Title: A universal Hamiltonian simulator: the full characterization

Date: Nov 11, 2013 04:00 PM

URL: http://pirsa.org/13110065

Abstract: We show that if the ground state energy problem of a classical spin model is NP-hard, then there exists a choice parameters of the model such that its low energy spectrum coincides with the spectrum of \emph{any} other model, and, furthermore, the corresponding eigenstates match on a subset of its spins. This implies that all spin physics, for example all possible universality classes, arise in a single model. The latter property was recently introduced and called ``Hamiltonian completeness", and it was shown that several different models had this property. We thus show that Hamiltonian completeness is essentially equivalent to the more familiar complexity-theoretic notion of NP-completeness. Additionally, we also show that Hamiltonian completeness implies that the partition functions are the same. These results allow us to prove that the 2D Ising model with fields is Hamiltonian complete, which is substantially simpler than the previous examples of complete Hamiltonians. Joint work with Toby Cubitt.

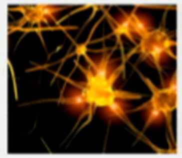
Universal Hamiltonian simulators

A full characterization

Gemma De las Cuevas
Toby Cubitt

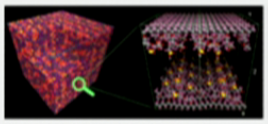
(in the arXiv soon) Perimeter Institute, Nov 11 2013

neurons in a brain



economical systems

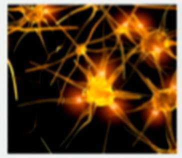
materials



Complexity

ecosystems

neurons in a brain



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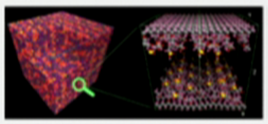

Complexity

ecosystems

"bottom-up approach"

Build a model which is

- simple enough to handle
- complex enough to capture some relevant phenomena

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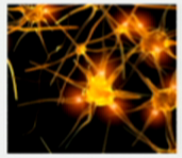

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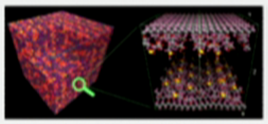

ecosystems

Complexity

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Build a model which is

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One family of such toy models are

Classical spin models

Classical spin models

Defined by

- classical degrees of freedom
- a cost function depending on the configuration of variables

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q levels

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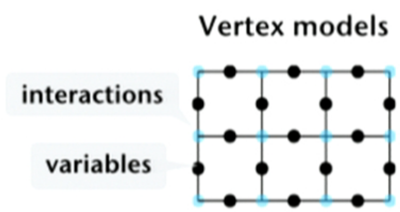
Classical spin models

The interaction pattern could be…

- **Associated to a graph G:**



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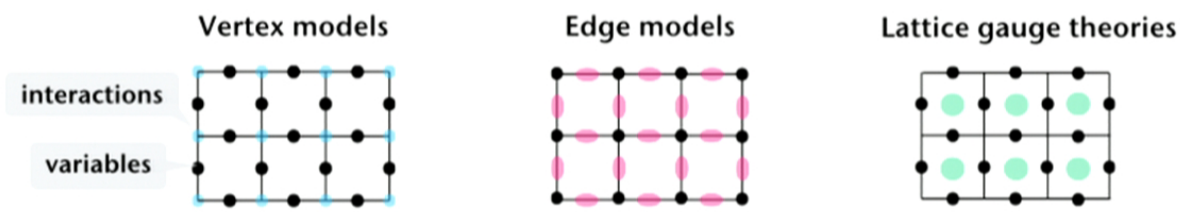

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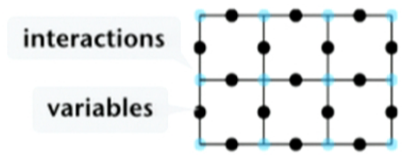
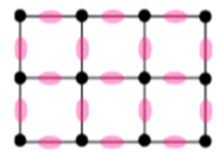
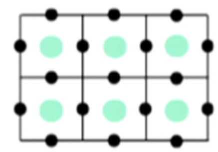
Vertex models



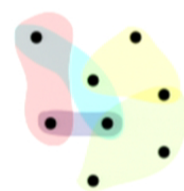
Edge models

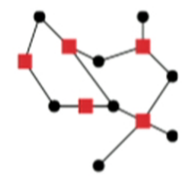


Lattice gauge theories



- More generally, one may need a Hypergraph or Factor graph





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✓ Broad and versatile class

✓ Can be used for different complex systems

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- A "spin model" is a family of Hamiltonians that are related in some way

  For example, "the 2D Ising model with fields" is the set of all Ising models defined on a 2D square lattice (of any size) with some coupling strengths and some fields.

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What do these models have in common?

Is it possible to "simulate" one model in terms of another?

Hamiltonian simulator

A Hamiltonian H simulates a Hamiltonian H', if there exists a choice of parameters of H so that

- the low-energy spectrum of H equals the spectrum of H', and
- the eigenstates of H' are reproduced in a subset of the eigenstates of H.

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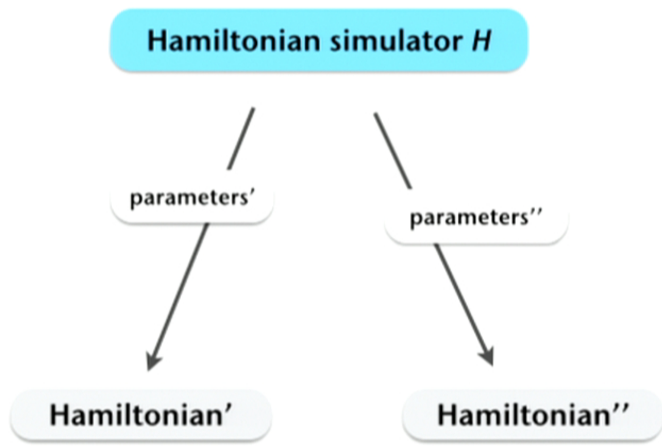

