Title: On Scale and Conformal Invariance in Four Dimensions

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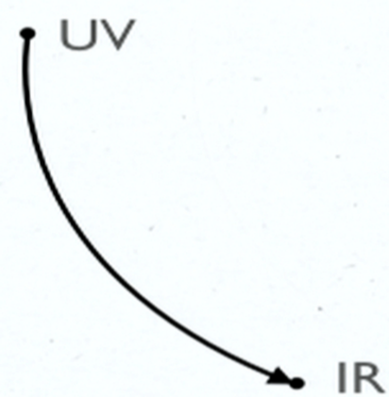
Abstract: I will be discussing
relation between scale and conformal symmetry in unitary Lorentz invariant QFTs in four dimensions.

The Grand Scheme of Things

General Properties of (unitary, Lorentz invariant) QFT

- General properties of renormalization group flow
  - Irreversibility of the RG flow, a/c-theorem
  - End points of the flow, scale vs conformal invariance
- Minkowskian vs. Euclidean language
  - Unitarity and Reflection Positivity
  - Properties of S-matrix



Zamolodchikov's c-theorem

One function to rule them all

- key object: 2pt function of the S.E. tensors $\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle$
- conservation $\partial_\mu T_{\mu\nu} = 0$ implies 2pt is governed by one function

$$\langle T_{\mu\nu}(p) T_{\rho\sigma}(-p)\rangle = (p_\mu p_\nu - p^2 g_{\mu\nu})(p_\rho p_\sigma - p^2 g_{\rho\sigma})\,\frac{f(s)}{p^4}$$

- in the conformal theory $f(s) \to c \cdot s$ where $s = p^2$
- positivity of $\Delta c = f'(\infty) - f'(0)$ follows from the sum rule

$$\Delta c = \int \frac{ds}{s^2}\,\Im f(s) \qquad \Delta c = \int d^2x\, x^2 \langle T(x) T(0)\rangle$$

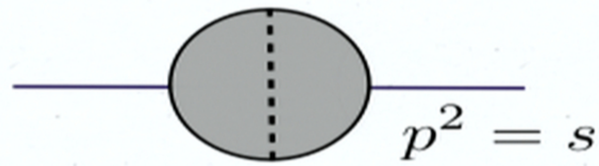
- existence of c-function $c(x)$ is a "miracle" (possibly attributed to coordinate space and only one function at play)

QFT in 2d: nice and simple

Many equally good ways to see the same

- many ways to prove c-theorem

$$\Delta c = \int \frac{ds}{s^2}\, \Im f(s)$$



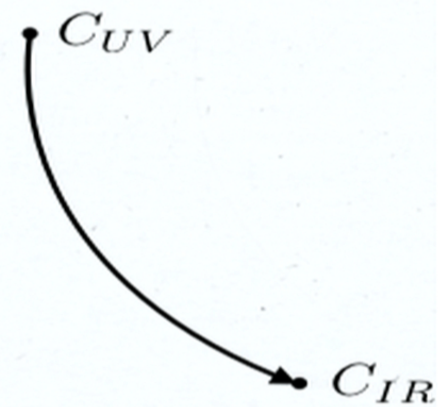
$$f(s) = \langle T(p) T(-p) \rangle$$

- Minkowski language = Euclidean language

$$\Im f \geq 0 \underset{\text{K.L.}}{\Longleftrightarrow} ||T\rangle|^2 \underset{\text{O.Sh.}}{\Longleftrightarrow} \langle T(x) T(0) \rangle \geq 0$$

- Scale invariance $\Rightarrow$ conformal invariance

$$\Im f(s) \sim s \Rightarrow \Delta c \to \infty \Rightarrow \langle TT \rangle \to \infty$$



Effective action language in 2d

One of *many* ways to formulate the proof in 2d

- generating functional of connected diagrams (effective action)

$$\mathcal{L} \to \mathcal{L} + \tau\, T \qquad\qquad W[\tau] = W[g_{\mu\nu} = \eta_{\mu\nu} e^{-2\tau}]$$

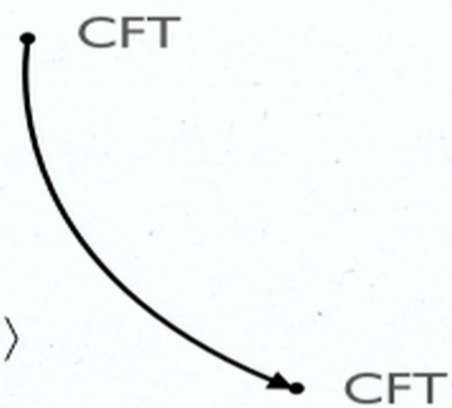
- locality of the effective action + form of the conformal anomaly

