Title: Quantum and Classical Anomalies

Date: Nov 08, 2013 10:30 AM

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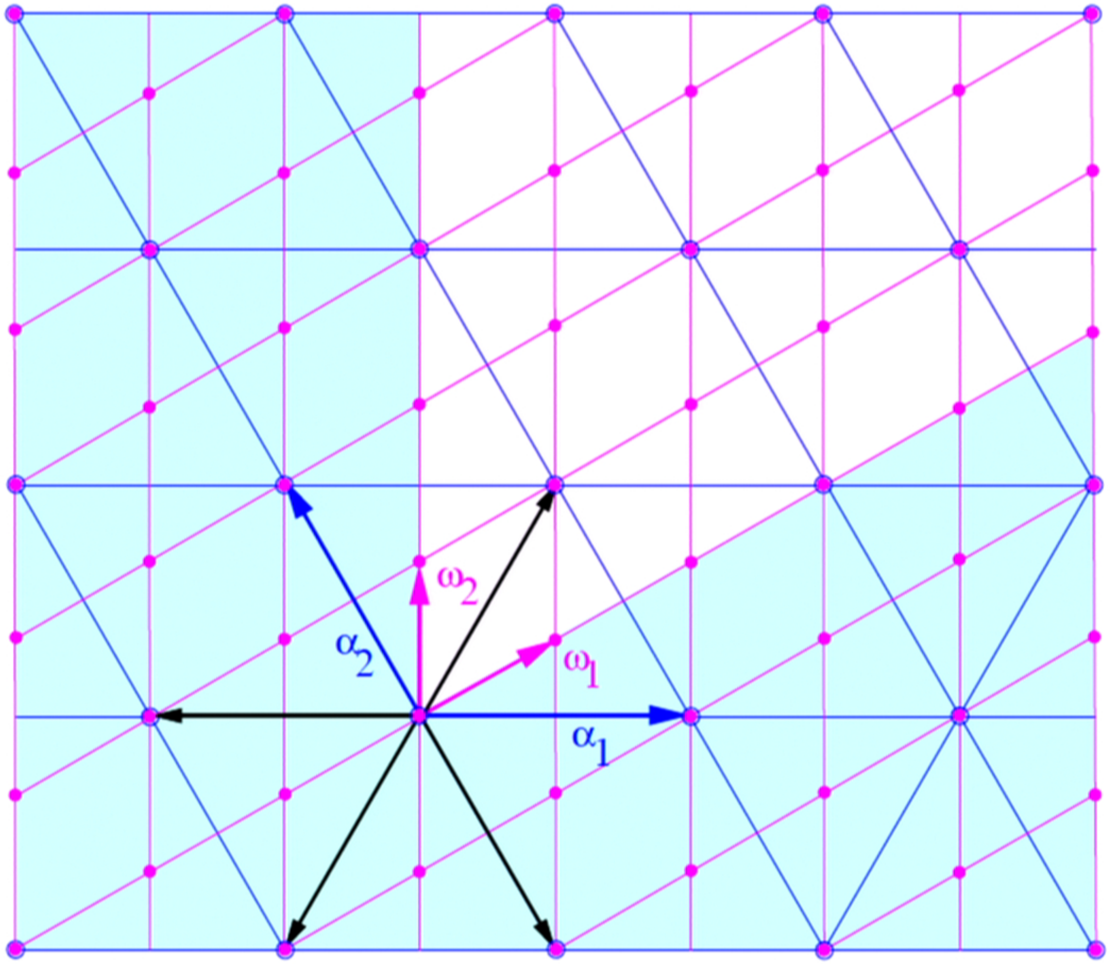
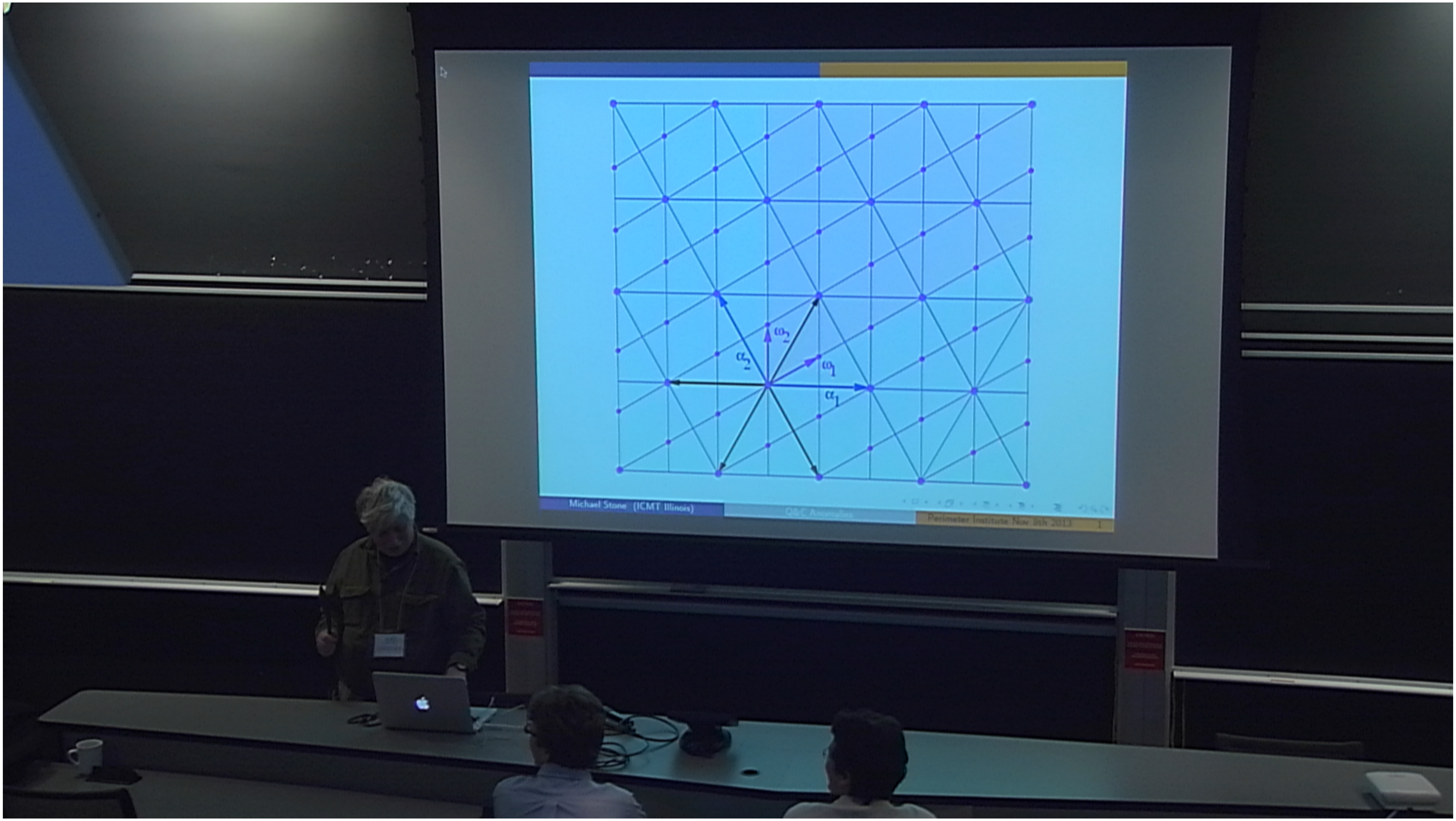
Abstract: I will begin reviewing the Callan-Harvey mechanism of anomaly inflow with particular focus on topological edge states and show how the inflow picture naturally converts the non-covariant "consistent" gauge anomaly of Bardeen and Zumino to the more physical "covariant" anomaly. I will then discuss some recent derivations of the covariant form of the gauge anomaly from classical phase space flows.


ω2
α2
ω1
α1


ω_2
α_2
ω_1
α_1
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Quantum and Classical Anomalies
Michael Stone
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Recent papers

- MS, *Gravitational Anomalies and Thermal Hall effect in Topological Insulators*, Phys. Rev. **B85** 184503 (2012).
- MS, *An Analogue of Hawking Radiation in the Quantum Hall Effect*, Class. Quantum. Grav. 30 085003 (2013).
- MS, V. Dwivedi, *A Classical Version of the Non-Abelian Gauge Anomaly,* Phys. Rev. **D88** 045012 (2013).
- V. Dwivedi, MS, *Classical chiral kinetic theory and anomalies in even space-time dimensions*, arXiv:1308.4576.


Outline
1 Quantum Anomalies
Chiral symmetry is dangerous
The Callan-Harvey effect
Consistent versus Covariant anomalies
Lessons learned
2 Classical Anomalies
Basic idea
Liouville's theorem
Co-adjoint orbits and the Wong equations
Classical and Quantum traces
Conclusions
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Goldstone & Wilczek 1981

Filling the ground state of the one-dimensional single-particle fermion Hamiltonian

$$\begin{aligned} H &= -i\sigma_3\partial_x + \Delta(x)\sigma_1 e^{i\sigma_3\theta(x,t)} \\ &\equiv -i\sigma_3\partial_x + \Delta(x)(\sigma_1\cos\theta + \sigma_2\sin\theta) \end{aligned}$$

leads to charge and currents

$$\begin{aligned} \langle\rho(x)\rangle_{\text{gnd}} &\simeq -\frac{1}{2\pi}\partial_x\theta(x), \\ \langle j(x)\rangle_{\text{gnd}} &\simeq \frac{1}{2\pi}\partial_t\theta(x) \end{aligned}$$

Chiral symmetry is dangerous!

- Chiral unitary transformation

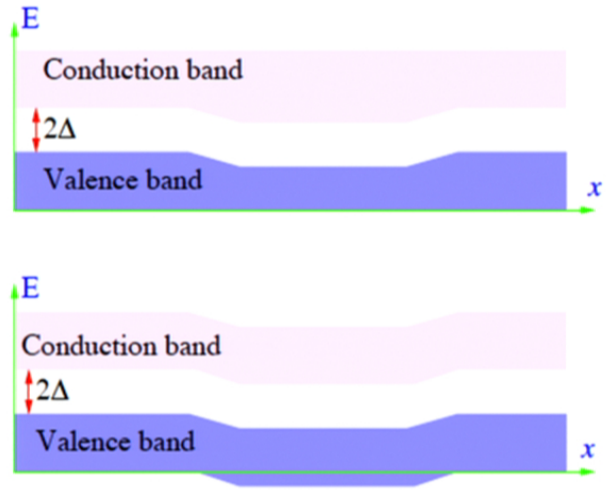
$$e^{i\sigma_3\theta/2}\left\{-i\sigma_3\partial_x + \Delta(x)\sigma_1 e^{i\sigma_3\theta}\right\}e^{-i\sigma_3\theta/2}$$
$$= \quad -i\sigma_3\partial_x + \Delta(x)\sigma_1 - \frac{1}{2}\partial_x\theta$$

- Suggests that a positive rate of twist in $\theta(x)$ is equivalent to the addition of a negative potential.

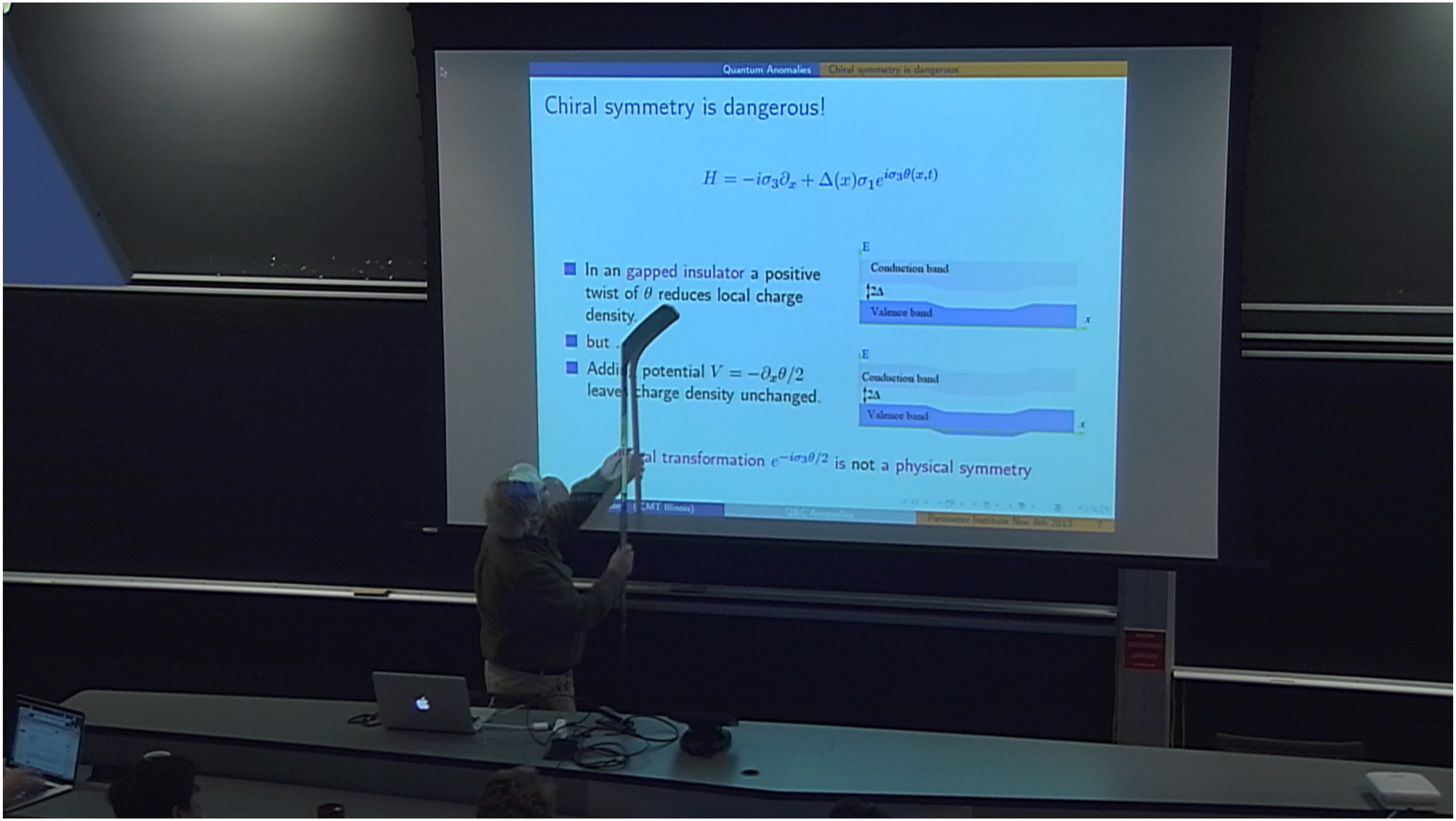
Chiral symmetry is dangerous!

