Title: Dimensional reduction in the sky

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Abstract: In several approaches to quantum-gravity, the spectral dimension of spacetime runs from the standard value of 4 in the infrared (IR) to a smaller value in the ultraviolet (UV). Describing this running in terms of deformed dispersion relations, I show that a striking cosmological implication is that that UV behavior leading to 2 spectral dimensions results in an exactly scale-invariant spectrum of vacuum scalar and tensor fluctuations. I discuss scenarios that break exact scale-invariance and show that the tensor to scalar ratio is fixed by the UV ratio between the speed of gravity and the speed of light. Cosmological perturbations in this framework display a wavelength-dependent speed of light, but by transforming to a suitable "rainbow frame" this feature can be removed, at the expense of modifying gravity. In particular it turns out that the following concepts are closely connected: scale-invariance of vacuum fluctuations, conformal invariance of the gravitational coupling, UV reduction to spectral dimension 2 in position space and UV reduction to Hausdorff dimension 2 in energy-momentum space.

Dimensional Reduction in the Sky

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based on work with G. Amelino-Camelia, Michele Arzano, João Magueijo

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Outline

- Running Spectral Dimension in Quantum Gravity
- Spectral Dimension Reduction from Modified Dispersion Relation
- Dimensional Reduction and Cosmology
- Implications for Momentum Space Dimensionality and Gravity
- Dimensional Reduction without a Preferred Frame

Heat diffusion

- On a Riemannian manifold:

$$\frac{\partial}{\partial s} K(\xi_0, \xi, s) + \Delta K(\xi_0, \xi, s) = 0$$

(heat equation for diffusion process from ξ_0 to ξ during diffusion time s)

- Return probability density

$$P(s) = \frac{1}{V} \int d\xi \sqrt{|g|} K(\xi, \xi, s)$$

$$K = < \xi | e^{-s\Delta} | \xi_0 > \quad \longrightarrow \quad P(s) = \frac{1}{V} \sum_j e^{-\lambda_j s}$$

(sum over eigenvalues of the Laplacian)

Spectral dimension and geometry

- The return probability density is related to the manifold geometrical properties

$$P(s) = \frac{1}{V(4\pi s)^{d/2}} \sum_n a_n s^n \qquad \text{(heat trace expansion)}$$

$$a_0 = \int \sqrt{|g|}, \quad a_1 \sim \int \sqrt{|g|} R, \quad a_2 \sim \int \sqrt{|g|} \left[5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2(R_{\mu\nu\rho\sigma})^2 \right], \ \ldots$$

- Spectral dimension

$$d_S(s) = -2 \frac{d \ln P(s)}{d \ln s}$$

flat space: $P(s) = (4\pi s)^{-d/2} \longrightarrow d_S(s) \equiv d$

in general: (Riem. manif.) $d_S(s) = d - 2\frac{\sum_{n=1} n\, a_n s^n}{\sum_{n=0} a_n s^n} \longrightarrow d_S(s \to 0) = d$, $d_S(s \to \infty) = 0$

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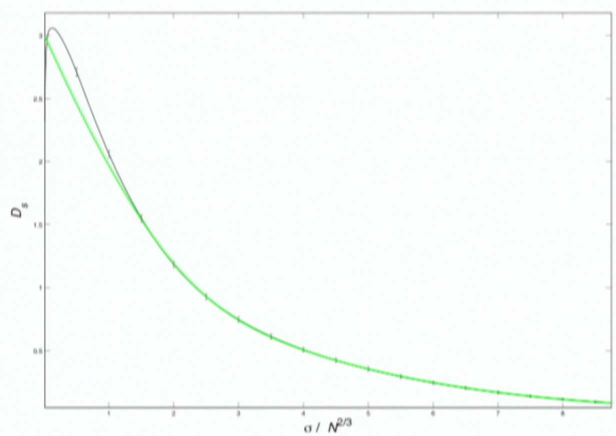
Spectral dimension in Quantum Gravity

