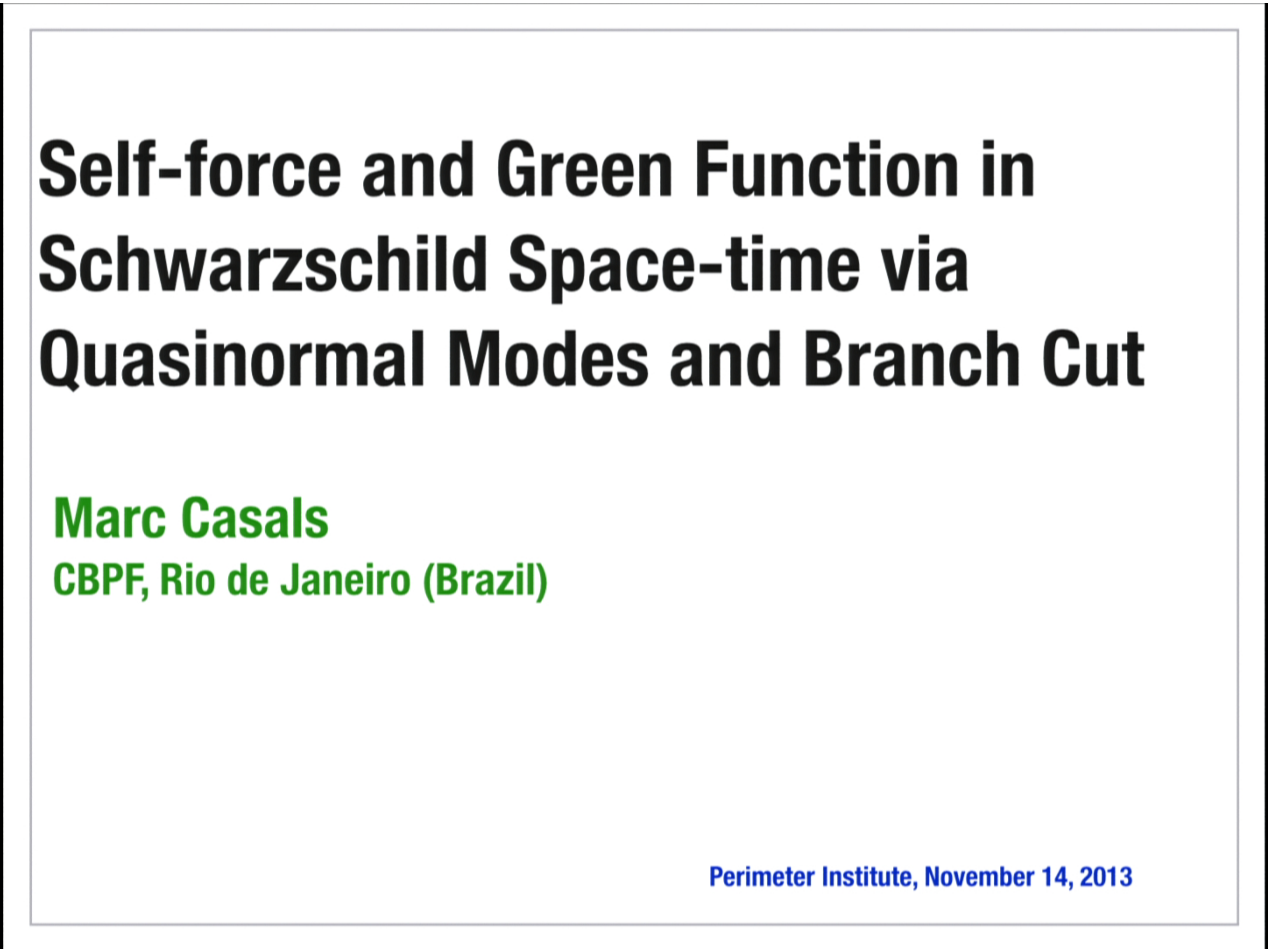
Title: Self-force and Green function in Schwarzschild spacetime via quasinormal modes and branch cut

Date: Nov 14, 2013 01:00 PM

URL: http://pirsa.org/13110067

Abstract: The modelling of gravitational wave sources is of timely interest given the exciting prospect of a first detection of gravitational waves by the new generation of detectors. The motion of a small compact object around a massive black hole deviates from a geodesic due to the action of its own field, giving rise to a self-force and the emission of gravitational waves. The self-force program has recently achieved important results using well-established methods. In this talk, we will present a different, novel method, where the self-force is calculated via the Green function of the wave equation that the field perturbation satisfies. We will present a calculation of the global Green function on Schwarzschild black hole spacetime. The calculation is carried out via a spectroscopy analysis of the Green function, which includes quasinormal modes and a branch cut in the complex-frequency plane. We will apply this analysis to calculate the self-force on a scalar charge and to reveal geometrical properties of wave propagation on a Schwarzschild background.



Self-force and Green Function in Schwarzschild Space-time via Quasinormal Modes and Branch Cut

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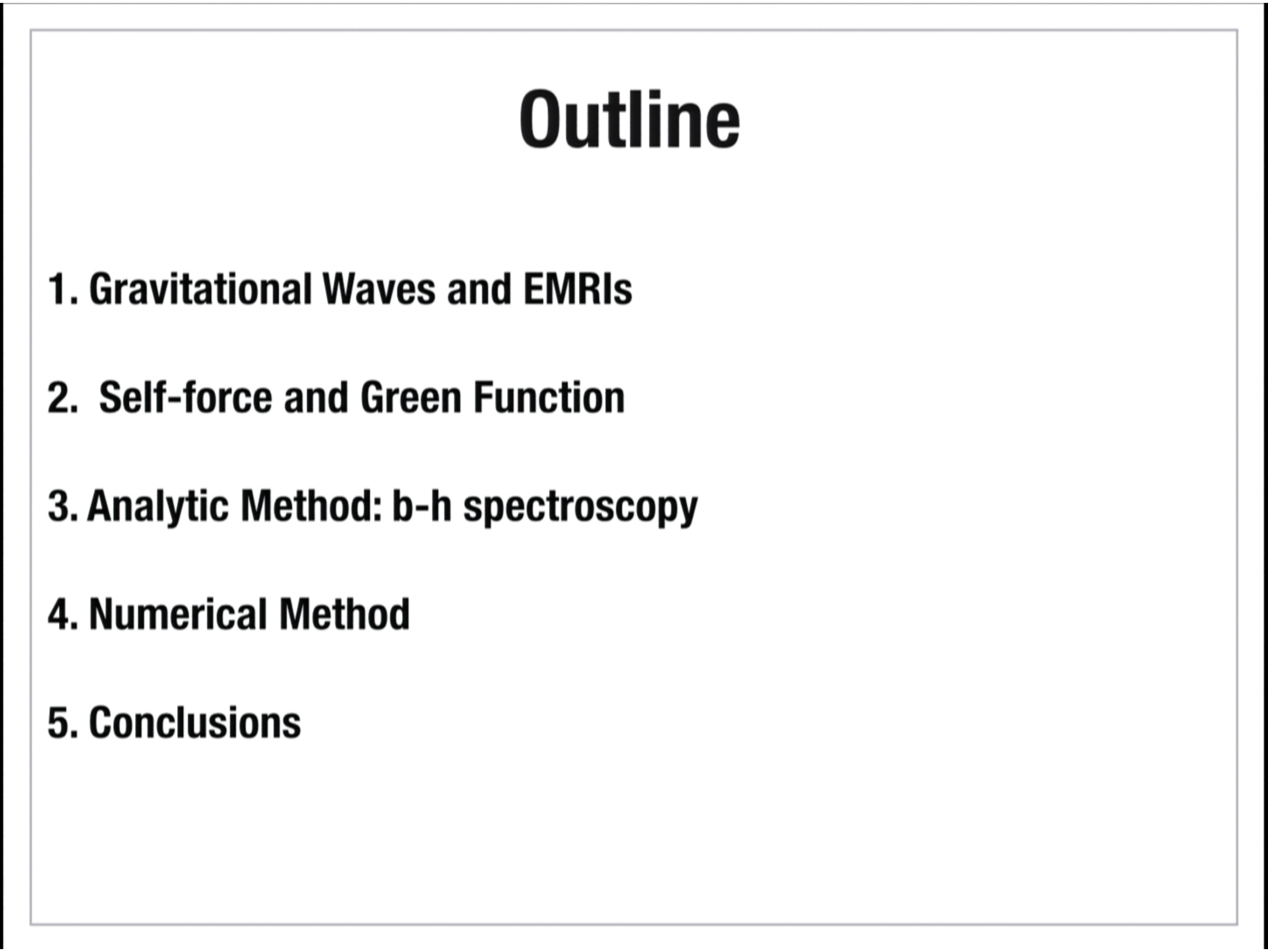
CBPF, Rio de Janeiro (Brazil)