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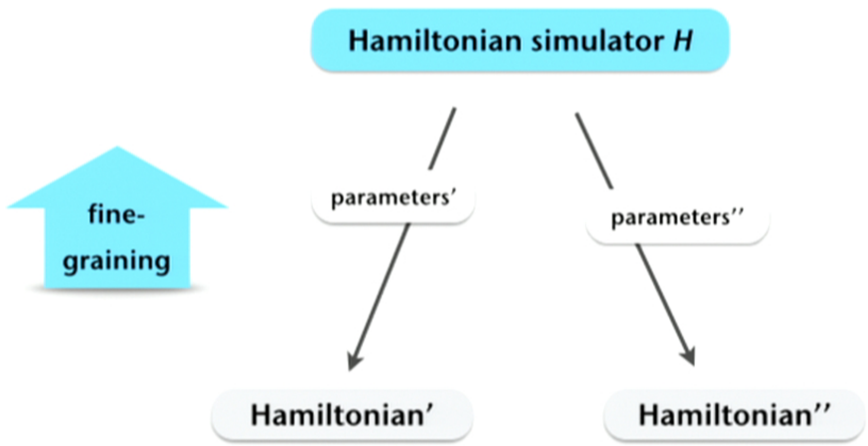

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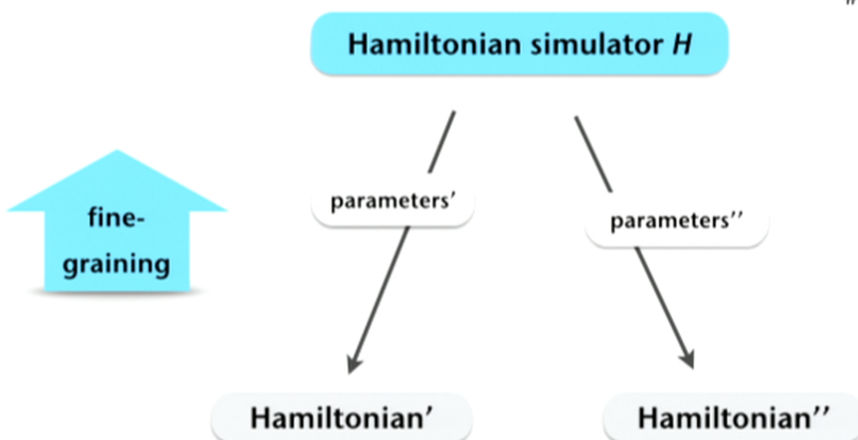
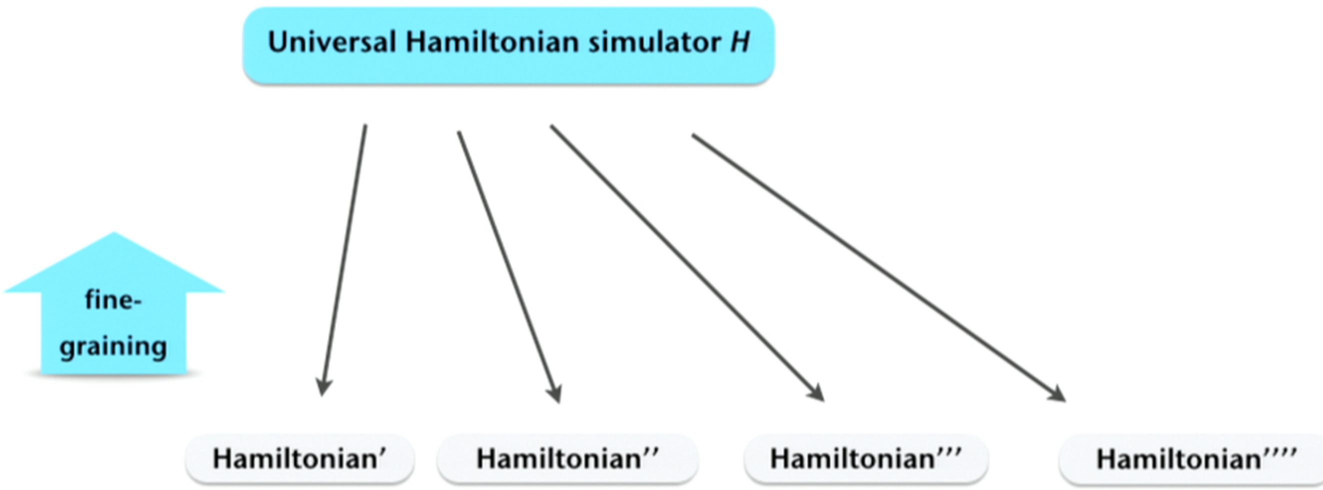
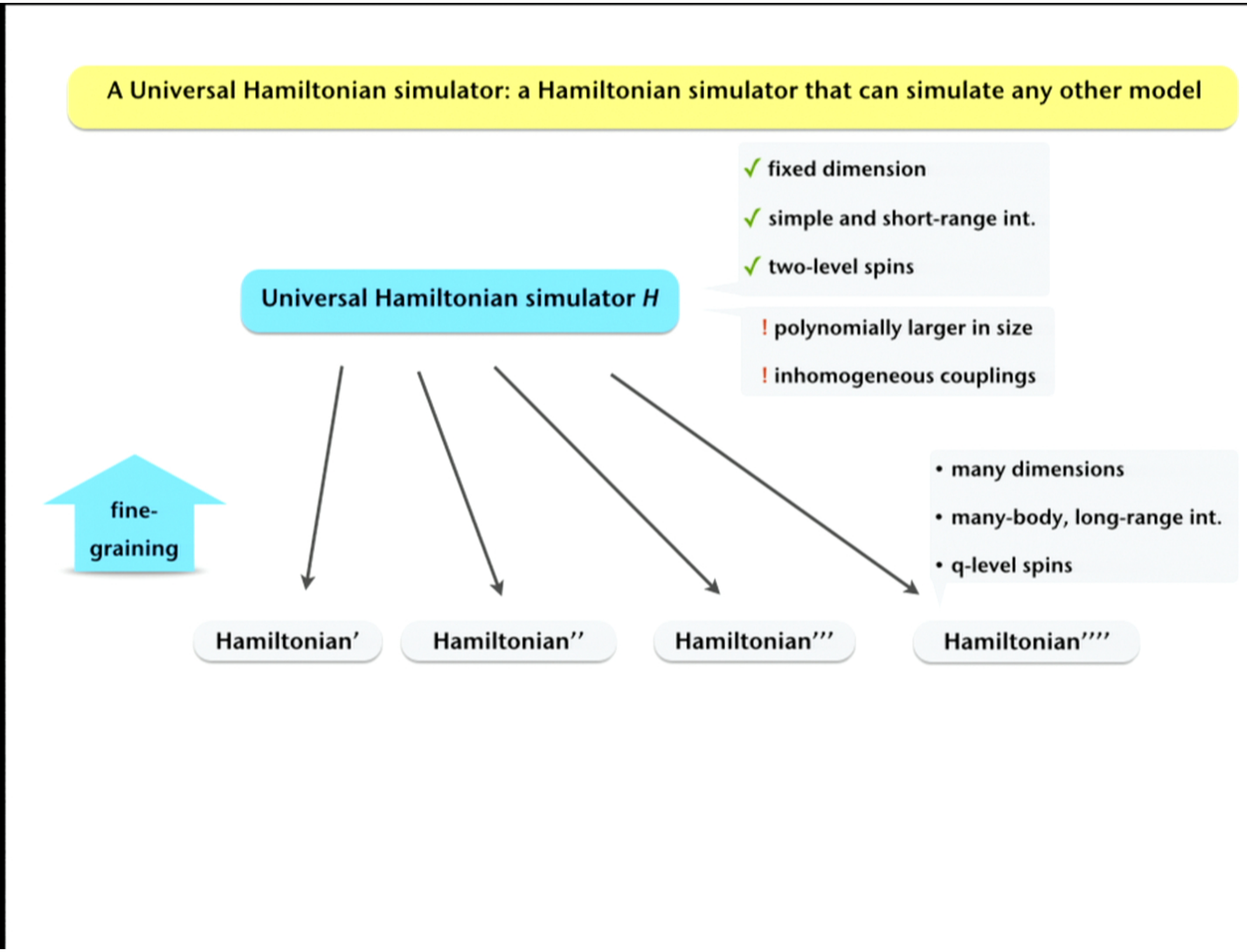