➡ simplest case: CFT $\to$ mass gap

$$W = \int d^2x\, \sqrt{g} \left(\underbrace{cR\,\Box^{-1} R}_{\text{Polyakov's action}} \quad + \quad \underbrace{aR + \mathcal{O}(R^2)}_{\text{General local action}} \right)$$

$$W[\tau] = \int d^2x\, \Delta c\, (\partial\tau)^2 + \mathcal{O}(\partial^3)$$

$$\langle T(x) T(0) \rangle = \frac{\delta^2 W}{\delta\tau(x)\delta\tau(0)} \qquad \Delta c = \int d^2x\, x^2 \langle T(x) T(0) \rangle$$



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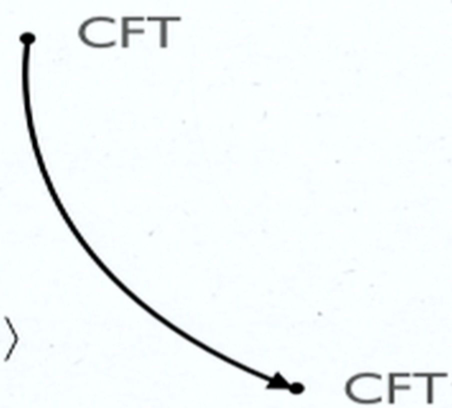


Polchinski theorem

Scale invariance $\Rightarrow$ conformal invariance in 2d

- scale invariance: $f(s) = k \cdot s$ or $f(s) = k \cdot s \log(s/\mu)$

  first option is in fact conformally invariant $\langle T(p)T(-p)\rangle \sim p^2$

  second option is NOT scale invariant because of

  $$\langle T_{\mu\nu}(p)T_{\rho\sigma}(-p)\rangle = (p_\mu p_\nu - p^2 g_{\mu\nu})(p_\rho p_\sigma - p^2 g_{\rho\sigma})\,\frac{f(s)}{p^4}$$

- role of unitarity is transparent in SFT $\Rightarrow$ CFT in 2d
- Polchinski mainly had to prove that scale invariance is what we intuitively think it is i.e. correlators are μ independent
- c-theorem and SFT $\Rightarrow$ CFT both based on the 2pt function $\langle TT\rangle$

C-theorem in (2n)d

Cardy conjecture

- conformal anomaly in 2d $T = \frac{c}{12} R$
- conformal anomaly in 4d $T = a\, E_4 + 3c\, W^2$
- conformal anomaly in (2n)d $T = c_{2n} E_{2n} + \underbrace{\ldots}_{\text{made of Weyl tensors}}$

- Cardy conjecture: $\langle T \rangle_{S^{2n}} \sim c_{2n}$ is decreasing from UV to IR

  in 4d $a_{UV} \geq a_{IR}$

➡ reasoning: $\partial \langle T \rangle_{S^4} / \partial r \sim - \int_{S^4} \langle T(0) T(x) \rangle d^4 x \leq 0$

a-theorem in 4d

What we knew (or believed) before Komargodski-Schwimmer

- since $T = a\, E_4 + \cdots \sim R^2$ the a-anomaly is not seen in the 2pt but requires 3 or 4pt function of the S.E. tensors
- unitarity of the 3pt function $\langle TTT \rangle$ does not imply positivity
- unitarity of $\langle T_{\mu\nu} T_{\mu\nu} T_{\mu\nu} T_{\mu\nu} \rangle$ (somehow) implies positivity;
  yet the 4pt function includes hundreds of functions and relate positivity to a' is a very difficult task
- no obvious advantage of Minkowski space over Euclidean or momentum over coordinate space

Unitarity in Euclidean space

An ode to Osterwalder and Schrader

- Unitarity in Minkowski = reflection positivity in Euclidean space

Minkowski: unitarity = positivity of norm $\left|\,|\phi \ldots \phi\rangle\right|^2 \geq 0$

Euclidean: unitarity = reflection positivity $\langle \phi \ldots \phi\, \phi \ldots \phi \rangle_{\text{full}} \geq 0$