Perimeter Institute, November 14, 2013

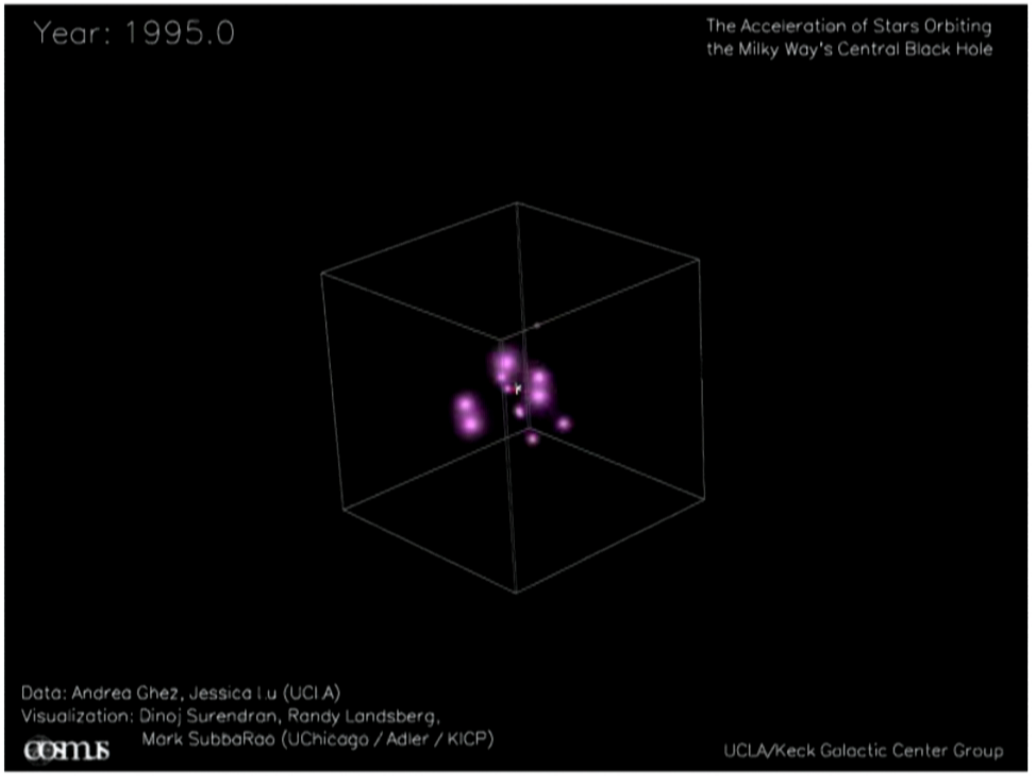


Outline

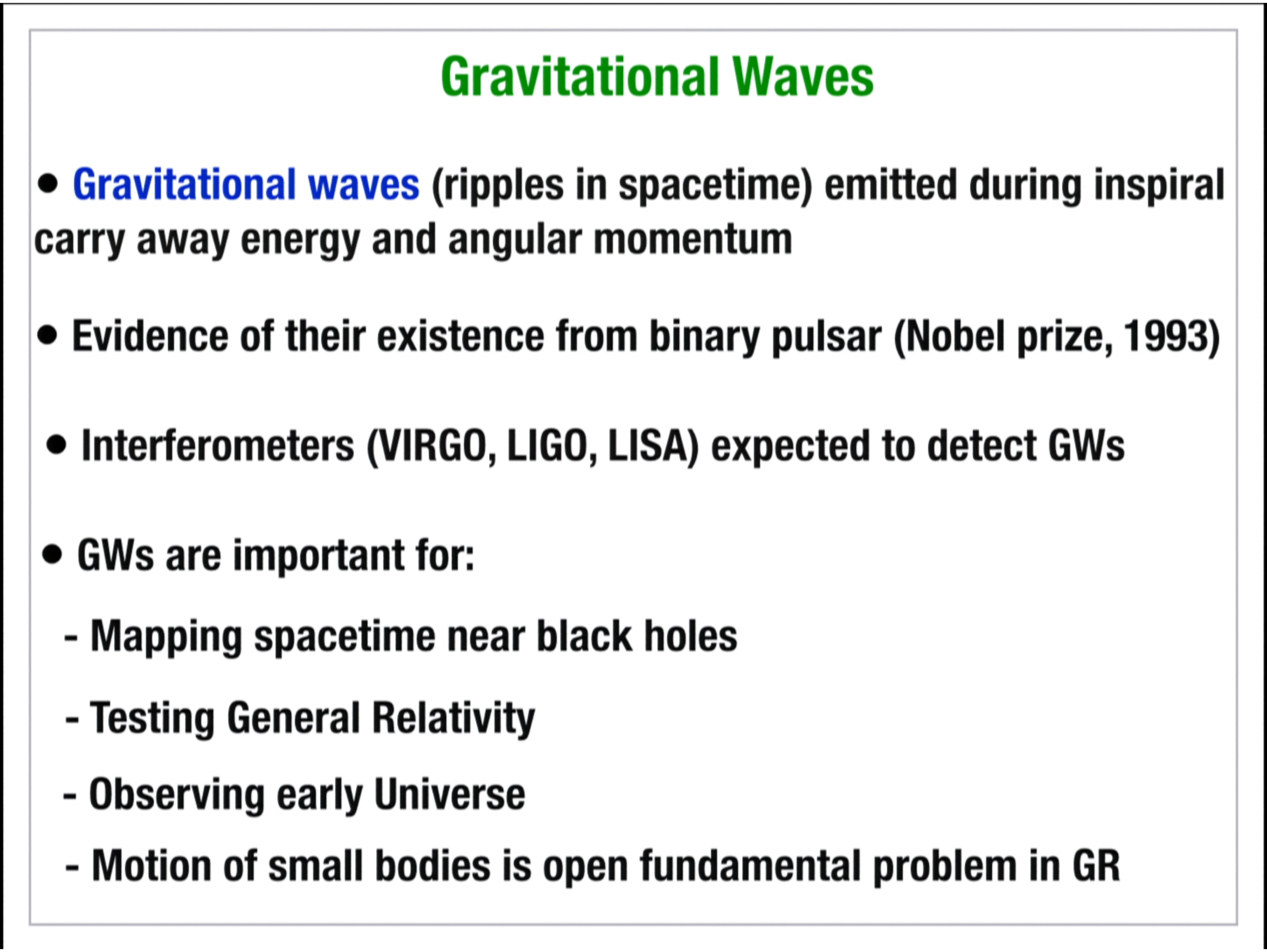
1. Gravitational Waves and EMRIs
2. Self-force and Green Function
3. Analytic Method: b-h spectroscopy
4. Numerical Method
5. Conclusions

Supermassive Black Holes

Supermassive (~4 million Solar masses) black hole at the centre of the Milky Way



Credit: UCLA



Gravitational Waves

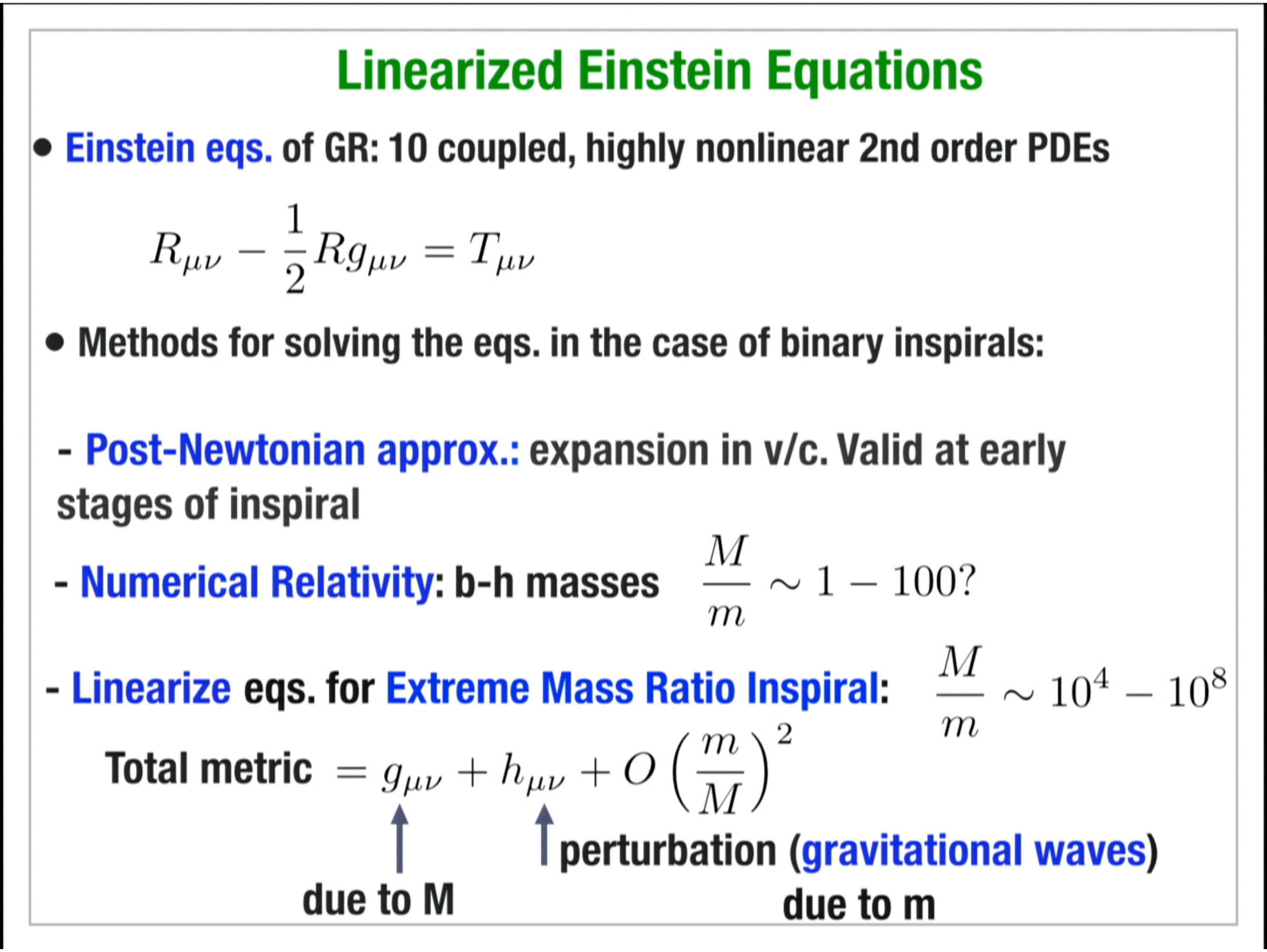
- **Gravitational waves** (ripples in spacetime) emitted during inspiral carry away energy and angular momentum

- Evidence of their existence from binary pulsar (Nobel prize, 1993)

- Interferometers (VIRGO, LIGO, LISA) expected to detect GWs

- GWs are important for:

  - Mapping spacetime near black holes
  - Testing General Relativity
  - Observing early Universe
  - Motion of small bodies is open fundamental problem in GR



Linearized Einstein Equations

- **Einstein eqs.** of GR: 10 coupled, highly nonlinear 2nd order PDEs

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$$

- Methods for solving the eqs. in the case of binary inspirals:

- **Post-Newtonian approx.:** expansion in v/c. Valid at early stages of inspiral

- **Numerical Relativity**: b-h masses $\frac{M}{m} \sim 1 - 100?$

- **Linearize** eqs. for **Extreme Mass Ratio Inspiral**: $\frac{M}{m} \sim 10^4 - 10^8$

Total metric $= g_{\mu\nu} + h_{\mu\nu} + O\left(\frac{m}{M}\right)^2$

$\uparrow$ due to M

$\uparrow$ perturbation (**gravitational waves**) due to m

Self-Force

- Inspiral of small mass ($\sim 10 M_\odot$) around super-massive Black Hole ($\sim 10^5 - 10^9 M_\odot$) deviates from geodesic due to the action of its own 'regularized' field: Self-Force

- Self-field is singular at the location of particle -> regularization obeying covariance and causality $h_{\alpha\beta} \rightarrow h^R_{\alpha\beta}$

- Alternative viewpoint: motion is geodesic in spacetime with metric of super-massive black-hole plus 'regularized' metric of small mass $g_{\alpha\beta} + h^R_{\alpha\beta}$

Self-Force

- This S-F is the gravitational equivalent of the **Abraham-Lorenz-Dirac** (1938) force on an accelerated electric charge in flat space-time:

perpendicular projector to velocity

$$ma^{\mu} = f^{\mu}_{ext} + \underbrace{\frac{2e^2}{3m} P^{\mu}{}_{\nu} \frac{df^{\nu}_{ext}}{d\tau}}_{\textbf{S-F}}$$

- Standard methods for calculating S-F are via the field:

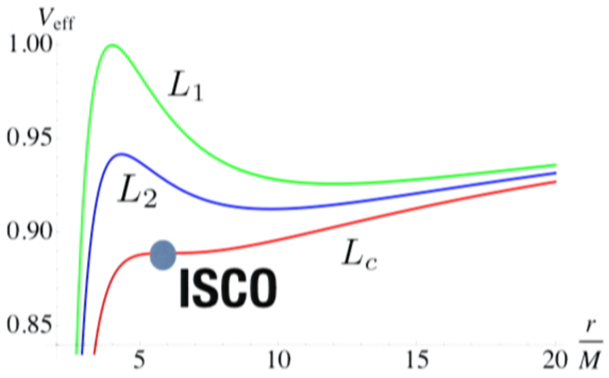
  (1) Mode-sum regularization (Barack et al.)