parameters needed to specify H and # spins where H acts are polynomially larger

A Universal Hamiltonian simulator: a Hamiltonian simulator that can simulate any other model

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✓ simple and short-range int.
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! inhomogeneous couplings
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Hamiltonian′
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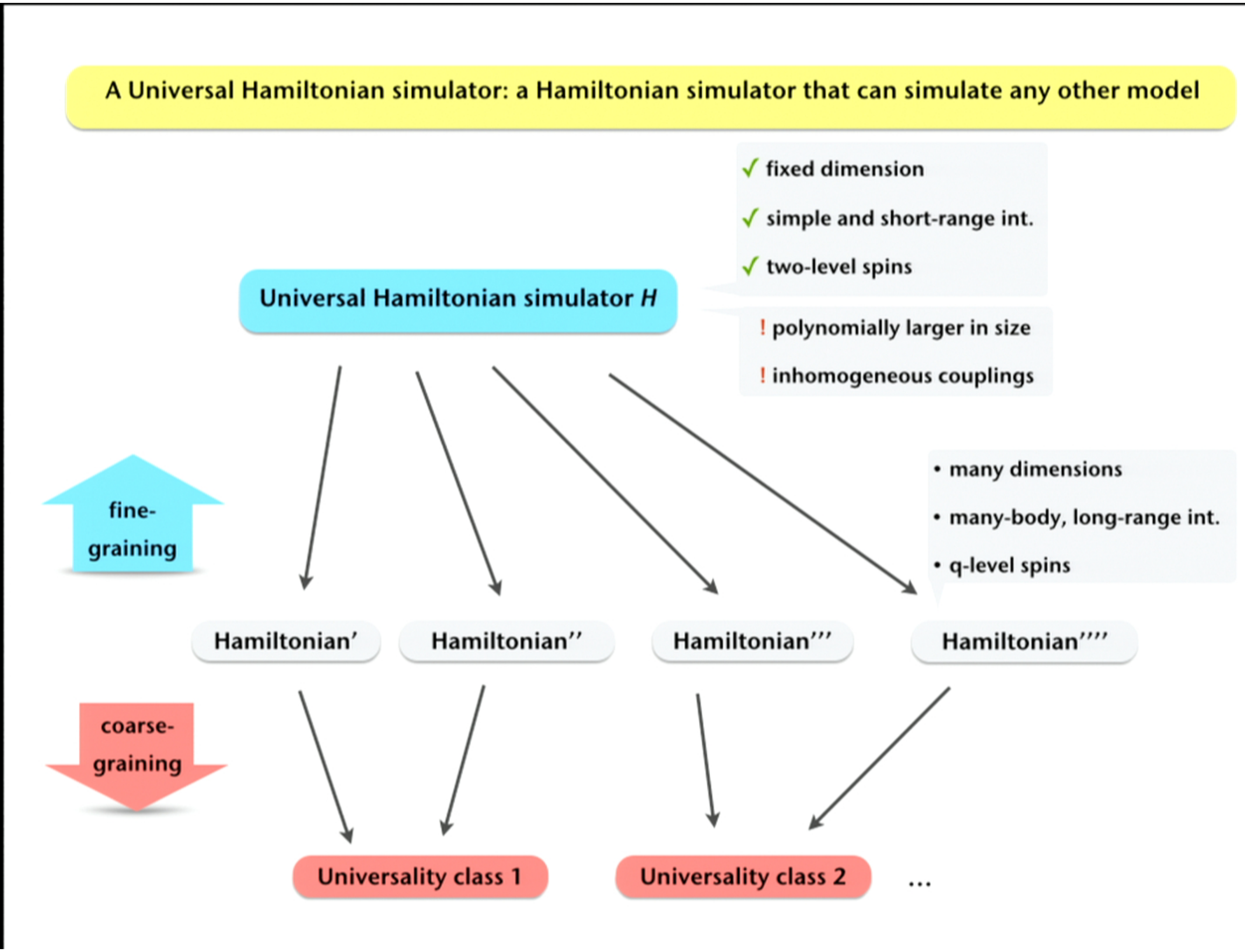
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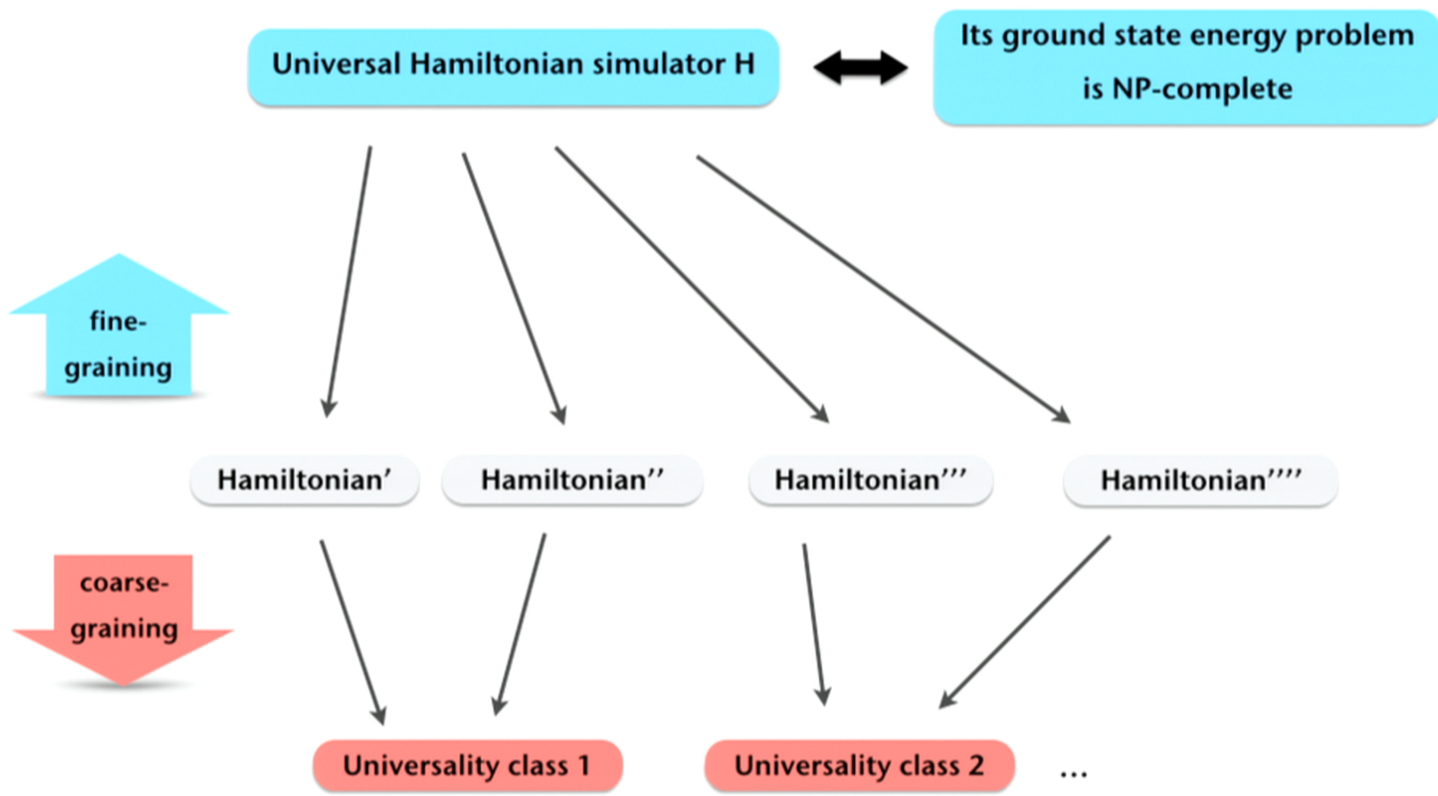
This work


