quantum mathematics.

- $\mathcal{L}_{set} = \{\in\}$, and $\mathcal{L}_A = \{=, 0, ', \dot{+}, \dot{\times}\}$; divided into *predicates* and *functions*.
- Other allowed symbols include variables $(x, y, z, \ldots)$, logical connectives $(\wedge, \vee, \neg, \forall, \exists)$, and parentheses.
- Precise formal rules for constructing *formal statements* using the allowed symbols
- Two approaches to mathematical logic — *formal deductions* and *semantics*.

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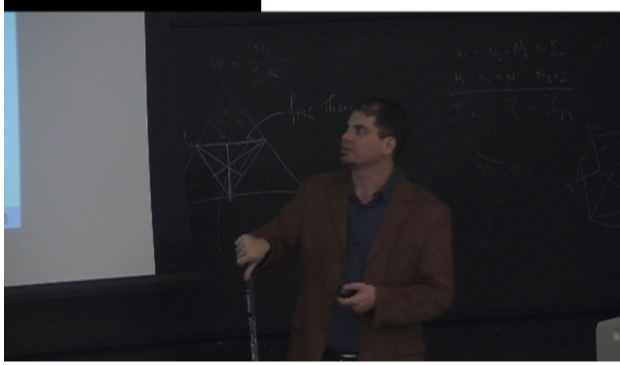

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Example: not a formal statement

$$(\forall\exists)z))x \in \neg(y$$

Example: formal statements for set theory and arithmetic

Sets: $(\exists y)(\forall x)[(x \in y) \wedge \neg(x \in y)]$ Arithmetic: $(\forall x)(x \dot{\times} 0 = 0)$

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Classical Semantics

We begin with a *special* set $\mathcal{A}$ of formal statements — the *axioms*. A *model* for these axioms consists of

- A *universe* of objects (which the variables run over)
- An *interpretation* of functions as operations on the universe.
- A *truth valuation* $[\![\cdot]\!]$ which gives the "truth value" (i.e. an element of $\{0, 1\}$) of any formal statement, and
  - $A \in \mathcal{A} \Rightarrow [\![A]\!] = 1.$
  - $[\![B]\!]$, $[\![B \wedge C]\!] = [\![B]\!] \wedge [\![C]\!]$,
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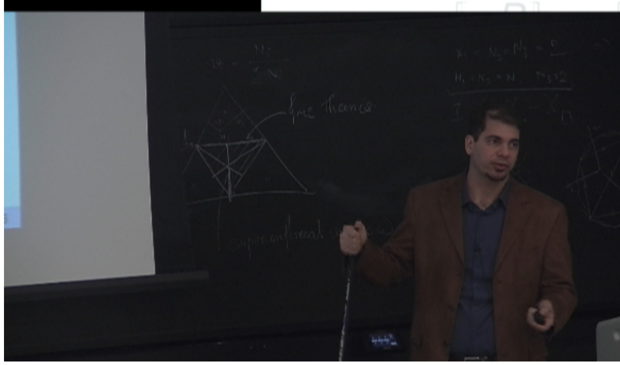

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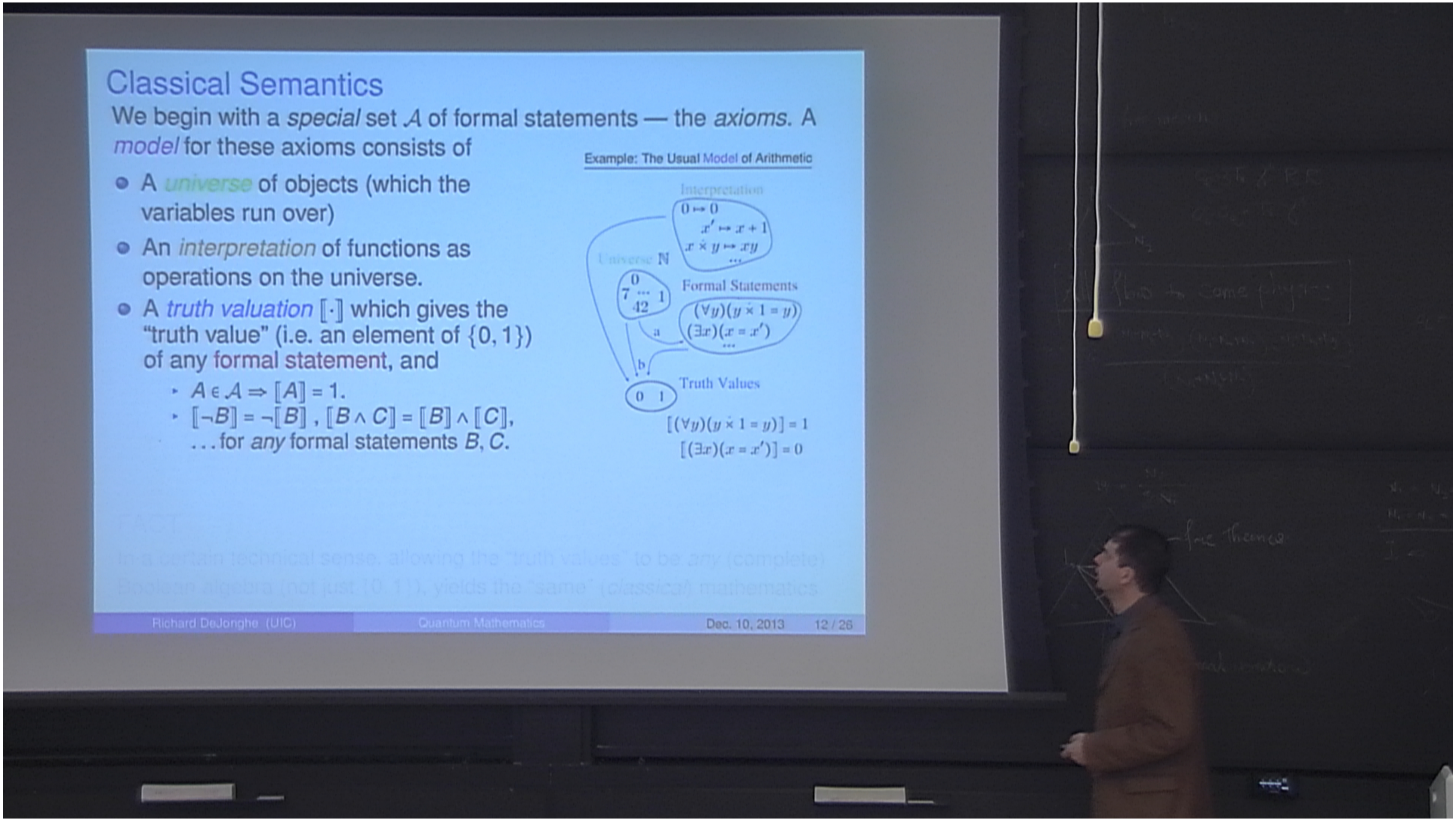

Classical Semantics
We begin with a special set $\mathcal{A}$ of formal statements — the axioms. A model for these axioms consists of
A universe of objects (which the variables run over)
An interpretation of functions as operations on the universe.
A truth valuation $[\![\cdot]\!]$ which gives the "truth value" (i.e. an element of $\{0, 1\}$) of any formal statement, and
$A \in \mathcal{A} \Rightarrow [\![A]\!] = 1$.
$[\![\neg B]\!] = \neg [\![B]\!]$, $[\![B \wedge C]\!] = [\![B]\!] \wedge [\![C]\!]$,
. . . for any formal statements B, C.
Example: The Usual Model of Arithmetic
Interpretation
$0 \mapsto 0$
$x' \mapsto x + 1$
$x \dot{\times} y \mapsto xy$
Universe $\mathbb{N}$
0 7 1 42
Formal Statements
$(\forall y)(y \dot{\times} 1 = y)$
$(\exists x)(x = x')$
a
b
Truth Values
0 1
$[\![(\forall y)(y \dot{\times} 1 = y)]\!] = 1$
$[\![(\exists x)(x = x')]\!] = 0$
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Richard DeJonghe (UIC)
Quantum Mathematics
Dec. 10, 2013
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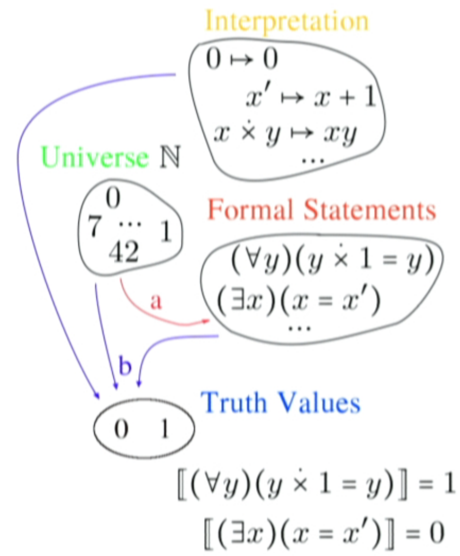

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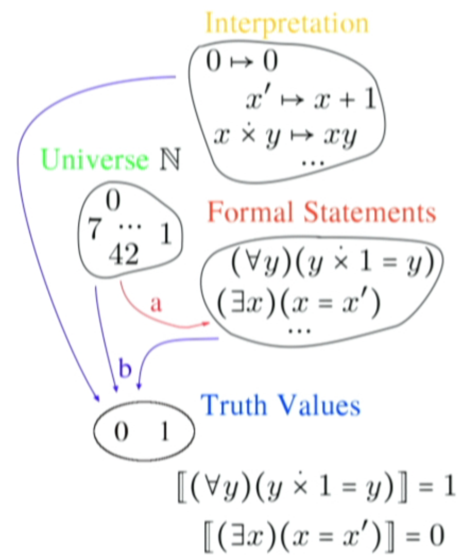

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Quantum Semantics

A *quantum model* for a set of axioms $\mathcal{A}$ is similar to a classical model, but the truth values are allowed to be *any* orthomodular lattice.

- We still have a *universe* of objects and an *interpretation* of functions as operations on the universe.
- The *truth valuation* $[\![\cdot]\!]$ now maps the formal statements into an orthomodular lattice — still require
  - $A \in \mathcal{A} \Rightarrow [\![A]\!] = 1.$
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Sets of axioms which are equivalent in classical mathematics (i.e. they have exactly the same models) may *not* be in quantum mathematics!