  (2) Effective source (suggested by Detweiler)

- Here we will present a method calculating S-F via Green function

Conservative S-F results in Schwarzschild with standard methods

- **Correction to frequency of the innermost stable circular orbit (an observable of GW astronomy) (Barack&Sago'09)**



$$\frac{\Delta\Omega_{ISCO}}{\Omega_{ISCO}} = 0.4870\frac{m}{M} \qquad \Omega \equiv \frac{d\varphi}{dt}$$

Conservative S-F results in Schwarzschild with standard methods

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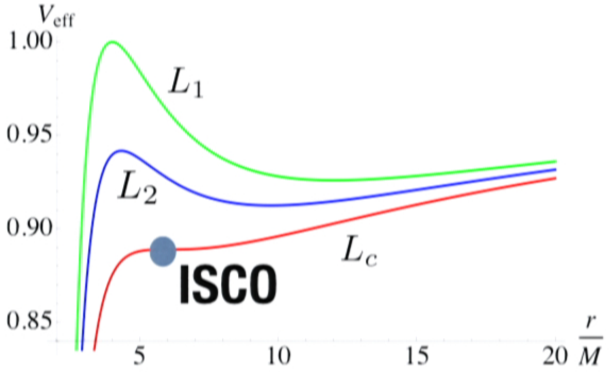
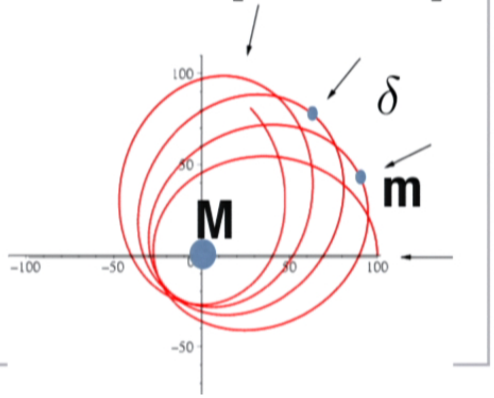


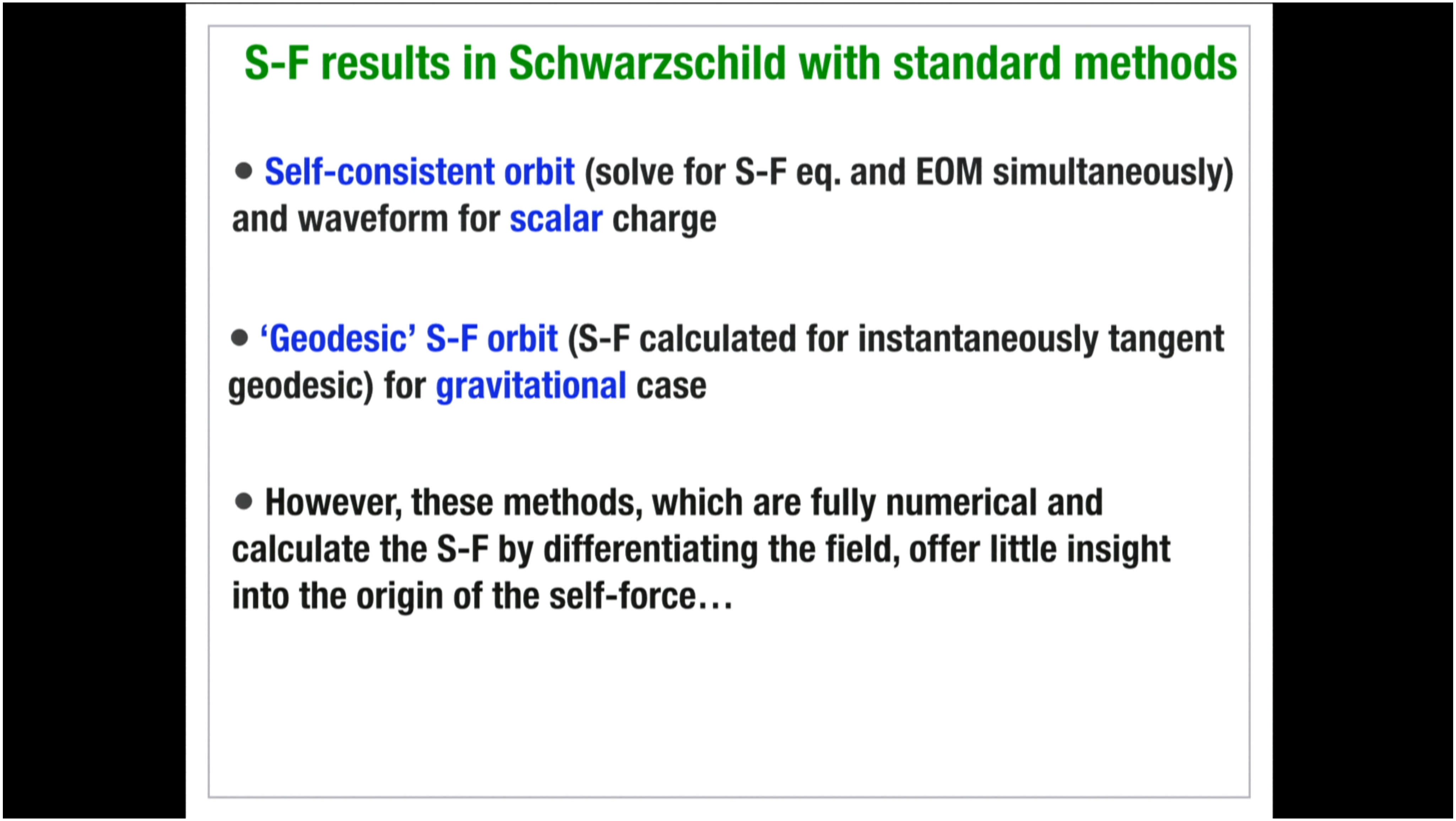
- **Correction to precession effect (rate of periastron advance for small eccentricity) (Barack,Damour&Sago'10)**

$$\delta = 2\pi\left[\rho^{-1/2} - 1\right]$$

$$\rho \equiv \frac{\omega_r^2}{\Omega^2} = \underset{\substack{\uparrow\\ \textbf{Kepler}}}{1} - \underset{\substack{\uparrow\\ \textbf{Einstein}}}{6(M_T\Omega)^{2/3}} + \underset{\substack{\uparrow\\ \textbf{S-F}}}{\frac{m}{M}\rho_{SF}} + O\left(\frac{m}{M}\right)^2$$



ω_r **: radial freq.** $\hat{\Omega}$ **: "mean" angular freq.**



S-F results in Schwarzschild with standard methods

- **Self-consistent orbit** (solve for S-F eq. and EOM simultaneously) and waveform for **scalar** charge

- **'Geodesic' S-F orbit** (S-F calculated for instantaneously tangent geodesic) for **gravitational** case

- However, these methods, which are fully numerical and calculate the S-F by differentiating the field, offer little insight into the origin of the self-force…

Self-force via Green Function

- **S-F for scalar charge (Quinn'00)**

$$F_\mu(\tau) = q^2 \int_{-\infty}^{\tau^-} d\tau' \; \nabla_\mu G_{ret}(z(\tau), z(\tau')) + \text{local}$$

- **Retarded Green function defined by**

$$\Box G_{ret}(x, x') = \delta_4(x, x')$$ **with causality b.c.**

- **Similar for emag (spin=1, DeWitt&Brehme'60) and gravitational (spin=2) fields (MiSaTaQuWa'97)**

- **Global structure of G_{ret} is crucial!**

Method of Matched Expansions

- **Non-local part of S-F:** $\int_{-\infty}^{\tau^-} d\tau' \, \nabla_\mu G_{ret}$

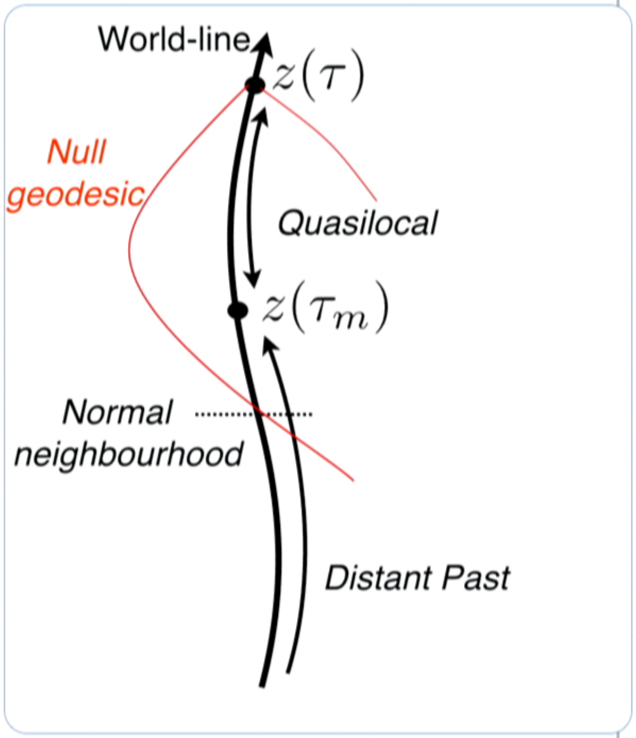
- **Matched expansions:** **choose** τ_m **:**

- before that point ('Quasilocal' region)

$$\int_{\tau_m}^{\tau^-} d\tau' \, \nabla_\mu G_{ret}$$

- after that point ('Distant Past')

$$\int_{-\infty}^{\tau_m} d\tau' \, \nabla_\mu G_{ret}$$



Method of Matched Expansions

- A priori no such τ_m need exist

- Anderson&Wiseman'05: weak-field approx. in DP in Schwarzschild. "Poor" convergence.

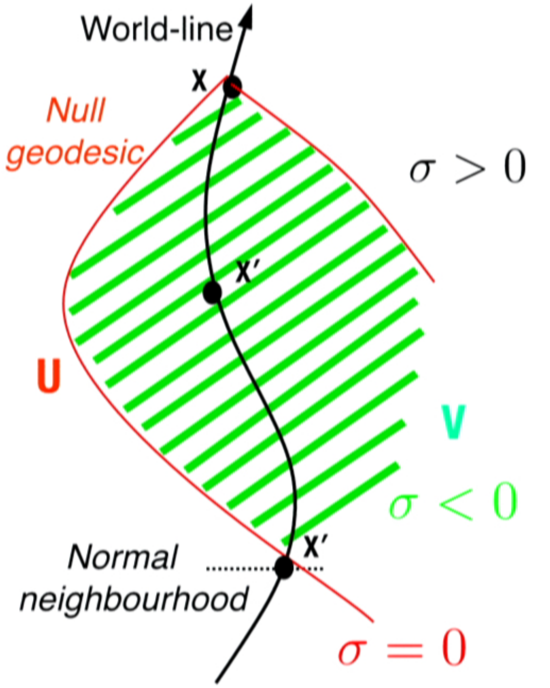
- Casals,Dolan,Ottewill,Wardell'09: successful application of method of matched expansions in Nariai space-time $dS_2 \times \mathbb{S}^2$

Quasilocal - Hadamard form

$$G_{ret}\left(x,x'\right)=\underbrace{\theta\left(\Delta t\right)}_{\neq 0 \text{ in the past}}\left\{\underbrace{U\left(x,x'\right)\delta\left(\sigma\right)}_{\neq 0 \text{ on light cone}}+\underbrace{V\left(x,x'\right)\theta\left(-\sigma\right)}_{\neq 0 \text{ inside light cone}}\right\}$$

- σ : **geodesic distance** between x & x’
- U & V regular
- Only valid in **normal neighbourhood**
- It renders **regularization** trivial

$$\int_{\tau_m}^{\tau^-} d\tau' \; \nabla_\mu G_{ret} = \int_{\tau_m}^{\tau} d\tau' \; \nabla_\mu V$$