Universal Hamiltonian simulator H
Its ground state energy problem is NP-complete
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Outline

- Prelude: computational complexity
- Universal Hamiltonian simulators: the full characterization
- Idea of the proof
- Conclusions

The ground state energy problem

Given a classical spin model with a certain Hamiltonian *H*, define this problem

Input: a constant *K* and a Hamiltonian *H*

Output: Yes if *H* has energy state with energy $< K$. Else, No.

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K, *H* → **Machine** → Yes / No

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- P is the class of problems which can be solved on a polynomial time Turing machine.

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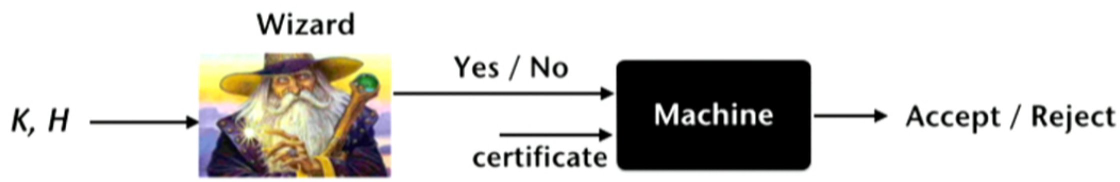

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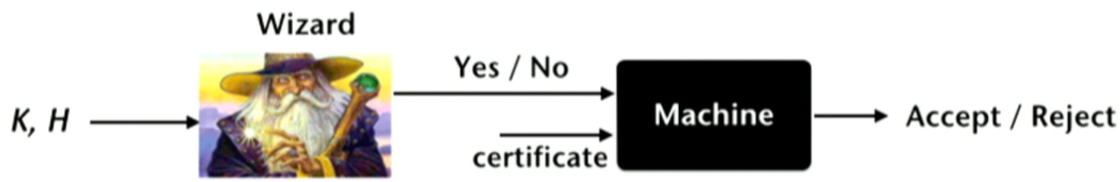
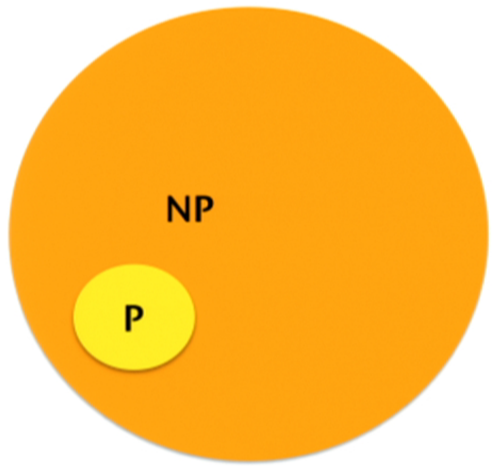
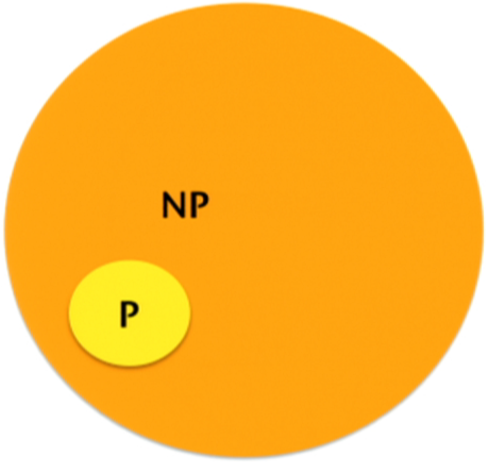
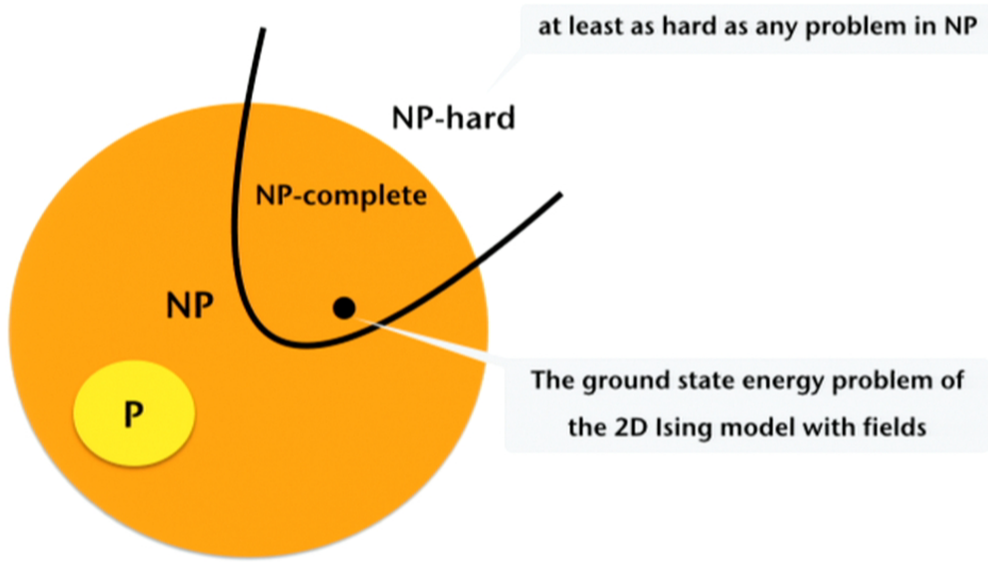