$$H = -i\sigma_3\partial_x + \Delta(x)\sigma_1 e^{i\sigma_3\theta(x,t)}$$

- In an gapped insulator a positive twist of θ reduces local charge density.
- but ...
- Adding potential $V = -\partial_x\theta/2$ leaves charge density unchanged.



Chiral transformation $e^{-i\sigma_3\theta/2}$ is not a physical symmetry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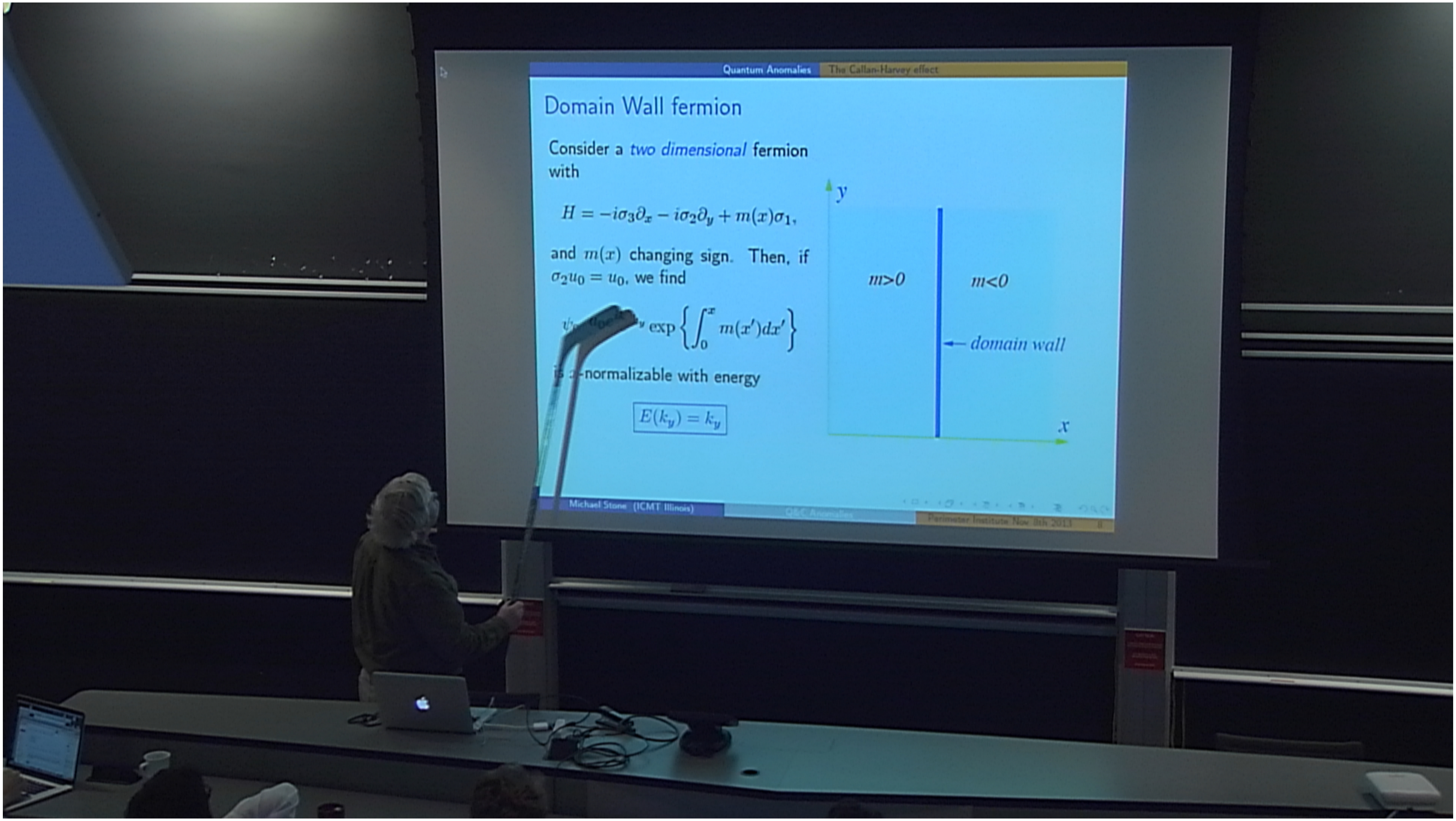
Quantum Anomalies
The Callan-Harvey effect
Domain Wall fermion
Consider a two dimensional fermion with
$H = -i\sigma_3\partial_x - i\sigma_2\partial_y + m(x)\sigma_1,$
and $m(x)$ changing sign. Then, if $\sigma_2 u_0 = u_0$, we find
-normalizable with energy
$E(k_y) = k_y$
y
m>0
m<0
domain wall
x
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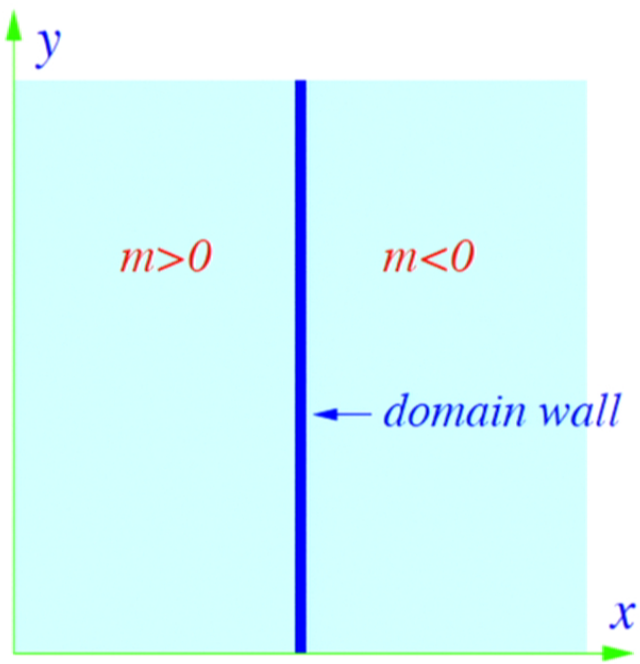
is x-normalizable with energy

$$\boxed{E(k_y) = k_y}$$



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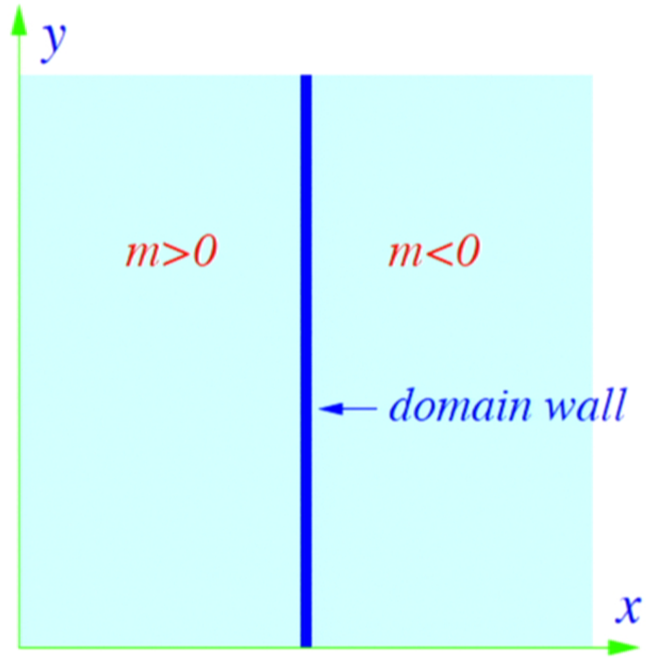


The Chiral Anomaly

Introduce an electric field parallel to the domain wall

$$H \to -i\sigma_3\partial_x - i\sigma_2(\partial_y - ieA_y) + m(x)\sigma_1.$$

with $A_y = -eEt$. Then

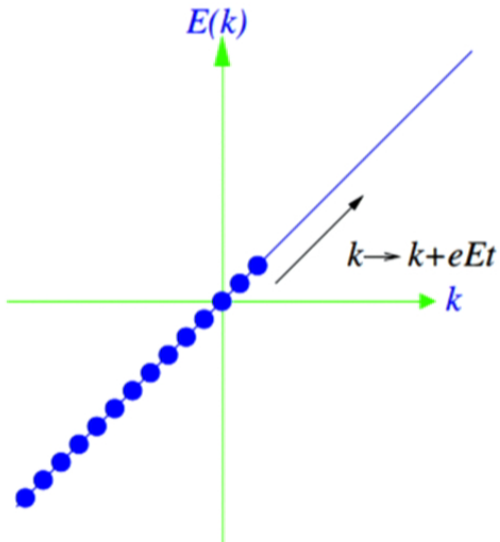
$$k_y \to k_y + eEt$$

and

$$j_0 = e\int_{-\infty}^{\infty}\frac{dk}{2\pi} : a_k^\dagger a_k :$$

gives the anomaly

$$\boxed{\frac{\partial j_0}{\partial t} + \frac{\partial j_y}{\partial y} = \frac{e^2 E}{2\pi}}$$



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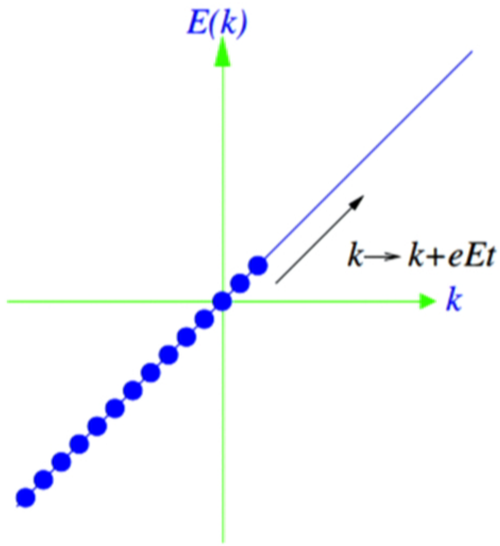


