Title: Bounding the Elliptope of Quantum Correlations & Proving Separability in Mixed States

Date: Nov 26, 2013 03:30 PM

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Abstract: <span>We present a method for determining the maximum possible violation of any linear Bell inequality per quantum mechanics. Essentially this amounts to a constrained optimization problem for an observableâ€™s eigenvalues, but the problem can be reformulated so as to be analytically tractable. This opens the door for an arbitrarily precise characterization of quantum correlations, including allowing for non-random marginal expectation values. Such a characterization is critical when contrasting QM to superficially similar general probabilistic theories. We use such marginal-involving quantum bounds to estimate the volume of all possible quantum statistics in the complete 8-dimensional probability space of the Bell-CHSH scenario, measured relative to both local hidden variable models as well as general no-signaling theories. See arXiv:1106.2169. Time permitting, weâ€™ll also discuss how one might go about trying to prove that a given mixed state is, in fact, not entangled. (The converse problem of certifying non-zero entanglement has received extensive treatment already.) Instead of directly asking if any separable representation exists for the state, we suggest simply checking to see if it â€œfitsâ€• some particular known-separable form. We demonstrate how a surprisingly valuable sufficient separability criterion follows merely from considering a highly-generic separable form. The criterion we generate for diagonally-symmetric mixed states is apparently completely tight, necessary and sufficient. We use integration to quantify the â€œvolumeâ€• of states captured by our criterion, and show that it is as large as the volume of states associated with the PPT criterion; this simultaneously proves our criterion to be necessary as well as the PPT criterion to be sufficient, on this family of states. The utility of a sufficient separability criterion is evidenced by categorically rejecting Dicke-model superradiance for entanglement generation schema. See arXiv:1307.5779.

Bounding the Elliptope of Quantum Correlations

& Proving Separability in Mixed States

Make sure you can read this text.

Notes will show up in the margins.

Elie Wolfe

11/26/2013

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What is the linear quantum bound?

e.g. Tsirelson Bound

What is the linear quantum bound?

e.g. Tsirelson Bound

Probability of correlated output

$$P[A_0 = B_0] + P[A_0 = B_1] + P[A_1 = B_0] - P[A_1 = B_1] \leq \begin{cases} 3 \rightarrow \text{LHVM} \\ 2+\sqrt{2} \rightarrow \text{QM} \\ 4 \rightarrow \text{NOSIG} \end{cases}$$

Well known

2

What is the linear quantum bound?

e.g. Tsirelson Bound

$$P[A_0 = B_0] + P[A_0 = B_1] + P[A_1 = B_0] - P[A_1 = B_1] \leq \begin{cases} 3 \rightarrow \text{LHVM} \\ 2+\sqrt{2} \rightarrow \text{QM} \\ 4 \rightarrow \text{NOSIG} \end{cases}$$

"Expected Values"

$$\langle A_i \rangle \equiv P[A_i = +1] - P[A_i = +1] \qquad P[A_i = +1] + P[A_i = +1] = 1$$

$$P[A_i = B_j] = P[A_i \cdot B_j = +1] \qquad P[A_i = B_j] \Rightarrow \frac{\langle A_i \cdot B_j \rangle + 1}{2}$$

One can easily switch between forms

2

What is the linear quantum bound?

More familiar form

e.g. Tsirelson Bound

$$\langle A_0 \cdot B_0 \rangle + \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \leq \begin{cases} 2 \rightarrow \text{LHVM} \\ 2\sqrt{2} \rightarrow \text{QM} \\ 4 \rightarrow \text{NOSIG} \end{cases}$$

Preference explained on slide 7

$$\langle A_i \rangle \equiv P[A_i = +1] - P[A_i = +1] \qquad P[A_i = +1] + P[A_i = +1] = 1$$

$$P[A_i = B_j] = P[A_i \cdot B_j = +1] \qquad P[A_i = B_j] \Rightarrow \frac{\langle A_i \cdot B_j \rangle + 1}{2}$$

What is the linear quantum bound?

e.g. Tsirelson Bound

$$\langle A_0 \cdot B_0 \rangle + \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \leq \begin{cases} 2 \rightarrow \text{LHVM} \\ 2\sqrt{2} \rightarrow \text{QM} \\ 4 \rightarrow \text{NOSIG} \end{cases}$$

Most general CHSH linear inequality

$$\begin{pmatrix} c_1 \langle A_0 \rangle + c_2 \langle A_1 \rangle + c_3 \langle B_0 \rangle + c_4 \langle B_1 \rangle + \\ + c_5 \langle A_0 \cdot B_0 \rangle + c_6 \langle A_0 \cdot B_1 \rangle + c_7 \langle A_1 \cdot B_0 \rangle + c_8 \langle A_1 \cdot B_1 \rangle \end{pmatrix} \leq ?$$

We want to find quantum limit for all weights

DESIDERATUM: To determine Quantum Bound for any weighted marginal-involving linear inequality

Motivation: Quantum Elliptope

› 4-d quantum elliptope solved (TLM: '87, '88, '03)

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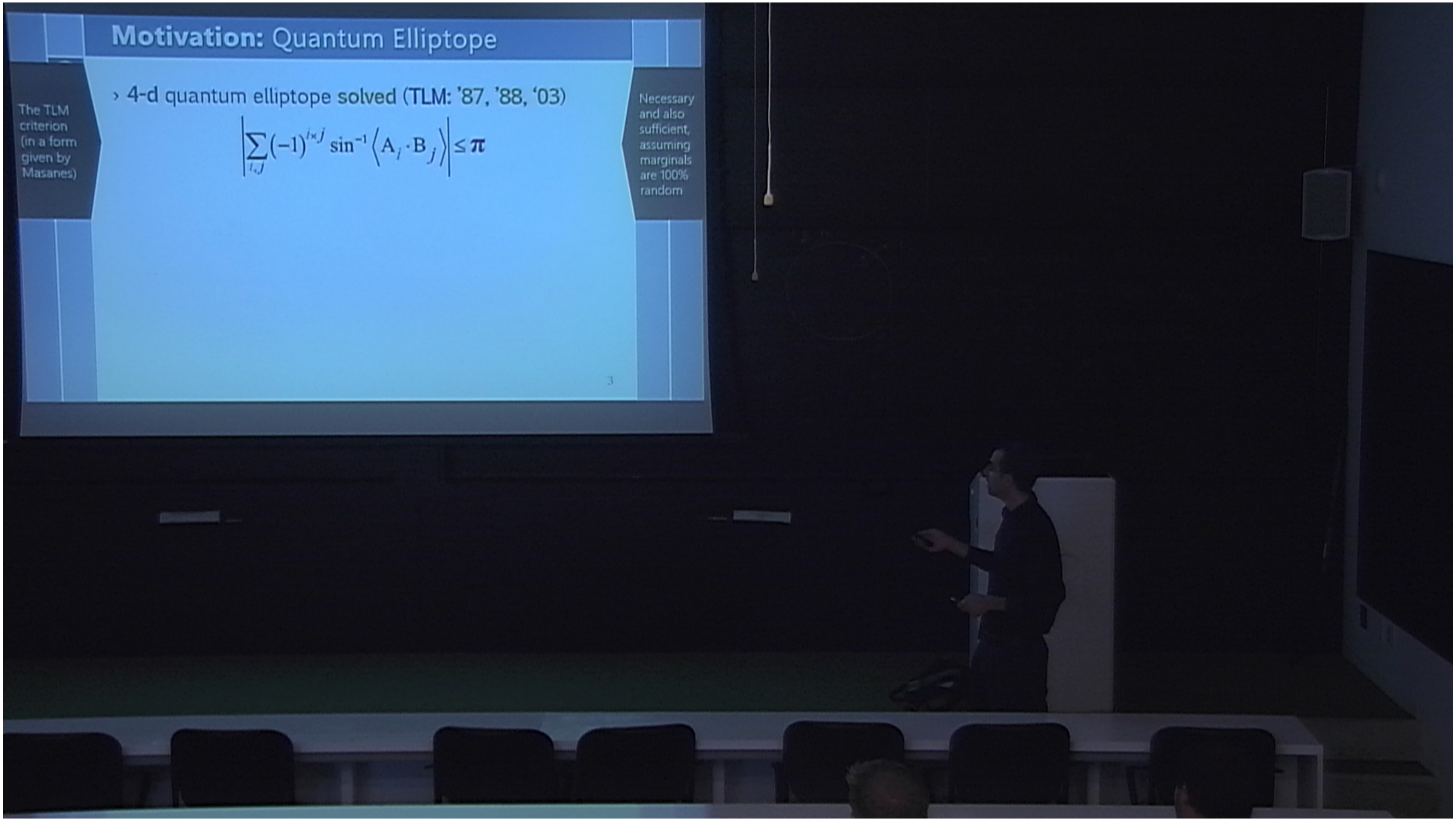


Motivation: Quantum Elliptope
› 4-d quantum elliptope solved (TLM: '87, '88, '03)
$$\left|\sum_{i,j}(-1)^{i\times j}\sin^{-1}\langle A_i\cdot B_j\rangle\right|\leq\pi$$
The TLM criterion (in a form given by Masanes)
Necessary and also sufficient, assuming marginals are 100% random
3

Motivation: Quantum Elliptope

Therefore this is an interesting problem!