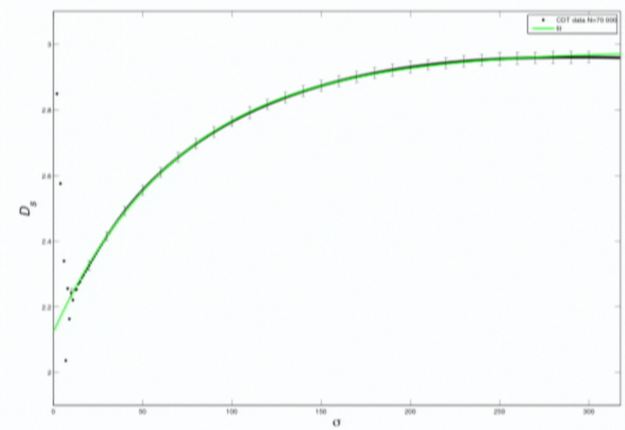
- most QG theories find running spectral dimension in the UV ($s \to 0$)

$$d_S^{(QG)}(s \to 0) \neq d$$

- IR limit probes global geometry, intermediate scales probe local (flat) geometry

example (3d CDT)





[D. Benedetti and J. Henson PRD 2009]

IR behavior

UV behavior

Spectral dimension in Quantum Gravity

- many QG theories favor UV running of spectral dimension to 2

  - Causal Dynamical Triangulation in 3d and 4d [J. Ambjorn, J. Jurkiewicz and R. Loll, PRL 2005] [D. Benedetti and J. Henson PRD 2009]

  - asymptotically safe gravity in 4d [D. F. Litim, PRL (2004)]

  - Horava-Lifshitz gravity in 4d with characteristic exponent z=3 [P. Horava, PRL 2009]

  - Loop Quantum Gravity [L. Modesto, CQG 2009]

⟶ Investigate cosmological implications using a toy model with same running in the UV

From dispersion relation to spectral dimension

- dispersion relation is the momentum space representation of the Laplacian

$$\omega^2 = f(k^2) \quad \longleftrightarrow \quad D_L = -\partial_t^2 - f(-\nabla^2)$$

$\longrightarrow$ spectral dimension is probed by a fictitious diffusion process governed by the "Wick rotated" Laplacian operator (in flat ST)

$$\left[\frac{\partial}{\partial s} + \left(-\partial_t^2 + f(-\nabla^2)\right)\right] K(\xi_0, \xi, s) = 0$$

- the return probability can be written as $P(s) = \int \frac{d^D k\, d\omega}{(2\pi)^{D+1}} e^{-s\left(\omega^2 + f(k^2)\right)}$

and the spectral dimension

$$d_S(s) = 2s \frac{\int d^D k\, d\omega \left[\omega^2 + f(k^2)\right] e^{-s\left(\omega^2 + f(k^2)\right)}}{\int d^D k\, d\omega\, e^{-s(\omega^2 + f(k^2))}}$$

[T. Sotiriou, M. Visser, S. Weinfurtner PRD 2011]

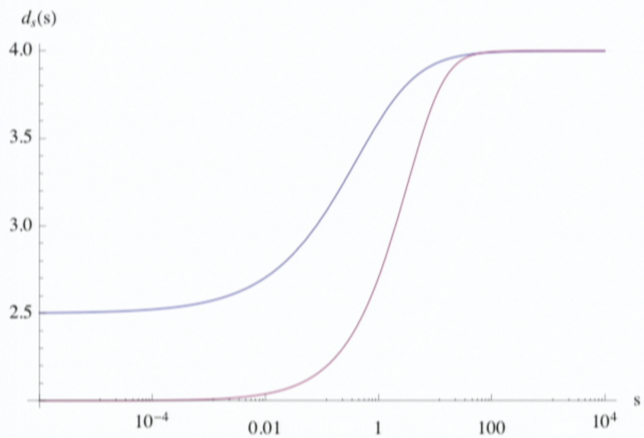
From MDR to RSD - an example

- the ansatz (Euclidean MDR)

$$\omega^2 + p^2 \left(1 + (\lambda p)^{2\gamma}\right) = 0$$

gives the general result

$$d_S(0) = 1 + \frac{D}{1+\gamma}$$



spectral dimension for D=3

blue: $\gamma = 1$

purple: $\gamma = 2$

($d_S(0) = 2$ for $\gamma = 2$ and D+1=3+1)

Cosmological perturbations

- second-order action of cosmological perturbations

$$S^{(2)} = \frac{1}{2}\int d^4x \left[v'^2 - (\partial_i v)^2 + \frac{a''}{a}v^2\right]$$

⟶ equation of motion in Fourier space

$$v'' + \left[c^2k^2 - \frac{a''}{a}\right]v = 0$$

- modes matching in de Sitter ST

solution of EOM at small scales: $v \sim \frac{e^{-ik\beta\eta c}}{\sqrt{ck}}$ $(\omega\eta >> 1)$

solution of EOM at large scales: $v \sim F(k)a$ $(\omega\eta << 1)$

⟶ $F(k) \sim \frac{1}{k^{3/2}}$

⟶ the power spectrum $P(k) \sim k^3 \left|\frac{v}{a}\right|^2$ is scale invariant