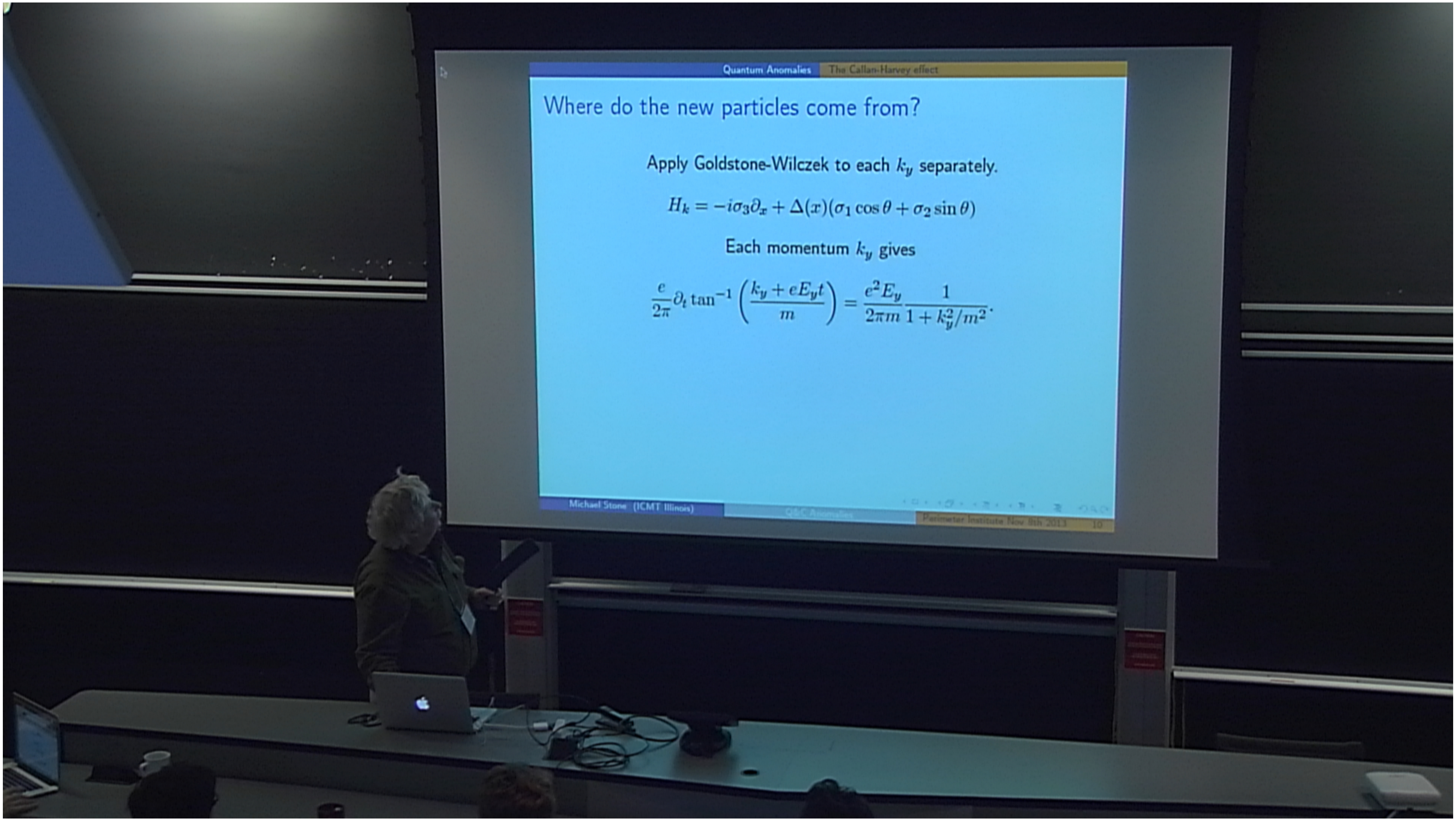
Where do the new particles come from?

Apply Goldstone-Wilczek to each k_y separately.

$$H_k = -i\sigma_3\partial_x + \Delta(x)(\sigma_1 \cos\theta + \sigma_2 \sin\theta)$$

Each momentum k_y gives

$$\frac{e}{2\pi}\partial_t \tan^{-1}\left(\frac{k_y + eE_y t}{m}\right) = \frac{e^2 E_y}{2\pi m}\frac{1}{1 + k_y^2/m^2}.$$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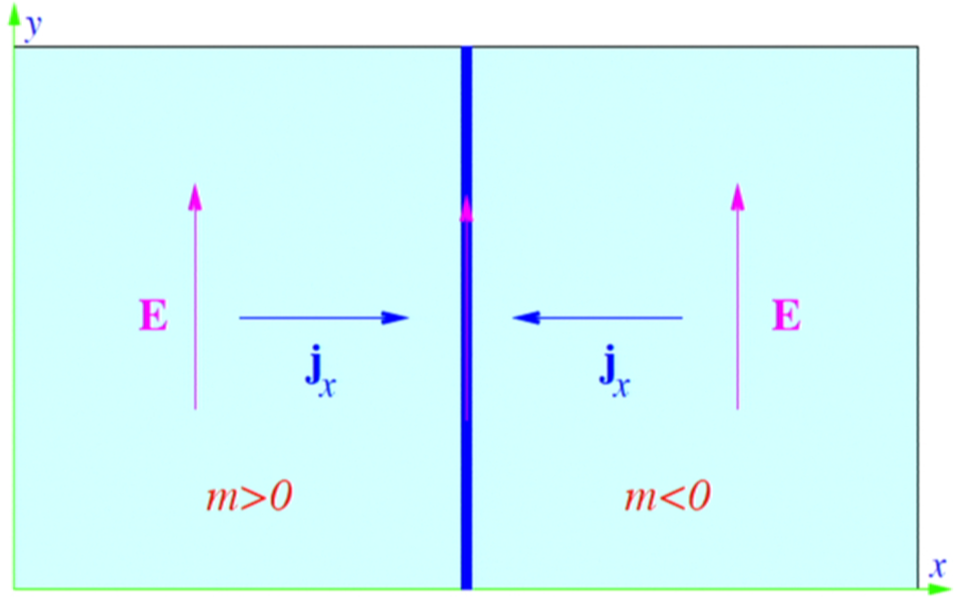
Where do the new particles come from?

Answer: They flow into the universe from outside



The Callan-Harvey effect (1985)

Condensed Matter Applications

- Momentum paradoxes in ^{3}He-A (MS-Garg-Muzikar '85; MS-Gaitan '87; Schakel-Battenburg '89,Volovik '89–)
- Quantum Hall effect (MS '91, Baez-Balachandran-Travesset-Stern '98)
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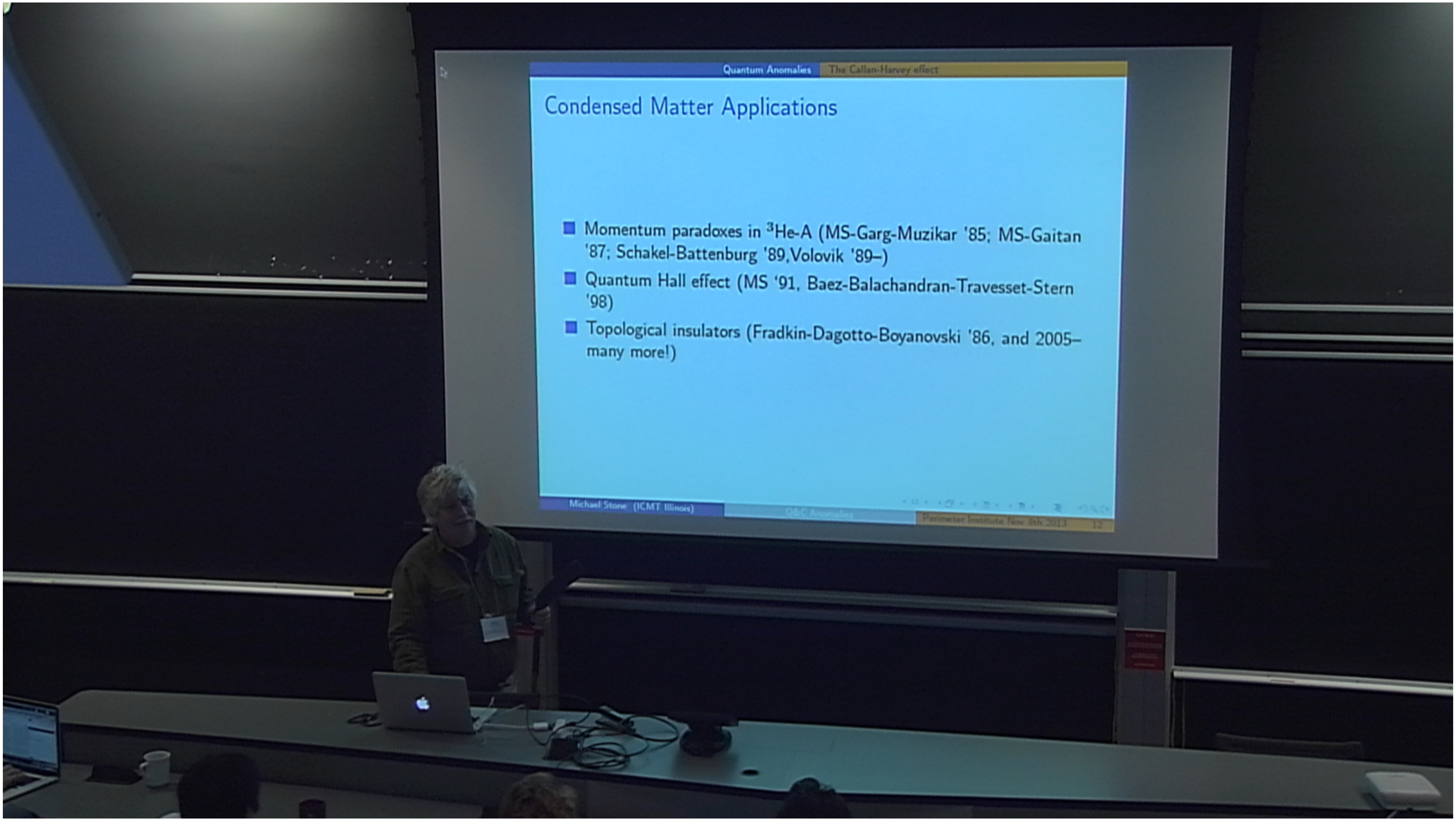
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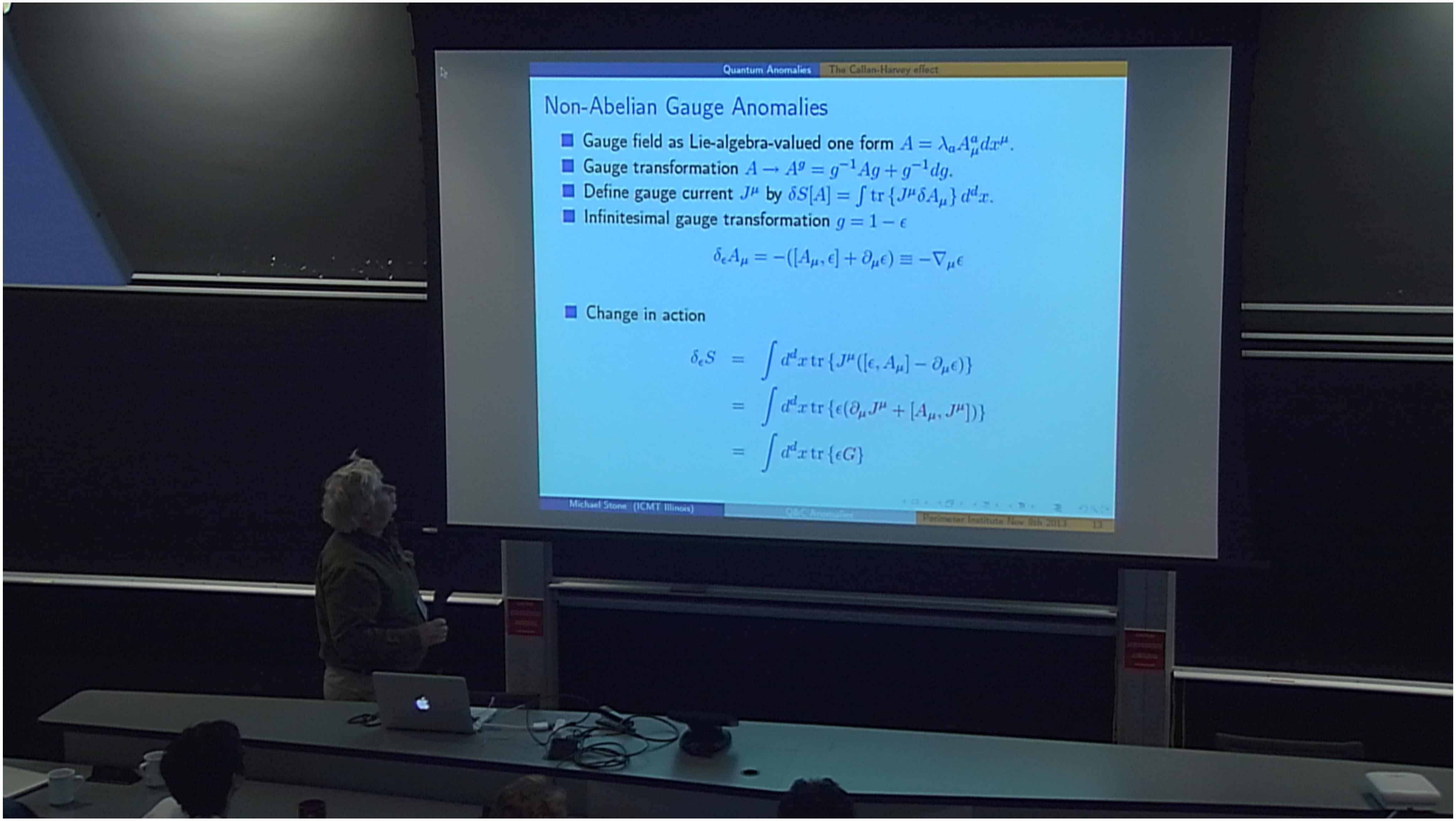
Quantum Anomalies
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Non-Abelian Gauge Anomalies
Gauge field as Lie-algebra-valued one form $A = \lambda_a A^a_\mu dx^\mu$.
Gauge transformation $A \to A^g = g^{-1}Ag + g^{-1}dg$.
Define gauge current J^μ by $\delta S[A] = \int \mathrm{tr}\{J^\mu \delta A_\mu\}\, d^dx$.
Infinitesimal gauge transformation $g = 1 - \epsilon$
$\delta_\epsilon A_\mu = -([A_\mu, \epsilon] + \partial_\mu \epsilon) \equiv -\nabla_\mu \epsilon$
Change in action
$\delta_\epsilon S = \int d^dx\, \mathrm{tr}\{J^\mu([\epsilon, A_\mu] - \partial_\mu \epsilon)\}$
$= \int d^dx\, \mathrm{tr}\{\epsilon(\partial_\mu J^\mu + [A_\mu, J^\mu])\}$
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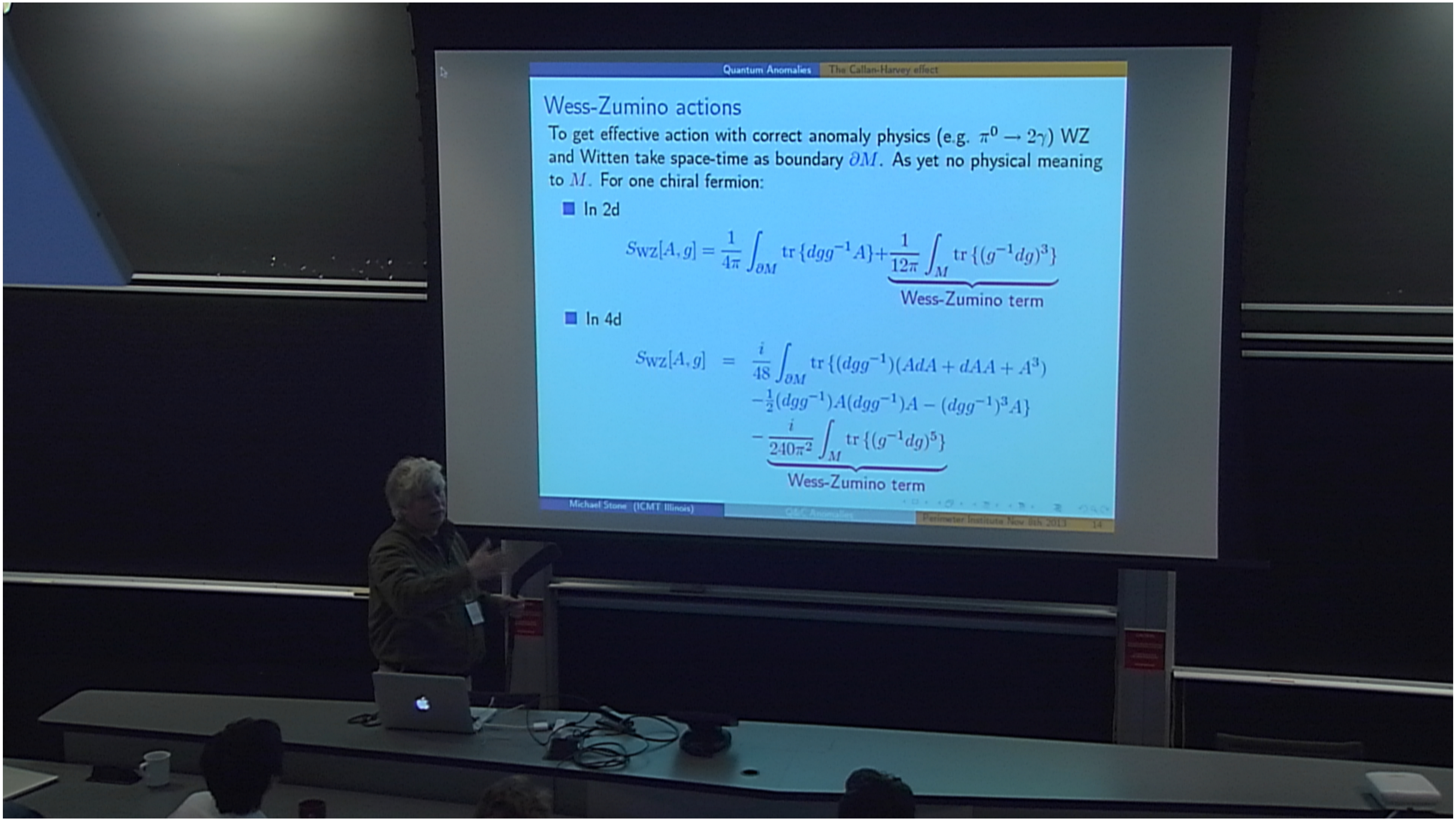
Quantum Anomalies
The Callan-Harvey effect
Wess-Zumino actions
To get effective action with correct anomaly physics (e.g. $\pi^0 \to 2\gamma$) WZ and Witten take space-time as boundary ∂M. As yet no physical meaning to M. For one chiral fermion:
In 2d
$S_{WZ}[A,g] = \frac{1}{4\pi}\int_{\partial M} \mathrm{tr}\{dgg^{-1}A\} + \underbrace{\frac{1}{12\pi}\int_M \mathrm{tr}\{(g^{-1}dg)^3\}}_{\text{Wess-Zumino term}}$
In 4d
$S_{WZ}[A,g] = \frac{i}{48}\int_{\partial M} \mathrm{tr}\{(dgg^{-1})(AdA + dAA + A^3) - \frac{1}{2}(dgg^{-1})A(dgg^{-1})A - (dgg^{-1})^3 A\} - \underbrace{\frac{i}{240\pi^2}\int_M \mathrm{tr}\{(g^{-1}dg)^5\}}_{\text{Wess-Zumino term}}$
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Wess-Zumino actions