Quasilocal - Hadamard form

- **Calculate V with, e.g., coordinate expansion using WKB**

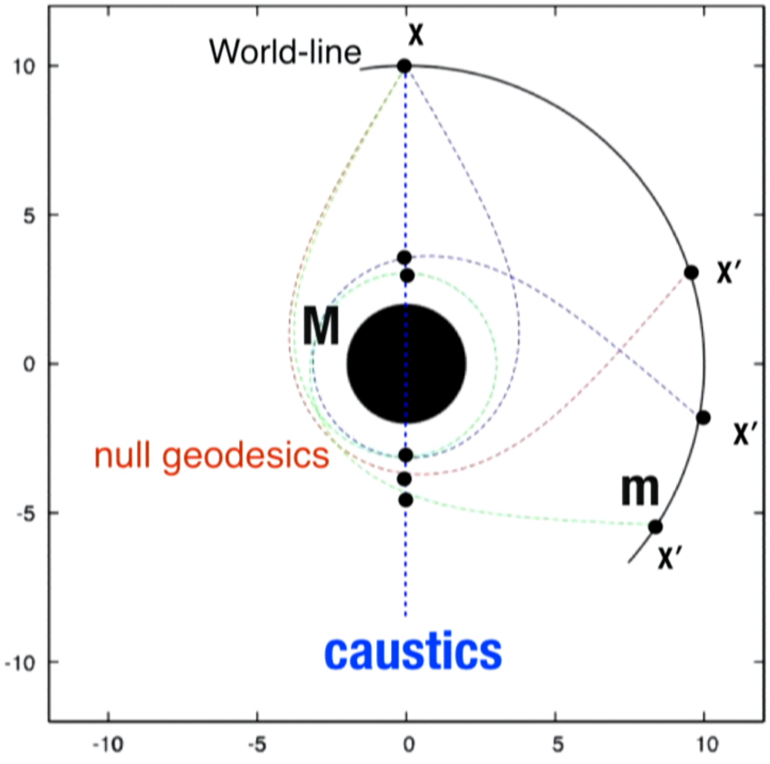
$$V(x,x') = \sum_{i,j,k=0}^{\infty} v_{ijk}(r)\ (t-t')^{2i}(1-\cos\gamma)^j(r-r')^k$$

Distant past - Singularities of Green function

- "Propagation of singularities theorems": outside normal nbd, $G_{ret}(x,x')$ is singular along null geodesics (ie, $\sigma = 0$)

- Form of singularity outside the normal nbd?

- **Caustics**: focus points/where light cone intersects itself

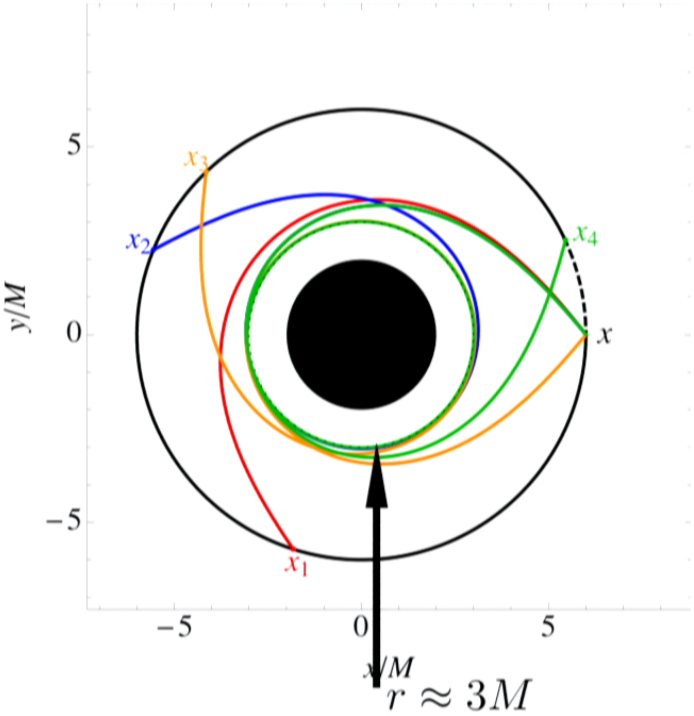


Timelike circular geodesic in Schwarzschild (r=10M)

Method of Matched Expansions

- Here we apply it to Schwarzschild. Scalar charge in a circular geodesics at r=6M (also did eccentric geod.)

Casals,Dolan,Ottewill&Wardell'13



Distant Past: Black Hole Spectroscopy

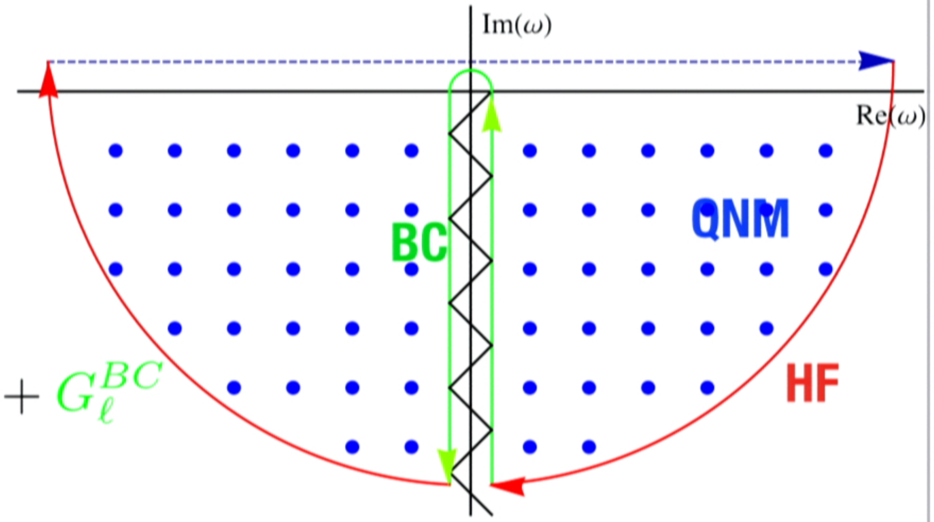
- **Multipolar decomposition:**

$$G_{ret}(x,x') = \frac{1}{rr'}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)G_\ell^{ret}(r,r';t)$$

- **Fourier transform:**

$$G_\ell^{ret}(r,r';t) \equiv \int_{-\infty+ic}^{\infty+ic} d\omega\ G_\ell(r,r';\omega)e^{-i\omega t}$$

Complex-Frequency Plane



- **Residue theorem:**

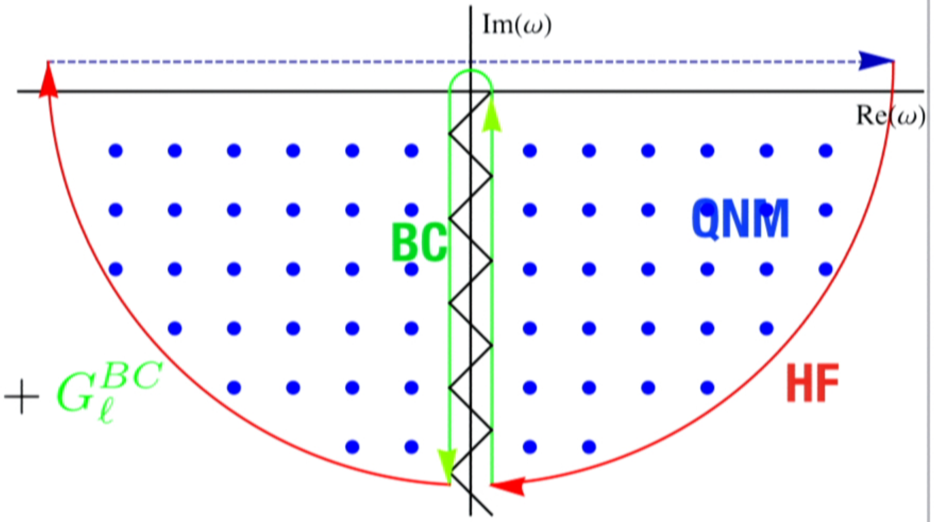
$$G_\ell^{ret} = G_\ell^{HF} + G_\ell^{QNM} + G_\ell^{BC}$$

G_ℓ^{HF} **Integral along high-frequency arc. Zero in Distant Past.**

G_ℓ^{QNM} **Sum over residues of poles (quasinormal modes)**

G_ℓ^{BC} **Integral around branch cut**

Complex-Frequency Plane



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$$G_\ell^{ret} = \xcancel{G_\ell^{HF}} + G_\ell^{QNM} + G_\ell^{BC}$$

G_ℓ^{HF}	**Integral along high-frequency arc. Zero in Distant Past.**
G_ℓ^{QNM}	**Sum over residues of poles (quasinormal modes)**
G_ℓ^{BC}	**Integral around branch cut**

Radial Equation

- **Green function modes:** $G_\ell(r,r';\omega) = \dfrac{R_\ell^{in}\left(r_<,\omega\right) R_\ell^{up}\left(r_>,\omega\right)}{W(\omega)}$

- $R_\ell^{in/up}$ **are slns. of radial ODE ('Regge-Wheeler eq.') for the perturbation:**

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right] R_\ell(r,\omega) = 0 \qquad V(r) = \left(1 - \frac{1}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{(1-s^2)}{r^3}\right]$$

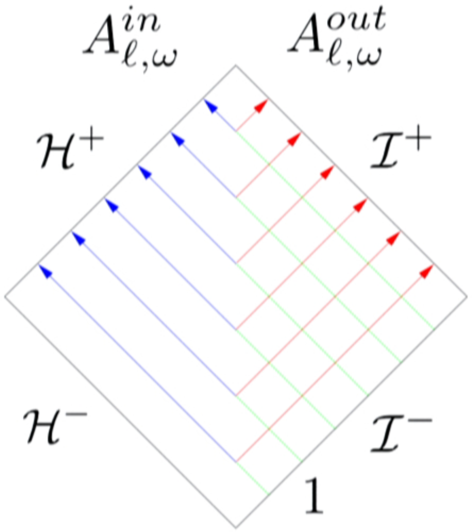
$$r_* = r_*(r) \in (-\infty, \infty) \qquad s = 0, 1, 2$$

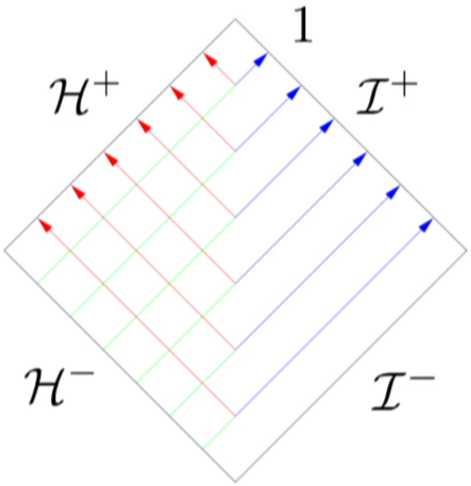
$$2M = 1$$

Radial solutions

- **Two lin. indep. slns.:**

$$R_\ell^{in} \underset{r_* \to -\infty}{\sim} e^{-i\omega r_*} \qquad R_\ell^{up} \underset{r_* \to \infty}{\sim} e^{+i\omega r_*}$$