- NP is the class of problems for which
    - if the answer is Yes, there exists a certificate which is easy to verify
    - if the answer is No, all certificates are rejected.


NP
P



- The ground state energy problem of all classical spin models is in NP: just take the spin configuration as the certificate.

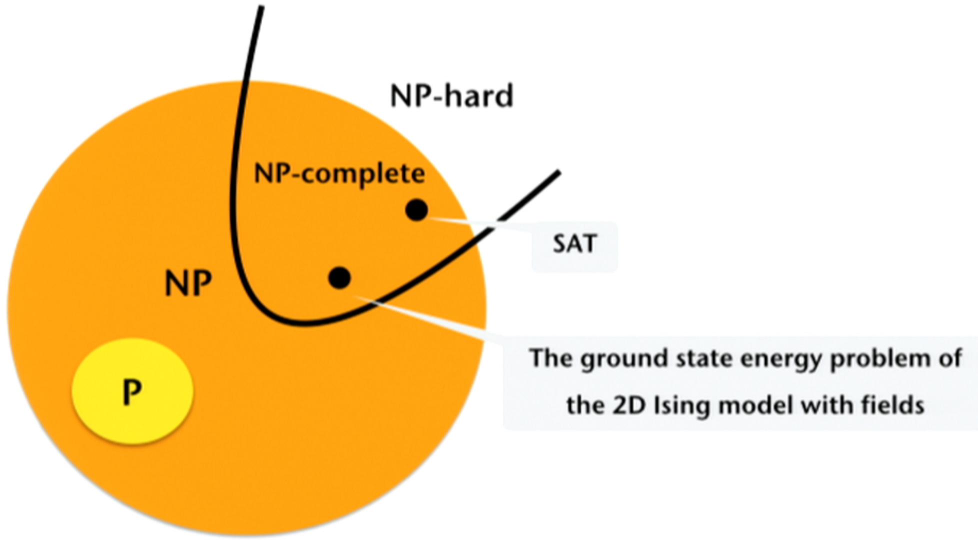


- The ground state energy problem of all classical spin models is in NP: just take the spin configuration as the certificate.

- For some classical spin models it is NP-hard.

  For example, for the 2D Ising model with fields $H_{2D}(\mathbf{s}) = -\sum_{(i,j)\in E} J_{ij} s_i s_j - \sum_{k\in V} h_k s_k$

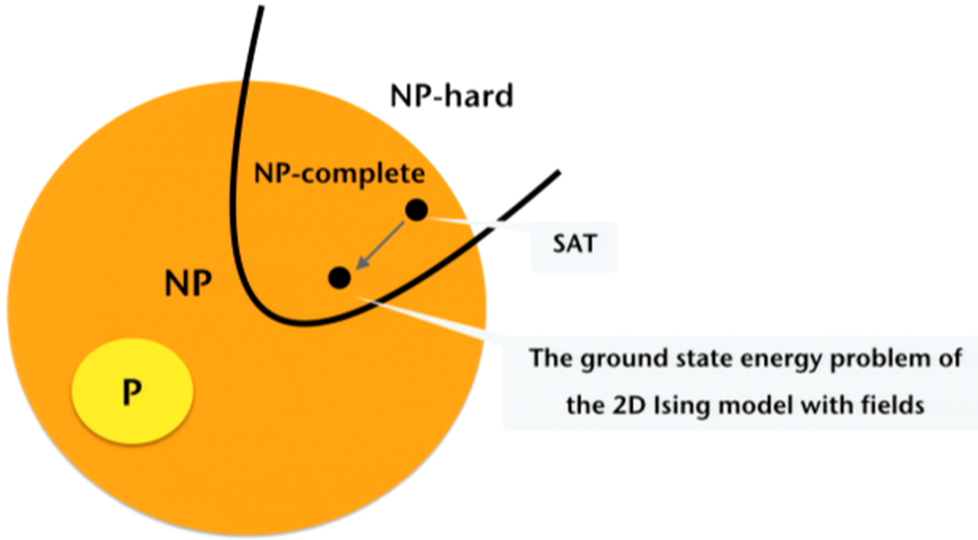
‣ F. Barahona, J Phys A 1982.



SAT

Input: a Boolean formula $\phi(x_1, \ldots, x_n)$

Output: Yes if there is an assignment x such that $\phi(x) = 1$. Else, No.