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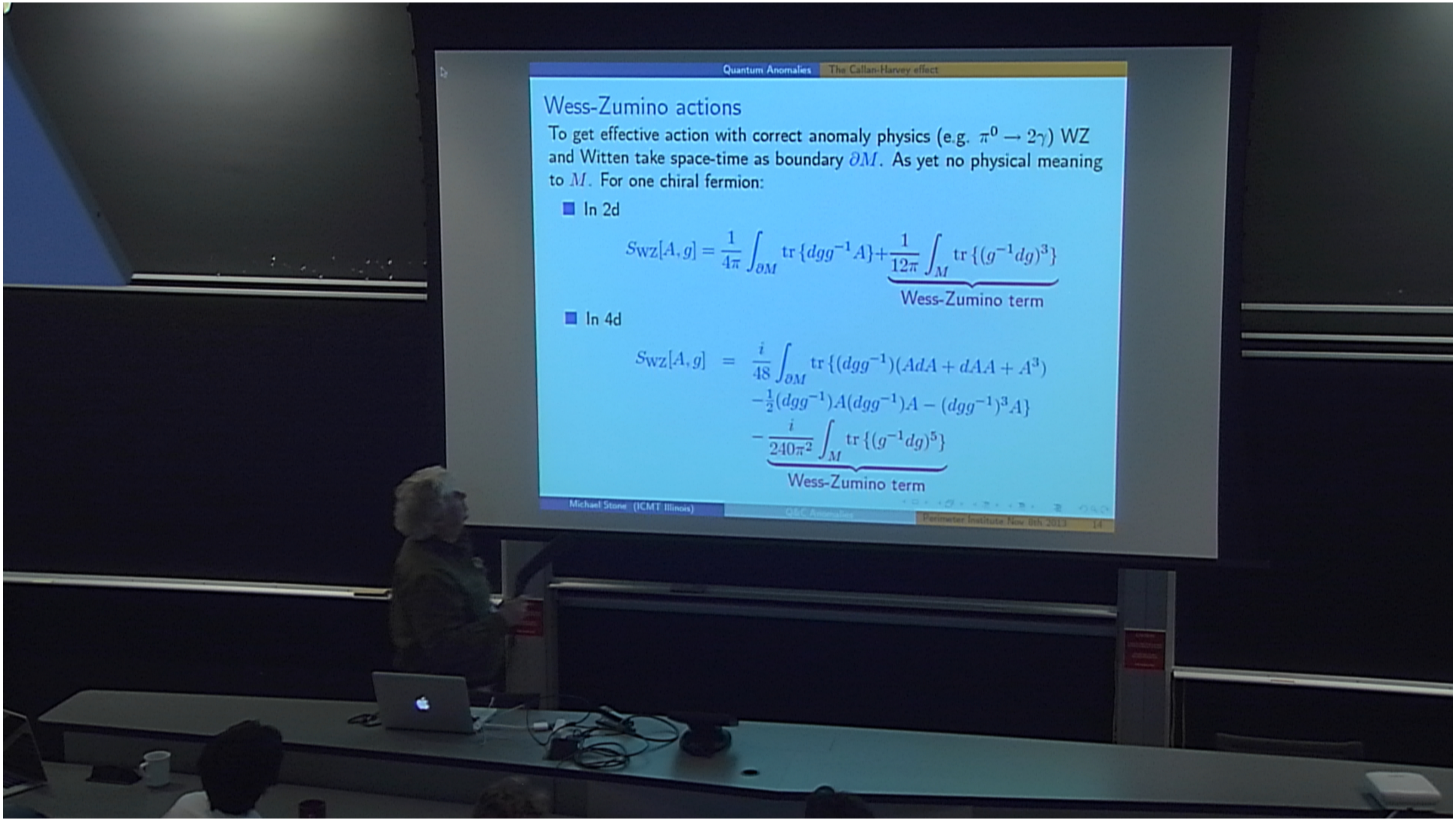
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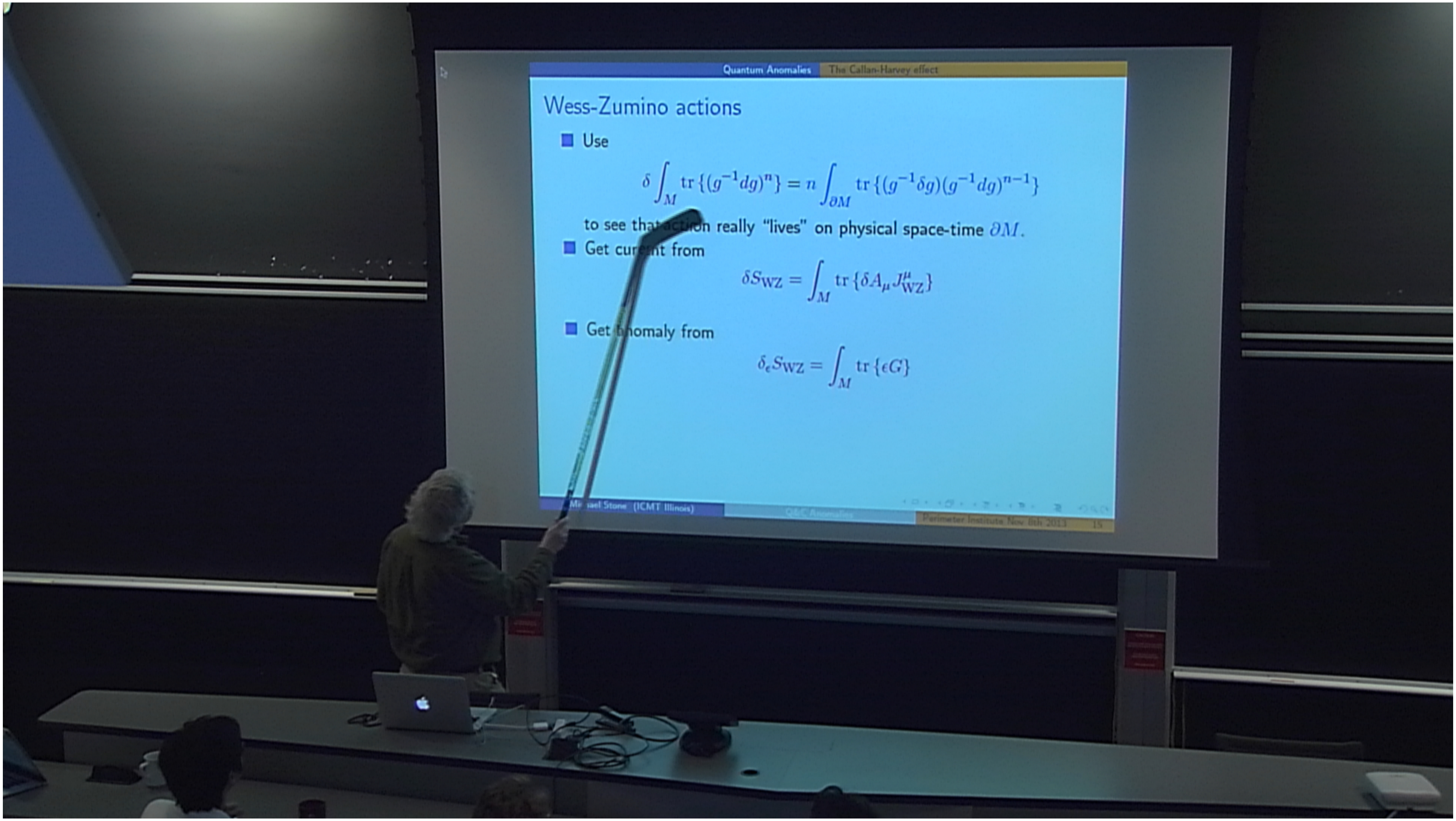
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Use
$\delta \int_M \mathrm{tr}\{(g^{-1}dg)^n\} = n \int_{\partial M} \mathrm{tr}\{(g^{-1}\delta g)(g^{-1}dg)^{n-1}\}$
to see that action really "lives" on physical space-time ∂M.
Get current from
$\delta S_{\mathrm{WZ}} = \int_M \mathrm{tr}\{\delta A_\mu J^\mu_{\mathrm{WZ}}\}$
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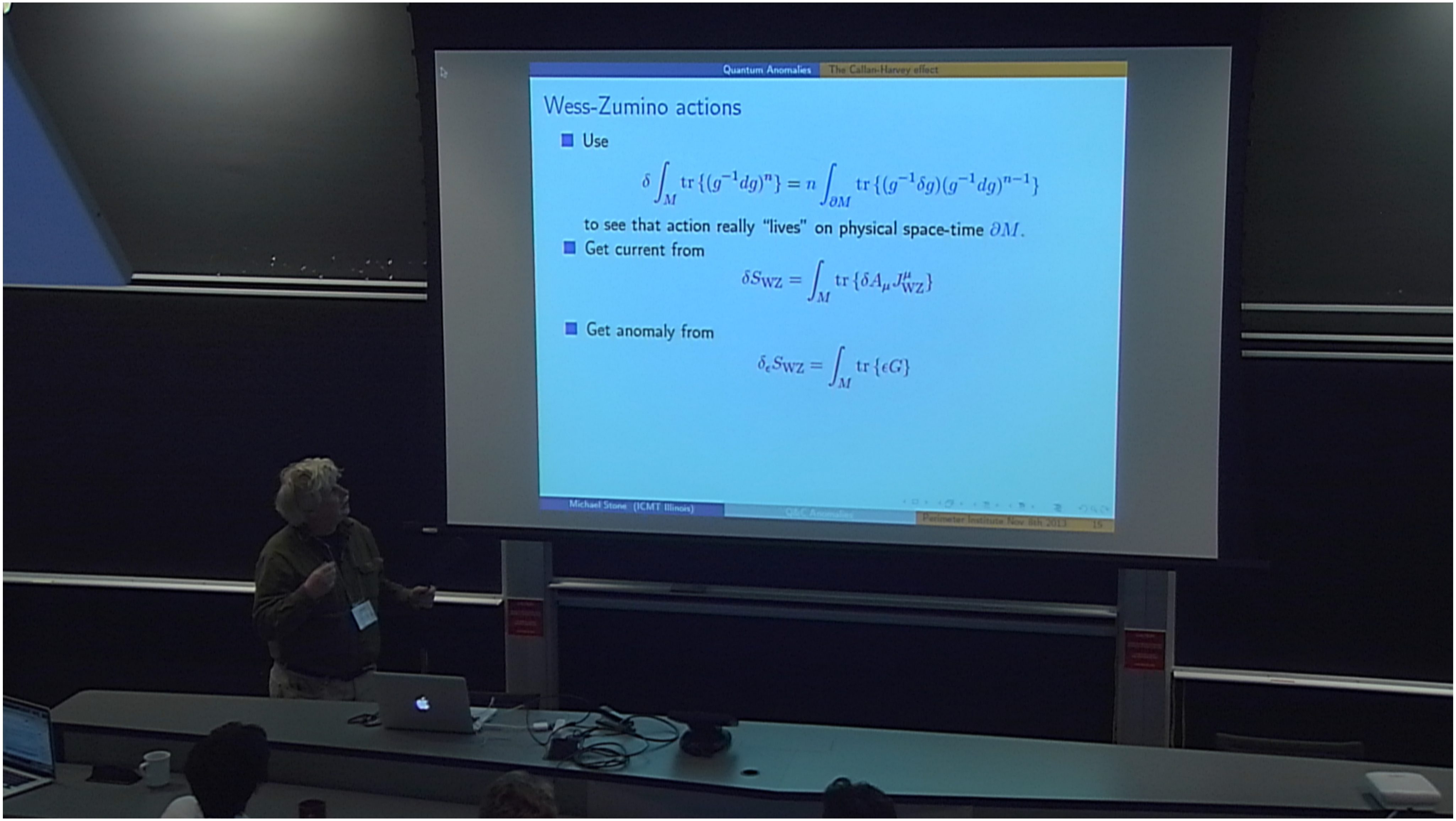
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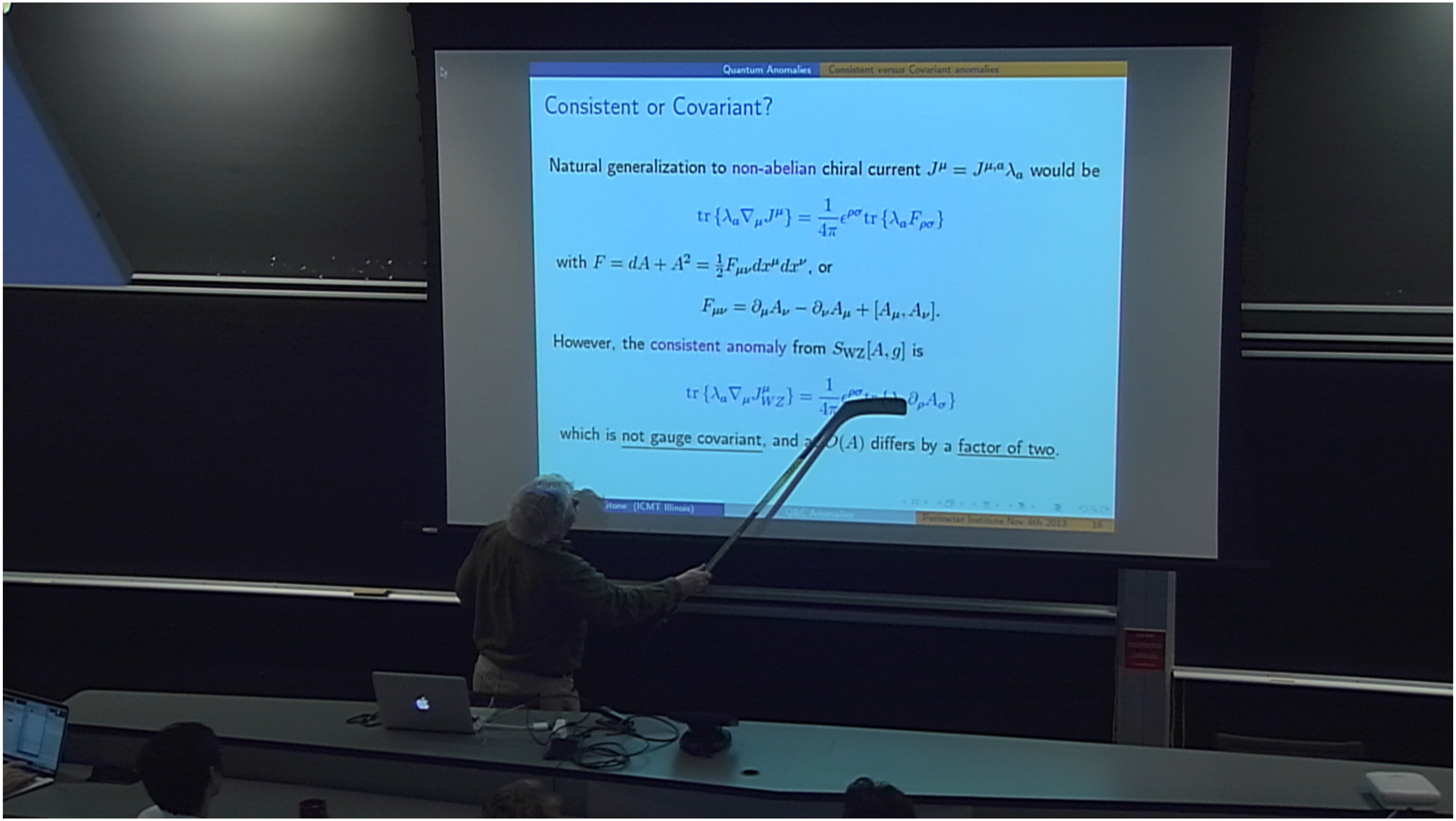
Quantum Anomalies
Consistent versus Covariant anomalies
Consistent or Covariant?
Natural generalization to non-abelian chiral current $J^\mu = J^{\mu,a}\lambda_a$ would be
$$\mathrm{tr}\{\lambda_a \nabla_\mu J^\mu\} = \frac{1}{4\pi}\epsilon^{\rho\sigma}\mathrm{tr}\{\lambda_a F_{\rho\sigma}\}$$
with $F = dA + A^2 = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$, or
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$
However, the consistent anomaly from $S_{\mathrm{WZ}}[A,g]$ is
$$\mathrm{tr}\{\lambda_a \nabla_\mu J^\mu_{WZ}\} = \frac{1}{4\pi}\epsilon^{\rho\sigma} \ldots \partial_\rho A_\sigma\}$$
which is not gauge covariant, and … $\mathcal{O}(A)$ differs by a factor of two.
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Consistent or Covariant?

