

Title: 13/14 PSI - Condensed Matter - Lecture 4

Date: Nov 14, 2013 10:45 AM

URL: <http://pirsa.org/13110030>

Abstract:

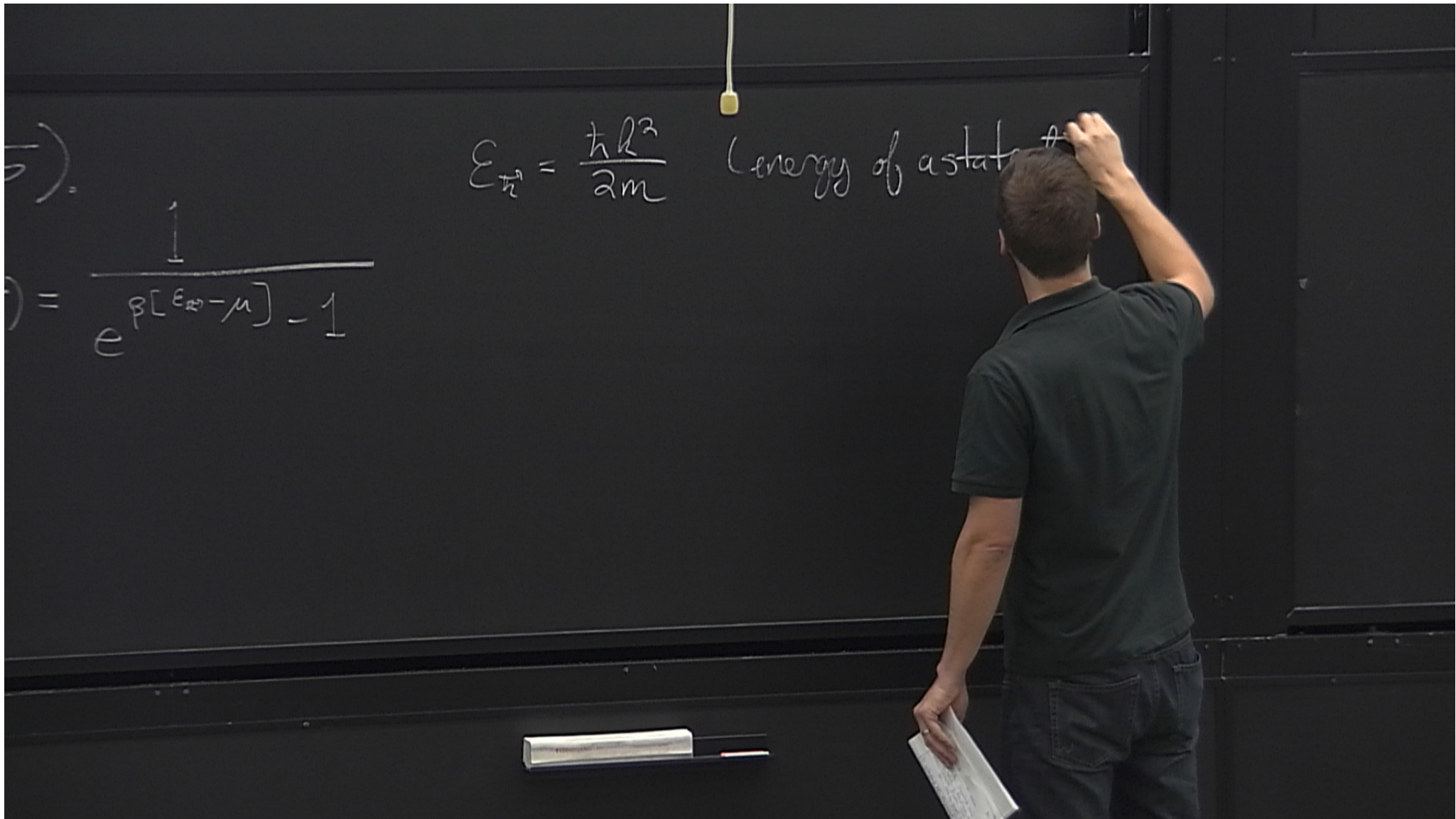
Bose-Einstein Condensate: (predicted)