- **4-d** quantum elliptope **solved** (**TLM**: '87, '88, '03)

$$\left|\sum_{i,j}(-1)^{i\times j}\sin^{-1}\left\langle A_i\cdot B_j\right\rangle\right|\leq \boldsymbol{\pi}$$

- **8-d** quantum elliptope **open problem** (NPA hierarchy)

NPA *has* given a result. We'll come back to this point.

Motivation: Quantum Elliptope

Linear bounds = facets. Can “fake” a curved boundary.

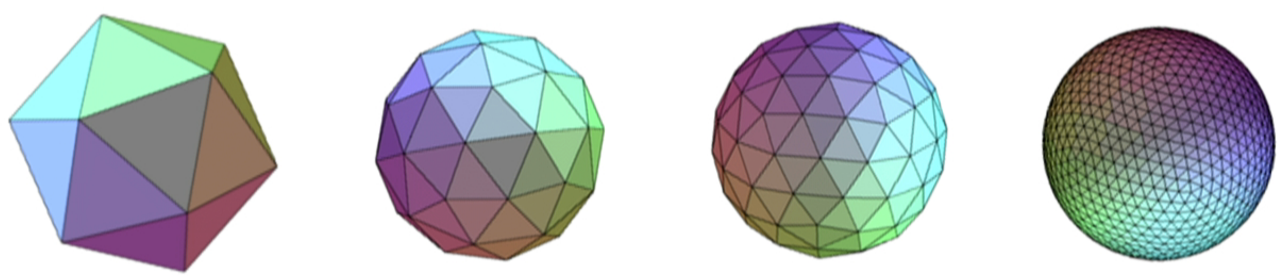
- **4-d** quantum elliptope **solved** (**TLM**: ’87, ’88, ‘03)

$$\left|\sum_{i,j}(-1)^{i\times j}\sin^{-1}\left\langle A_i\cdot B_j\right\rangle\right|\le\pi$$

- **8-d** quantum elliptope **open problem** (NPA hierarchy)

PROPOSAL: Generate very many Tsirelson inequalties to circumscribe the quantum elliptope with a collection of facets

See *Supporting Hyperplane Theorem*



Why an 8 dimensional space?

$A_0 B_0$	$\overline{A_0} B_0$	$A_1 B_0$	$\overline{A_1} B_0$
$A_0 \overline{B_0}$	$\overline{A_0}\,\overline{B_0}$	$A_1 \overline{B_0}$	$\overline{A_1}\,\overline{B_0}$
$A_0 B_1$	$\overline{A_0} B_1$	$A_1 B_1$	$\overline{A_0} B_0$
$A_0 \overline{B_1}$	$\overline{A_0}\,\overline{B_1}$	$A_1 \overline{B_1}$	$\overline{A_1}\,\overline{B_1}$

Every corner square sums to 1.

Drop four more by equivalent ways to get the marginals.

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Why an 8 dimensional space?

A_0B_0	$\overline{A_0}B_0$	A_1B_0	$\overline{A_1}B_0$
$A_0\overline{B_0}$	$\overline{A_0}\,\overline{B_0}$	$A_1\overline{B_0}$	$\overline{A_1}\,\overline{B_0}$
A_0B_1	$\overline{A_0}B_1$	A_1B_1	$\overline{A_0}B_0$
$A_0\overline{B_1}$	$\overline{A_0}\,\overline{B_1}$	$A_1\overline{B_1}$	$\overline{A_1}\,\overline{B_1}$

k parties, **m** $\frac{\text{settings}}{\text{party}}$, **d** $\frac{\text{outputs}}{\text{setting}}$: **Dimension** $= \left(\left(\mathbf{d}-1\right)\mathbf{m}+1\right)^{\mathbf{k}} - 1$

Here's a general formula for the dimension of the elliptope

Derived yesterday. Ask me later for a proof.

Primary Source

Recall that this is what we are tying to solve

$$\begin{pmatrix} c_1 \langle A_0 \rangle + c_2 \langle A_1 \rangle + c_3 \langle B_0 \rangle + c_4 \langle B_1 \rangle + \\ + c_5 \langle A_0 \cdot B_0 \rangle + c_6 \langle A_0 \cdot B_1 \rangle + c_7 \langle A_1 \cdot B_0 \rangle + c_8 \langle A_1 \cdot B_1 \rangle \end{pmatrix} \le ?$$

For now we are only interesting in the QM limit.

Primary Source

$$\left(\begin{array}{c} c_1\langle A_0\rangle + c_2\langle A_1\rangle + c_3\langle B_0\rangle + c_4\langle B_1\rangle + \\ + c_5\langle A_0\cdot B_0\rangle + c_6\langle A_0\cdot B_1\rangle + c_7\langle A_1\cdot B_0\rangle + c_8\langle A_1\cdot B_1\rangle \end{array}\right) \leq ?$$

Citations imply an existing result in the literature

ALREADY SOLVED?

1980 **B.S. Cirel'son** Cited by 753...

"Quantum generalizations of Bell's inequality." Letters in Mathematical Physics 4, 93-100.

"Theorem 2" has a formula for the general CHSH quantum bound

Primary Source

$$\begin{pmatrix} c_1\langle A_0\rangle + c_2\langle A_1\rangle + c_3\langle B_0\rangle + c_4\langle B_1\rangle + \\ +c_5\langle A_0\cdot B_0\rangle + c_6\langle A_0\cdot B_1\rangle + c_7\langle A_1\cdot B_0\rangle + c_8\langle A_1\cdot B_1\rangle \end{pmatrix} \leq ?$$

ALREADY SOLVED?

1980 **B.S. Cirel'son** Cited by 753...
"Quantum generalizations of Bell's inequality."
Letters in Mathematical Physics 4, 93-100.

From: Elie Wolfe <*elupus@gmail.com*> To: Tsirelson Boris <*tsirel@post.tau.ac.il*> **Feb 22 2011**

Dr. Tsirelson, ...I am **having difficulty** understanding and implementing the condition that you expressed... Could point me to other papers discussing this work, in particular the **paper which contains your proof** of this theorem**?** ~Elie

...but I couldn't get it to work.

So I asked for help.

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Tsirelson's Story

From: Tsirelson Boris <tsirel@post.tau.ac.il> **Feb 23 2011**
to: Elie Wolfe <elupus@gmail.com>

Dear Elie, ...my formulas appear to be wrong. It means, some computational error was made by me many years ago. Now, if you want to, you can try to retrace the calculations, find the error and fix it. Unfortunately, the proof was never published (by me), and the calculations are lost. Why did not I publish it? Well, I was very discouraged that no one was interested at all in these results. Nowadays it seems strange, but check the literature of 1980-1990: quite few related works (only L. Landau, and Summers, Werner). And in addition, that time I was in Alia refusal in Soviet Union; it was difficult to me to publish anything at all, the more so, something "strange, neither mathematics nor physics, of very little interest". *Best wishes, --Boris*

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Measures=Observables, Limit=Eigenvalue

This is the recipe for determine linear quantum bounds.

- Define **only local** quantum measurements with ± 1 eigenvalues.
- The joint observations are **composed operators.** Hence the joy of

$$\langle \mathrm{A}_i \cdot \mathrm{B}_j \rangle$$

This is why we use expected values.

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Measures=Observables, Limit=Eigenvalue

- Define **only local** quantum measurements with ±1 eigenvalues.
- The joint observations are **composed operators.** Hence the joy of

$$\langle A_i \cdot B_j \rangle$$

- For qubits we draw from the **Pauli Matrices**
- Measurements for **different parties must commute.**

$$\langle A_i \rangle \to \langle A_i \otimes \mathbf{1} \rangle \qquad \langle B_j \rangle \to \langle \mathbf{1} \otimes B_j \rangle \qquad \langle A_i \cdot B_j \rangle \to \langle A_i \otimes B_j \rangle$$

- For **dichotomic** scenarios, one degree of freedom is sufficient.