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which is not gauge covariant, and at $O(A)$ differs by a factor of two.

Bardeen polynomial

Redefine the current by adding c-number term

$$J^\mu \to J^\mu_{\rm WZ} + X^\mu,$$

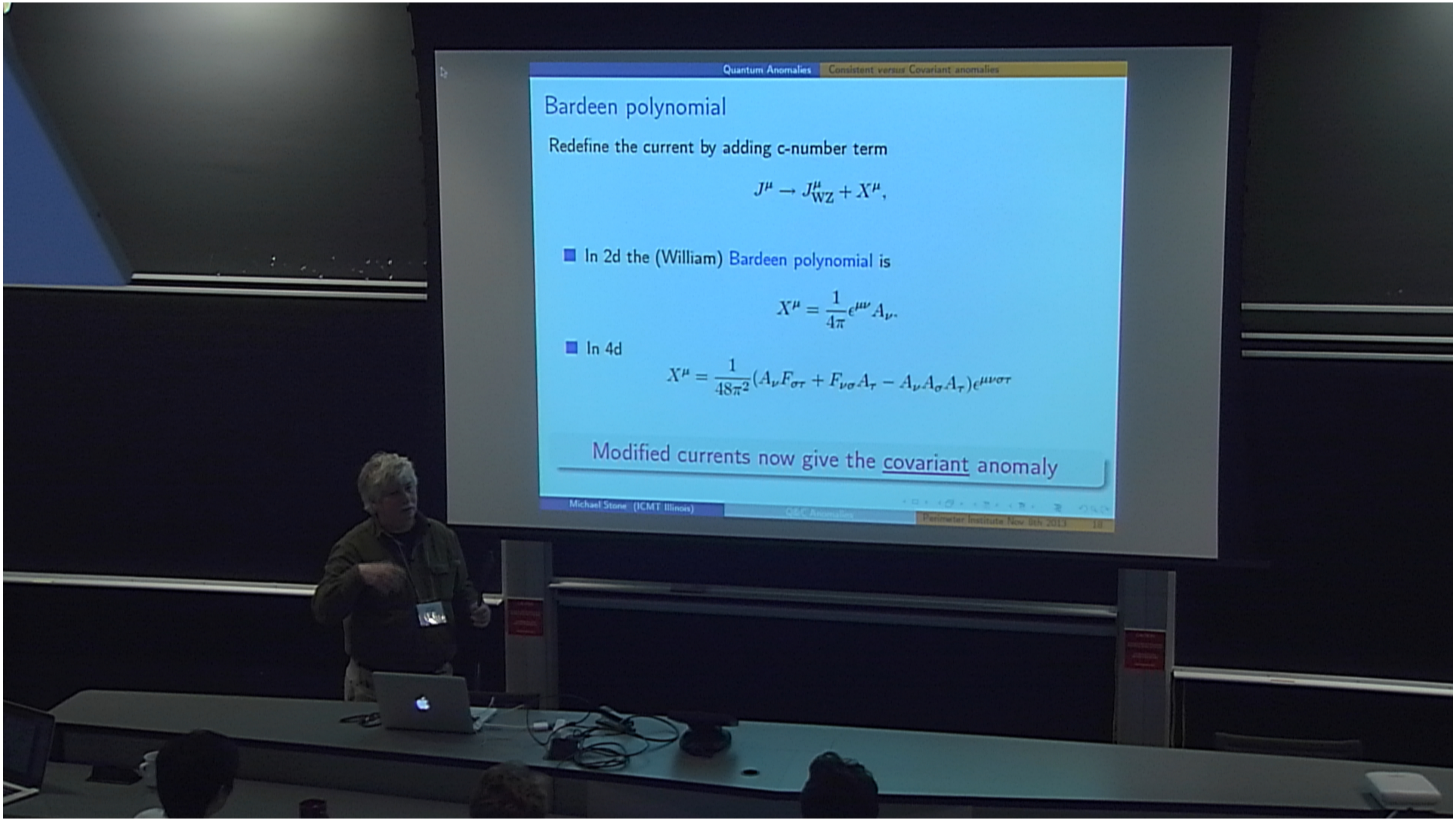
- In 2d the (William) Bardeen polynomial is

$$X^\mu = \frac{1}{4\pi}\epsilon^{\mu\nu}A_\nu.$$

- In 4d

$$X^\mu = \frac{1}{48\pi^2}(A_\nu F_{\sigma\tau} + F_{\nu\sigma}A_\tau - A_\nu A_\sigma A_\tau)\epsilon^{\mu\nu\sigma\tau}$$