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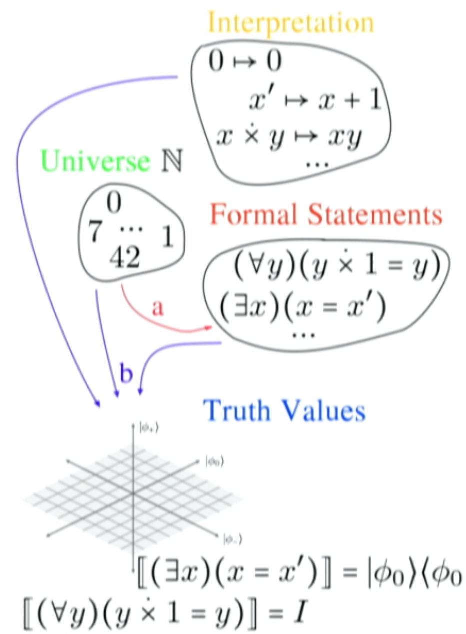
Example: A possible Quantum Model of Arithmetic



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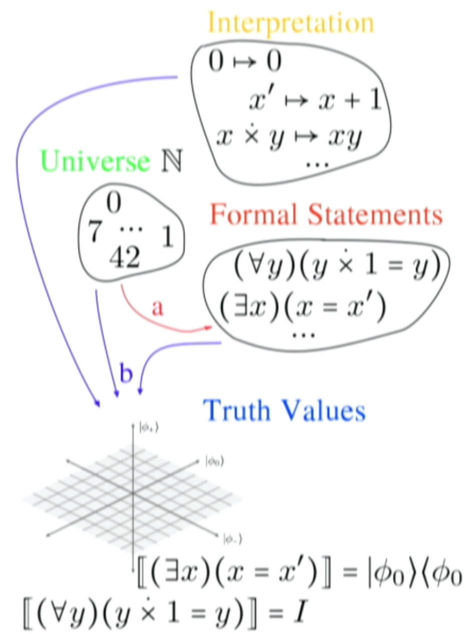

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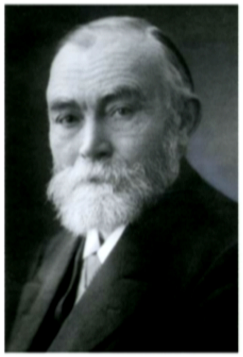
Axiomatic Set Theory

Set theory provides a foundation for virtually all of modern mathematics.

- In pure set theory *every* object is a set.
- In its original formulation by Cantor and further developed by Frege, set theory used the "axiom of abstraction" — for any formal statement $\psi(x)$ (with variable x), one could form the set $\{x : \psi(x)\}$
- Bertrand Russel then considered the set $S = \{x : x \notin x\}$ and arrived at the paradox which carries his name.
- This led to the *Zermelo-Fraenkel axioms with choice* for set theory.

Georg Cantor
Mar. 3 1845 — Jan. 6 1918



Gottlob Frege
Nov. 8 1848 — July 26 1925

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ZFC axioms for set theory

ZFC1 *Extensionality:* $(\forall x)(\forall y)[x = y \rightarrow (\forall z)(x \in z \leftrightarrow y \in z)]$.

ZFC2 *Pairing:* $(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow u = x \vee u = y)$.

ZFC3 *Separation Schema:* For ψ any wff,

$$(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow u \in x \wedge \psi(u, y)).$$

ZFC4 *Union:* $(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow (\exists z)(u \in z \wedge z \in x))$.

ZFC5 *Power Set:* $(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow u \subseteq x)$.

ZFC6 *Infinity:* $(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \rightarrow y \cup \{y\} \in x))$.

ZFC7 *Replacement Schema:* For ψ any wff,

$$[(\forall x)(\forall y)(\forall z)(\psi(x, y) \wedge \psi(x, z) \rightarrow y = z)] \rightarrow (\forall x)(\exists z)(\forall u)[u \in z \leftrightarrow (\exists y)(y \in x \wedge \psi(y, u)).$$

ZFC8 *Regularity:* $(\forall x)[x \neq \emptyset \rightarrow (\exists y)(y \in x \wedge y \cap x = \emptyset)]$.

ZFC9 *Choice:*

$$(\forall z)\big([(\forall x)(\forall y)(x \in z \rightarrow x \neq \emptyset) \wedge (x \in z \wedge y \in z \wedge x \neq y \rightarrow x \cap y = \emptyset)] \rightarrow (\exists s)(\forall t)[t \in z \rightarrow (\exists u)(s \cap t = \{u\})]\big).$$

Universes of Sets

- The standard model of set theory is the *classical universe* $\mathfrak{V}$, which consists of every possible set one can construct starting from the empty set.

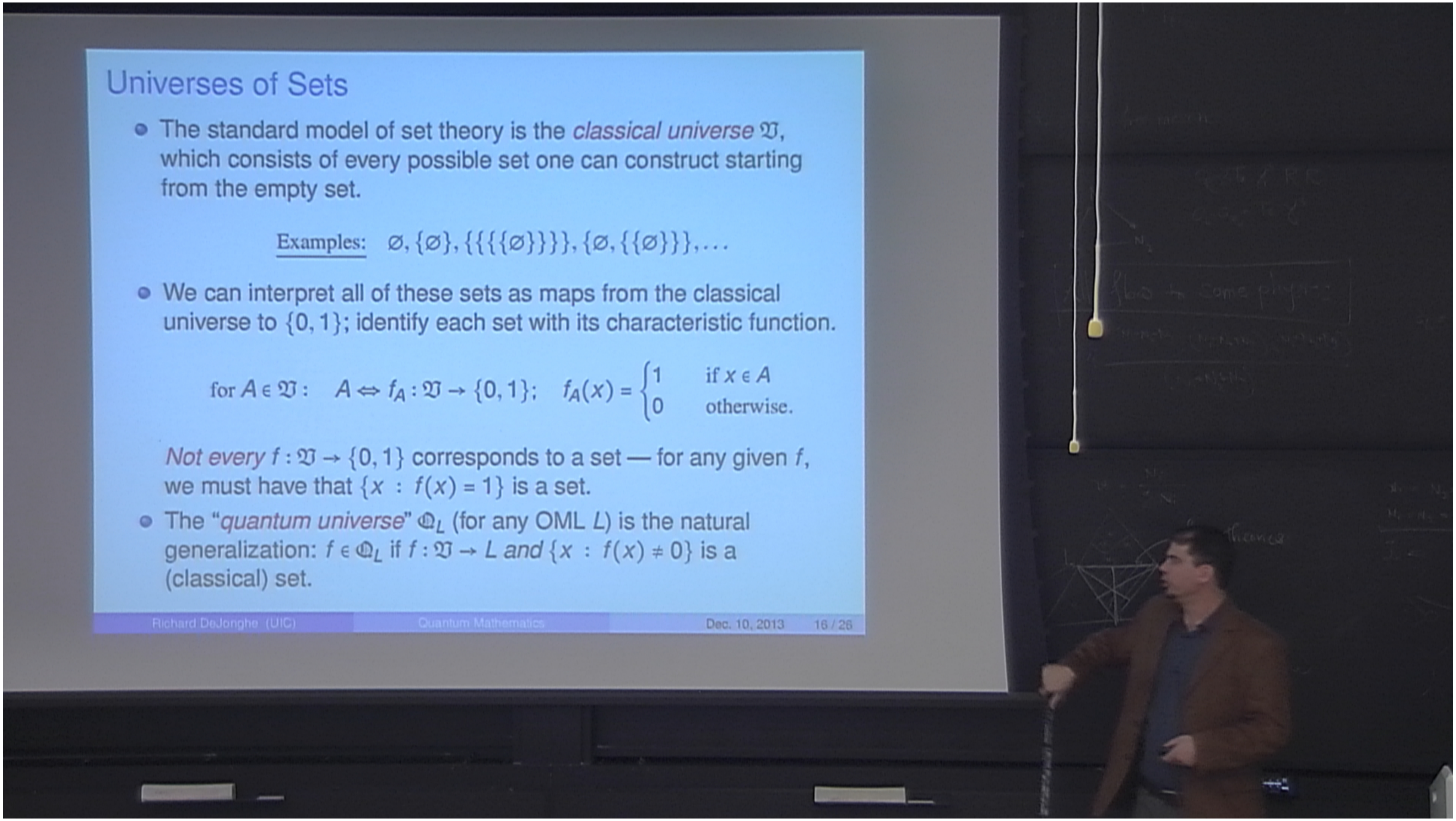
$$\underline{\text{Examples:}} \quad \varnothing, \{\varnothing\}, \{\{\{\{\varnothing\}\}\}\}, \{\varnothing, \{\{\varnothing\}\}\}, \ldots$$

- We can interpret all of these sets as maps from the classical universe to $\{0, 1\}$; identify each set with its characteristic function.

$$\text{for } A \in \mathfrak{V}: \quad A \Leftrightarrow f_A : \mathfrak{V} \to \{0,1\}; \quad f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Not every $f : \mathfrak{V} \to \{0, 1\}$ corresponds to a set — for any given f, we must have that $\{x \ : \ f(x) = 1\}$ is a set.