Cosmological perturbations with MDR

- start from same EOM for perturbations

$$v'' + \left[c^2k^2 - \frac{a''}{a}\right] v = 0$$

- if dispersion relation is modified then c is k-dependent

$$\omega^2 + p^2\left(1 + (\lambda p)^{2\gamma}\right) = 0 \quad \xrightarrow{(\gamma=2)} \quad c \sim (\lambda p)^2 \sim \left(\frac{\lambda k}{a}\right)^2$$

- solution of EOM at small scales: $v \sim \dfrac{e^{-ik\beta\eta c}}{\lambda k^{3/2}} a$

$\longrightarrow$ the power spectrum $P(k) \sim k^3 \left|\dfrac{v}{a}\right|^2$ is already scale invariant before the mode exits the horizon and for any equation of state

Connection with observations

- Planck results: scale invariance is not exact ($n_S = 0.960 \pm 0.007$)

[Planck Collaboration arXiv:1303.5082 [astro-ph.CO]]

⟶ we can match $P(k) \sim k^3 \left|\frac{v}{a}\right|^2 \equiv k^{n_s - 1}$ with $\gamma \gtrsim 2$

or with a slow running, e.g. $\gamma(p) = 2 - \frac{2}{1 + C \ln\left(1 + (\lambda p)^2\right)}$

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from modes matching: $\left|\frac{v}{a}\right| = \frac{1}{kc} a^{-2}$ @ $ck\eta \sim 1$

for generic γ: $c = \left(\frac{k}{a}\right)^{\gamma}$

scale factor depends on equation of state: $a(\eta) = \eta^{1/\epsilon - 1}$ $\left(\epsilon = \frac{3}{2}(1+w)\right)$

$$\longrightarrow \quad n_s - 1 = \frac{\epsilon(\gamma - 2)}{\gamma - \epsilon - 1}$$

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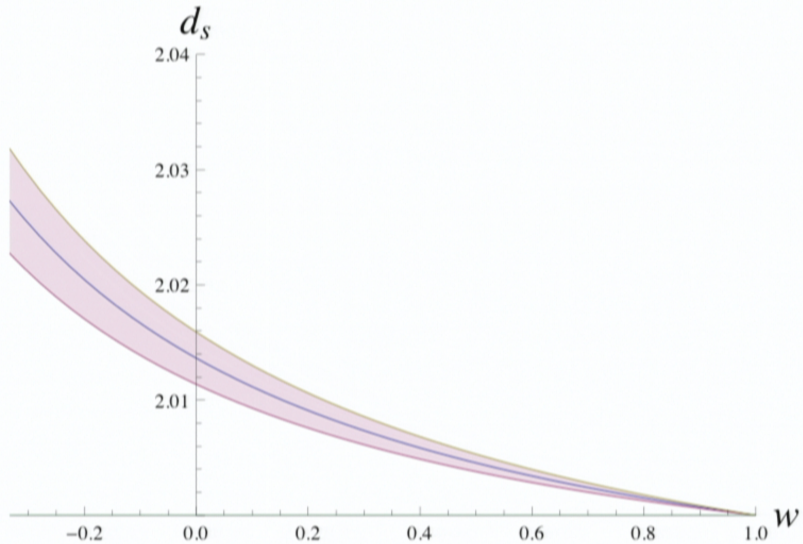

Values of d_S and w allowed by Planck 1-sigma constraint

Connection with observations

- Amplitude of the spectrum is linked to the coefficient of the correction in the dispersion relation

if $\omega^2 + p^2\left(1 + (\lambda p)^4\right) = 0$

$$\longrightarrow \qquad A_s \equiv \sqrt{k^3|F/a|^2} = \left(\frac{L_p}{\lambda}\right) \sim 10^{-5}$$

- Tensor to scalar ratio is linked to ratio between speed of scalar modes and of tensor modes

if $c_t/c_s = b$

$$\longrightarrow \qquad r \equiv \frac{A_t}{A_s} = b < 0.1$$

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Momentum space dimensional reduction

- change momentum variables to make the dispersion relation trivial

$$\omega^2 + p^2 \left(1 + (\lambda p)^4\right) = 0 \qquad \longrightarrow \qquad \tilde{p} = p\sqrt{1 + (\lambda p)^{2\gamma}}$$