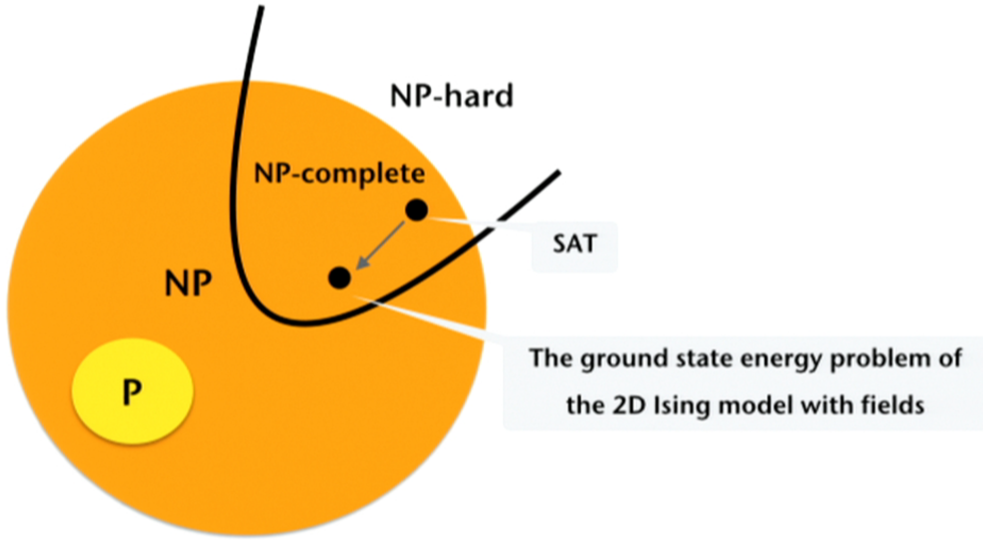
SAT

Given a Boolean formula ϕ ,
is it satisfiable?

poly reduction →

Ground state energy problem

Given a Hamiltonian H and a constant K,
is there a state with energy below K?



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Ground state energy problem

Given a Hamiltonian *H* and a constant *K*, is there a state with energy below *K*?

ϕ is satisfiable ⟺ *H* has a state with energy below *K*

H and *K* depend on ϕ
and are defined on polynomially more variables

Main result

A spin model is a universal Hamiltonian simulator ⟷ Its Ground state energy problem is NP-complete

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- There exists a faithful reduction from SAT to its Ground state energy problem, and
- it is "closed".

If two Hamiltonians are in the model, their sum is too.

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- Corollary:

The 2D Ising model with fields is a universal Hamiltonian simulator.

Idea of the proof

Proof

A spin model is a universal Hamiltonian simulator

➡ Easy

- There exists a faithful reduction from SAT to its Ground state energy problem, and
- It is "closed".

Take the the 2D Ising with fields:

- The reductions SAT to Vertex Cover to Ground state energy problem are faithful.
- Use the crossing gadget to reduce Vertex Cover to Planar Vertex Cover, and the ferromagnetic Ising interactions to force the configurations of neighboring spins to be equal in the ground state.

Proof

A spin model is a universal Hamiltonian simulator ⬅

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Closure

We prove the result for a single k-body Hamiltonian term.
Finally we add all these terms and use closure of the model.

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Closure

We prove the result for a single k-body Hamiltonian term.

Finally we add all these terms and use closure of the model.

Faithful reduction

We use 3 faithful reductions from SAT.

We assume that the resulting Hamiltonian has ground state energy 0 and gap at least 1.

Each will introduce new auxiliary spins.

Setting the stage

- Take the canonical basis for Boolean functions on k bits: $\{e_i(x) : \mathbb{Z}_2^k \to \mathbb{Z}_2\}$

$$e_i(x) = 1 \text{ iff } x = i$$

For example, for $k = 2$ and $x = 3$ we have

$$e_1(3) = 0$$

$$e_2(3) = 0$$

$$e_3(3) = 1$$ "flag" of the state 3

$$e_4(3) = 0$$

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- Expand the Hamiltonian in terms of this basis $H'(x) = \sum_{i=1}^{2^k} E_i e_i(x)$

- Define the XOR operation: $a \odot b := \overline{a \oplus b}$

a	b	$a \odot b$
0	0	1
0	1	0
1	0	0
1	1	1

It is 1 if a=b

First faithful reduction: It forces a flag spin to "say" in which configuration the physical spins are.

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The formula $e_i(\sigma_S) \odot \sigma_{b_i}$ is satisfiable ⟷ **The Hamiltonian H_1 has ground state with 0 energy.**

The satisfying assignment

$e_i(\sigma_S) \odot \sigma_{b_i} = 1$

⟷

Ground state configuration

$\sigma_{b_i} = e_i(\sigma_S)$

$\sigma_{A_i} \in \Sigma_{A_i}$

For example, for $\sigma_S = 3$

$$\sigma_{b_1} = e_1(3) = 0$$
$$\sigma_{b_2} = e_2(3) = 0$$
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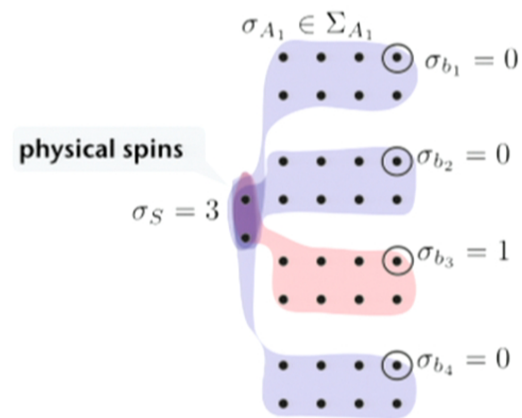
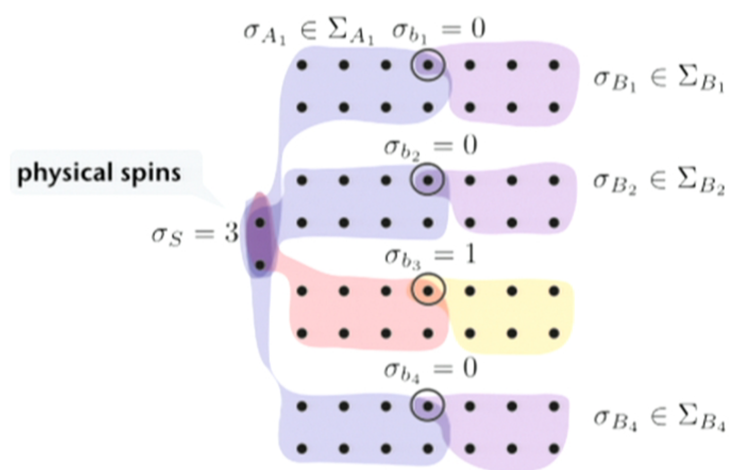
$$\sigma_{b_4} = e_4(3) = 0$$



Second faithful reduction: It forces the flag spin to be in 0.