The use of tensor product Hilbert spaces may not always be ok. See: Tsirelson's Problem

Variation between the two measurements is all that matters.

Measures=Observables, Limit=Eigenvalue

We use X and Y bases b/c empty along the diagonals

$$A_k$$

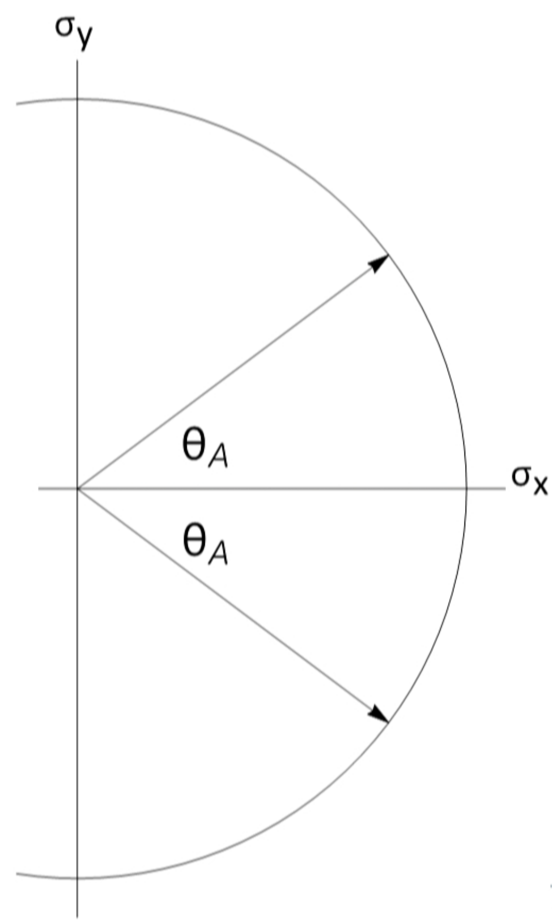
$$\Downarrow$$

$$\cos(\theta_A)\sigma_{\mathbf{x}} + (-1)^k \sin(\theta_A)\sigma_{\mathbf{y}}$$

$$\Downarrow$$

$$\begin{bmatrix} 0 & e^{-\mathbf{i}(-1)^k \theta_A} \\ e^{\mathbf{i}(-1)^k \theta_A} & 0 \end{bmatrix}$$

- For **dichotomic** scenarios, one degree of freedom is sufficient.



Reflection symmetry makes is easier, but is not needed.

Find the Largest Eigenvalue

Key point: Translate the linear inequality into a quantum operator

$$\langle A_i \rangle \to \langle A_i \otimes \mathbf{1} \rangle \qquad \langle B_j \rangle \to \langle \mathbf{1} \otimes B_j \rangle \qquad \langle A_i \cdot B_j \rangle \to \langle A_i \otimes B_j \rangle$$

$$\langle Z \rangle = \left\langle \begin{matrix} c_1 A_0 \otimes \mathbf{1} + c_2 A_1 \otimes \mathbf{1} + c_3 \mathbf{1} \otimes B_0 + c_4 \mathbf{1} \otimes B_1 + \\ c_5 A_0 \otimes B_0 + c_6 A_0 \cdot B_1 + c_7 A_1 \otimes B_0 + c_8 A_1 \otimes B_1 \end{matrix} \right\rangle$$

We must maximize this operator over all possible quantum states

Find the Largest Eigenvalue

$$\langle A_i \rangle \to \langle A_i \otimes \mathbf{1} \rangle \qquad \langle B_j \rangle \to \langle \mathbf{1} \otimes B_j \rangle \qquad \langle A_i \cdot B_j \rangle \to \langle A_i \otimes B_j \rangle$$

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Express the matrix → obtain **characteristic polynomial**

→ **solve** for the roots → find **largest root** →

Roots are just the eigenvals

Wait, solve for the roots?? Right...

Find the Largest Eigenvalue

$$\langle A_i \rangle \to \langle A_i \otimes \mathbf{1} \rangle \qquad \langle B_j \rangle \to \langle \mathbf{1} \otimes B_j \rangle \qquad \langle A_i \cdot B_j \rangle \to \langle A_i \otimes B_j \rangle$$

$$\langle Z \rangle = \left\langle \begin{matrix} c_1 A_0 \otimes \mathbf{1} + c_2 A_1 \otimes \mathbf{1} + c_3 \mathbf{1} \otimes B_0 + c_4 \mathbf{1} \otimes B_1 + \\ c_5 A_0 \otimes B_0 + c_6 A_0 \cdot B_1 + c_7 A_1 \otimes B_0 + c_8 A_1 \otimes B_1 \end{matrix} \right\rangle$$

Express the matrix → obtain **characteristic polynomial** → **solve** for the roots → find **largest root** → **vary** over θ_A & θ_B (to span all polynomial coefficients).

Roots are just the eigenvals

Wait, solve for the roots?? Right...

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A Better Way / An Analytic Approach

- Solving for the many ($\#_{\text{marginal-observables}}=4$) roots of the polynomial is **HARD**.
- Simultaneous maximization over multiple degrees of freedom ($\theta_A,\theta_B\ldots=2$) is **HARD**.

A Better Way / An Analytic Approach

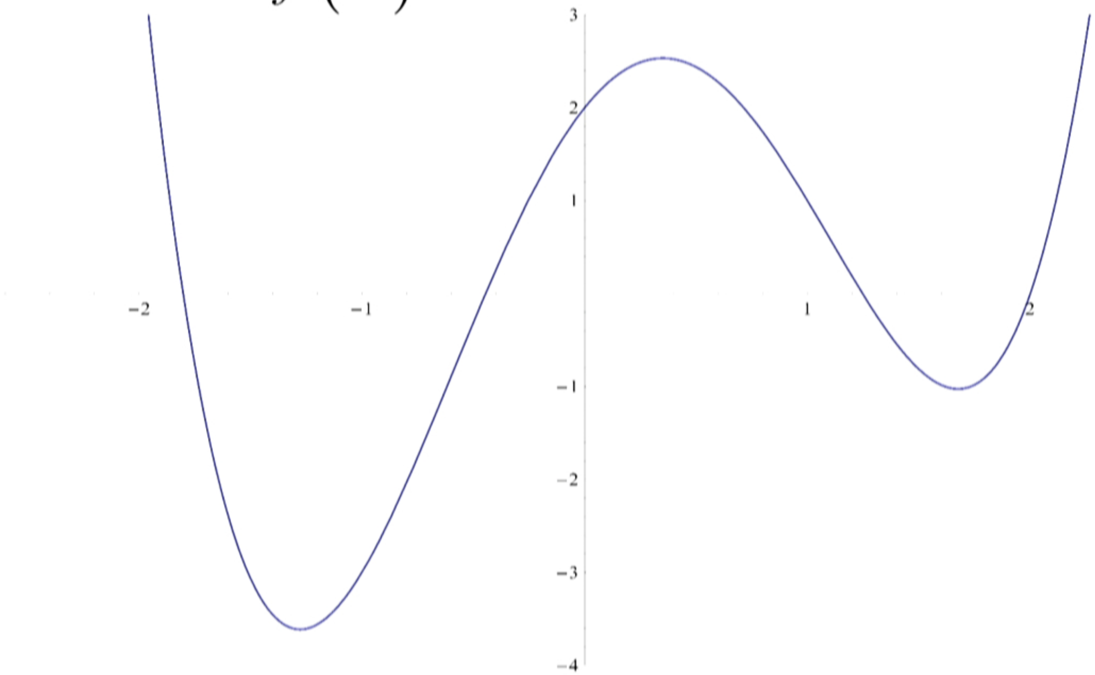
- Solving for the many ($\#_{\text{marginal-observables}}=4$) roots of the polynomial is **HARD**.
- Simultaneous maximization over multiple degrees of freedom ($\theta_A,\theta_B\ldots=2$) is **HARD**.

- Advantage: **Hyperbolic Polynomial** (all roots real)
- Use **Intermediate Value Theorem** to tightly define the region *larger* than the largest root.

Intermediate Value Theorem

Just a random quartic with four real roots.