- The "*quantum universe*" $\mathfrak{Q}_L$ (for any OML L) is the natural generalization: $f \in \mathfrak{Q}_L$ if $f : \mathfrak{V} \to L$ *and* $\{x \ : \ f(x) \neq 0\}$ is a (classical) set.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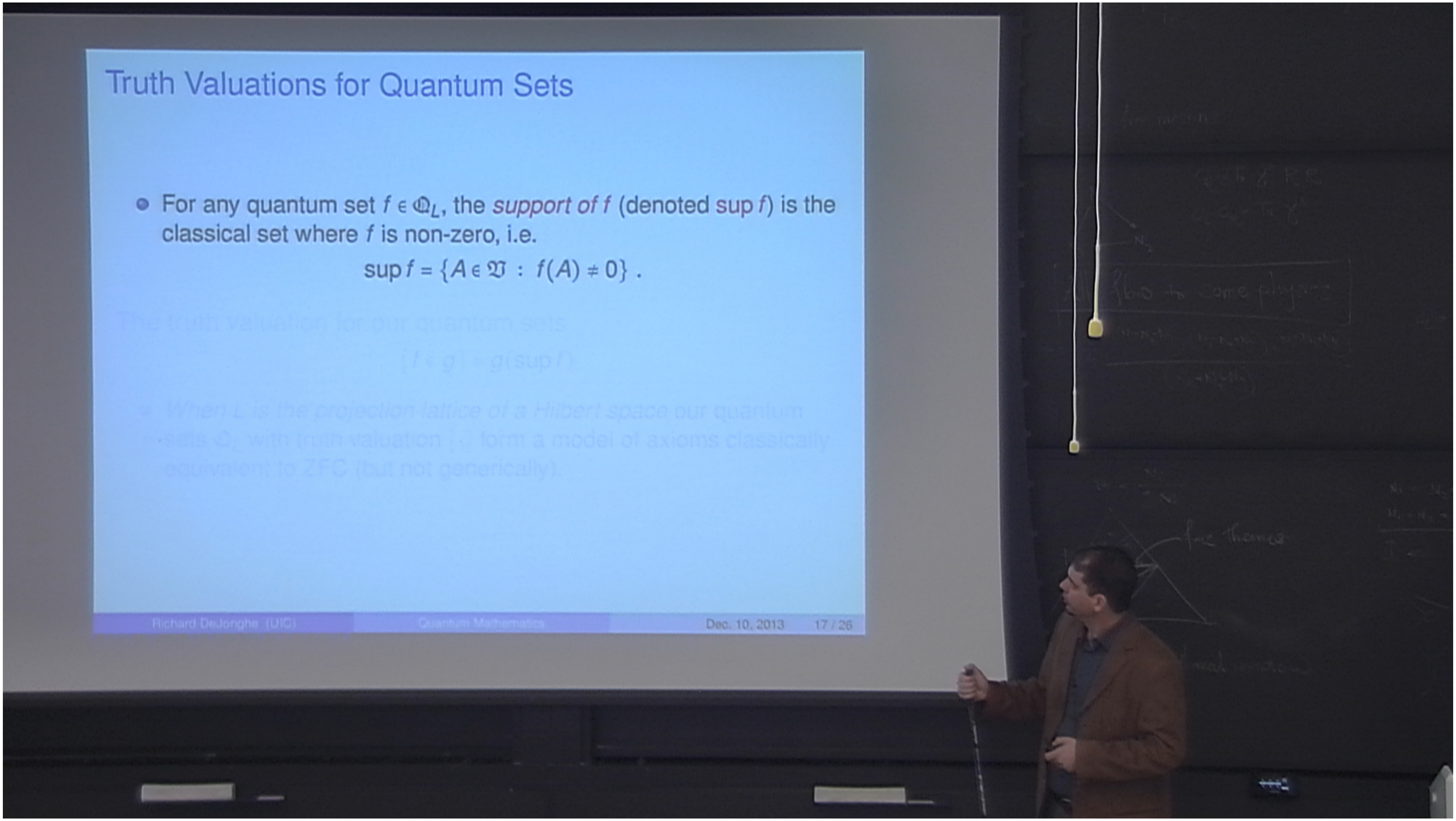
Truth Valuations for Quantum Sets
For any quantum set $f \in \mathfrak{Q}_L$, the *support of f* (denoted $\sup f$) is the classical set where f is non-zero, i.e.
$\sup f = \{A \in \mathfrak{V} : f(A) \neq 0\}$.
The truth valuation for our quantum sets
$[f \in g] = g(\sup f)$
When L is the projection lattice of a Hilbert space our quantum sets $\mathfrak{Q}_L$ with truth valuation $[\cdot]$ form a model of axioms classically equivalent to ZFC (but not generically).
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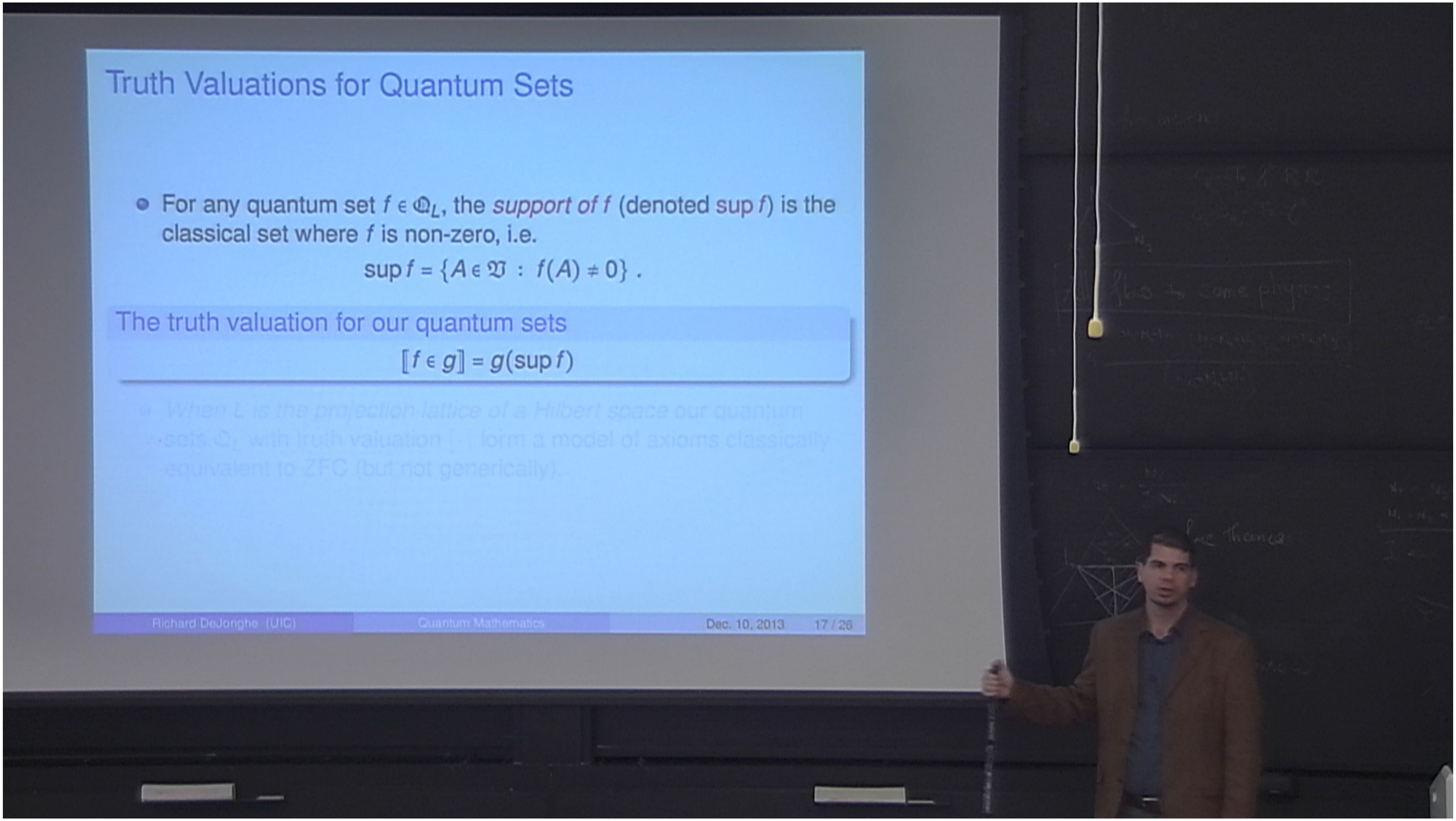
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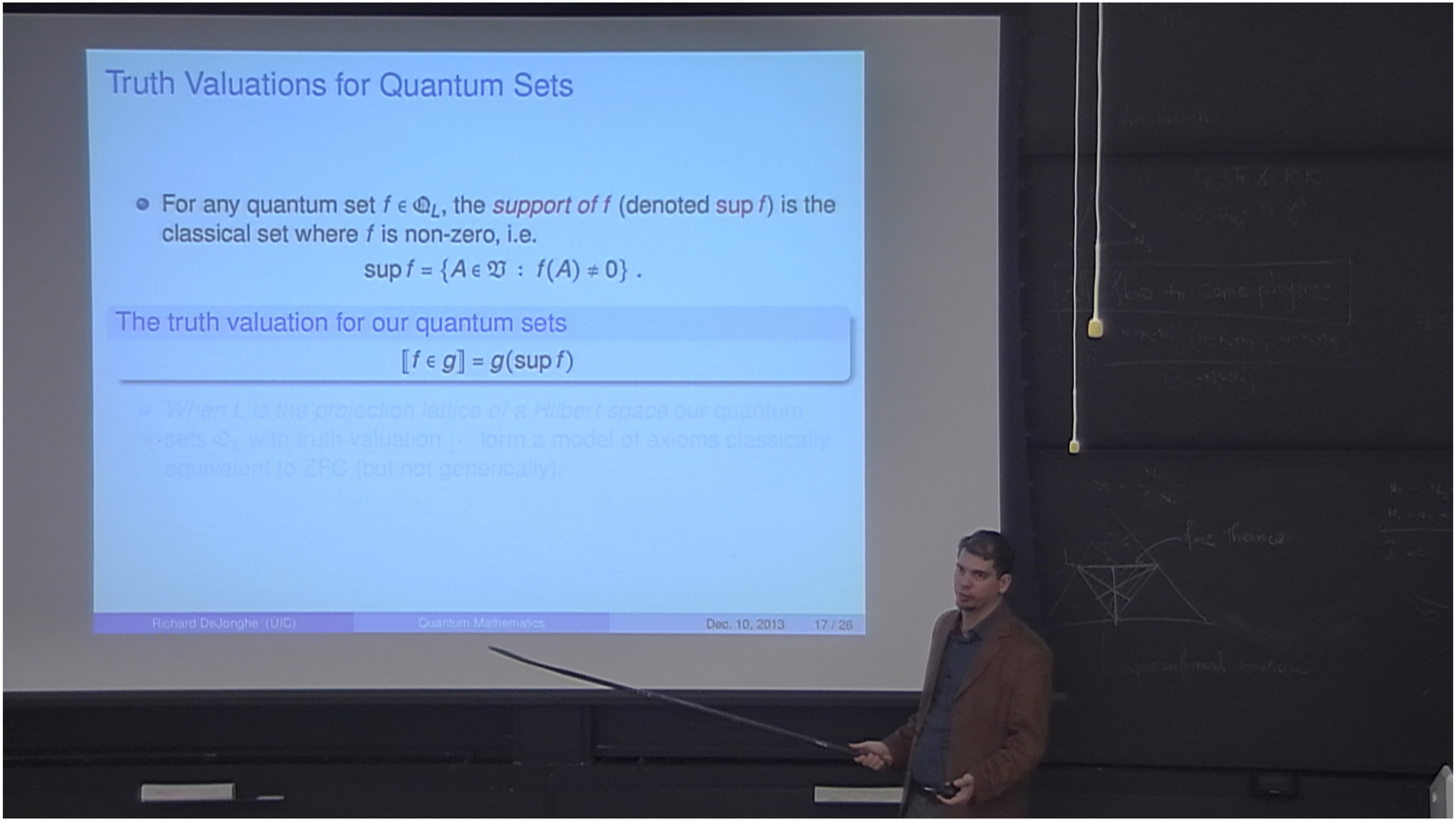
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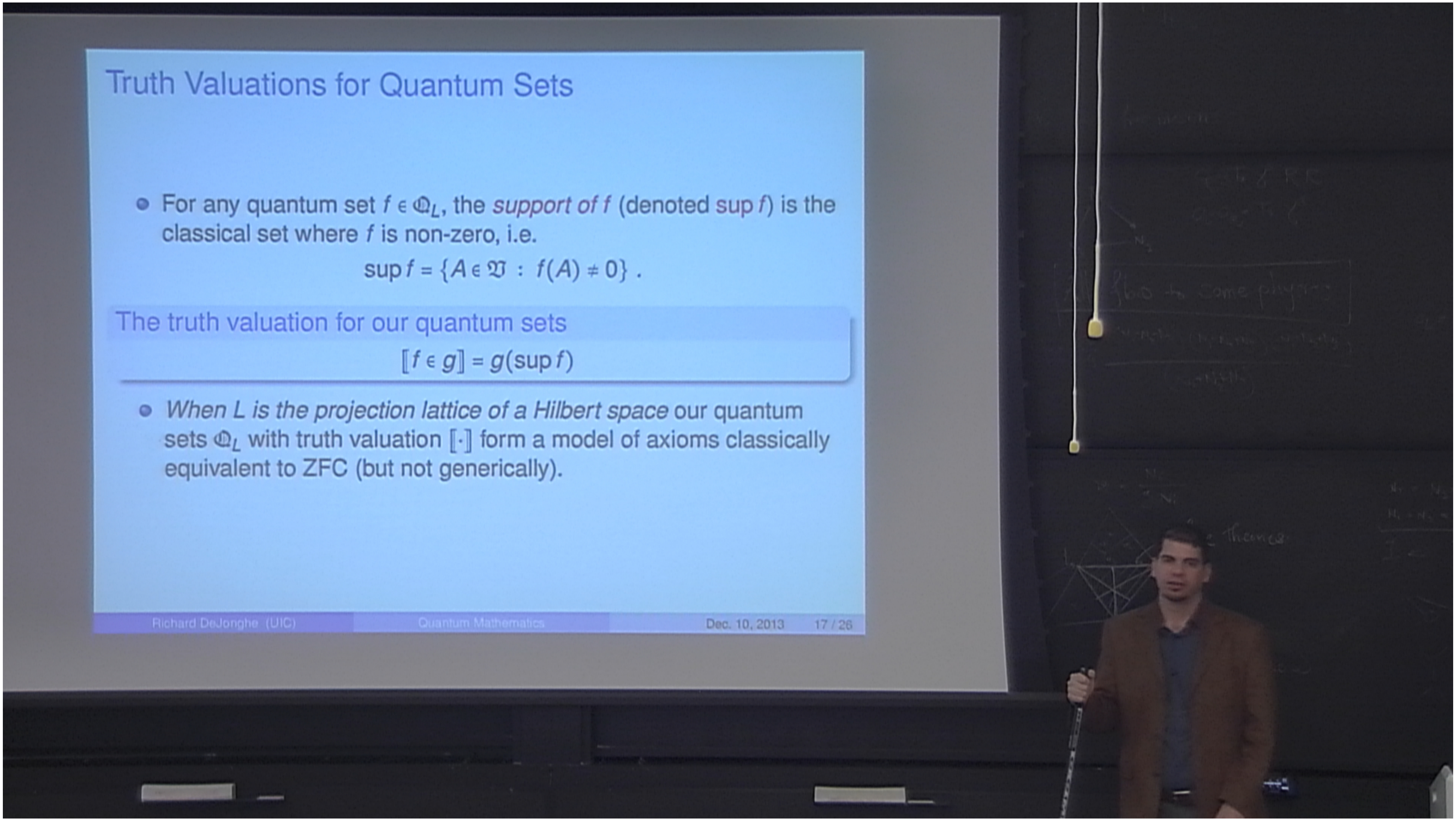
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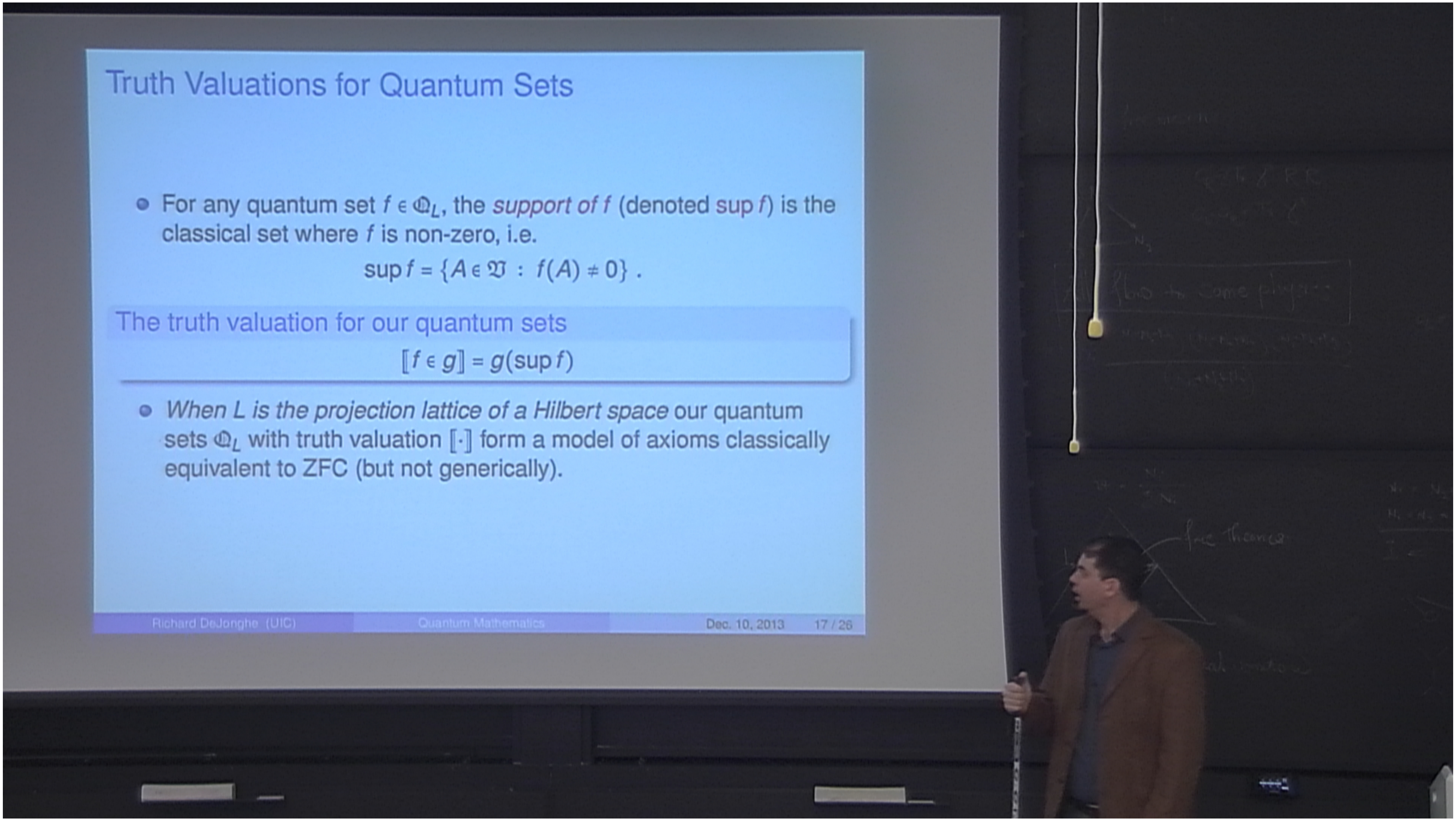