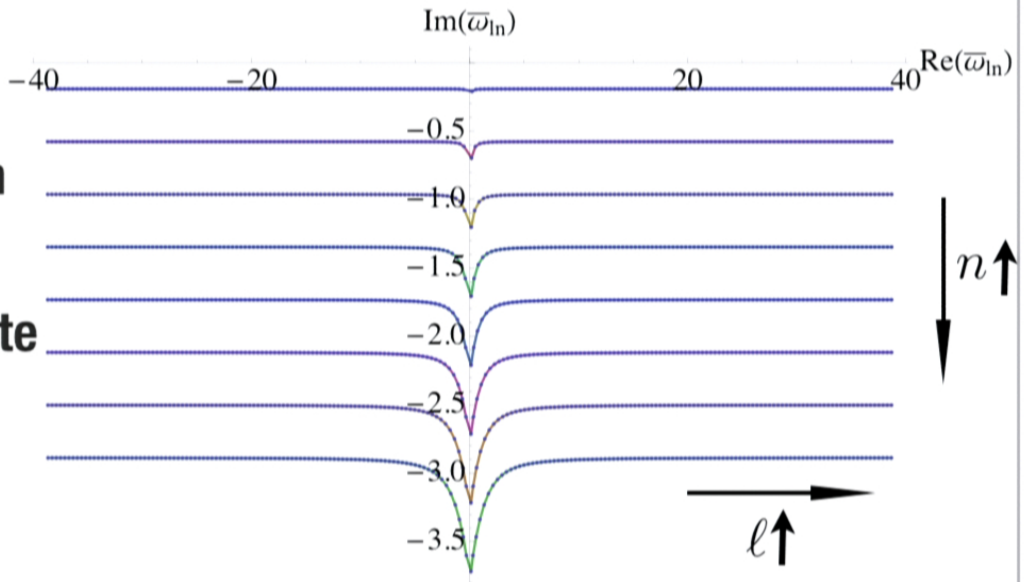
Quasinormal Modes

- **QNM frequencies**: simple poles of $G_\ell = \dfrac{R_\ell^{in}(r_<, \omega)\, R_\ell^{up}(r_>, \omega)}{W(\omega)}$ in the complex-ω plane: $W(\omega_{ln}) = 0$

- **Boundary conditions:** $\underset{r_* \to -\infty}{e^{-i\omega_{\ell n} r_*} \sim R_\ell^{in}} \propto \underset{r_* \to \infty}{R_\ell^{up} \sim e^{+i\omega_{\ell n} r_*}}$

$\mathrm{Re}(\omega_{\ell n})$: freq. of oscillation

$\mathrm{Im}(\omega_{\ell n})$: decay rate



Quasinormal Modes

- **QNM sum:**

$$G_\ell^{QNM}(r,r';\Delta t) = \sum_{n=0}^{\infty} \mathrm{Re}\left(\frac{R_\ell^{in}(r,\omega) R_\ell^{in}(r',\omega)}{\omega A_{\ell,\omega}^{out} \frac{\partial A_{\ell,\omega}^{in}}{\partial \omega}} e^{-i\omega\Delta t} \right)\Bigg|_{\omega=\omega_{\ell,n}}$$

- **n-sum convergent for** $\Delta t \gtrapprox |r_*| + |r'_*|$

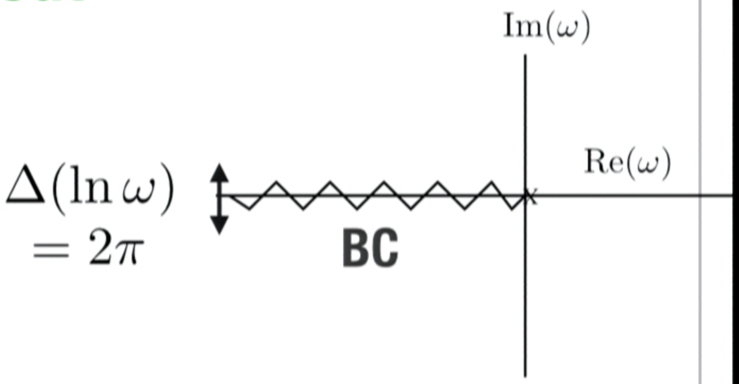
- ℓ **-sum leads to divergences at light-crossing times**

Branch Cut

- **Ahem...what is a BC??**

Ex: $\ln\omega = \ln|\omega| + i\arg(\omega)$

$\arg(\omega) \in (-\pi, \pi]$



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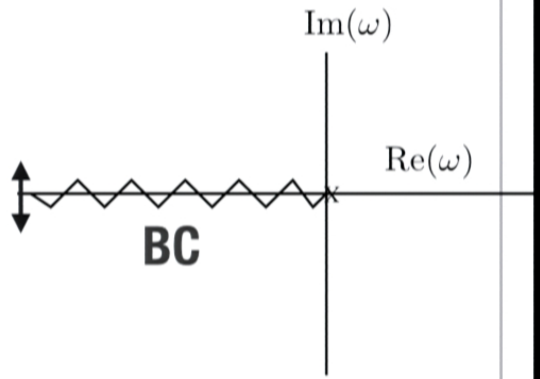
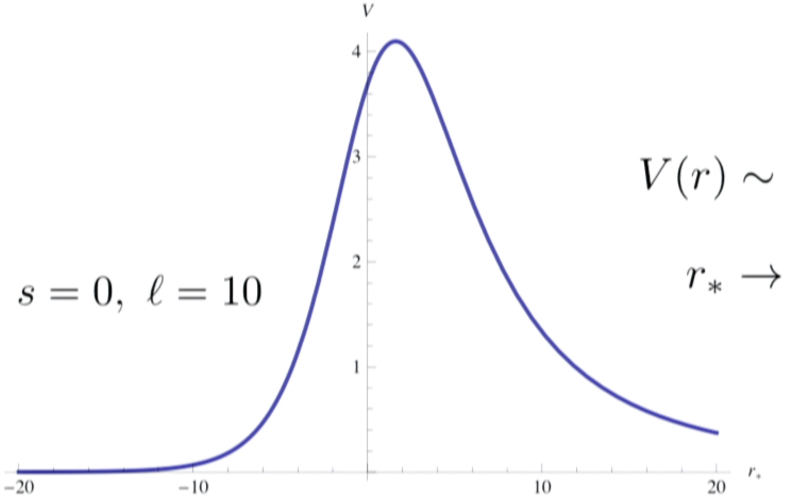


$\Delta(\ln\omega)$
$= 2\pi$

- **BC is due to non-exponential decay of potential at radial infinity:**



$s = 0,\ \ell = 10$

$$V(r) \sim \frac{\ell(\ell+1)}{r_*^2} + \frac{2\ell(\ell+1)r_h \ln(r_*/r_h)}{r_*^3}$$

$$r_* \to \infty$$

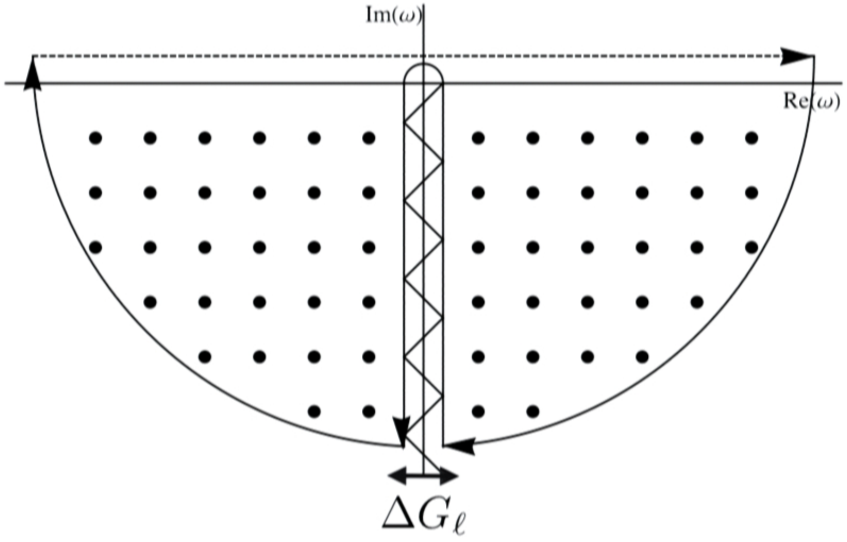
Branch Cut

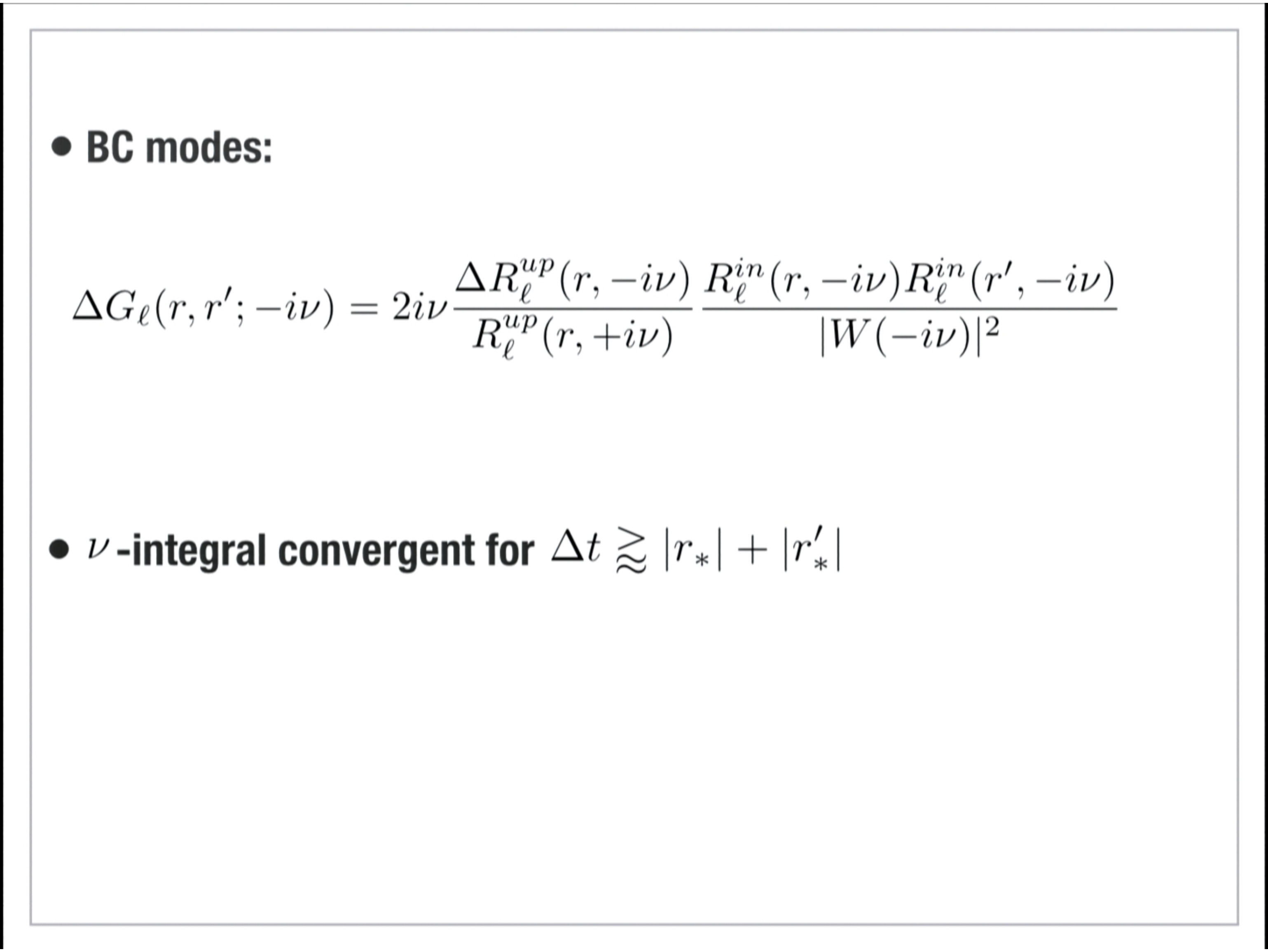
- **BC integral** $G_\ell^{BC}(r,r';t) = \int_0^\infty d\nu\ \Delta G_\ell(r,r';-i\nu)e^{-\nu t}$