The formula $\sigma_{b_i} \odot 0$ is satisfiable ⟷ The Hamiltonian H_2 has ground state with 0 energy.

The satisfying assignment $\sigma_{b_i} = 0$ ⟷ Ground state configuration: $\sigma_{b_i} = 0$, $\sigma_{B_i} \in \Sigma_{B_i}$



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The non-satisfying assignment $\sigma_{B_i} = \sigma^\star \in \Sigma_{B_i}$ gives energy $\kappa_i = H_2^i(\sigma_{b_i} = 1, \sigma_{B_i} = \sigma^\star) \geq 1$

Other non-satisfying assignments correspond to other excited states of H_2 .

Also for partition functions

- Our result also holds for partition functions:

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  It requires a modified construction so that the degeneracy

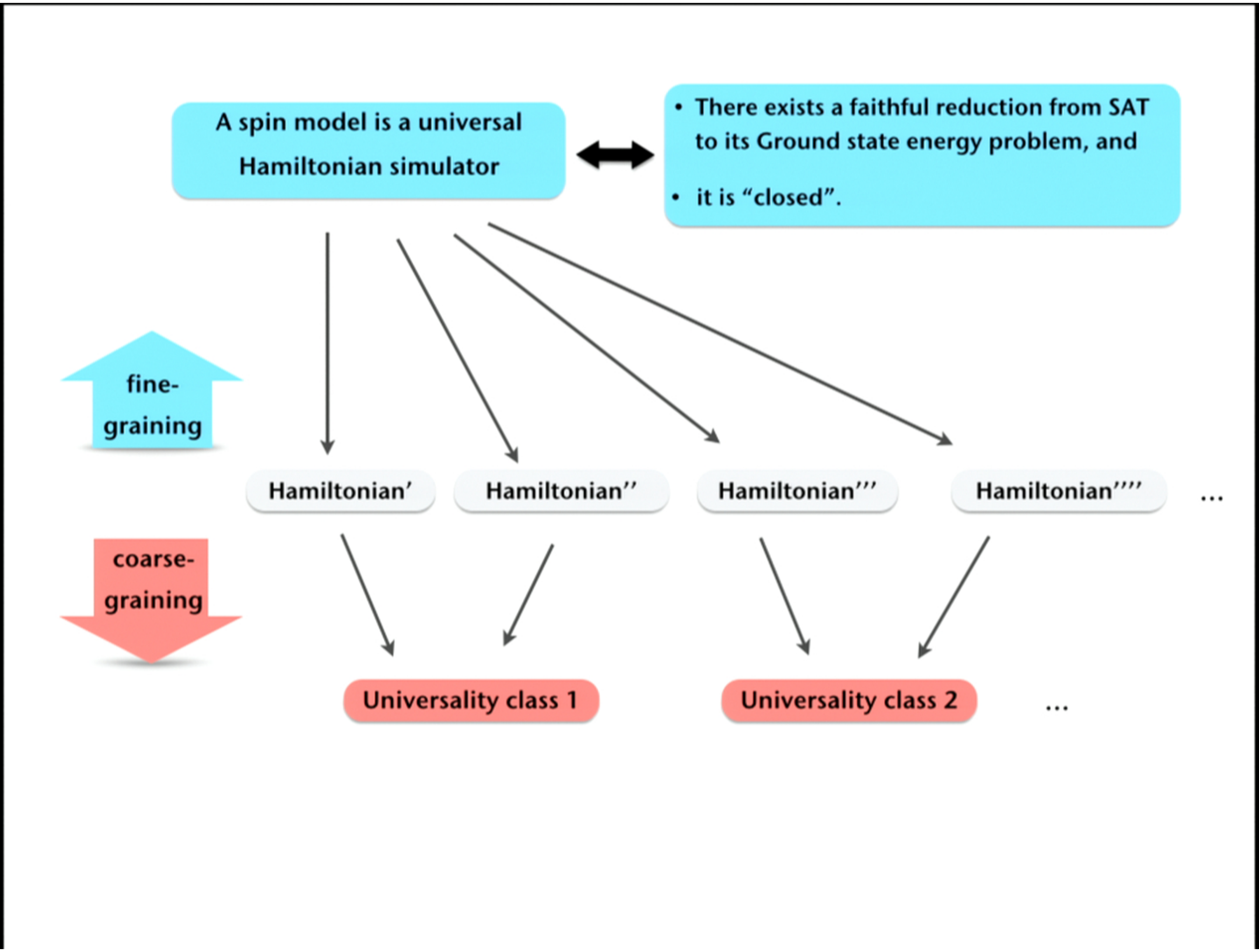
  of the auxiliary spins is the same for all energy levels.

- This implies that we recover previous "completeness results"

  The complete model was found to be (in each case a different proof was used):

  - 2D Ising with fields with imaginary couplings
    - M. Van den Nest, W. Dür, H. J. Briegel, PRL 2008
  - 3D Ising with for a restricted class of models
    - GDLC, W. Dür, M. Van den Nest, H. J. Briegel, JSTAT 2009.
  - 4D Ising lattice gauge theory
    - GDLC, W. Dür, H. J. Briegel, M. A. Martin-Delgado, NJP 2010, PRL 2010.