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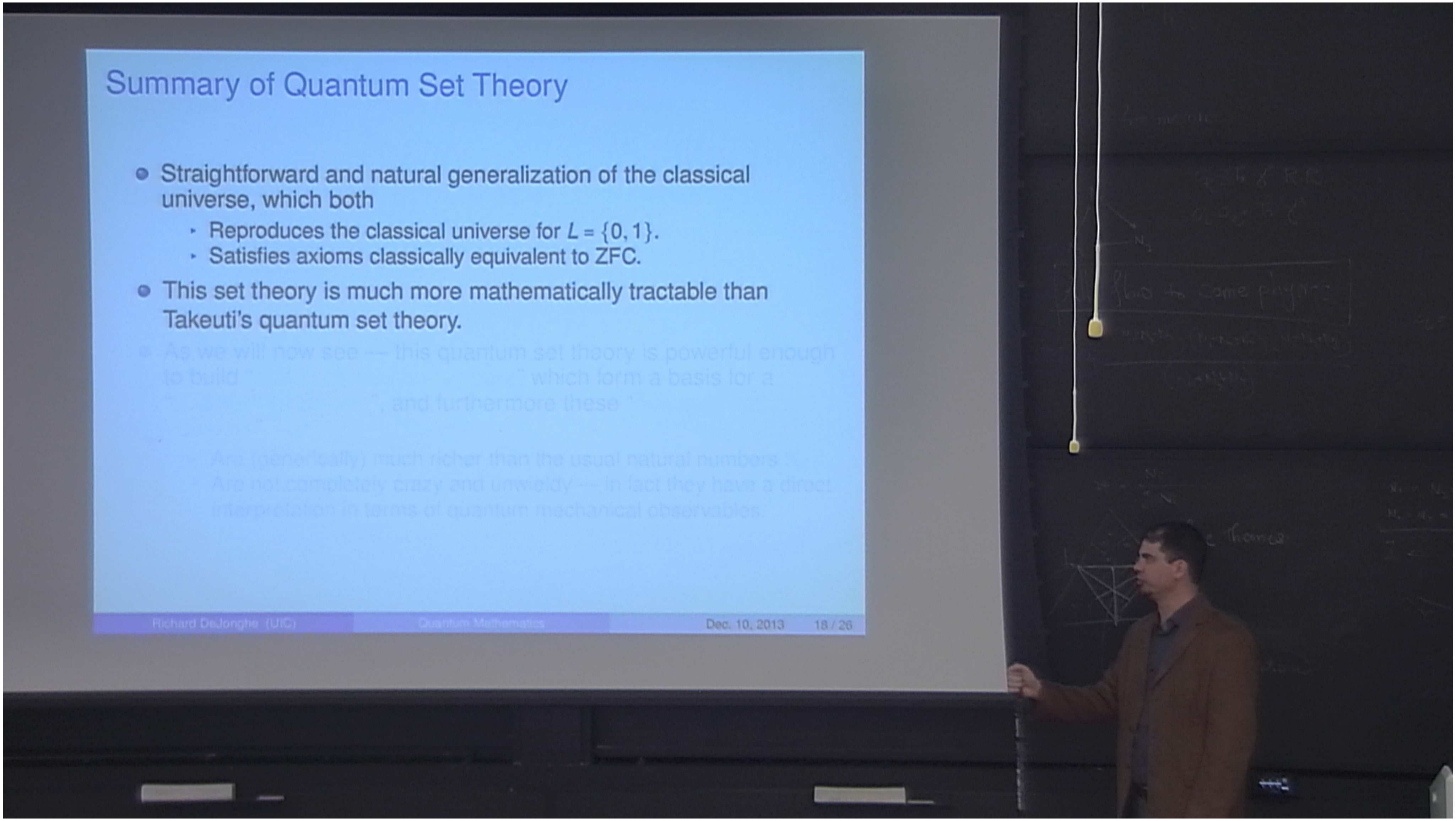
Summary of Quantum Set Theory
Straightforward and natural generalization of the classical universe, which both
Reproduces the classical universe for $L = \{0, 1\}$.
Satisfies axioms classically equivalent to ZFC.
This set theory is much more mathematically tractable than Takeuti's quantum set theory.
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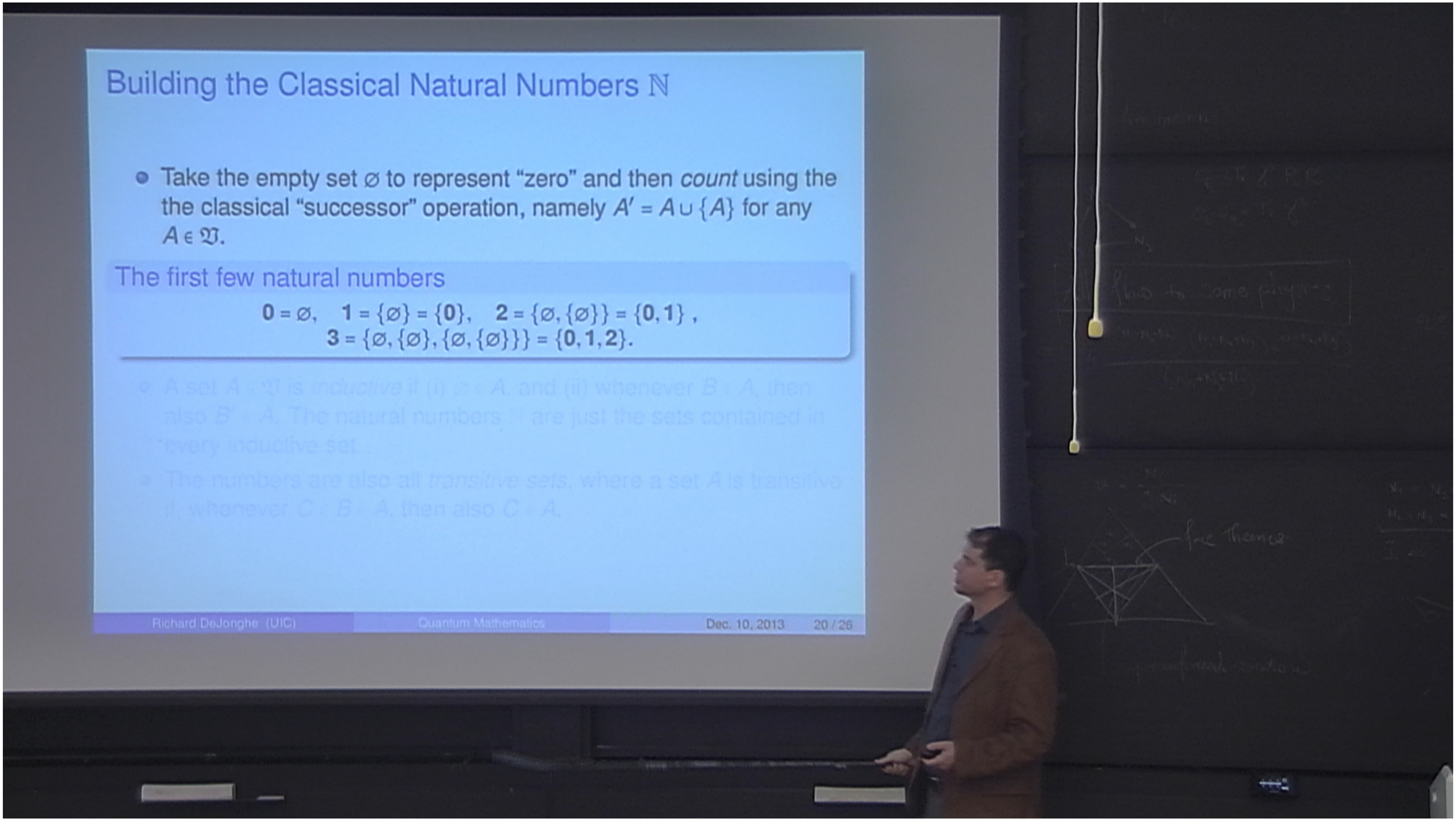
Building the Classical Natural Numbers ℕ
Take the empty set ∅ to represent "zero" and then count using the the classical "successor" operation, namely A′ = A ∪ {A} for any A ∈ 𝔙.
The first few natural numbers
0 = ∅, 1 = {∅} = {0}, 2 = {∅, {∅}} = {0, 1} ,
3 = {∅, {∅}, {∅, {∅}}} = {0, 1, 2}.
A set A ∈ 𝔙 is inductive if (i) ∅ ∈ A, and (ii) whenever B ∈ A, then also B′ ∈ A. The natural numbers ℕ are just the sets contained in every inductive set.
The numbers are also all transitive sets, where a set A is transitive if, whenever C ∈ B ∈ A, then also C ∈ A.
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Building the Classical Natural Numbers $\mathbb{N}$