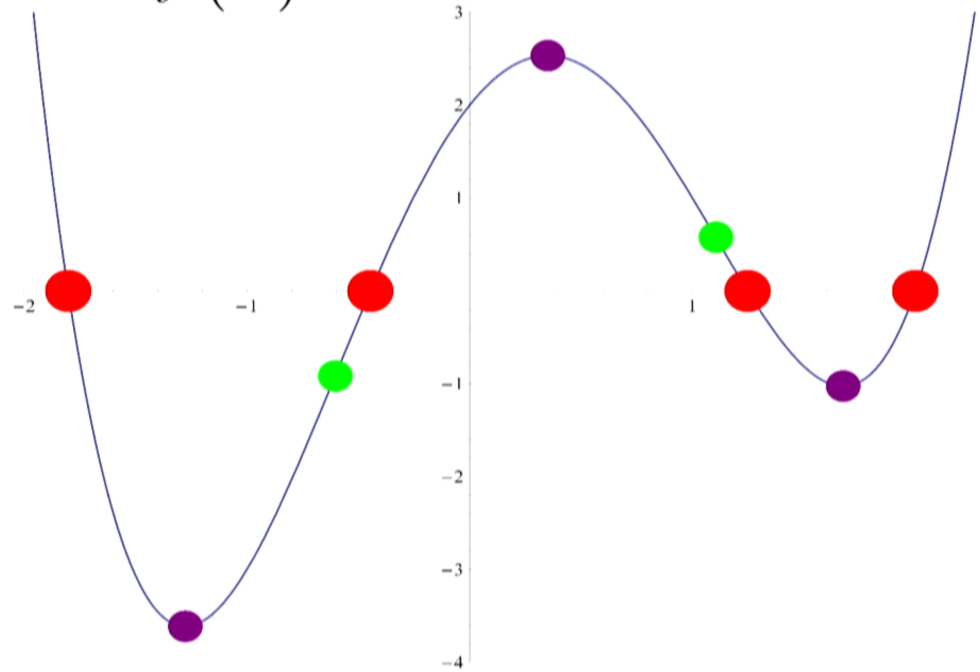
$$f(m) = m^4 - m^3 - 4m^2 + 3m + 2$$



We can always set the leading coefficient to one.

Intermediate Value Theorem

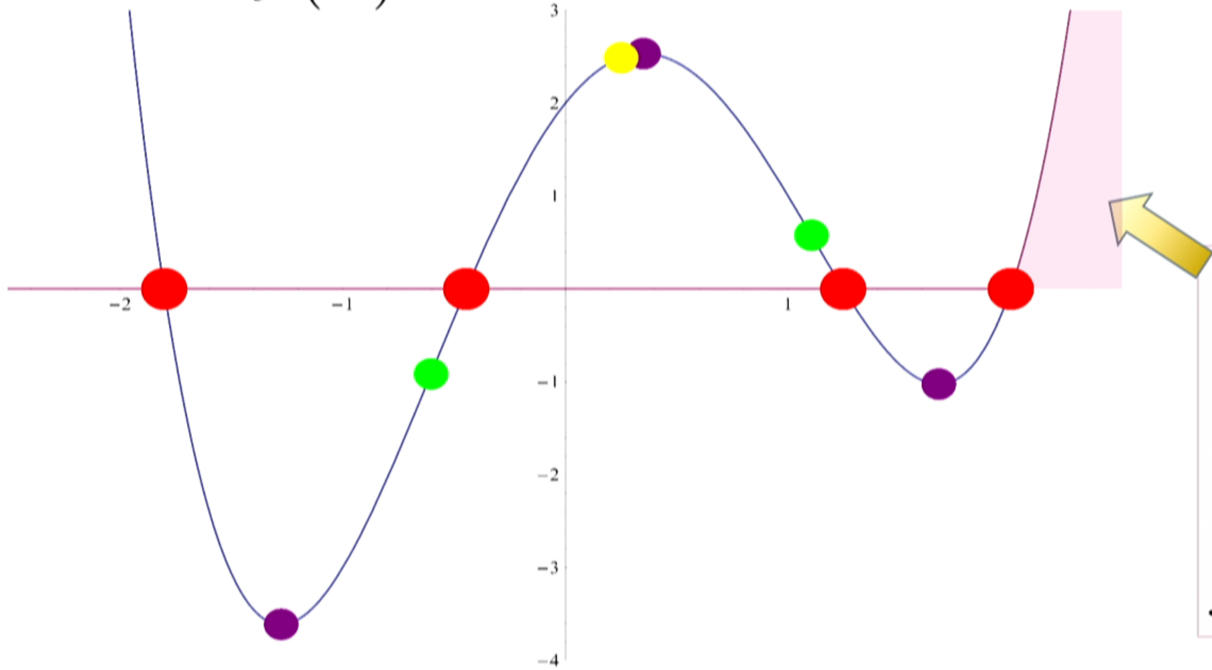
$$f(m) = m^4 - m^3 - 4m^2 + 3m + 2$$



Intermediate Value Theorem

$$f(m) = m^4 - m^3 - 4m^2 + 3m + 2$$

All sign changes in the function and in its derivatives happen BEFORE largest root.

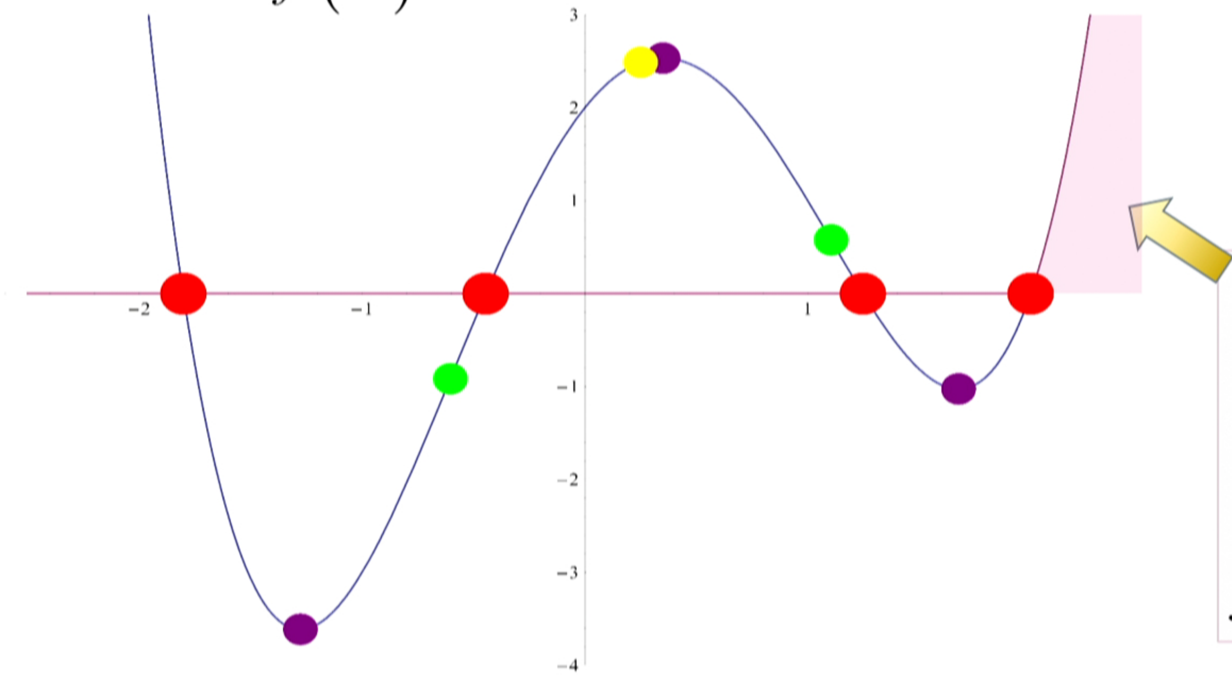


$$f(m) > 0$$
$$f'(m) > 0$$
$$f''(m) > 0$$
$$f'''(m) > 0$$

Look at behavior near +infinity. Must be the same!!

Intermediate Value Theorem

$$f(m) = m^4 - m^3 - 4m^2 + 3m + 2$$



$$f(m) > 0$$
$$f'(m) > 0$$
$$f''(m) > 0$$
$$f'''(m) > 0$$

From maximize over multiple...

...to MINimize over just one!

Holds for **any Hermitian** Eigenvalue Problem!

"ForAll" Constrained Optimization

Note how this gives weights to a marginal

$$\langle A_0 \rangle + \langle A_0 \cdot B_0 \rangle + \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \leq ?$$

We'll work this out together now.

"ForAll" Constrained Optimization

$$\langle A_0 \rangle + \langle A_0 \cdot B_0 \rangle + \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \leq ?$$

First step: convert to an operator.

$$M = \begin{bmatrix} 0 & \beta^\dagger \\ \beta & 0 \end{bmatrix}, \; \beta = 2\begin{bmatrix} \frac{\cos(\theta_A) + i\sin(\theta_A)}{2} & \cos(\theta_A + \theta_B) + i\sin(\theta_A - \theta_B) \\ \cos(\theta_A - \theta_B) + i\sin(\theta_A + \theta_B) & \frac{\cos(\theta_A) + i\sin(\theta_A)}{2} \end{bmatrix}$$

Specific form is not important.