Modified currents now give the covariant anomaly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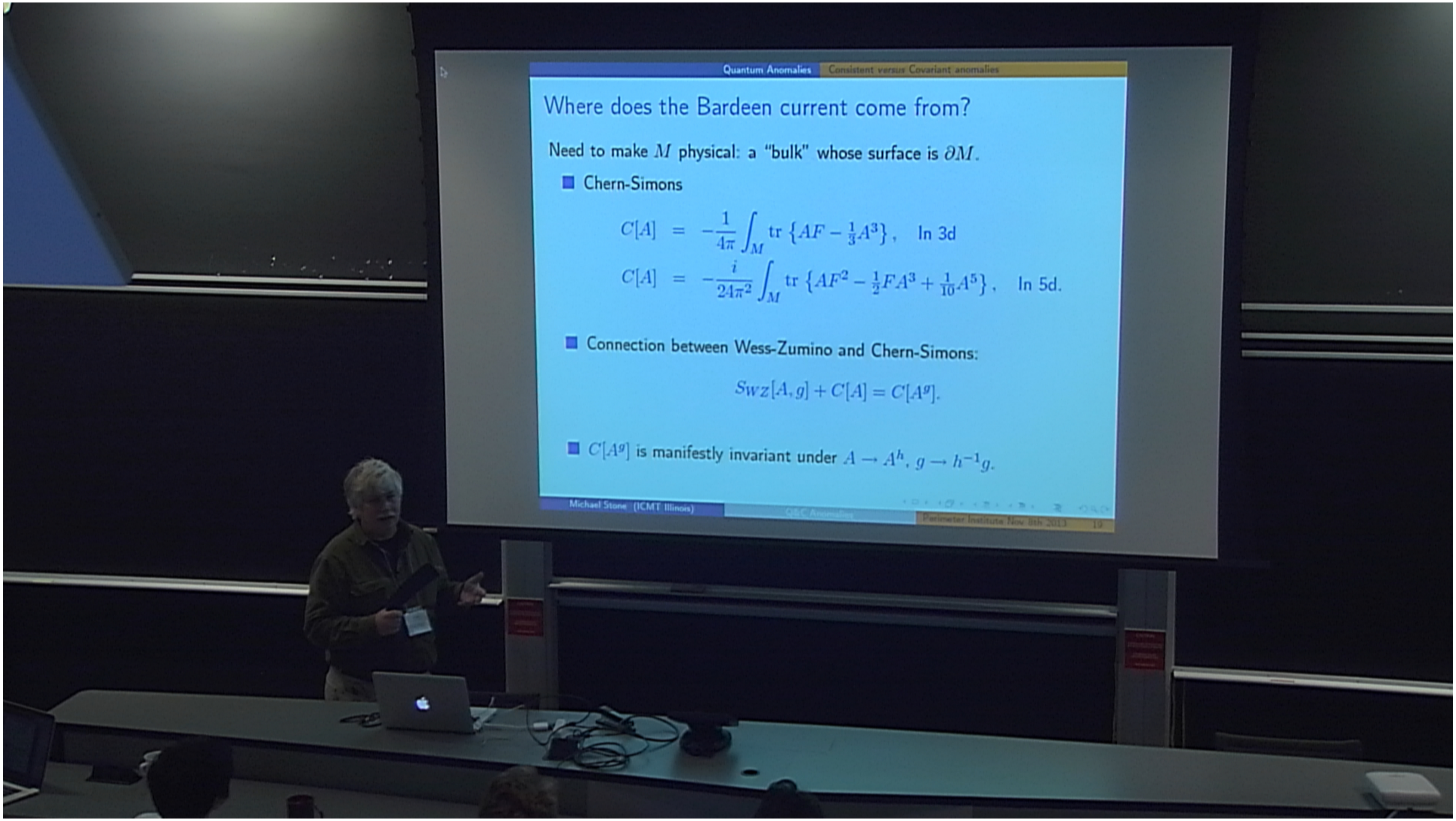
Quantum Anomalies
Consistent versus Covariant anomalies
Where does the Bardeen current come from?
Need to make M physical: a "bulk" whose surface is ∂M.
Chern-Simons
$C[A] = -\frac{1}{4\pi}\int_M \mathrm{tr}\,\{AF - \frac{1}{3}A^3\}$, In 3d
$C[A] = -\frac{i}{24\pi^2}\int_M \mathrm{tr}\,\{AF^2 - \frac{1}{2}FA^3 + \frac{1}{10}A^5\}$, In 5d.
Connection between Wess-Zumino and Chern-Simons:
$S_{WZ}[A,g] + C[A] = C[A^g]$.
$C[A^g]$ is manifestly invariant under $A \to A^h$, $g \to h^{-1}g$.
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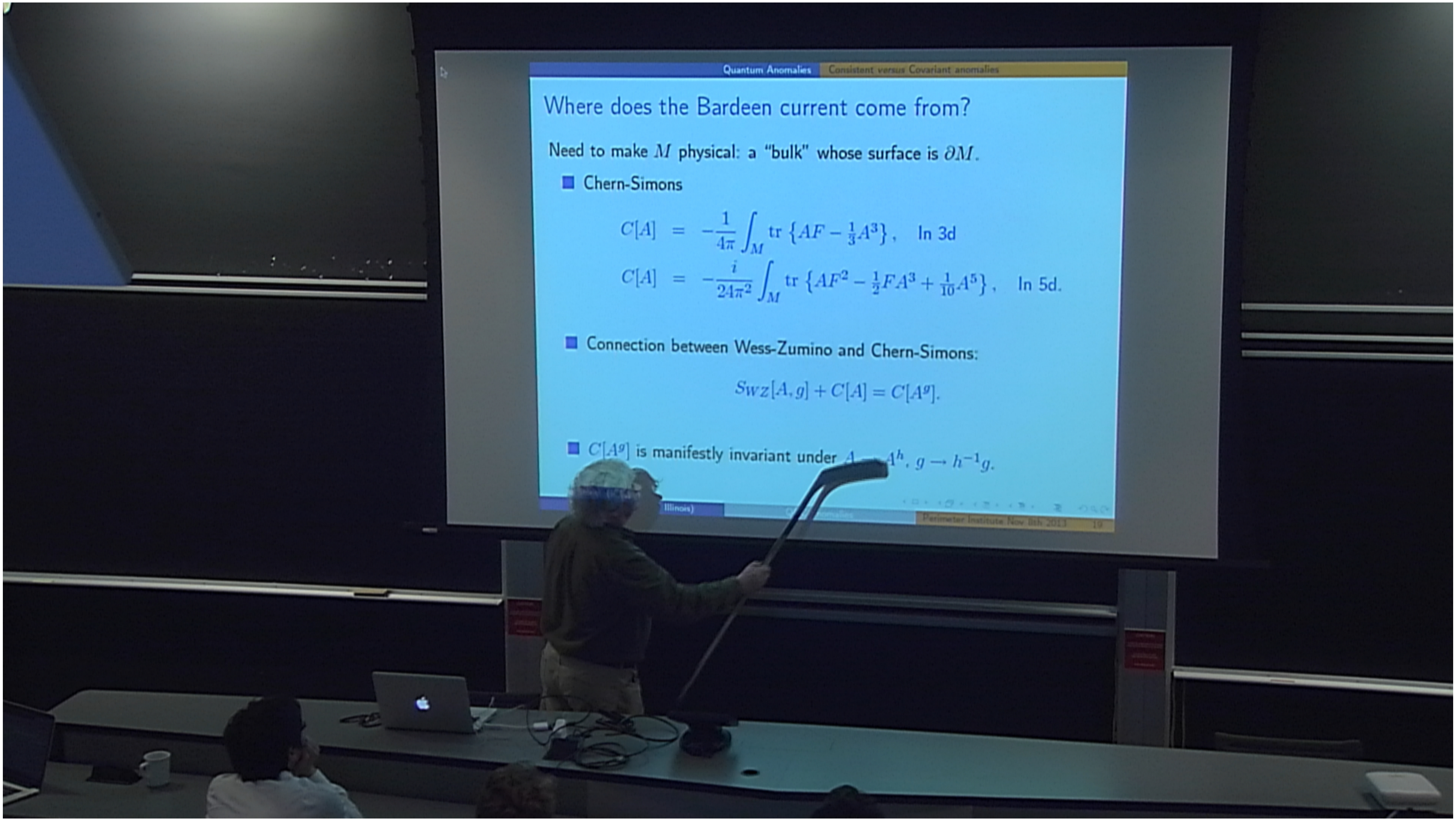

Quantum Anomalies
Consistent versus Covariant anomalies
Where does the Bardeen current come from?
Need to make M physical: a "bulk" whose surface is ∂M.
Chern-Simons
$C[A] = -\frac{1}{4\pi}\int_M \mathrm{tr}\left\{AF - \frac{1}{3}A^3\right\}$, In 3d
$C[A] = -\frac{i}{24\pi^2}\int_M \mathrm{tr}\left\{AF^2 - \frac{1}{2}FA^3 + \frac{1}{10}A^5\right\}$, In 5d.
Connection between Wess-Zumino and Chern-Simons:
$S_{WZ}[A,g] + C[A] = C[A^g]$.
$C[A^g]$ is manifestly invariant under … A^h, $g \to h^{-1}g$.
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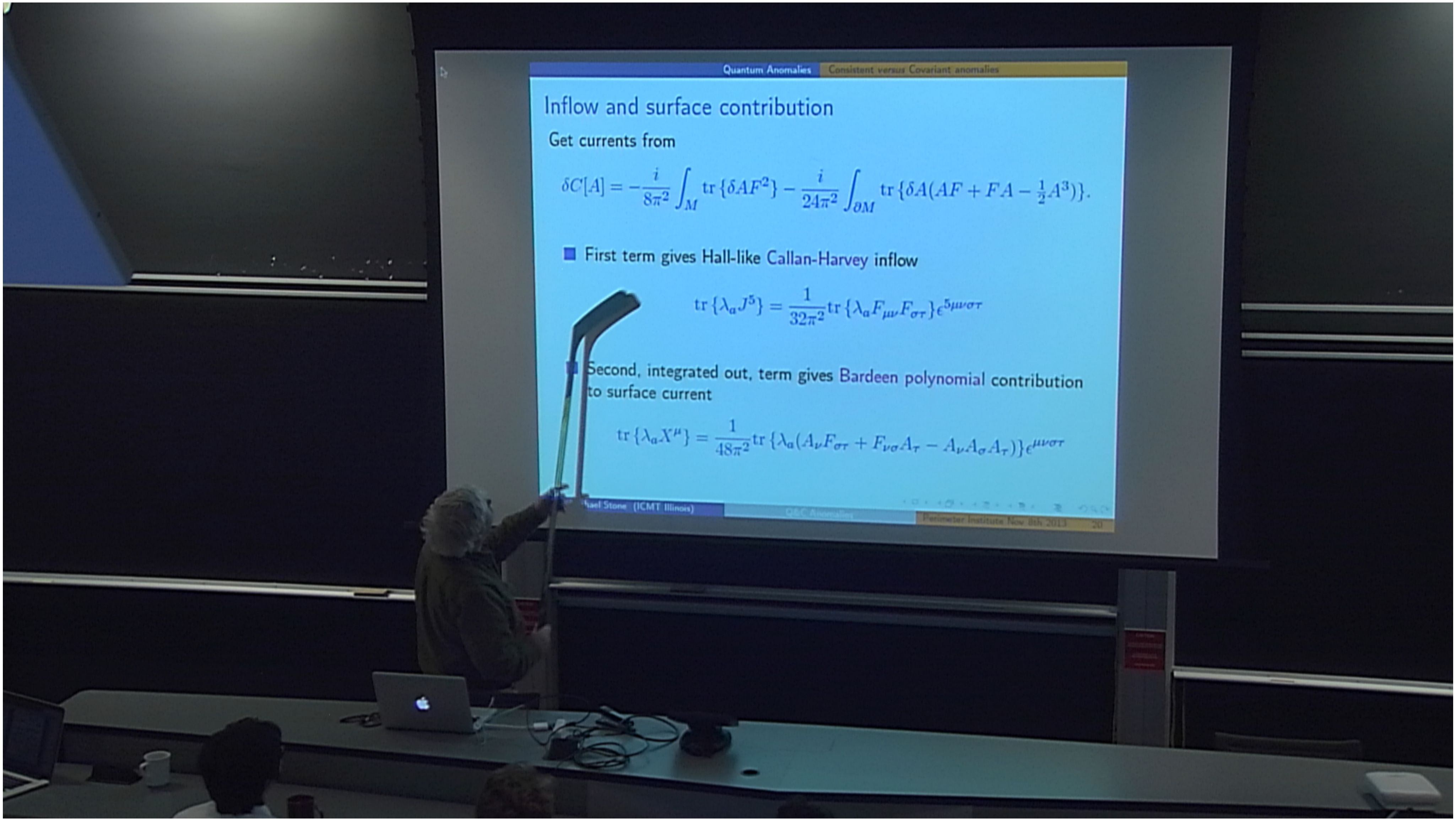
Quantum Anomalies
Consistent versus Covariant anomalies
Inflow and surface contribution
Get currents from
$\delta C[A] = -\frac{i}{8\pi^2}\int_M \mathrm{tr}\{\delta A F^2\} - \frac{i}{24\pi^2}\int_{\partial M} \mathrm{tr}\{\delta A(AF + FA - \frac{1}{2}A^3)\}.$
First term gives Hall-like Callan-Harvey inflow
$\mathrm{tr}\{\lambda_a J^5\} = \frac{1}{32\pi^2}\mathrm{tr}\{\lambda_a F_{\mu\nu}F_{\sigma\tau}\}\epsilon^{5\mu\nu\sigma\tau}$
Second, integrated out, term gives Bardeen polynomial contribution to surface current
$\mathrm{tr}\{\lambda_a X^\mu\} = \frac{1}{48\pi^2}\mathrm{tr}\{\lambda_a(A_\nu F_{\sigma\tau} + F_{\nu\sigma}A_\tau - A_\nu A_\sigma A_\tau)\}\epsilon^{\mu\nu\sigma\tau}$
(ICMT Illinois)
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Inflow and surface contribution