- Take the empty set $\varnothing$ to represent "zero" and then *count* using the the classical "successor" operation, namely $A' = A \cup \{A\}$ for any $A \in \mathfrak{V}$.

The first few natural numbers

$$\mathbf{0} = \varnothing, \quad \mathbf{1} = \{\varnothing\} = \{\mathbf{0}\}, \quad \mathbf{2} = \{\varnothing, \{\varnothing\}\} = \{\mathbf{0}, \mathbf{1}\},$$
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- A set $A \in \mathfrak{V}$ is *inductive* if (i) $\varnothing \in A$, and (ii) whenever $B \in A$, then also $B' \in A$. The natural numbers $\mathbb{N}$ are just the sets contained in every inductive set.
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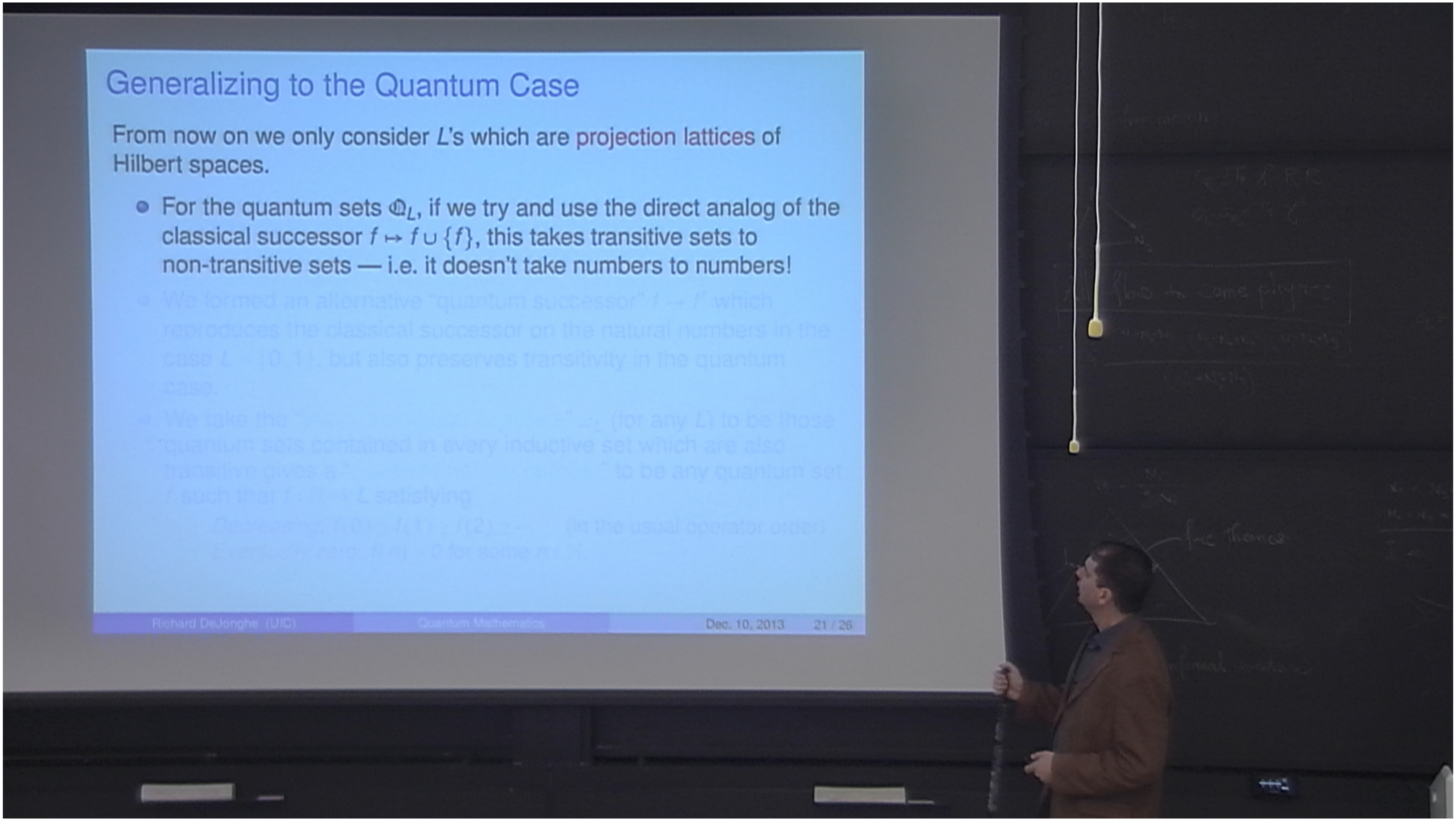
Generalizing to the Quantum Case
From now on we only consider L's which are projection lattices of Hilbert spaces.
For the quantum sets $\mathbb{Q}_L$, if we try and use the direct analog of the classical successor $f \mapsto f \cup \{f\}$, this takes transitive sets to non-transitive sets — i.e. it doesn't take numbers to numbers!
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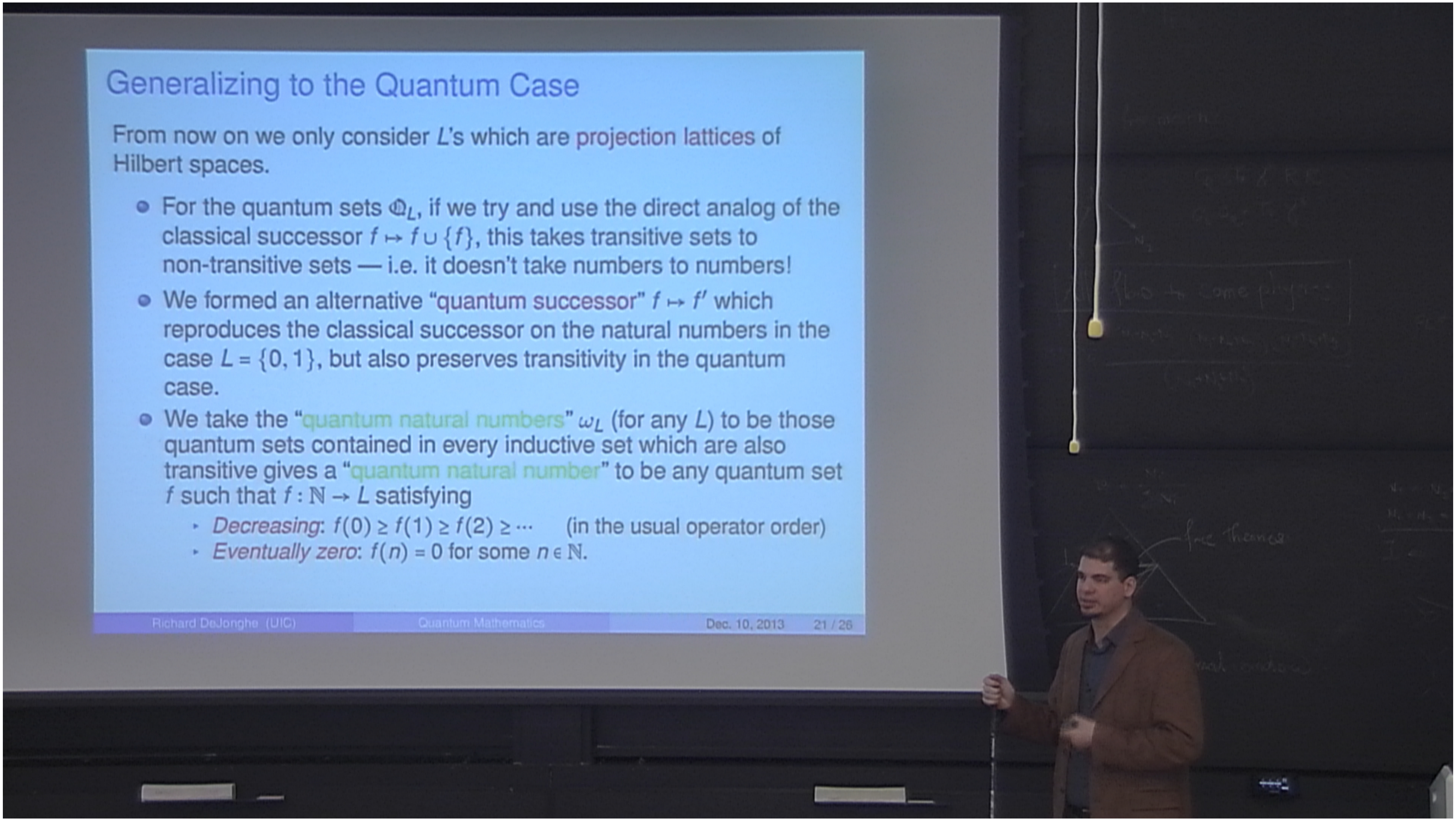
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We formed an alternative "quantum successor" $f \mapsto f'$ which reproduces the classical successor on the natural numbers in the case $L = \{0, 1\}$, but also preserves transitivity in the quantum case.