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"ForAll" Constrained Optimization

$$\langle A_0 \rangle + \langle A_0 \cdot B_0 \rangle + \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \leq ?$$

Constrain: The C.P. and all derivatives must be positive.

$$\forall_{\theta_A, \theta_B} : 0 \leq \begin{cases} m^4 - 10m^2 + 9 - 64\left(4\cos^2\theta_B - 1\right)\left(\sin^2\theta_B\right)\left(\cos^2\theta_A\right)\left(\sin^2\theta_A\right) \\ m^3 - 5m \\ 3m^2 - 5 \\ m \end{cases}$$

This is what is important!

$$M = \begin{bmatrix} 0 & \beta^\dagger \\ \beta & 0 \end{bmatrix}, \quad \beta = 2\begin{bmatrix} \frac{\cos(\theta_A) + i\sin(\theta_A)}{2} & \cos(\theta_A + \theta_B) + i\sin(\theta_A - \theta_B) \\ \cos(\theta_A - \theta_B) + i\sin(\theta_A + \theta_B) & \frac{\cos(\theta_A) + i\sin(\theta_A)}{2} \end{bmatrix}$$

"ForAll" Constrained Optimization

We drop weak conditions

$$\forall_{\theta_A,\theta_B} : m^4 - 10m^2 + 9 \geq 64\left(4\cos^2\theta_B - 1\right)\left(\sin^2\theta_B\right)\left(\cos^2\theta_A\right)\left(\sin^2\theta_A\right) \quad \text{and} \quad m \geq \sqrt{5}$$

$$\forall_{\theta_A,\theta_B} : 0 \leq \begin{cases} m^4 - 10m^2 + 9 - 64\left(4\cos^2\theta_B - 1\right)\left(\sin^2\theta_B\right)\left(\cos^2\theta_A\right)\left(\sin^2\theta_A\right) \\ m^3 - 5m \\ 3m^2 - 5 \\ m \end{cases}$$

Only these two matter now

"ForAll" Constrained Optimization

We want to maximize the right hand side

$$\forall_{\theta_A,\theta_B} : m^4 - 10m^2 + 9 \geq 64\left(4\cos^2\theta_B - 1\right)\left(\sin^2\theta_B\right)\left(\cos^2\theta_A\right)\left(\sin^2\theta_A\right) \quad \text{and} \quad m \geq \sqrt{5}$$

setting $\cos\theta_A = \sin\theta_A = 1/\sqrt{2}$ makes $64\left(\cos^2\theta_A\right)\left(\sin^2\theta_A\right) \to 16$

$$\Downarrow$$

$$\forall_{\theta_A} : m^4 - 10m^2 + 9 \geq 16\left(4\cos^2\theta_B - 1\right)\left(\sin^2\theta_B\right) \quad \text{and} \quad m \geq \sqrt{5}$$

We want worst-case scenario, needing largest **m**

"ForAll" Constrained Optimization

$$\forall_{\theta_A,\theta_B} : m^4 - 10m^2 + 9 \geq 64\left(4\cos^2\theta_B - 1\right)\left(\sin^2\theta_B\right)\left(\cos^2\theta_A\right)\left(\sin^2\theta_A\right) \quad \text{and} \quad m \geq \sqrt{5}$$

$$\forall_{\theta_A} : m^4 - 10m^2 + 9 \geq 16\left(4\cos^2\theta_B - 1\right)\left(\sin^2\theta_B\right) \quad \text{and} \quad m \geq \sqrt{5}$$

$$\Downarrow$$

$$\forall_{\cos^2\theta_B} : m^4 - 10m^2 + 9 \geq 16\left(4\cos^2\theta_B - 1\right)\left(1 - \cos^2\theta_B\right)$$

$$\Downarrow$$

$$\frac{\partial\left(4\cos^2\theta_B - 1\right)\left(1 - \cos^2\theta_B\right)}{\partial\cos^2\theta_B} = 0$$

- $\cos^2\theta_B \to 5/8$
- $6\left(4\cos^2\theta_B - 1\right)\left(1 - \cos^2\theta_B\right) \to \mathbf{9}$

Easier to maximize of Cosine squared than θ

Right hand side is at-most 9. Easy!

"ForAll" Constrained Optimization

Back-subsitute

$$m^4 - 10m^2 + 9 \geq \mathbf{9} \quad \text{and} \quad m \geq \sqrt{5}$$

Which condition is stronger?

$$\forall_{\theta_A} : m^4 - 10m^2 + 9 \geq 16\left(4\cos^2\theta_B - 1\right)\left(\sin^2\theta_B\right) \quad \text{and} \quad m \geq \sqrt{5}$$

$$\Downarrow$$

$$\forall_{\cos^2\theta_B} : m^4 - 10m^2 + 9 \geq 16\left(4\cos^2\theta_B - 1\right)\left(1 - \cos^2\theta_B\right)$$

$$\Downarrow$$

$$\frac{\partial\left(4\cos^2\theta_B - 1\right)\left(1 - \cos^2\theta_B\right)}{\partial\cos^2\theta_B} = 0$$

- $\cos^2\theta_B \rightarrow 5/8$
- $6\left(4\cos^2\theta_B - 1\right)\left(1 - \cos^2\theta_B\right) \rightarrow \mathbf{9}$

11

“ForAll” Constrained Optimization

PI

$$m^4 - 10m^2 + 9 \geq \mathbf{9} \quad \text{and} \quad m \geq \sqrt{5}$$

$$\Downarrow$$

$$m^4 - 10m^2 \geq 0 \quad \text{and} \quad m \geq \sqrt{5}$$

$$\Downarrow$$

$$m^2 \geq 10 \quad \text{and} \quad m \geq \sqrt{5}$$

And hence we have derived the quantum bound!

$$\langle A_0 \rangle + \langle A_0 \cdot B_0 \rangle + \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \leq \sqrt{\mathbf{10}}$$

This is a non-trivial bound!

Results: New Quantum Bounds

This top inequality is the one we just derived.

$$\Big(\langle A_0\rangle + \langle A_0\cdot B_0\rangle + \langle A_0\cdot B_1\rangle + \langle A_1\cdot B_0\rangle - \langle A_1\cdot B_1\rangle\Big) \le \begin{cases} 3 & \to \text{LHVM} \\ \sqrt{10} & \to \text{QM} \\ 4 & \to \text{NOSIG} \end{cases}$$

$$\Big(\langle A_0\rangle + \langle A_1\rangle - \langle B_0\rangle + \langle A_0\cdot B_0\rangle + \langle A_0\cdot B_1\rangle + \langle A_1\cdot B_0\rangle - \langle A_1\cdot B_1\rangle\Big) \le \begin{cases} 3 & \to \text{LHVM} \\ 3 & \to \text{QM} \\ 4 & \to \text{NOSIG} \end{cases}$$

TAKE HOME MESSAGE:

Simple marginal-involving bounds

Upgrade: Nonlinear Quantum Bounds

$$\forall \; |x| \leq 2:$$

$$\left(x\langle A_0 \rangle + \langle A_0 \cdot B_0 \rangle + \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \right) \leq \begin{cases} |x| + 2 & \rightarrow \text{LHVM} \\ \sqrt{2x^2+8} & \rightarrow \text{QM} \\ 4 & \rightarrow \text{NOSIG} \end{cases}$$

SURPRISE!

$$\left(x\langle A_0 \rangle + x\langle A_1 \rangle - x\langle B_0 \rangle + \langle A_0 \cdot B_0 \rangle + \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \right) \leq \begin{cases} |x| + 2 \\ \forall_{|x| \leq 1} : \dfrac{\sqrt{(2-x^2)(4-3x^2)} - x^2}{1-x^2} \\ \forall_{1 \leq |x|} : |x| + 2 \\ 4 \end{cases}$$

Analytic technique work for non-numbers as well!