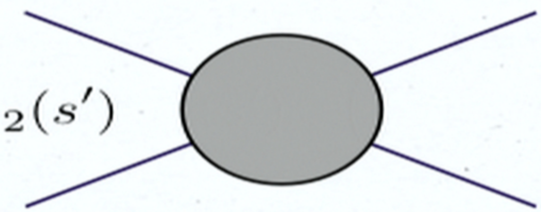
- Other manifestations of unitarity in Minkowski space?

positivity of ϵ

Dubovsky et al. 2008

$$\mathcal{L} = (\partial\phi)^2 + \frac{\epsilon}{\Lambda^4}\,(\partial\phi)^4 + \ldots$$

$$\mathcal{A}_{2\to 2}(s) = \epsilon\frac{s^2}{\Lambda^4} + \mathcal{O}(s^4) = \int \frac{ds'}{(s'-s)}\,\Im\,\mathcal{A}_{2\to 2}(s')$$



$p_1 = -p_3 \quad p_2 = -p_4$

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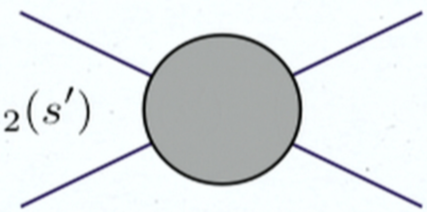

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Unitarity in Euclidean space II

We do not know how to formulate certain things in the Euclidean space

- Is reflection positivity **all** unitarity implies in Euclidean space?
  - yes
    there is a way to prove $\epsilon \geq 0$ using reflection positivity
  - no
    O.-Sch. is necessary but not sufficient
  - yes, but..
    low energies in Minkowski $\Leftrightarrow$ all energies in Euclidean description

In practical sense Minkowski space turns out to be much more powerful

a-theorem in 4d

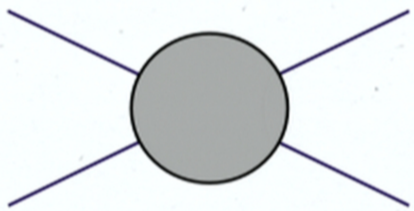
highlights of Komargodski-Schwimmer approach

- Effective action for dilaton

  $\mathcal{L} \to \mathcal{L} + \tau T$ $\qquad W[\tau] = W[g_{\mu\nu} = \eta_{\mu\nu} e^{-2\tau}]$

- Promoting dilaton to a dynamical particle

  scattering of $2 \to 2$



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- Covariance of $W[\tau]$
- On-shell condition for dilaton
  $\Box\varphi = \Box f(1 - e^{-\tau}) = 0$
- On-shell action (scattering) depends only on $\Delta a = a_{UV} - a_{IR}$

$$W[\tau] = \int (\partial\varphi)^2 + 2\Delta a(\partial\tau)^4 + \mathcal{O}(\partial^6)$$

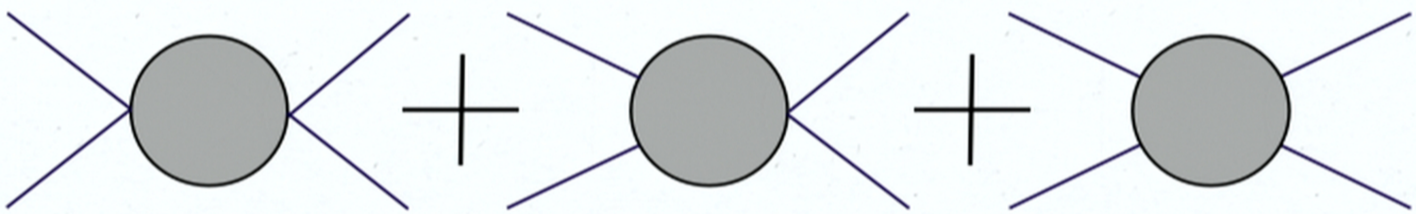
Scale vs Conformal in 4d

perturbative prove near CFT, Luty, Polchinski, Rattazzi

- Scale invariance implies $2 \to 2$ scattering is trivial

  otherwise the dispersion relation diverges $\mathcal{A}''_{2\to 2}(0) = \int_0^\infty \frac{ds}{s^3} \Im \mathcal{A}_{2\to 2}(s) \geq 0$

- In the vicinity of CFT ($T = 0$) 2pt is a leading contribution



$\langle TT \rangle$ $\qquad$ $\langle TTT \rangle$ $\qquad$ $\langle TTTT \rangle$

$\mathcal{A}_{2\to 2} \sim \langle TT \rangle$ $\qquad$ $T \simeq \beta \mathcal{O}$ $\qquad$ $\Im \mathcal{A}_{2\to 2} \sim \beta^2 \,| \, |\mathcal{O}\rangle|^2$

- connection with Hugh Osborn's work on a-theorem?