We take the "quantum natural numbers" ω_L (for any L) to be those quantum sets contained in every inductive set which are also transitive gives a "quantum natural number" to be any quantum set f such that $f : \mathbb{N} \to L$ satisfying
Decreasing: $f(0) \geq f(1) \geq f(2) \geq \cdots$ (in the usual operator order)
Eventually zero: $f(n) = 0$ for some $n \in \mathbb{N}$.
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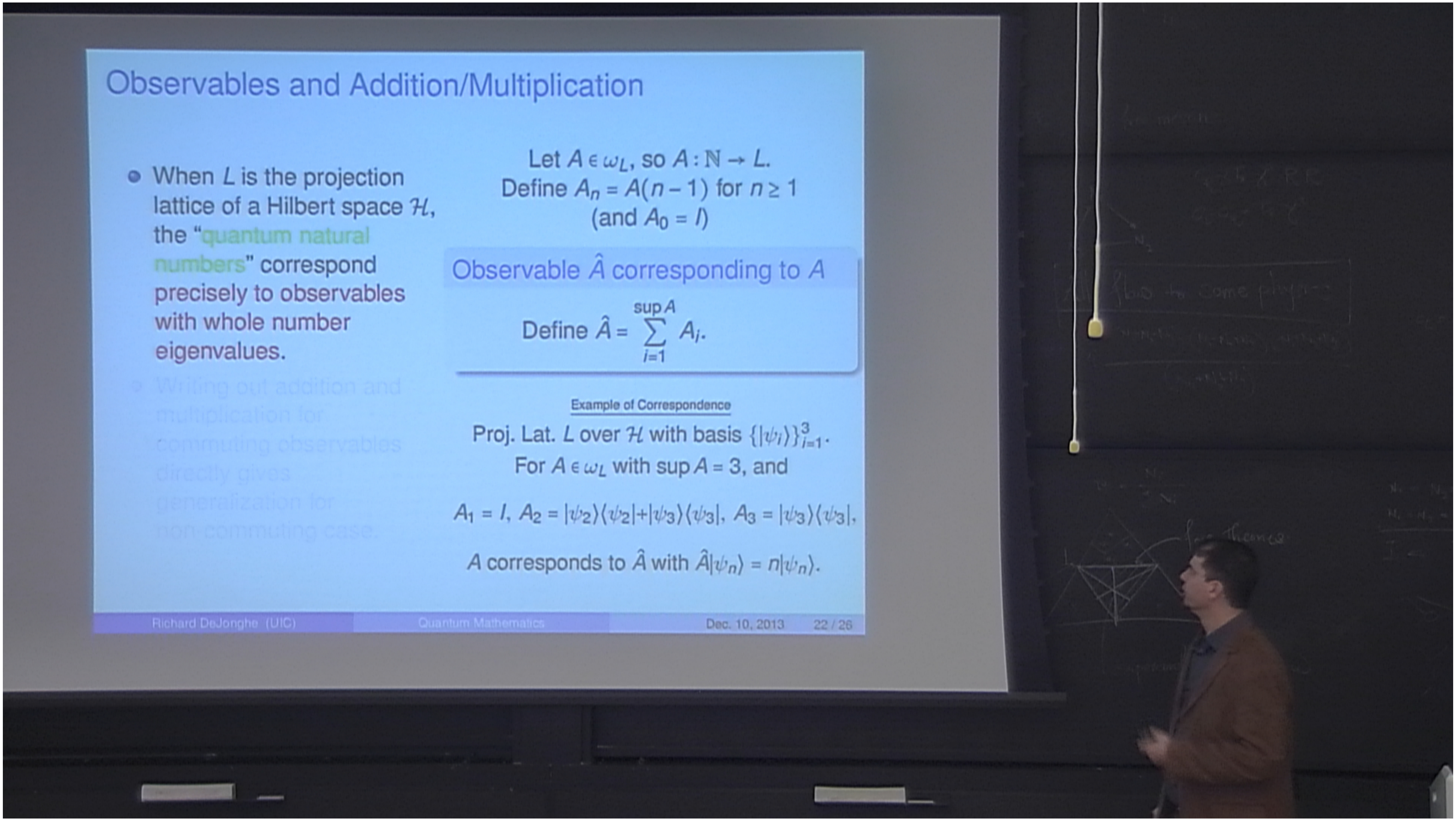
Observables and Addition/Multiplication
When L is the projection lattice of a Hilbert space $\mathcal{H}$, the "quantum natural numbers" correspond precisely to observables with whole number eigenvalues.
Writing out addition and multiplication for commuting observables directly gives generalization for non-commuting case.
Let $A \in \omega_L$, so $A : \mathbb{N} \to L$.
Define $A_n = A(n-1)$ for $n \geq 1$
(and $A_0 = I$)
Observable $\hat{A}$ corresponding to A
Define $\hat{A} = \sum_{i=1}^{\sup A} A_i$.
Example of Correspondence
Proj. Lat. L over $\mathcal{H}$ with basis $\{|\psi_i\rangle\}_{i=1}^{3}$.
For $A \in \omega_L$ with $\sup A = 3$, and
$A_1 = I$, $A_2 = |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|$, $A_3 = |\psi_3\rangle\langle\psi_3|$,
A corresponds to $\hat{A}$ with $\hat{A}|\psi_n\rangle = n|\psi_n\rangle$.
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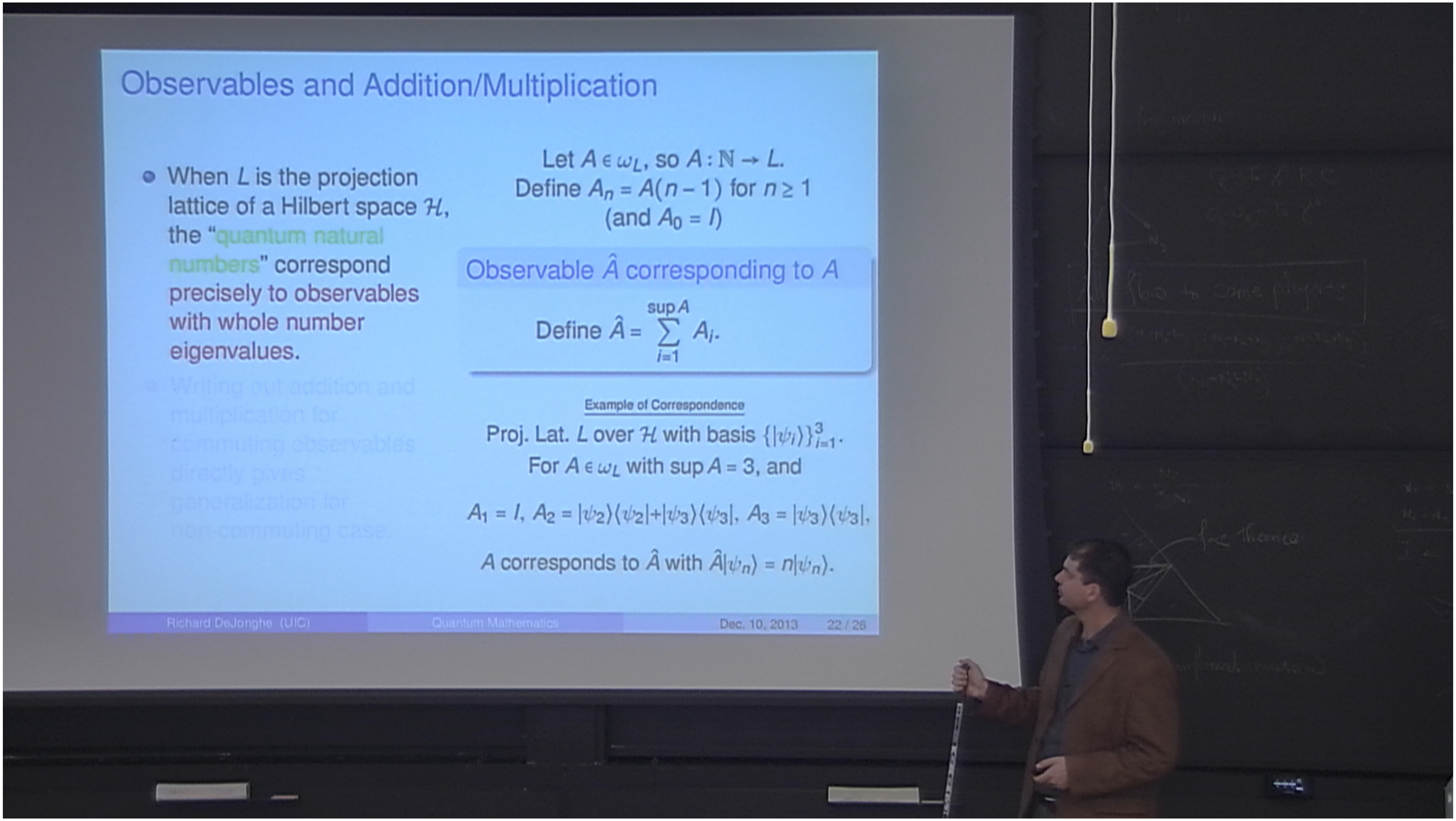
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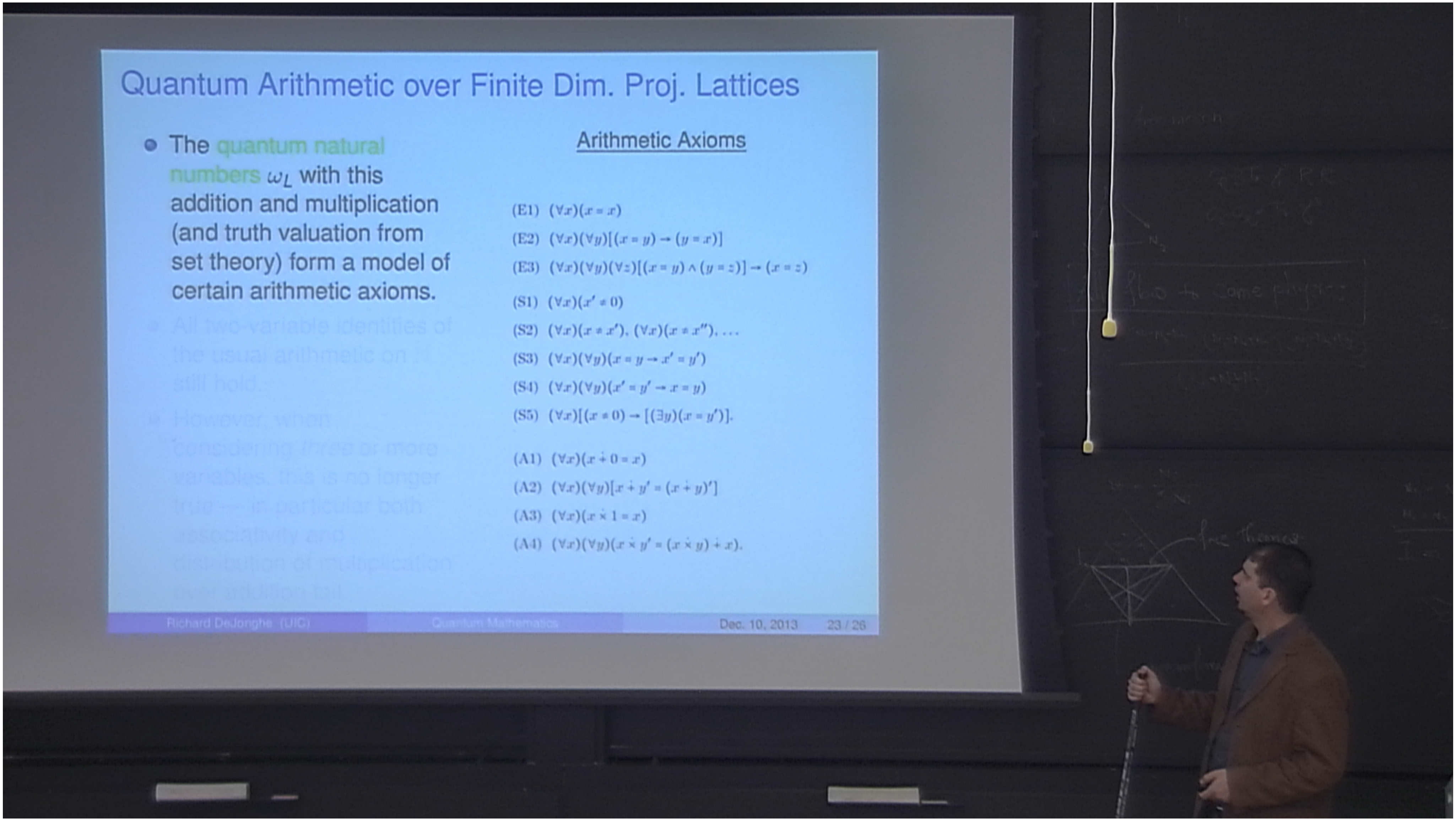