Sample Contrast with NPA

$$\langle A_0 \rangle + \langle A_1 \rangle - \langle B_0 \rangle + \langle A_0 \cdot B_0 \rangle + \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \leq \mathbf{3}$$

A linear quantum bound, deemed QB_3

Sample Contrast with NPA

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$$f_{j,k}\equiv\sin^{-1}\left(\frac{\langle A_jB_k\rangle-\langle A_j\rangle\langle B_k\rangle}{\sqrt{\left(1-\langle A_j\rangle^2\right)\left(1-\langle B_k\rangle^2\right)}}\right),\qquad \left|f_{0,1}+f_{0,1}+f_{1,0}-f_{1,1}\right|\le\pi$$

NPA_1, criterion for first level of hierarchy

Sample Contrast with NPA

$$\langle A_0\rangle+\langle A_1\rangle-\langle B_0\rangle+\langle A_0\cdot B_0\rangle+\langle A_0\cdot B_1\rangle+\langle A_1\cdot B_0\rangle-\langle A_1\cdot B_1\rangle\le \mathbf{3}$$

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Here is a lightly-biased scenario, a test point in 8-space

$$\langle A_0\rangle=\langle A_1\rangle=\mathbf{1/3}\ ,\qquad \langle B_0\rangle=\langle B_1\rangle=\mathbf{0}$$

$$\langle A_0\cdot B_0\rangle=\langle A_0\cdot B_1\rangle=\langle A_1\cdot B_0\rangle=-\langle A_1\cdot B_1\rangle=\mathbf{2/3}$$

Question is, is it quantum? Could this ever happen?

Sample Contrast with NPA

$$\langle A_0\rangle+\langle A_1\rangle-\langle B_0\rangle+\langle A_0\cdot B_0\rangle+\langle A_0\cdot B_1\rangle+\langle A_1\cdot B_0\rangle-\langle A_1\cdot B_1\rangle\leq \mathbf{3}$$

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NPA_1 says it's ok... but QB_3 rejects it!

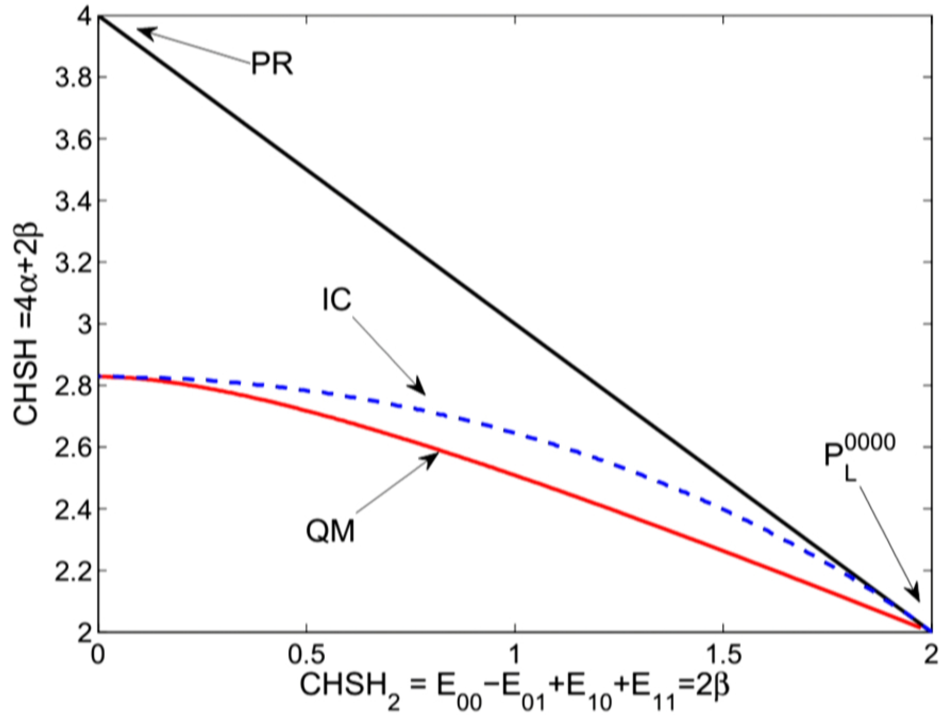
EXTENSION #1

"Recovering part of the quantum boundary from information causality "

PRA 80, 040103 (2009)

Jonathan Allcock
Nicolas Brunner
Marcin Pawlowski
Valerio Scarani

The authors sought to compare their IC bound to the true QM bound. They used **NPA_1**.



EXTENSION #2

"How much larger are quantum correlations than classical ones?"

PRA 72, 012113 (2005)

Adán Cabello

Cabello solved for the fraction of statistics compatible with QM **in the 4-dimensional** correlations **subspace**

$\mathrm{NOSIG}^{(8)}$	$\mathbf{TLM}^{(4)}$	$\mathrm{LHVM}^{(4)}$
1	$\frac{3\pi^2}{32} \approx \mathbf{0.925}$	$\frac{2}{3} \approx 0.667$

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But the 8-dimensional volume for the total quantum elliptope...

NOSIG$^{(8)}$	NPA$^{(8)}$	$\approx$ **QM**$^{(8)}$	LHVM$^{(4)}$
$=1088$	≈ 1086	$\lesssim \mathbf{1084}$	$=1024$

SUMMARY SLIDE

- Quantum Linear Inequalities are (again) State of the Art
- Quantum Linear Inequalities are Readily Derived
- Hermitian-Matrix Eigenvalue-Maximization can always be reformulated as single-variable constrained-optimization due to the Intermediate Value Theorem

$$\left(\begin{array}{l} \langle A_0 \rangle + \langle A_1 \rangle - \langle B_0 \rangle + \langle A_0 \cdot B_0 \rangle + \\ \langle A_0 \cdot B_1 \rangle + \langle A_1 \cdot B_0 \rangle - \langle A_1 \cdot B_1 \rangle \end{array} \right) \leq 3$$

Certifying Separability of Mixed States

Two Easy Ideas

Appearances can be deceiving

$$|\Phi_\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

Here are some highly entangled Bell states...

Appearances can be deceiving

$$|\Phi_\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

$$|\Phi_+\rangle\langle\Phi_+| = \frac{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|}{2}$$

$$|\Phi_-\rangle\langle\Phi_-| = \frac{|00\rangle\langle 00| - |11\rangle\langle 00| - |00\rangle\langle 11| + |11\rangle\langle 11|}{2}$$

they can be expressed as product states...

Appearances can be deceiving

$$|\Phi_\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

$$|\Phi_+\rangle\langle\Phi_+| = \frac{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|}{2}$$

$$|\Phi_-\rangle\langle\Phi_-| = \frac{|00\rangle\langle 00| - |11\rangle\langle 00| - |00\rangle\langle 11| + |11\rangle\langle 11|}{2}$$

$$|\Phi_+\rangle\langle\Phi_+| + |\Phi_-\rangle\langle\Phi_-| = |00\rangle\langle 00| + |11\rangle\langle 11|$$

and when mixed are seen to be separable!

We want to certify full separability

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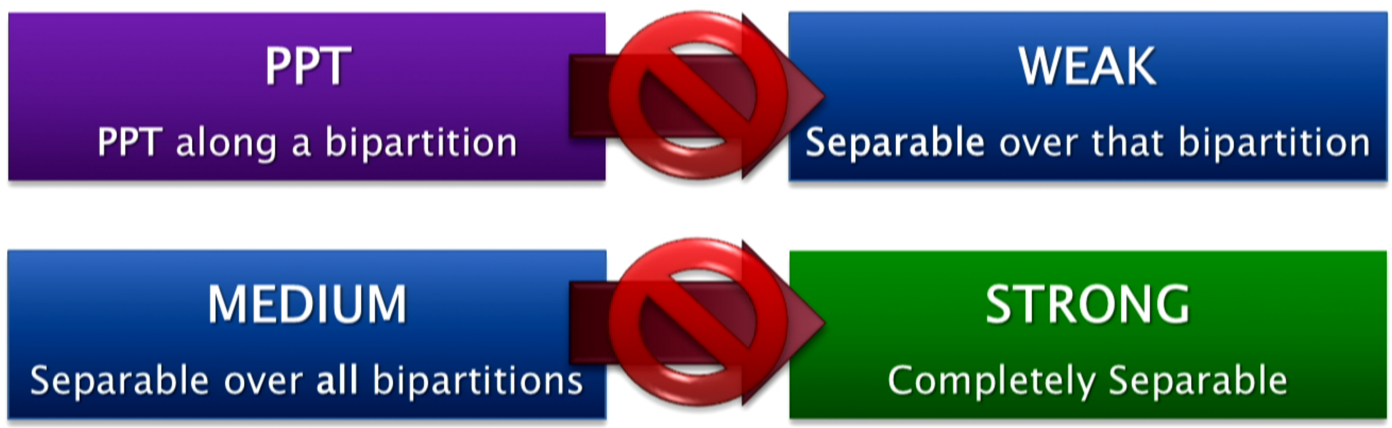
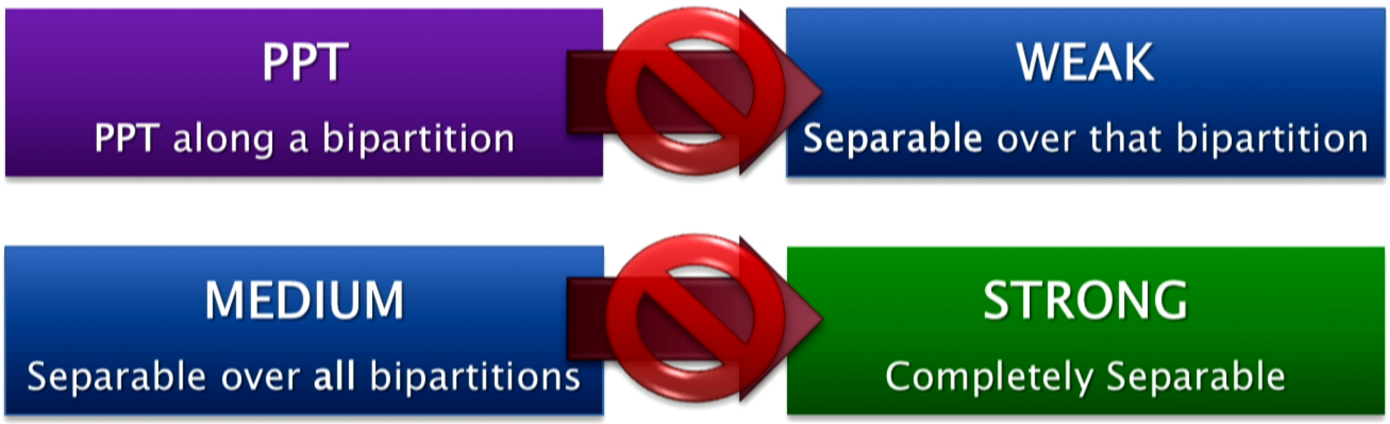