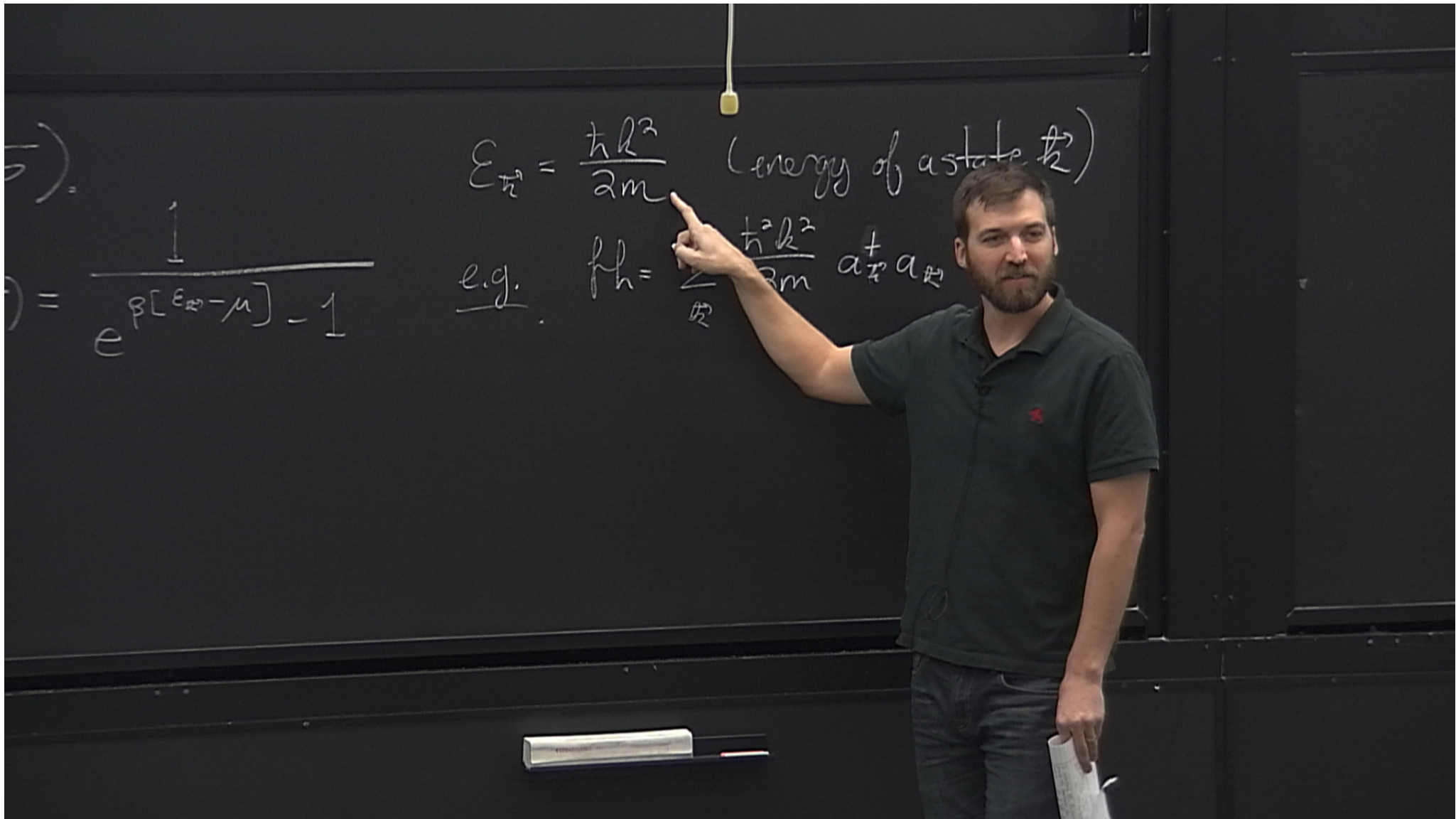
Bose-Einstein Condensate: (predicted 1924, observed 1995)