- need to change the momentum space measure accordingly

$$p^{D-1} dp \sim \tilde{p}^{\frac{D-1-\gamma}{1+\gamma}} d\tilde{p} \qquad \text{(in the UV)}$$

⟶ energy-momentum space (Haussdorf) dimension is effectively modified in the UV:

$$d_{E,\tilde{p}} = 2 + \frac{D-1-\gamma}{1+\gamma} = 1 + \frac{D}{1+\gamma}$$

the UV e-m space dimension matches the UV spectral dimension of spacetime

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Gravity in dimensionally reduced momentum space

- momentum space dimensional reduction affects equation for perturbations (in the linearizing variables)

quadratic action for perturbations:

$$S_2 = \int d\eta d^3 k a^2 \left[\zeta'^2 + c^2 k^2 \zeta^2\right]$$

change of variables (for comoving momenta):

$$\tilde{k} = k\sqrt{1 + (\lambda k)^{2\gamma}} \qquad k^2 dk \sim \tilde{k}^{\frac{2-\gamma}{1+\gamma}} d\tilde{k}$$

$$\longrightarrow \qquad S_2 = \int d\eta d\tilde{k}\, \tilde{k}^{\frac{2-\gamma}{1+\gamma}} a^2 \left[\zeta'^2 + \frac{\tilde{k}^2}{a^{2\gamma}} \zeta^2\right]$$

in these variables the effective speed of light is $c \sim a^{-\gamma}$, we redefine time units to make it trivial

Gravity in dimensionally reduced momentum space

- action in the linearizing units

$$S_2 = \int d\tau d\tilde{k}\, k^{\frac{2-\gamma}{1+\gamma}} z^2 \left[\dot{\zeta}^2 + \tilde{k}^2 \zeta^2\right] \qquad (z = a^{1-\frac{\gamma}{2}})$$

- EOM for perturbations: $(\zeta = -v/z)$

$$\ddot{v} + \left[\tilde{k}^2 - \frac{\ddot{z}}{z}\right] v = 0$$

for $\gamma = 2$ z is time independent ($z \equiv 1$)

⟶ the effect of expansion disappears and the theory is effectively conformal invariant, regardless of equation of state

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Dimensional reduction without a preferred frame

- until now we have considered a dispersion relation that can be valid in only one preferred frame

  does this mean that running to two spectral dimensions implies breakdown of relativistic symmetries?

- interplay between dispersion relation and measure can be used to build a relativistic theory

  in fact one can make the return probability density invariant by introducing a non-trivial measure on momentum space

  but will the theory run to two?

Dimensional reduction without a preferred frame

- Example: curved momentum space with de Sitter metric (Euclidean)

measure: $d\mu(E,p) = \sqrt{-g}dEd^3p = e^{3\ell E}dEd^3p$

Laplacian: $\mathcal{C}_\ell(1+\ell^{2\gamma}\mathcal{C}_\ell^\gamma)$ with $\mathcal{C}_\ell = \frac{4}{\ell^2}\sinh^2\left(\frac{\ell E}{2}\right) + e^{\ell E}|\vec{p}|^2$

$$\longrightarrow \qquad P(s) \sim \int dE dp\, p^2 e^{3\ell E} e^{-s\mathcal{C}_\ell\left(1+\ell^{2\gamma}\mathcal{C}_\ell^\gamma\right)}$$

change of variables to make the dispersion relation trivial in the UV:

$$\tilde{E} = e^{\ell E/2}/\ell = r\cos(\theta),\ \tilde{p} = e^{\ell E/2}p = r\sin(\theta) \longrightarrow P(s) \sim \int dr r^5 e^{-sr^{2(\gamma+1)}}$$

$$\hat{r} = r^{\gamma+1} \qquad \longrightarrow \qquad P(s) \sim \int d\hat{r}\, \hat{r}^{\frac{6}{1+\gamma}-1} e^{-s\hat{r}^2}$$

$$d_S(0) = \frac{6}{1+\gamma}$$

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Conclusions

- Running to spectral dimension of 2 in the UV is a common feature of many Quantum Gravity theories
- Scale invariant spectrum for primordial perturbations *if the running goes to 2*
- The framework allows also for achieving quasi-scale invariance and can be related to tensor-to-scalar ratio
- UV running of spectral dimension is associated to UV running of momentum space Hausdorff dimension to the *same value*
- Running of Hausdorff dimension of momentum space is associated to conformally invariant theory
- First explicit example of relativistic theory with running spectral dimension