We want to certify full separability

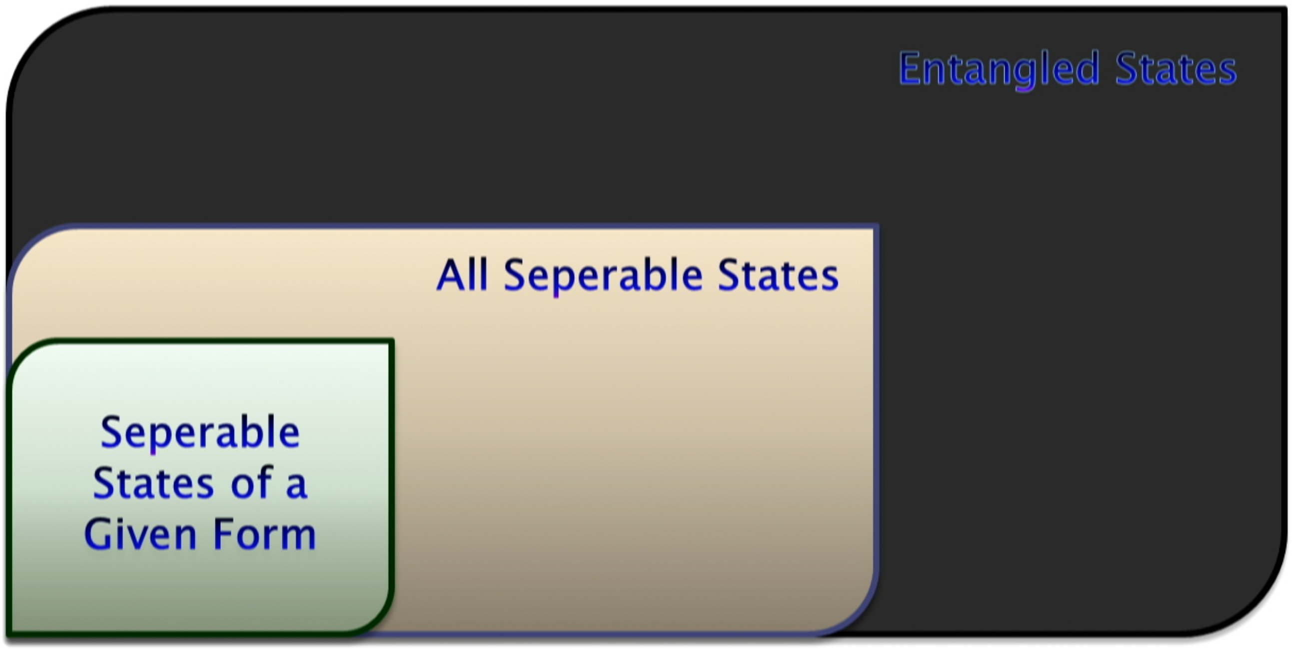


Last resort:

Try to **find an explicitly separable** decomposition

Easy Idea #1: **Build & Check**

Does the mixed state fit the **form** of some family of separable states?



Diagonally Symmetric States

GDS are (usually) entangled.

$$\rho_{\text{GDS}} = \sum_{n_0=0}^{N} \Pi_{n_0} \left|\text{Dicke}_{n_0}\right\rangle\left\langle\text{Dicke}_{n_0}\right|$$

Just an example.

Diagonally Symmetric States

GDS are (usually) entangled.

SDS are made to be separable

$$\rho_{\mathrm{GDS}} = \sum_{n_0=0}^{N} \Pi_{n_0} \left|\mathrm{Dicke}_{n_0}\right\rangle\left\langle\mathrm{Dicke}_{n_0}\right|$$

$$\rho_{\mathrm{SDS}} = \frac{N!}{n_0!\left(N-n_0\right)!} \sum_{n_0=0}^{N} \sum_{j=1}^{j_{\max}} x_j {y_j}^{n_0} (1-y_j)^{(N-n_0)} \left|\mathrm{Dicke}_{n_0}\right\rangle\left\langle\mathrm{Dicke}_{n_0}\right|$$

Does it fit?

Diagonally Symmetric States

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∴ a tailor-made sufficient separability criterion:

$$\rho_{\mathrm{GDS}} \in \rho_{\mathrm{SDS}} \quad \text{iff} \quad \exists_{x_j, y_j} \forall_{n_0} : \ \Pi_{n_0} = \frac{N! \sum_{j=1}^{j_{\max}} x_j y_j^{\,n_0} (1-y_j)^{(N-n_0)}}{n_0!(N-n_0)!}$$

$$\text{such that} \quad \forall_j \quad : \ 0 \le x_j \le 1 \ \wedge \ 0 \le y_j \le 1$$

The entangled states might fit the separable form!

Convexity conditions are important.

Four-Qubit Explicit Example

For diagonally symmetric states, this system of equations checks fit

$$\Pi_{n_0} = \frac{4!\sum_{j=1}^{3} x_j y_j^{\,n_0} (1-y_j)^{(4-n_0)}}{n_0!\,(4-n_0)!}$$

Derived on previous slide.

Four-Qubit Explicit Example

$$\Pi_{n_0} = \frac{4!\sum_{j=1}^{3} x_j y_j^{\,n_0} (1-y_j)^{(4-n_0)}}{n_0!(4-n_0)!}$$

$$\Pi_5 = x_1 (y_1)^4 + x_2 (y_2)^4$$

$$\frac{\Pi_4}{4} = x_1 (y_1)^3 (1-y_1) + x_2 (y_2)^3 (1-y_2)$$

$$\frac{\Pi_3}{6} = x_1 (y_1)^2 (1-y_1)^2 + x_2 (y_2)^2 (1-y_2)^2$$

$$\frac{\Pi_2}{4} = x_1 (y_1)(1-y_1)^3 + x_2 (y_2)(1-y_2)^3$$

$$\Pi_1 = x_1 (1-y_1)^4 + x_2 (1-y_2)^4 + x_3$$

Five equations for a system of four qubits.

Always N+1 equations. Should be same # of degrees of freedom!

Four-Qubit Explicit Example

$$\Pi_{n_0} = \frac{4!\sum_{j=1}^{3} x_j y_j^{\,n_0} (1-y_j)^{(4-n_0)}}{n_0!(4-n_0)!}$$

Our form gives x & y in pairs.

$$\Pi_5 = x_1(y_1)^4 + x_2(y_2)^4$$

$$\frac{\Pi_4}{4} = x_1(y_1)^3(1-y_1) + x_2(y_2)^3(1-y_2)$$

$$y_{j=(N+1)/2} \equiv 0$$

$$\frac{\Pi_3}{6} = x_1(y_1)^2(1-y_1)^2 + x_2(y_2)^2(1-y_2)^2$$

$$\frac{\Pi_2}{4} = x_1(y_1)(1-y_1)^3 + x_2(y_2)(1-y_2)^3$$

5th parameter

$$\Pi_1 = x_1(1-y_1)^4 + x_2(1-y_2)^4 + x_3$$

So when odd # of equations we set last y to zero.