Quantum Arithmetic over Finite Dim. Proj. Lattices
The quantum natural numbers ω_L with this addition and multiplication (and truth valuation from set theory) form a model of certain arithmetic axioms.
All two-variable identities of the usual arithmetic on $\mathbb{N}$ still hold.
However, when considering three or more variables, this is no longer true — in particular both associativity and distribution of multiplication over addition fail.
Arithmetic Axioms
(E1) $(\forall x)(x = x)$
(E2) $(\forall x)(\forall y)[(x = y) \to (y = x)]$
(E3) $(\forall x)(\forall y)(\forall z)[(x = y) \wedge (y = z)] \to (x = z)$
(S1) $(\forall x)(x' \neq 0)$
(S2) $(\forall x)(x \neq x'), (\forall x)(x \neq x''), \ldots$
(S3) $(\forall x)(\forall y)(x = y \to x' = y')$
(S4) $(\forall x)(\forall y)(x' = y' \to x = y)$
(S5) $(\forall x)[(x \neq 0) \to [(\exists y)(x = y')]$.
(A1) $(\forall x)(x \dot{+} 0 = x)$
(A2) $(\forall x)(\forall y)[x \dot{+} y' = (x \dot{+} y)']$
(A3) $(\forall x)(x \dot{\times} 1 = x)$
(A4) $(\forall x)(\forall y)(x \dot{\times} y' = (x \dot{\times} y) \dot{+} x)$.
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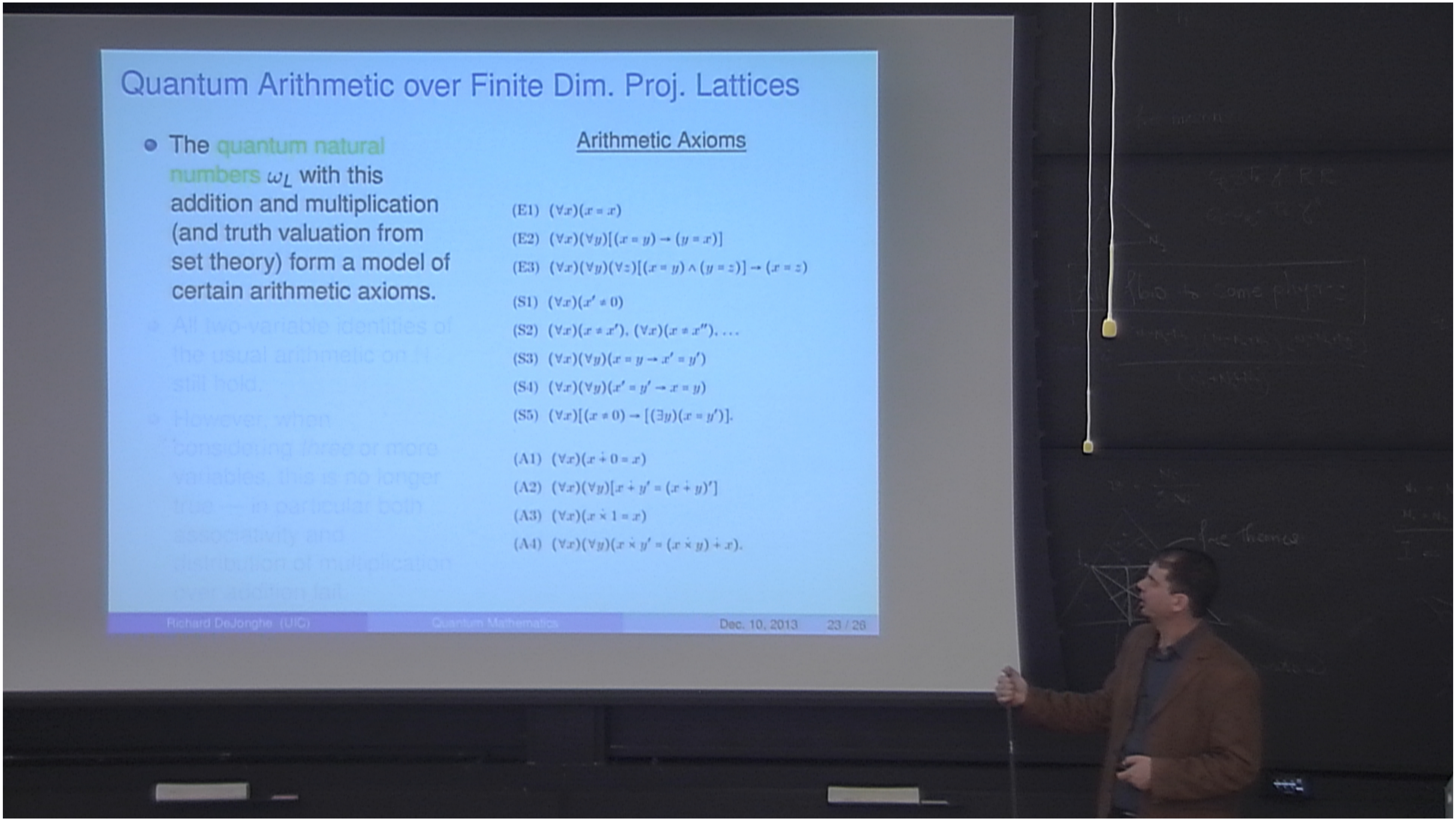

Quantum Arithmetic over Finite Dim. Proj. Lattices
The quantum natural numbers ω_L with this addition and multiplication (and truth valuation from set theory) form a model of certain arithmetic axioms.
Arithmetic Axioms
(E1) $(\forall x)(x = x)$
(E2) $(\forall x)(\forall y)[(x = y) \to (y = x)]$
(E3) $(\forall x)(\forall y)(\forall z)[(x = y) \wedge (y = z)] \to (x = z)$
(S1) $(\forall x)(x' \neq 0)$
(S2) $(\forall x)(x \neq x')$, $(\forall x)(x \neq x'')$, ...
(S3) $(\forall x)(\forall y)(x = y \to x' = y')$
(S4) $(\forall x)(\forall y)(x' = y' \to x = y)$
(S5) $(\forall x)[(x \neq 0) \to [(\exists y)(x = y')]$.
(A1) $(\forall x)(x \dot{+} 0 = x)$
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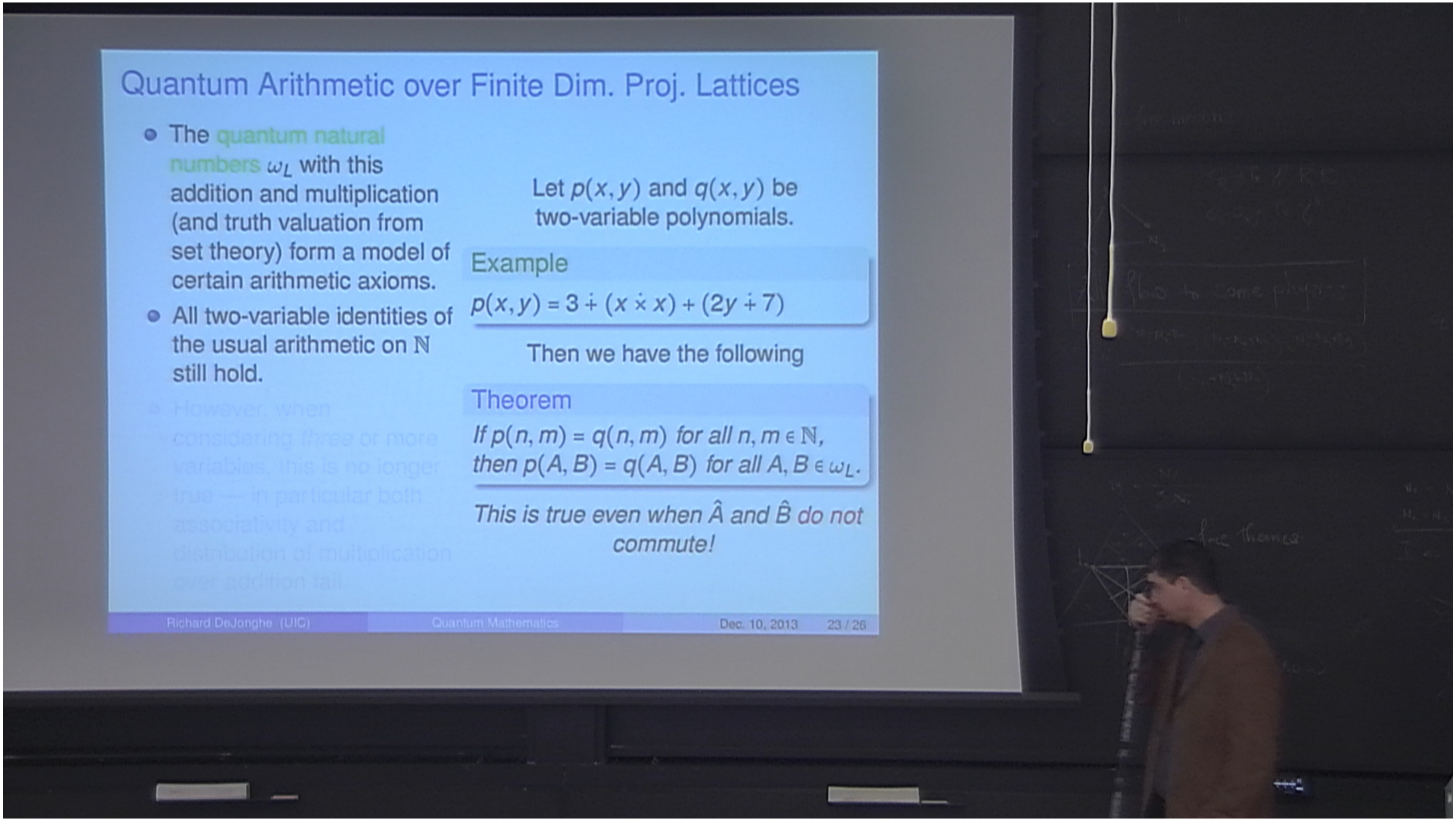
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However, when considering three or more variables, this is no longer true — in particular both associativity and distribution of multiplication over addition fail.
Let $p(x, y)$ and $q(x, y)$ be two-variable polynomials.
Example
$p(x, y) = 3 \dot{+} (x \dot{\times} x) \dot{+} (2y \dot{+} 7)$
Then we have the following
Theorem
If $p(n, m) = q(n, m)$ for all $n, m \in \mathbb{N}$, then $p(A, B) = q(A, B)$ for all $A, B \in \omega_L$.
This is true even when $\hat{A}$ and $\hat{B}$ do not commute!
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Towards an Interpretation in the Projection Lattices