$\omega = -i\nu$

$$\downarrow$$

$$\Delta R_\ell^{up}(r,-i\nu) \equiv \lim_{\epsilon\to 0}\left[R_\ell^{up}(r,\epsilon - i\nu) - R_\ell^{up}(r,-\epsilon - i\nu)\right]$$





- **BC modes:**

$$\Delta G_\ell(r, r'; -i\nu) = 2i\nu \frac{\Delta R_\ell^{up}(r, -i\nu)}{R_\ell^{up}(r, +i\nu)} \frac{R_\ell^{in}(r, -i\nu) R_\ell^{in}(r', -i\nu)}{|W(-i\nu)|^2}$$

- ν **-integral convergent for** $\Delta t \gtrapprox |r_*| + |r'_*|$

Methods for QNMs and BC

- **Large-**$|\omega|$ asymptotics by analytic continuation to **complex-r** plane

- **Small-**$|\omega|$ asymptotics by method of MST (match series of hypergeometric functions and series of Coulomb functions)

- **Mid-**$|\omega|$ by using series of **confluent hypergeometric functions**

$$R_\ell^{up} \propto \sum_{n=0}^{\infty} a_n (1-2\nu)_n \; U(s+1-2\nu+n, 2s+1, -2\nu r)$$

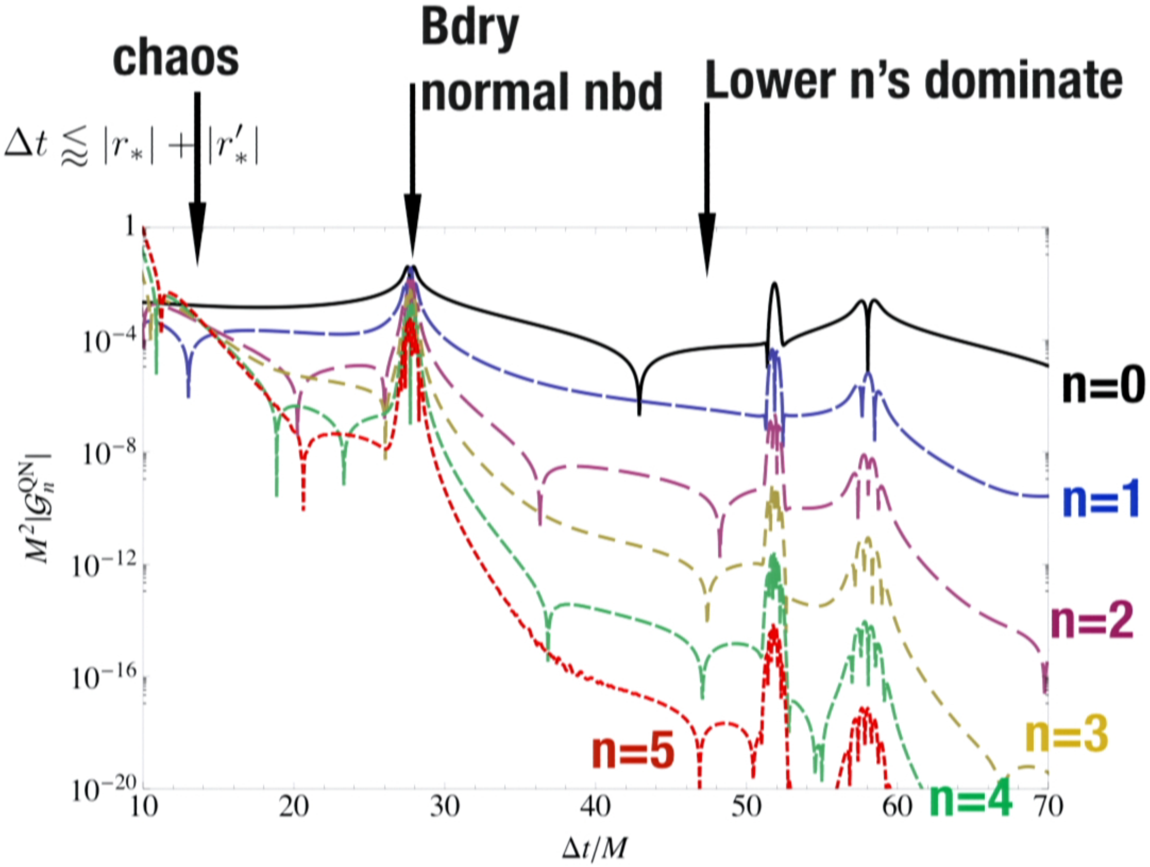
New series on BC:

$$\Delta R_\ell^{up} \propto \sum_{n=0}^{\infty} a_n \frac{(-1)^n \Gamma(1+n-2\nu) U(s-n+2\nu, 2s+1, 2\nu r)}{\Gamma(1+s+n-2\nu)\Gamma(1-s+n-2\nu)}$$

this can be evaluated ***on* the NIA**!

Results: QNMs

- **QNMs for different n's**



chaos

$\Delta t \lessapprox |r_*| + |r'_*|$

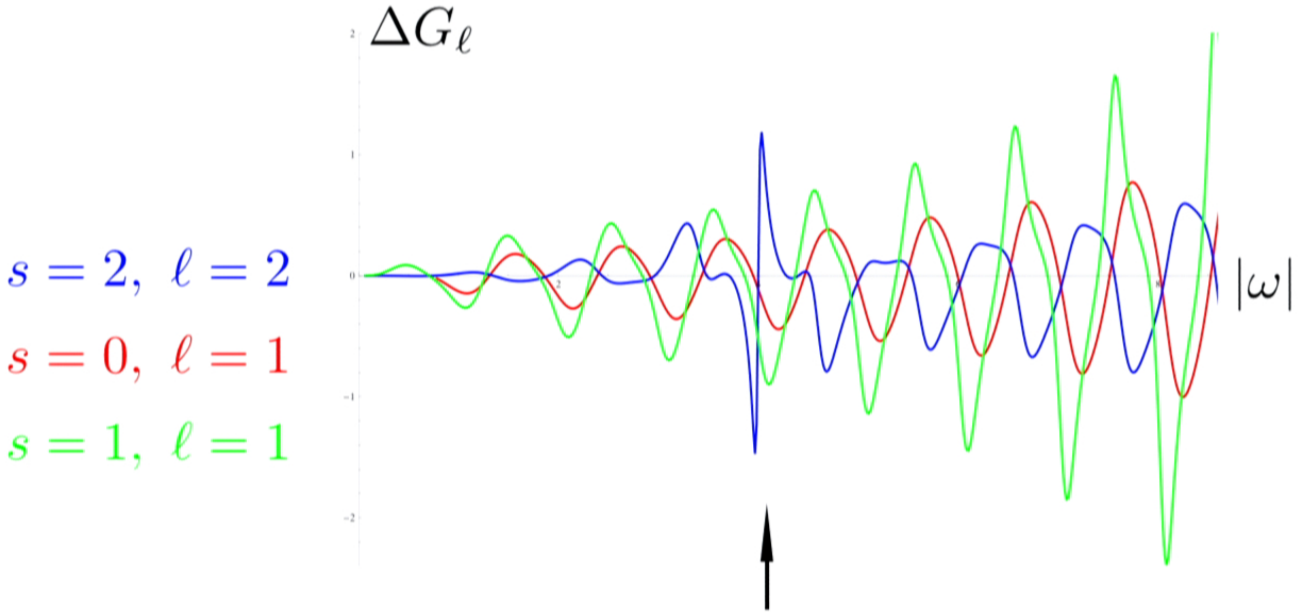
Bdry

normal nbd

Lower n's dominate

Results: BC

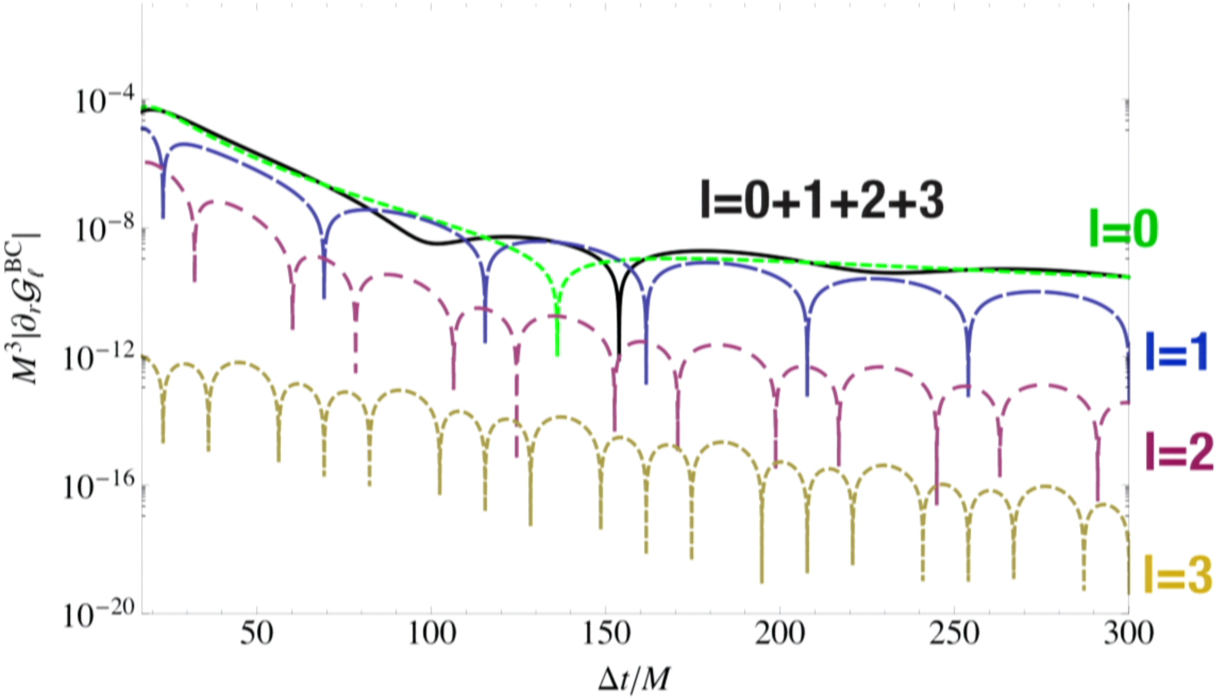
- **First ever analytic calculation (Casals&Ottewill'13)**



algebraically-special freq.