Get currents from

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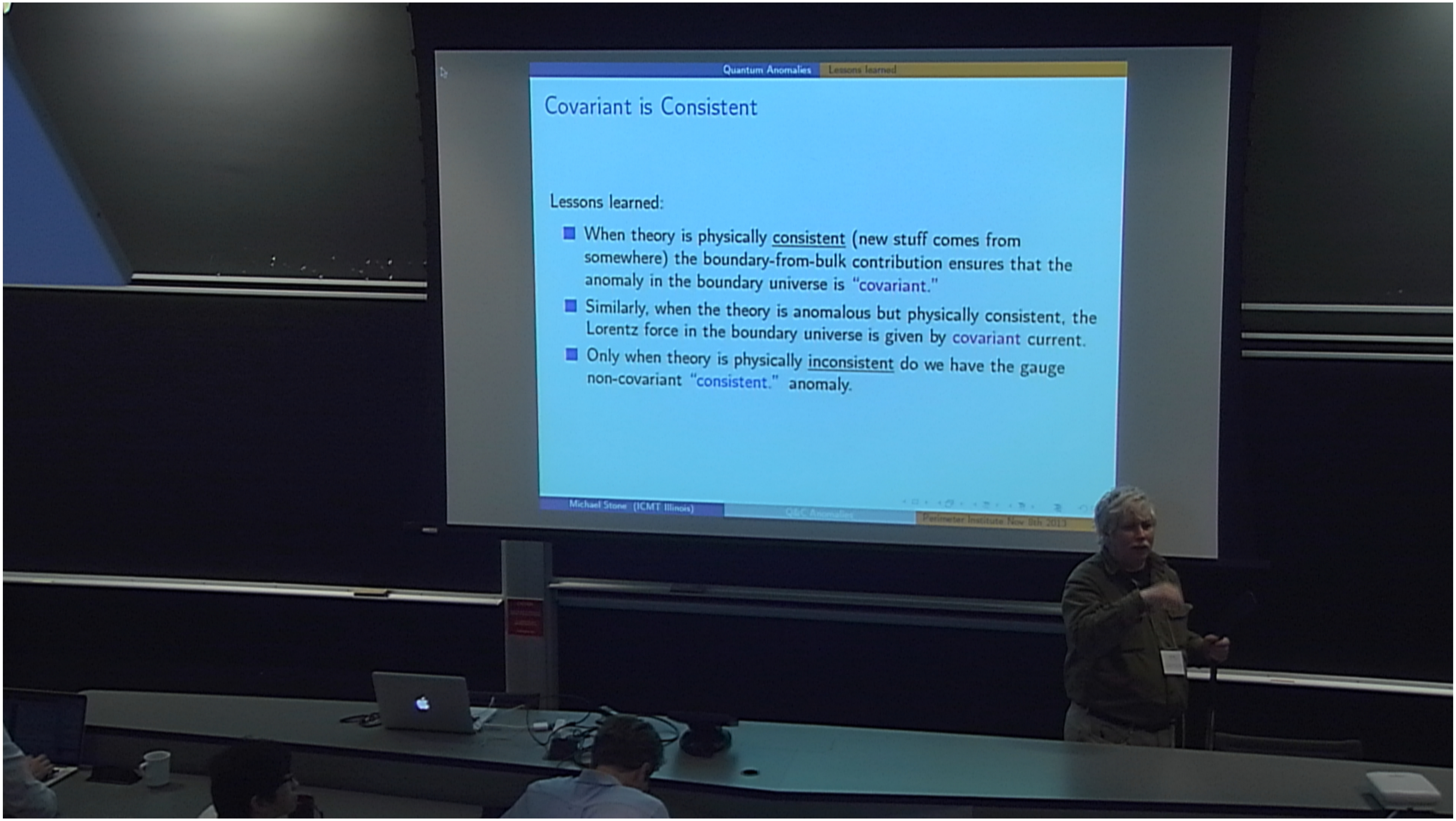
Covariant is Consistent

Lessons learned:

- When theory is physically consistent (new stuff comes from somewhere) the boundary-from-bulk contribution ensures that the anomaly in the boundary universe is "covariant."
- Similarly, when the theory is anomalous but physically consistent, the Lorentz force in the boundary universe is given by covariant current.
- Only when theory is physically inconsistent do we have the gauge non-covariant "consistent." anomaly.


Quantum Anomalies
Lessons learned
Covariant is Consistent
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Michael Stone (ICMT Illinois)
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Classical Anomalies

- Although the chiral anomaly is usually described as a quantum effect, there are classical mechanical versions.
- For massless fermions, use the incompressibility of phase-space flow to track flux through diabolical point.
- Works for both Abelian and non-Abelian theories in any dimension.

A diabolo

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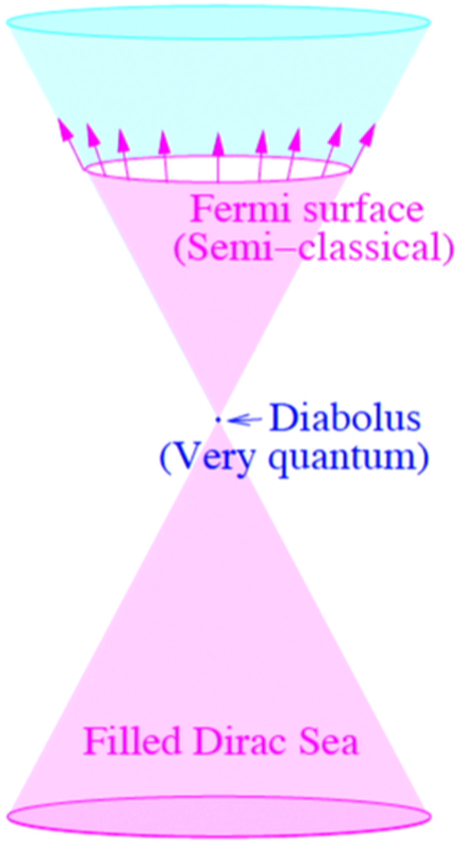


Classical Mechanics

- Action for $\boldsymbol{\xi} \equiv (\xi^1, \ldots \xi^{2n})$.

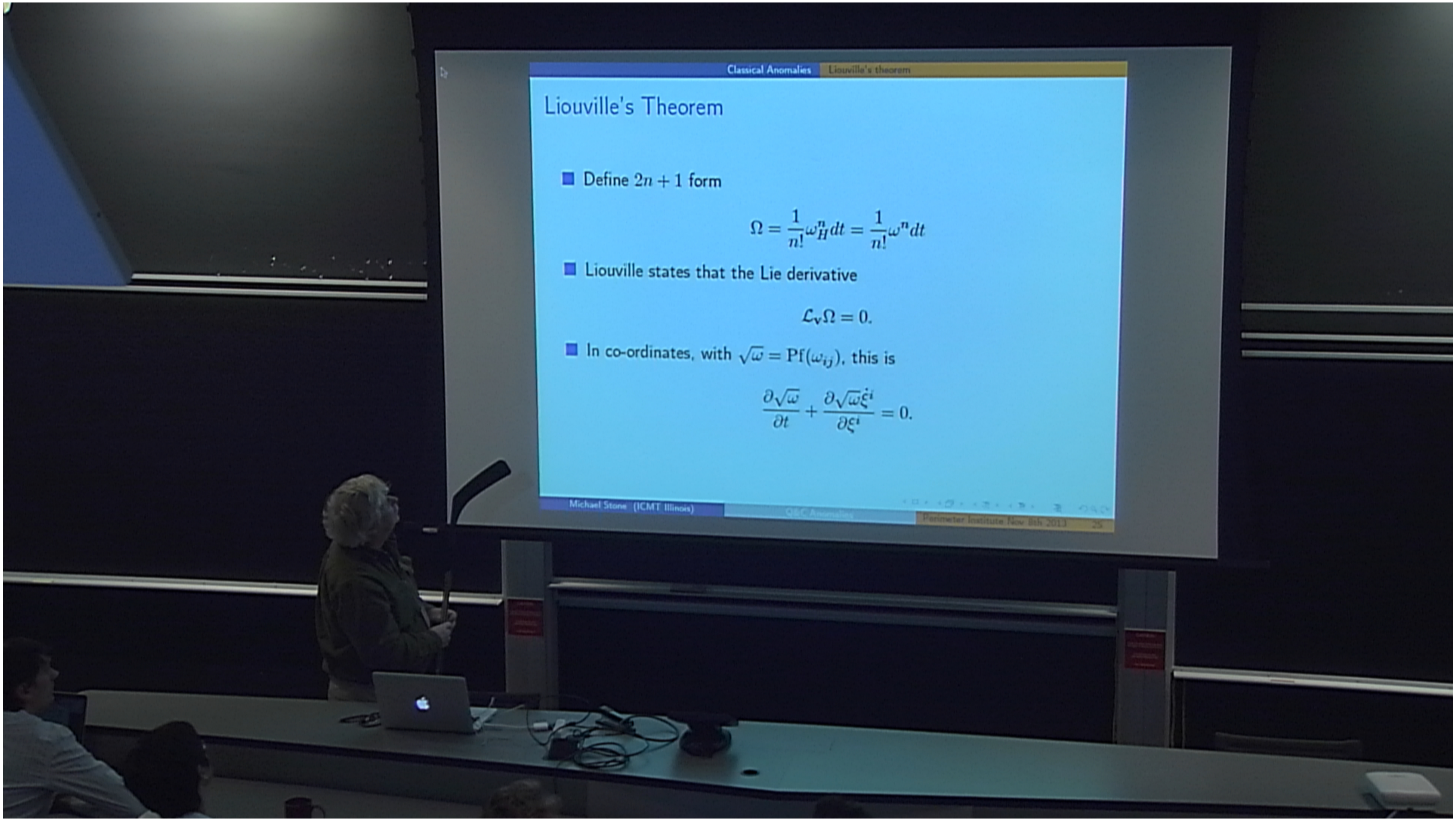
$$S[\boldsymbol{\xi}] = \int \left\{ \sum_{i=1}^{2n} \eta_i(\boldsymbol{\xi}, t)\dot{\xi}^i - H(\boldsymbol{\xi}, t) \right\} dt.$$

- Equations of motion

$$\left(\frac{\partial \eta_j}{\partial \xi^i} - \frac{\partial \eta_i}{\partial \xi^j}\right)\dot{\xi}^j = \left(\frac{\partial H}{\partial \xi^i} + \frac{\partial \eta_i}{\partial t}\right).$$

- Symplectic matrix

$$\omega_{ij} = \frac{\partial \eta_j}{\partial \xi^i} - \frac{\partial \eta_i}{\partial \xi^j}.$$


Classical Anomalies
Liouville's theorem
Liouville's Theorem
Define $2n+1$ form
$\Omega = \frac{1}{n!}\omega_H^n dt = \frac{1}{n!}\omega^n dt$
Liouville states that the Lie derivative
$\mathcal{L}_{\mathbf{v}}\Omega = 0.$
In co-ordinates, with $\sqrt{\omega} = \mathrm{Pf}(\omega_{ij})$, this is
$\frac{\partial\sqrt{\omega}}{\partial t} + \frac{\partial\sqrt{\omega}\dot{\xi}^i}{\partial\xi^i} = 0.$
Michael Stone (ICMT Illinois)