- This new sum ($\dot{+}$) and new product ($\dot{\times}$) respect both eigenvectors and eigenvalues!
- Observables with whole number eigenvalues have a natural interpretation as a "sequence of filters". The new sum ($\dot{+}$) and product ($\dot{\times}$) are the *unique* operations that respect both eigenvectors as well as this filter interpretation

(and satisfy one additional technical requirement).

Let $A, B \in \omega_L$ with L a projection lattice of a Hilbert space $\mathcal{H}$. Then

Theorem

For any $|\psi\rangle \in \mathcal{H}$ such that $A|\psi\rangle = a|\psi\rangle$ and $B|\psi\rangle = b|\psi\rangle$, we have that $(A \dot{+} B)|\psi\rangle = (a+b)|\psi\rangle$ and $(A \dot{\times} B)|\psi\rangle = ab|\psi\rangle$.

Theorem

Let c be an eigenvalue of $A \dot{+} B$. Then $c = a + b$ where a is an eigenvalue of A and b is an eigenvalue of B.

and similarly for the product ($\dot{\times}$).

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Any $A \in \omega_L$ is associated with the decreasing sequence of projectors
$$A_1 \geq A_2 \geq \cdots \geq A_n \geq A_{n+1} = 0.$$
We can think of these A_i's as a "sequence of filters", i.e. interpret a measurement of A as a sequence of measurements "filtering" by A_1, then A_2, etc. Then our product and sum respect this filter interpretation

Theorem

Let $|\psi\rangle \in \mathcal{H}$ such that $A_j|\psi\rangle = |\psi\rangle$ and $B_k|\psi\rangle = |\psi\rangle$. Then
$$(A \dot{+} B)_{j+k}|\psi\rangle = |\psi\rangle$$
$$(A \dot{\times} B)_{jk}|\psi\rangle = |\psi\rangle$$

Conclusions

- We have developed a new quantum set theory using quantum logic that
  - Generalizes the classical set theoretic universe in a simple way, yielding a mathematically elegant and tractable theory
  - Easily constructs "quantum natural numbers" that are tied to quantum observables in a natural way
- We have constructed an arithmetic on these "quantum natural numbers" which
  - not only "respects eigenvectors", but also "respects eigenvalues"
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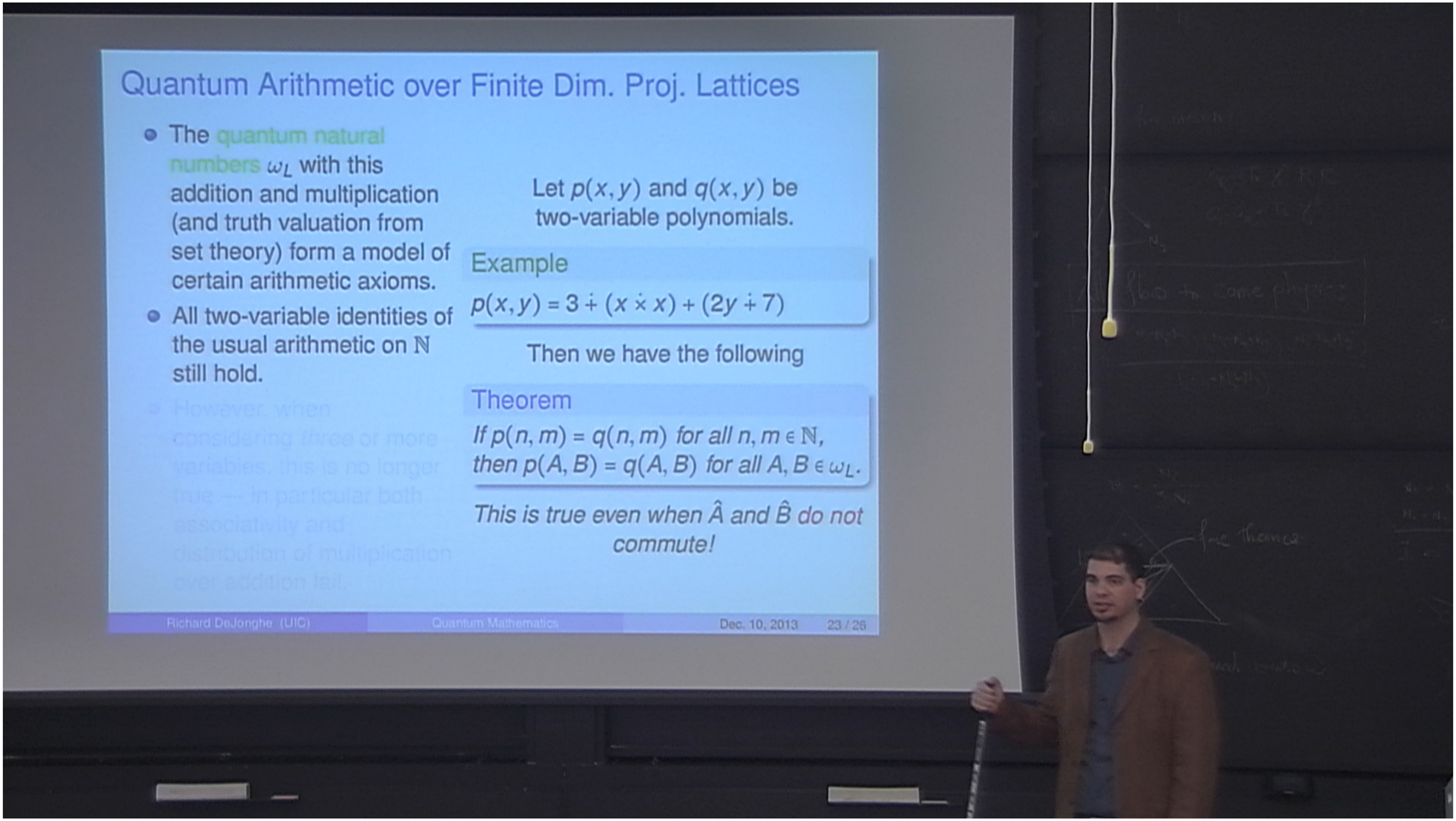
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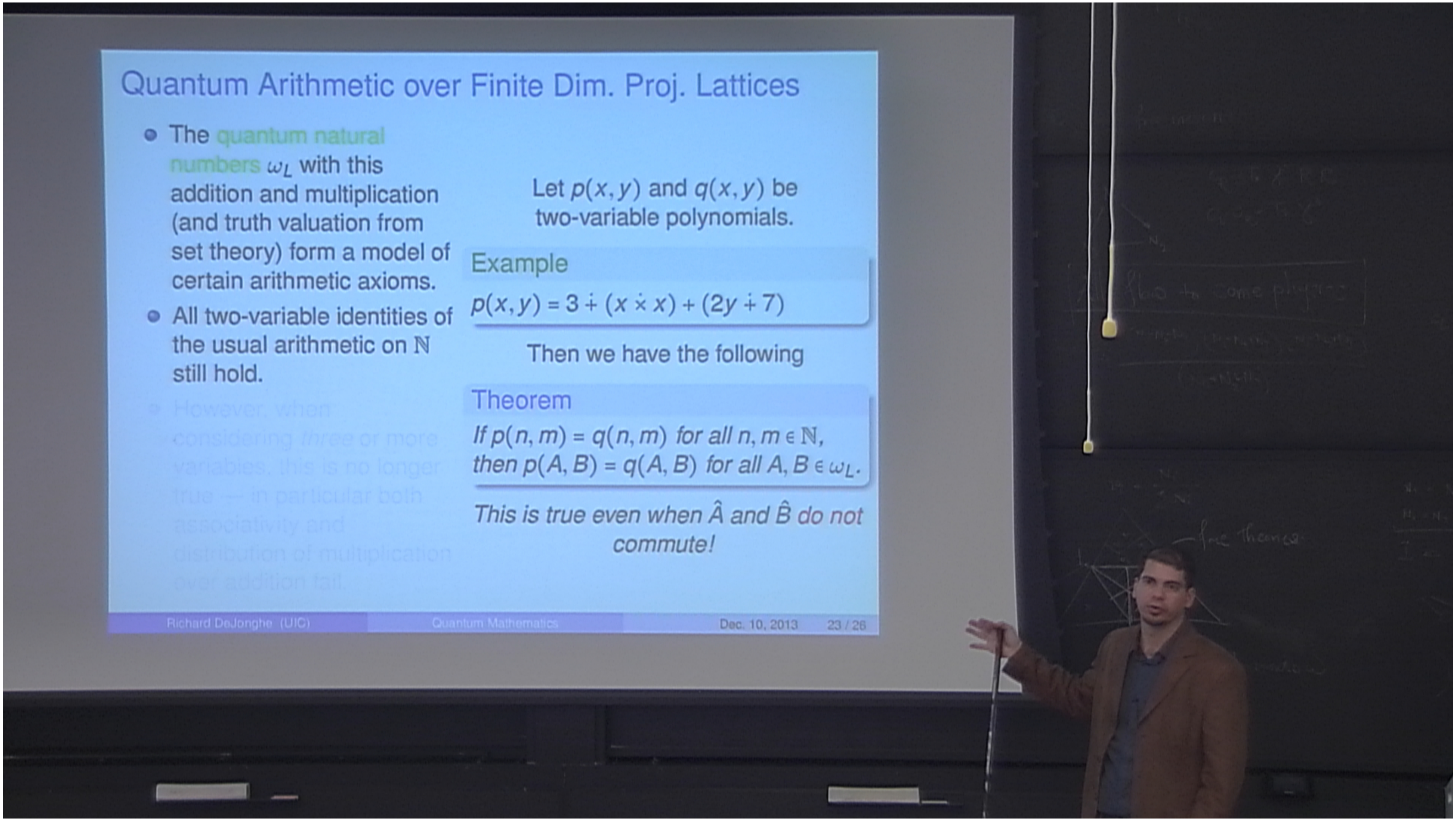

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