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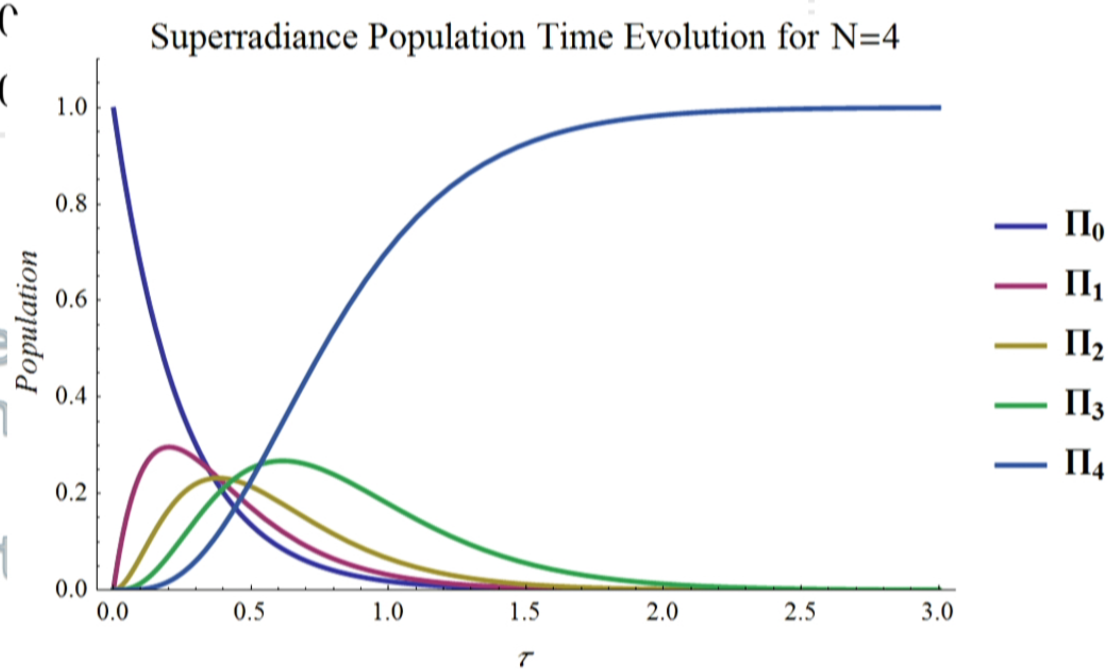
Dicke Model Superradiance

$$\Pi_{n_0}'[\tau] = -\underbrace{(n_0+1)(N-n_0)\Pi_{n_0}[\tau]}_{\text{Decay Rate}} + \underbrace{n_0(N-n_0+1)\Pi_{(n_0-1)}[\tau]}_{\text{Refill Rate}}$$

$$\Pi_{n_0}[0] = \begin{cases} 1 & n_0 = 0 \\ 0 & n_0 > 0 \end{cases}$$

Does superradiance time evolution lead to entangled stat



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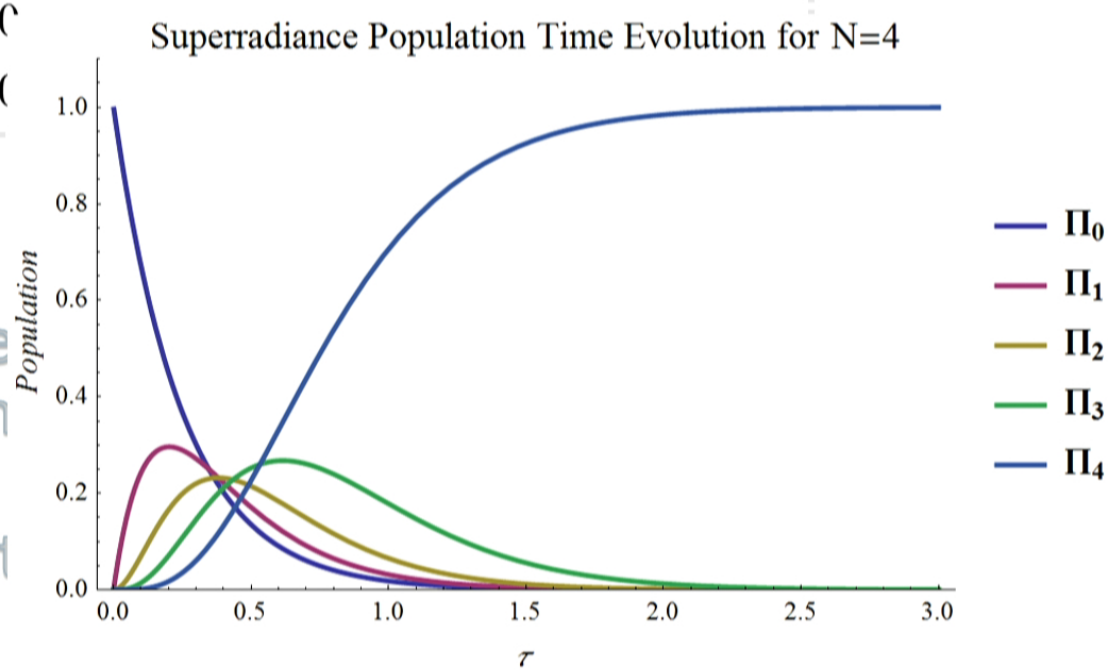

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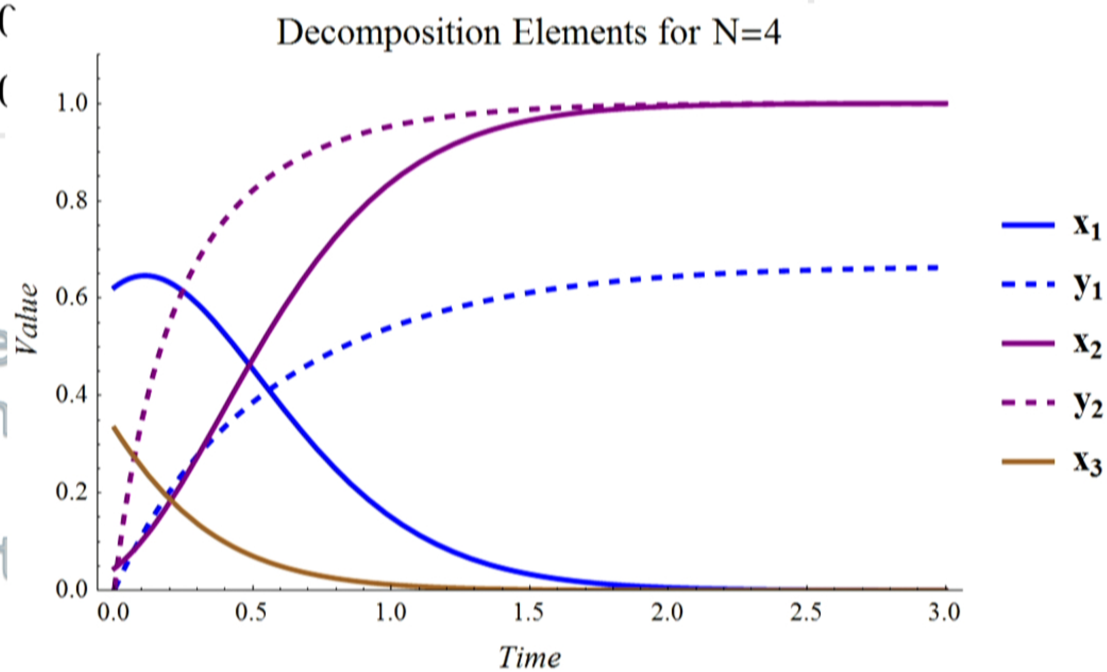

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Easy Idea #2: **Equality by Integration**

Did we do a good job?

So we have a sufficient separability criterion...

How "good" is it? Could it be **tight**? Compare to PPT

Trying to ballpark the fraction of separable states it certifies...

Easy Idea #2: **Equality by Integration**

So we have a sufficient separability criterion...

How "good" is it? Could it be **tight**? Compare to PPT

INTEGRATE THE **VOLUME OF STATES** PER CRITERION

- PPT: We define a 0/1 indicator function on the state
- SDS: We use a Jacobian transform to determine a volume element

Volume SDS = Volume PPT ∴ Necessary & Sufficient!

Incredible!

ALL separable states certified by **build-&-check!**

Two birds one stone: PPT also proven sufficient on this family of states.

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