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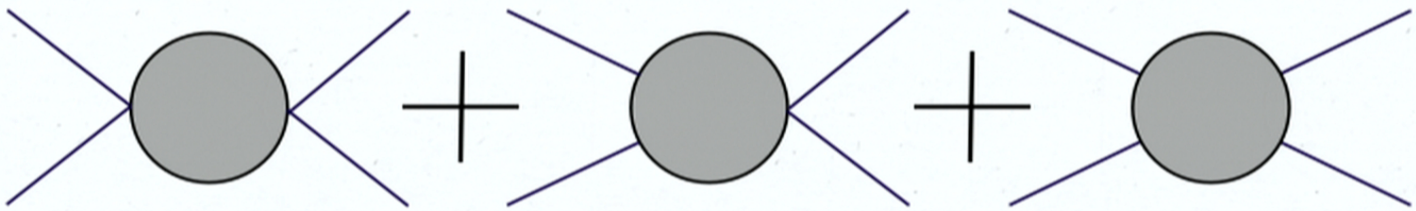

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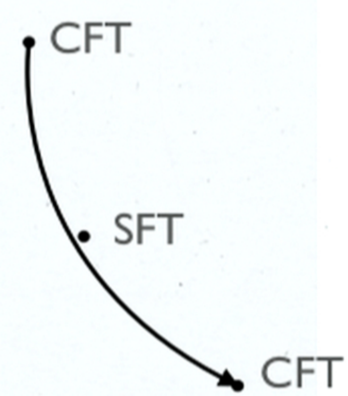
Scale vs Conformal in 4d

AD, Komargodski, Schwimmer, Theisen

- sum rule for $\mathcal{A}_{n\to n}$
- positivity of $\Im\,\mathcal{A}_{n\to n}(s)$

$$\Delta a = \int_0^\infty \frac{ds}{s^3}\,\Im\,\mathcal{A}_{n\to n}(s)$$

- scale invariance $\mathcal{A}_{n\to n}(s) \sim s^2$
  - in SFT all $n \to n$ dilaton scattering amplitudes are trivial!
  - all cuts are trivial $\langle \text{anything}|\varphi \ldots \varphi\rangle = 0$
  - all scattering of $\varphi-$particles is trivial



S-matrix of dilaton is trivial!

Scale vs Conformal in 4d

AD, Komargodski, Schwimmer, Theisen

- all amplitudes φ of vanish on-shell $\Rightarrow$ coupling $\int \varphi T$ vanishes on-shell: $\int \varphi \Box L$
- S-matrix of φ is trivial $\Rightarrow$ after a change of variables φ is a trivial field $\int \frac{1}{2}(\partial\varphi)^2 + \varphi T + \cdots = \int \frac{1}{2}(\partial(\varphi + L))^2 - L\Box L + \ldots$

◉ if $T = \Box L$ there is an improvement such that S.E. tensor is traceless and the theory is CFT

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◉ if $T = \Box L$ there is an improvement such that S.E. tensor is traceless and the theory is CFT

Counterexample

SFT + unitarity is NOT CFT

- Free gauge 2-form in 4d is dual to a free massless scalar with a shift symmetry $\phi \equiv \phi + \mathrm{const}$
  - all dilaton scattering amplitudes are trivial (our main assertion holds)
  - there is $L = \phi^2$ such that $T = \Box L$ but this operator is not present in the original theory

There is no local traceless S.E. tensor in this theory!

This theory is not a CFT (no primary for $\partial\phi$)

Outline

What have we learned?

- an argument: SFT+unitarity in 4d imply CFT

our "proof" is based on the properties of S-matrix which are intuitive but not completely understood on formal level (interplay of two languages)

- a-theorem is related to "SFT $\Rightarrow$ CFT"

in 2d and 4d both c-theorem and "SFT$\Rightarrow$ CFT" rely on the same technique (but this technique is dimension-specific)

- we still might be missing the underline picture!

how to prove a-theorem in 6d? scattering of 6 τ or other known tricks do not help with $\langle TTTTTT \rangle$

- new ways to use unitarity?

scattering amplitudes, energy correlators, etc.