Conclusions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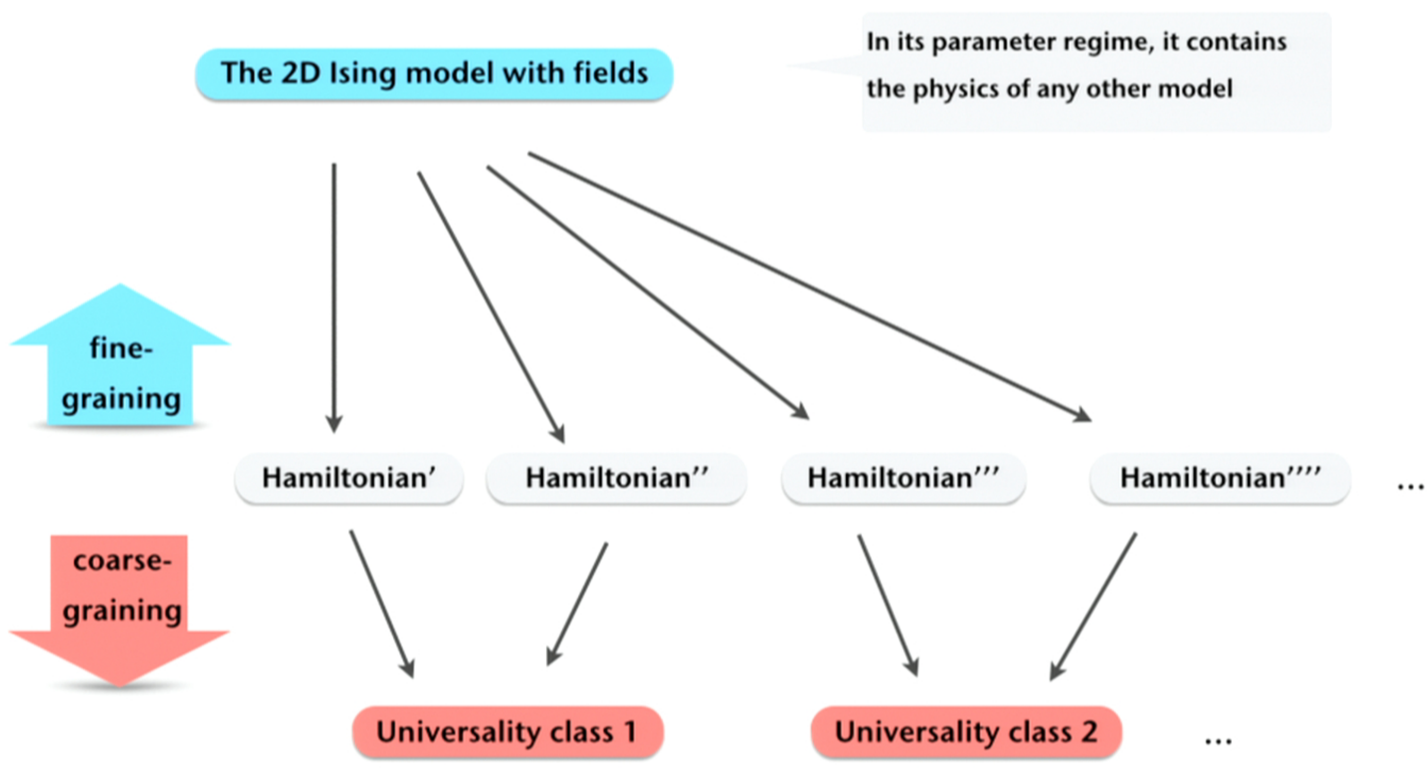
The 2D Ising model with fields
In its parameter regime, it contains the physics of any other model
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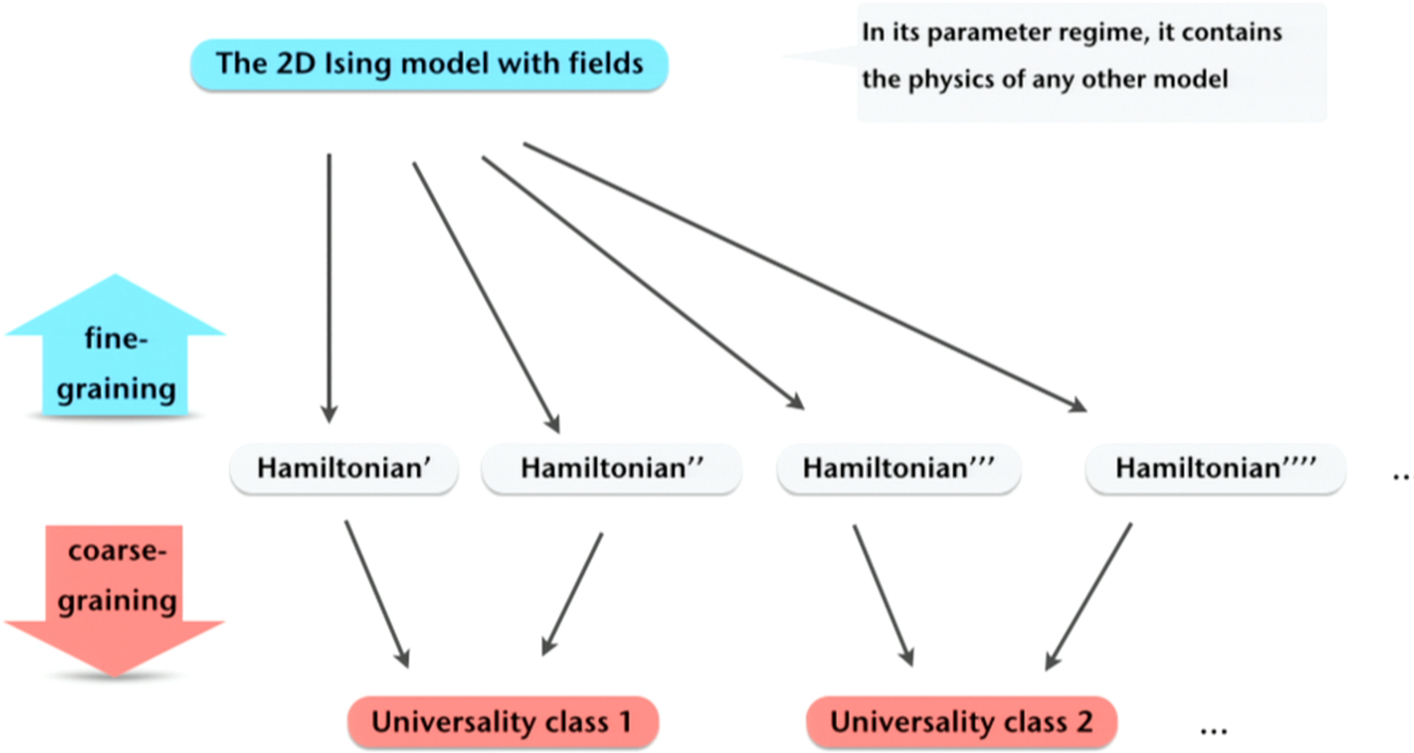

- Mappings on demand to devise simpler simulation schemes?
- Generalizations to quantum Hamiltonians?

Thank you!