Free gas of Bosons follows the following distribution

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ng distribution $n_{\vec{k}}(T) = \frac{1}{e^{\beta[\epsilon_{\vec{k}} - \mu]} - 1}$





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follows the following distribution

$$n_{\vec{k}}(T) = \frac{1}{e^{\beta \epsilon_{\vec{k}}} - 1}$$

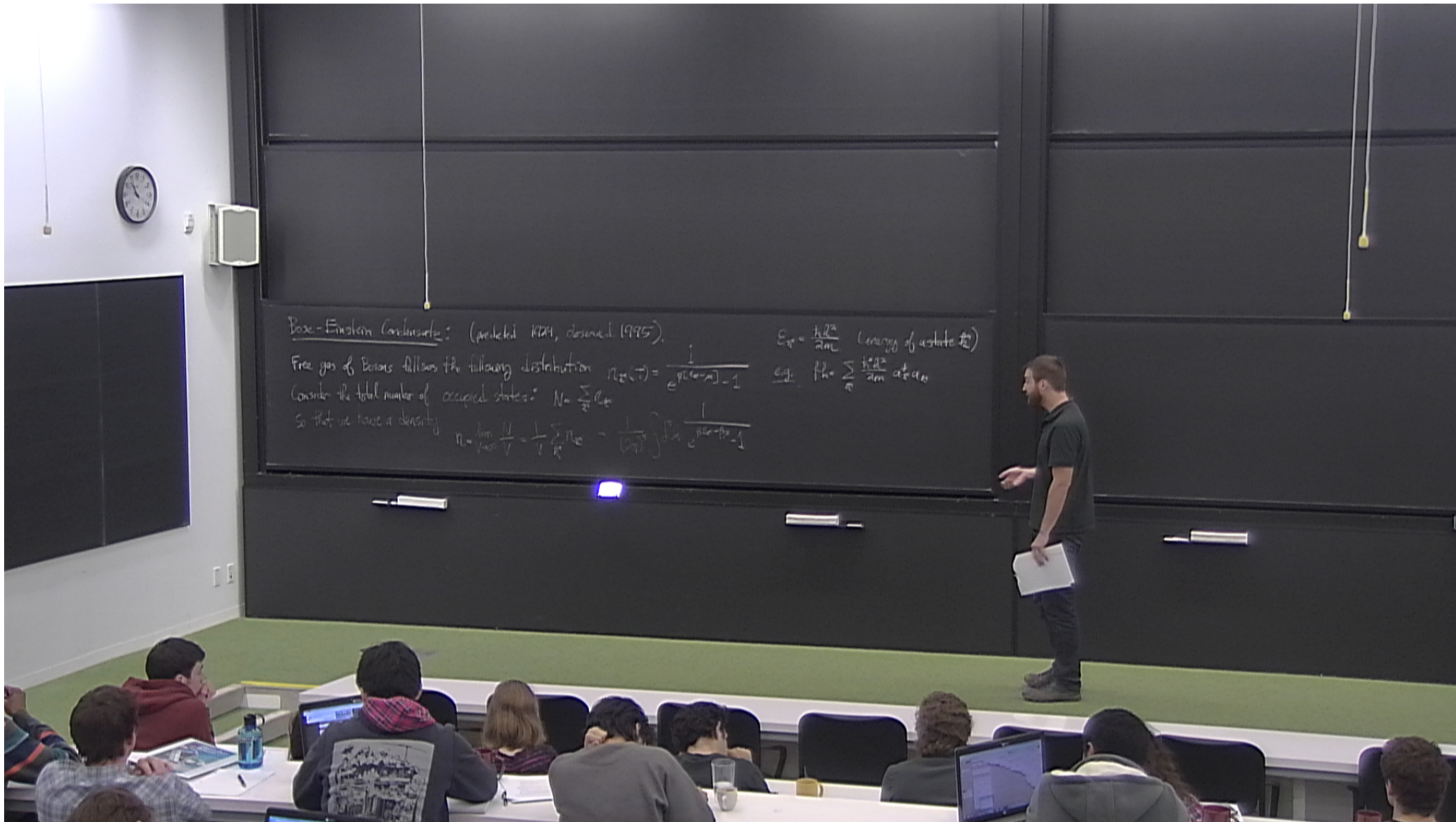
$$\epsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}$$