Results: Branch Cut

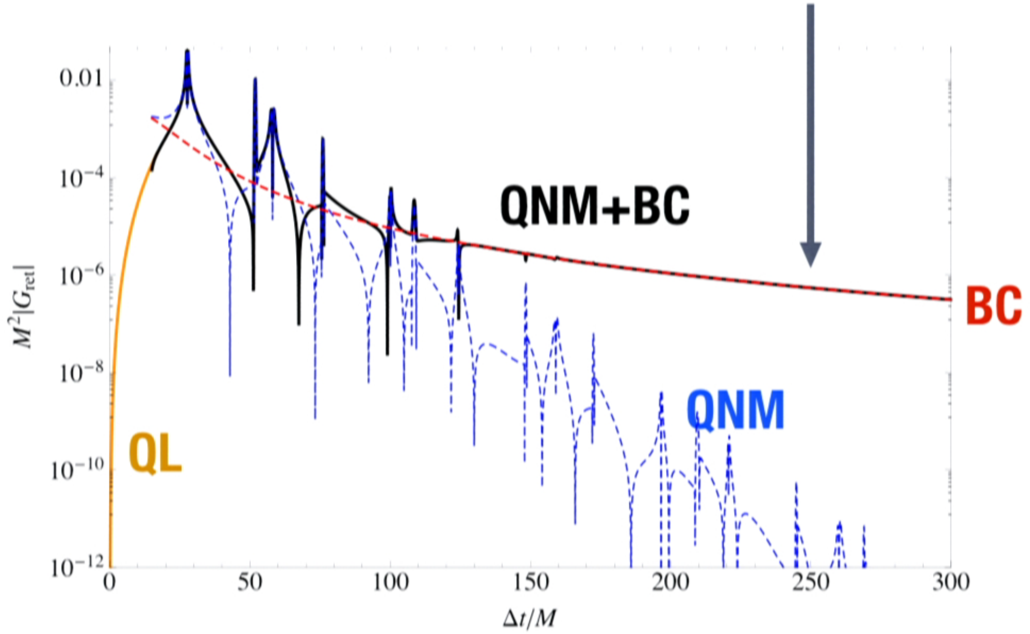
- Different BC ℓ-modes



- ‘Wagging of the tail’ due to $\ell = 1$

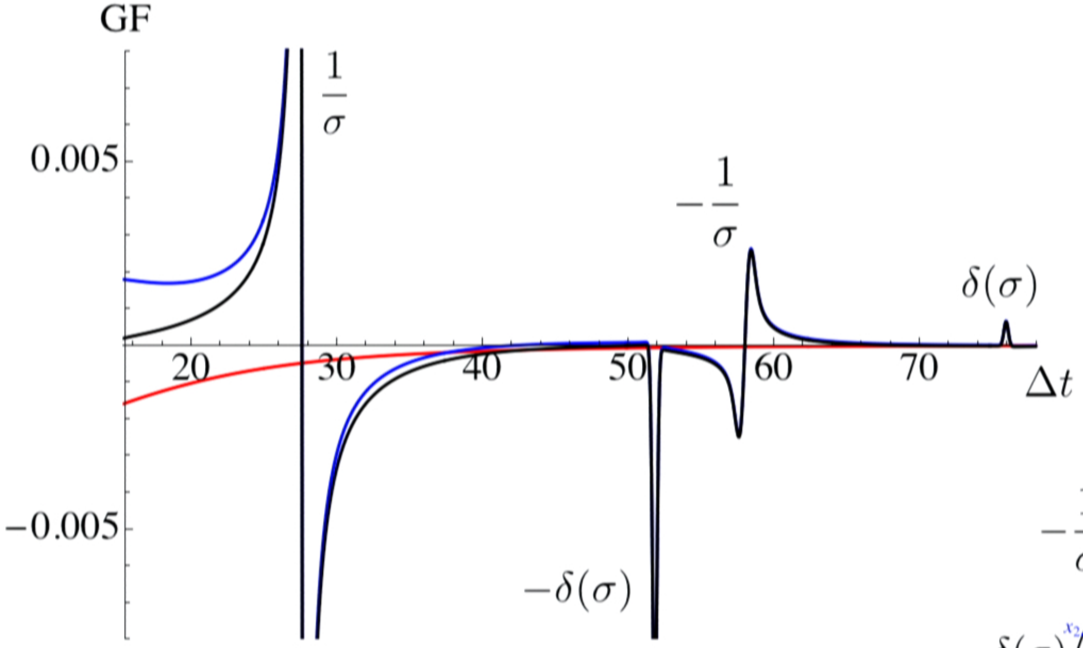
Results: Green Function

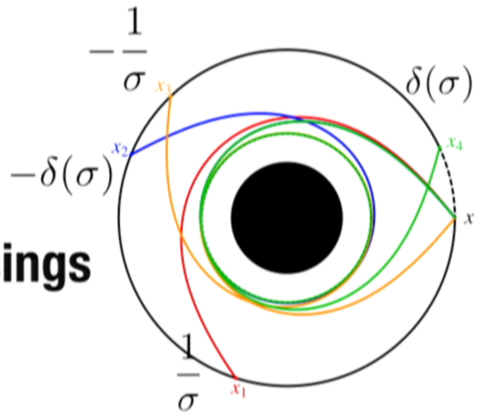
- **BC**: late-time tail decay $t^{-3}, \quad t^{-5}\ln t$



Newly found logarithmic tail decay (Casals&Ottewill'12)

Results: Green Function



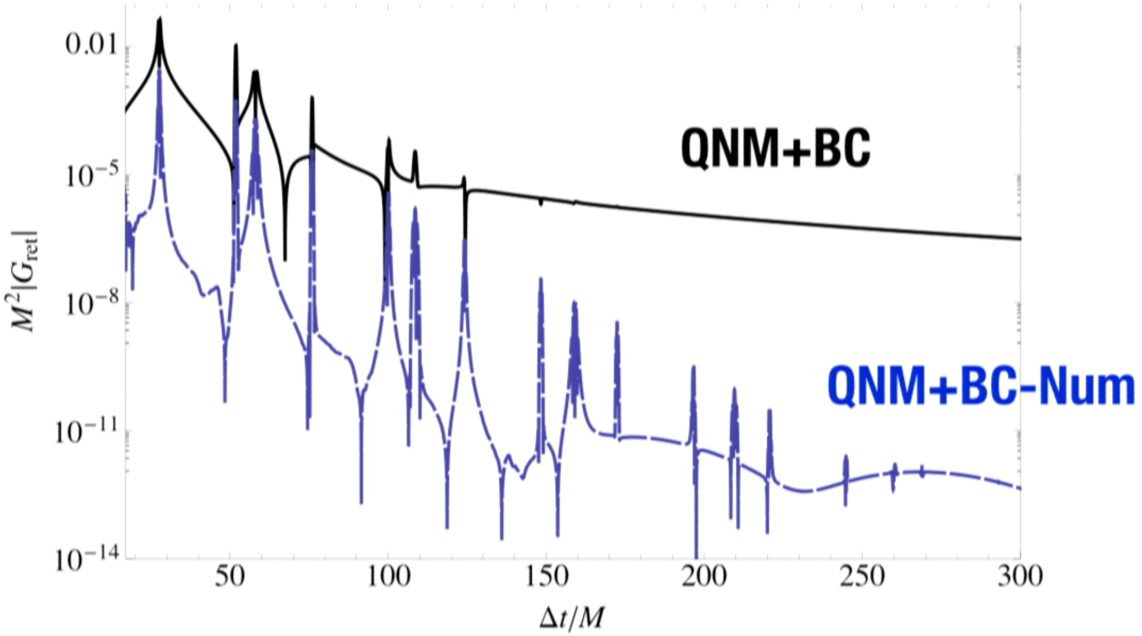


- **QNMs**: four-fold singularities at light-crossings (Dolan&Ottewill'11)

$$G_{ret} \sim \delta(\sigma),\ \frac{1}{\sigma},\ -\delta(\sigma),\ -\frac{1}{\sigma}$$

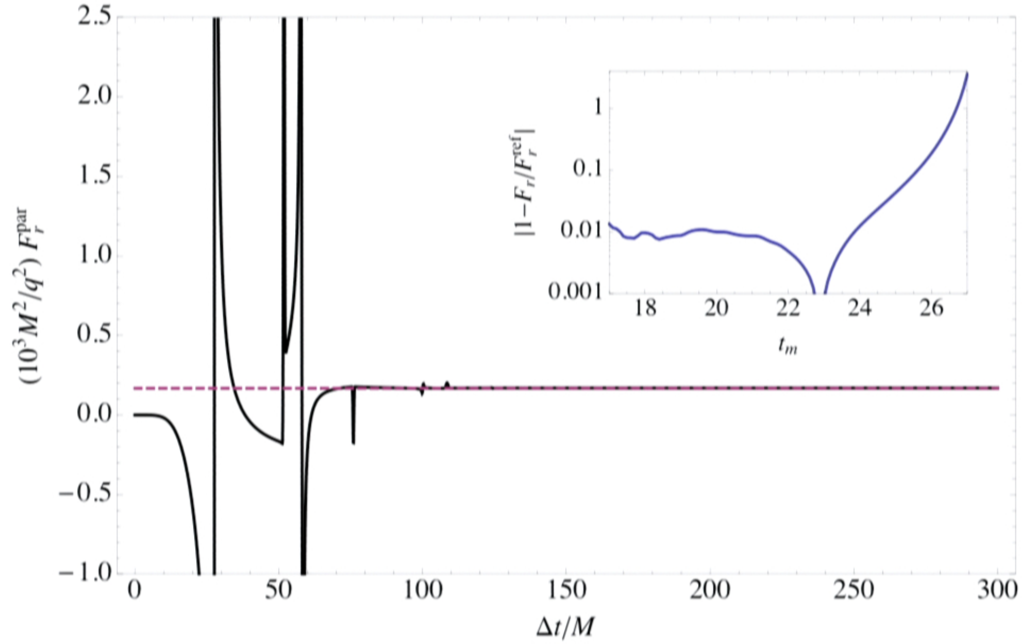
Green Function Validation

- **Validation of QNM+BC against an 'exact' numerical GF**



Results: Self-Force

- **'Partial S-F'** $F_\mu^{par} \equiv q^2 \nabla_\mu \int_{\tau-\Delta\tau}^{\tau^-} d\tau' \; G_{ret}$



- **Value 'settles' after 3rd light-crossing**
- **Rel.err.** $\approx 1\%$ **for** $t_m \in (17M, 23M)$

New Numerical method

(Casals,Dolan,Galley,Ottewill,Wardell,Zenginoglu: in preparation)