Liouville's Theorem

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Example

- Equation of motion

$$\begin{aligned} \dot{\mathbf{k}} &= \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B} \\ \dot{\mathbf{x}} &= \hat{\mathbf{k}} + \dot{\mathbf{k}} \times \mathbf{b} \end{aligned}$$

- The $\dot{\mathbf{k}} \times \mathbf{b}$ term is the anomalous velocity (Karplus-Luttinger '54) .
- Liouville measure

$$\sqrt{\omega} = 1 + \mathbf{b} \cdot \mathbf{B} \quad \text{(Q. Niu } \textit{et al.} \text{ '95-).}$$

- Berry monopole $\nabla \cdot \mathbf{b} = 2\pi\delta^3(\mathbf{k})$ gives source to Liouville's theorem

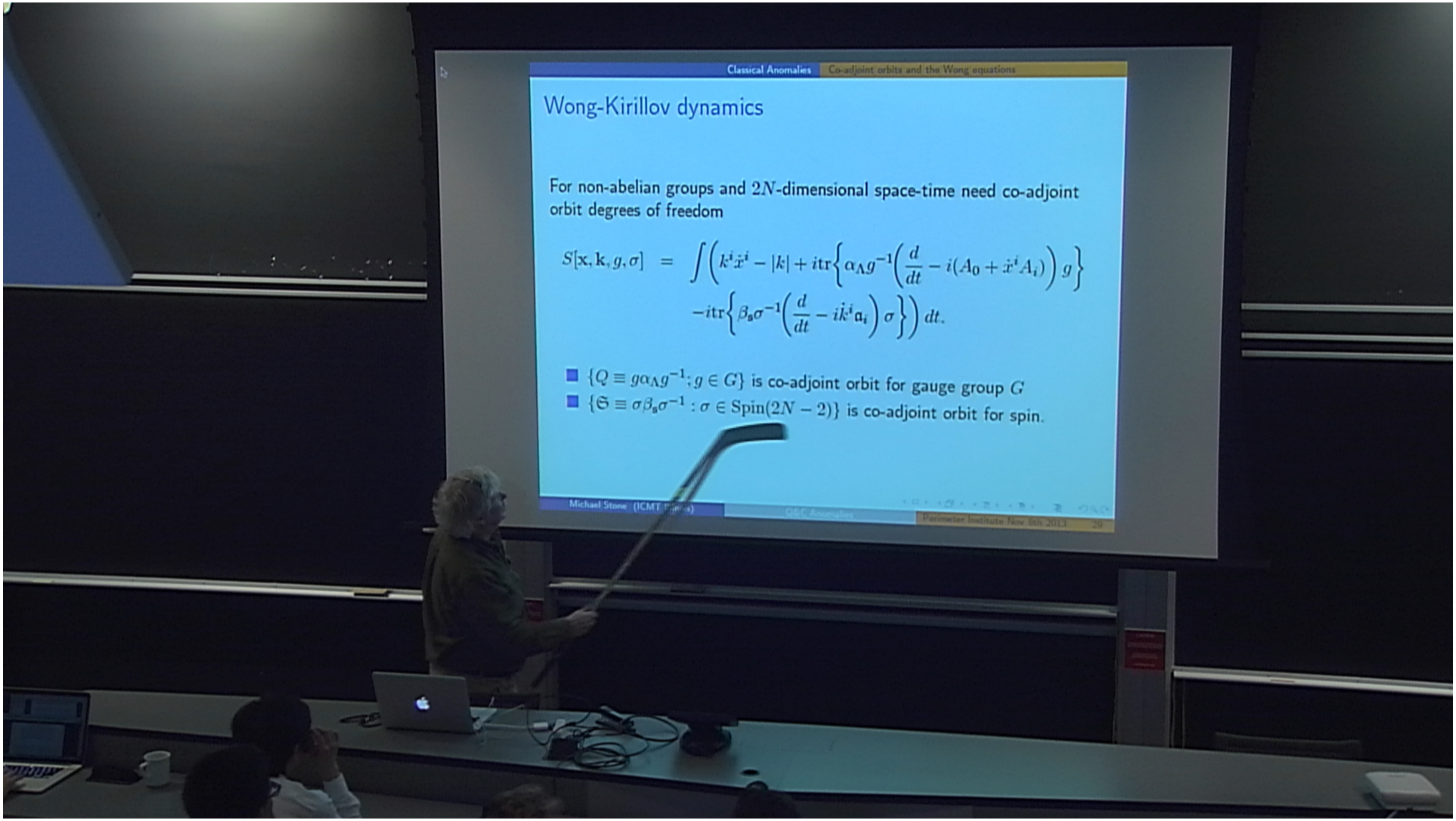
$$\boxed{\frac{\partial\sqrt{\omega}}{\partial t} + \frac{\partial\sqrt{\omega}\dot{k}^i}{\partial k^i} + \frac{\partial\sqrt{\omega}\dot{x}^i}{\partial x^i} = 2\pi\delta^3(\mathbf{k})(\mathbf{E} \cdot \mathbf{B})}$$

Wong-Kirillov dynamics

For non-abelian groups and $2N$-dimensional space-time need co-adjoint orbit degrees of freedom

$$S[\mathbf{x},\mathbf{k},g,\sigma] \;=\; \int\Bigg(k^i\dot{x}^i - |k| + i\mathrm{tr}\Big\{\alpha_\Lambda g^{-1}\Big(\frac{d}{dt} - i(A_0 + \dot{x}^i A_i)\Big)\,g\Big\} - i\mathrm{tr}\Big\{\beta_{\mathfrak{s}}\sigma^{-1}\Big(\frac{d}{dt} - i\dot{k}^i\mathfrak{a}_i\Big)\,\sigma\Big\}\Bigg)\,dt.$$

- $\{Q \equiv g\alpha_\Lambda g^{-1}; g \in G\}$ is co-adjoint orbit for gauge group G
- $\{\mathfrak{S} \equiv \sigma\beta_{\mathfrak{s}}\sigma^{-1} : \sigma \in \mathrm{Spin}(2N-2)\}$ is co-adjoint orbit for spin.


Classical Anomalies
Co-adjoint orbits and the Wong equations
Wong-Kirillov dynamics
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Quantum *versus* Classical Anomaly

In $d = 2N$ space-time

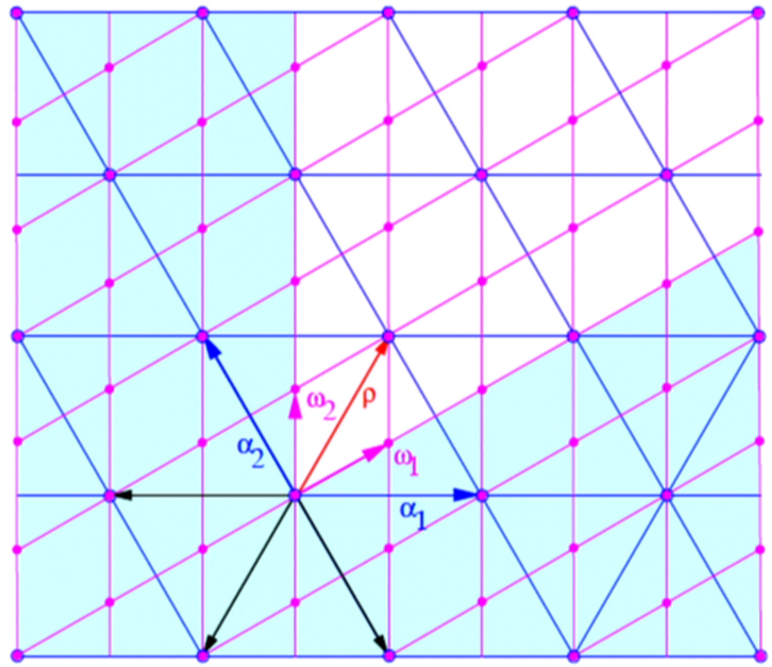
- Quantum

$$\mathrm{tr}\left\{\lambda_a \nabla_\mu J^\mu\right\} = \frac{1}{(4\pi)^N N!}\mathrm{str}_\Lambda\{\hat{\lambda}_a F_{\mu_1\mu_2}\cdots F_{\mu_{N-1}\mu_N}\}\epsilon^{\mu_1\cdots\mu_N}$$

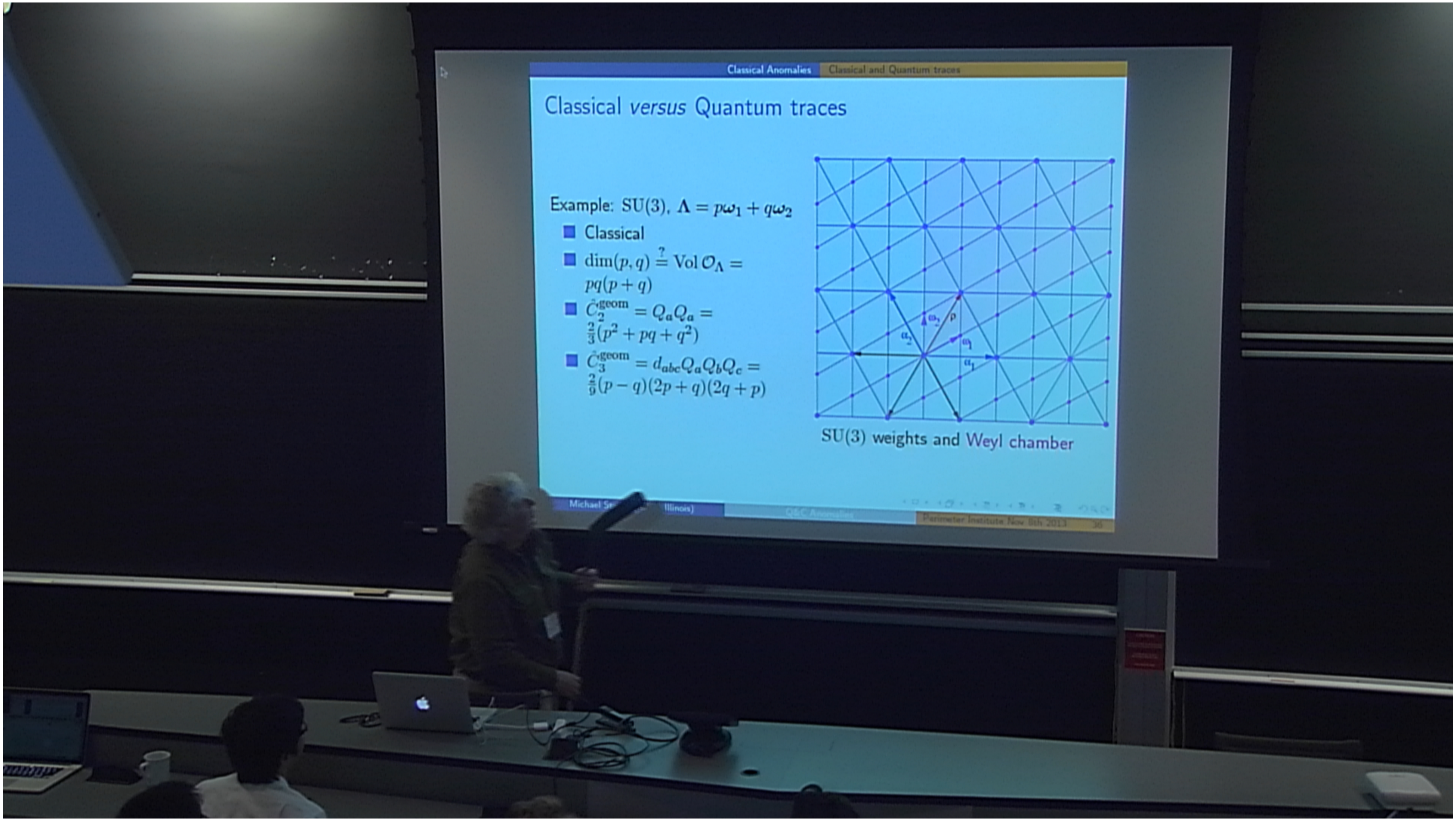
Classical *versus* Quantum traces

Example: $\mathrm{SU}(3)$, $\boldsymbol{\Lambda} = p\boldsymbol{\omega}_1 + q\boldsymbol{\omega}_2$

- Classical
- $\dim(p,q) \stackrel{?}{=} \mathrm{Vol}\,\mathcal{O}_\Lambda = pq(p+q)$
- $\hat{C}_2^{\mathrm{geom}} = Q_a Q_a = \frac{2}{3}(p^2 + pq + q^2)$
- $\hat{C}_3^{\mathrm{geom}} = d_{abc} Q_a Q_b Q_c = \frac{2}{9}(p-q)(2p+q)(2q+p)$



$\mathrm{SU}(3)$ weights and Weyl chamber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