- **Kirchhoff integral: evolution of initial data in space-time of b-h**

$$u(x) = \int_{t=0} \left[G_{ret}(x, x') \dot{u}^{ic}(\vec{x}') - u^{ic}(\vec{x}') \partial_t G_{ret}(x, x') \right] g^{tt}(x') d^3 \vec{x}'$$

$$\Box u = 0$$

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$$\Box u = 0$$

- **New method: numerical evolution of a 'Peaked Gaussian':**

zero

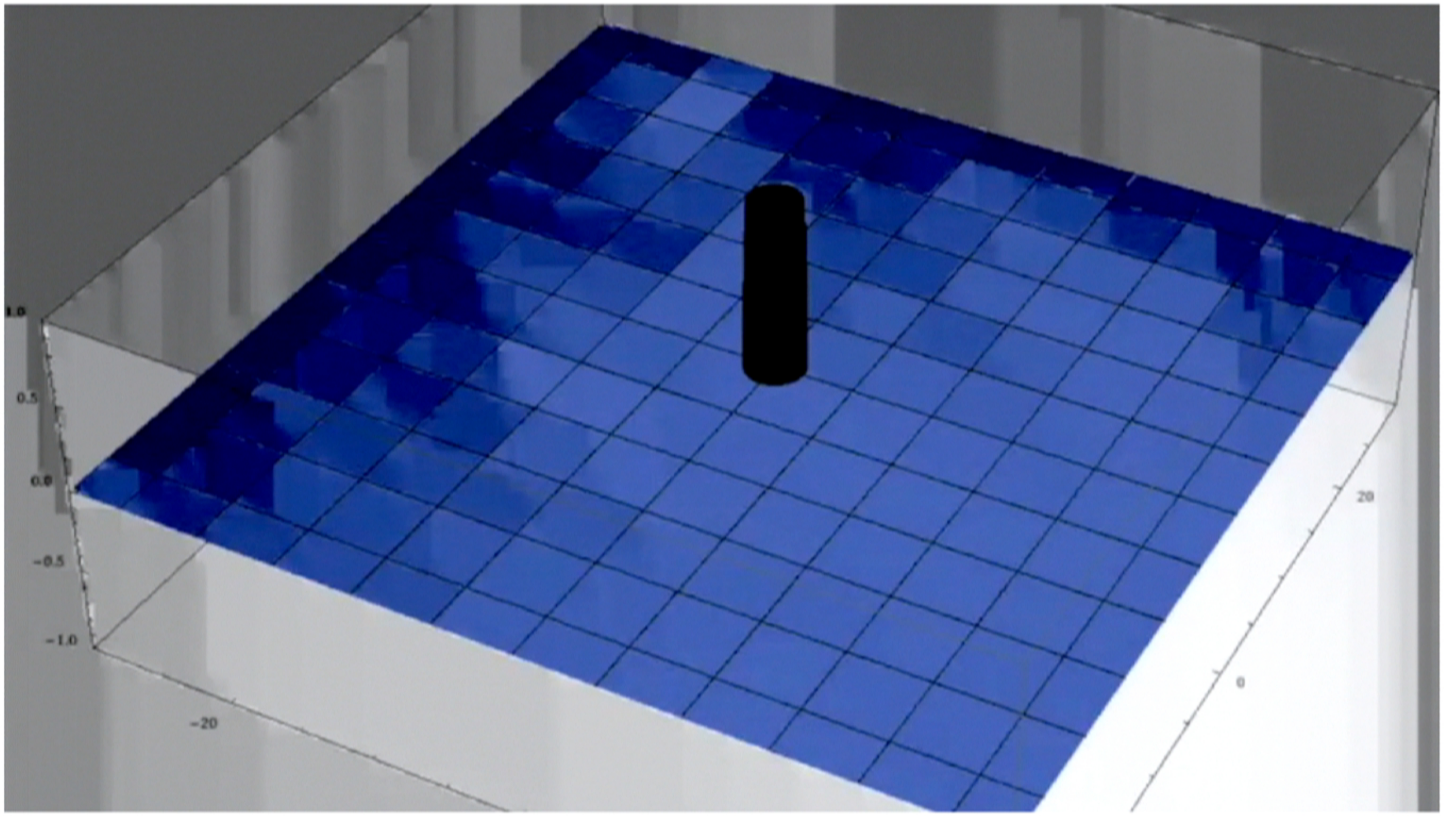
$$G_{ret}(x,x'')$$

$$\frac{1}{(2\pi w^2)^{3/2}} e^{-|\vec{x}'-\vec{x}''|^2/(2w^2)} \approx \delta_3(x'-x'')$$

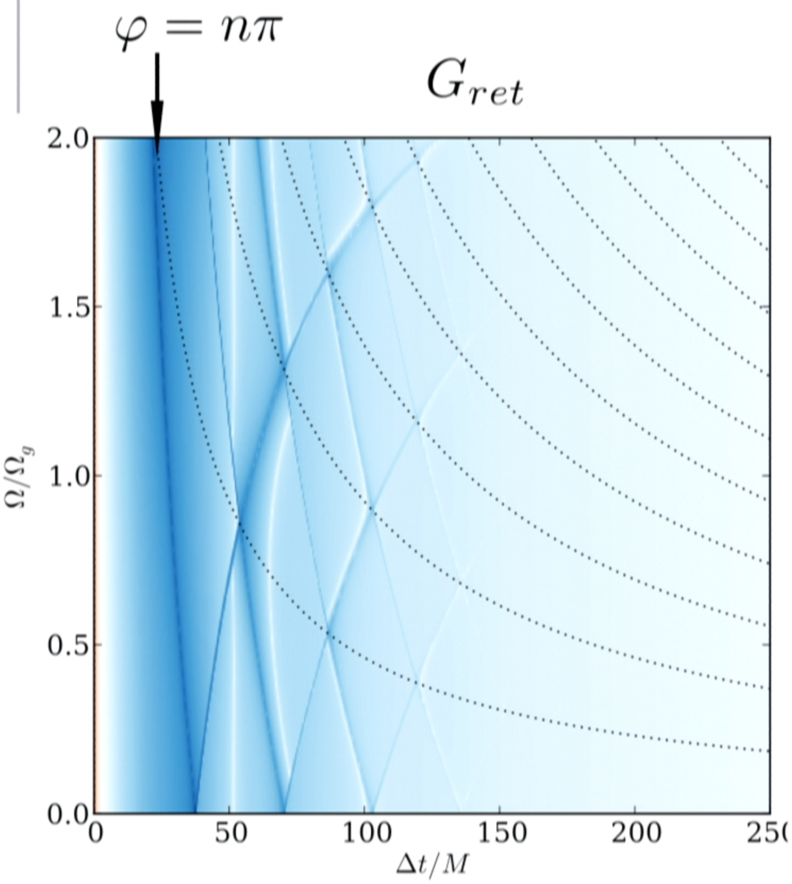
$$w \ll M$$

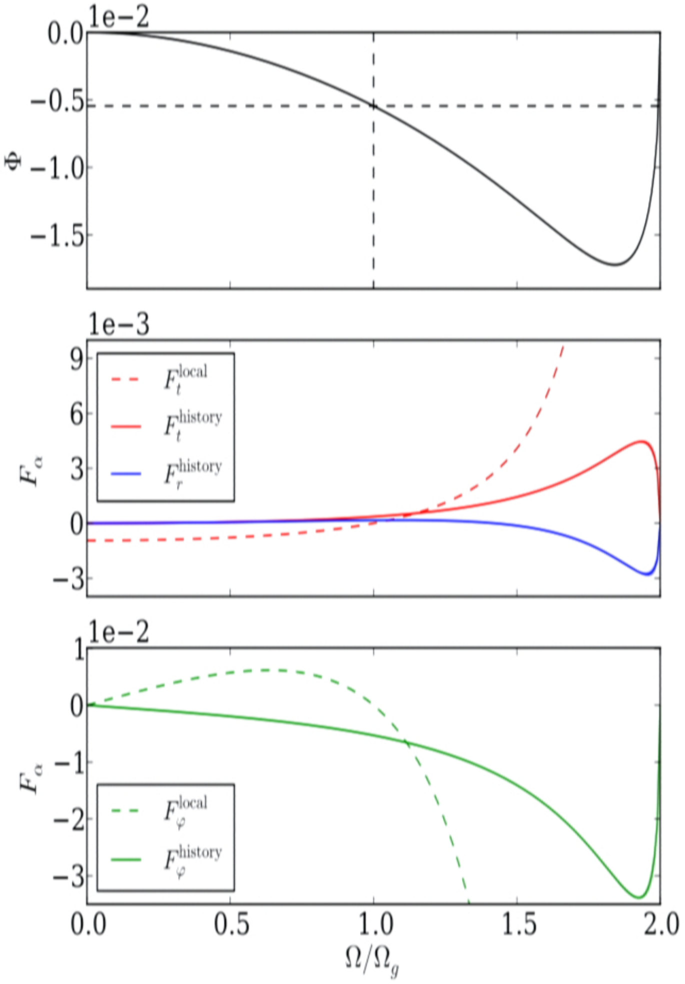
Results: Numerical Method

Evolution of peaked Gaussian around equator of Schwarzschild b-h

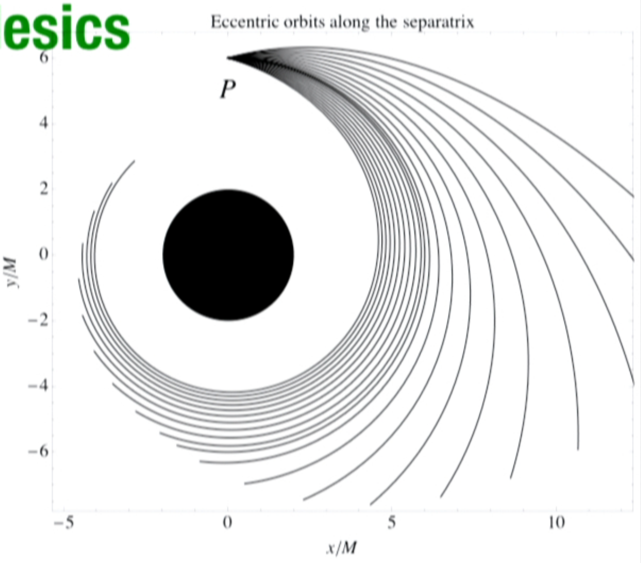


Results: S-F circular orbit



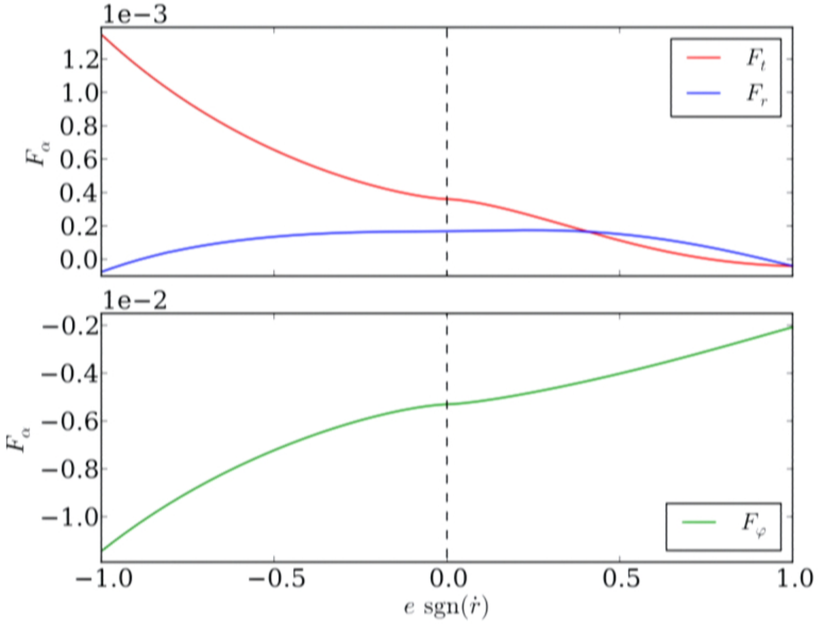


Results: S-F eccentric geodesics


Eccentric orbits along the separatrix

Sample of eccentric geodesics passing through same point P (r=10M)

- **Scalar S-F at P**



Summary

- Method of matched expansions successful in Schwarzschild
- Advantages:

- Trivial regularization

- Physical insight

- How good an approx. using n=0 for QNM and l=0 for BC ?

- Once r-indep quantities are calculated, only requires solving radial ODE

- Once GF calculated for all pairs of points, SF can be obtained for any orbit