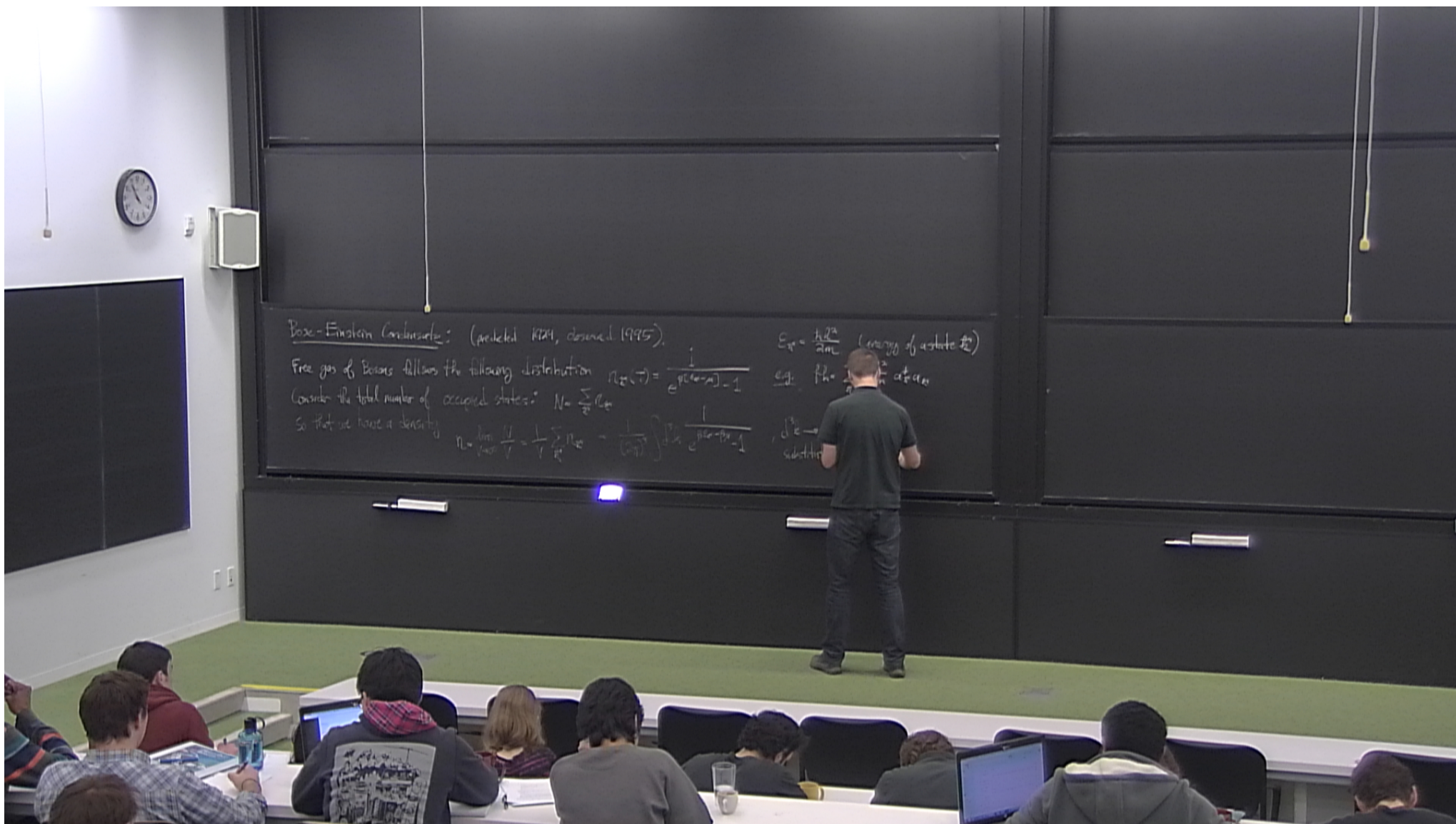
e.g. f_{ph}

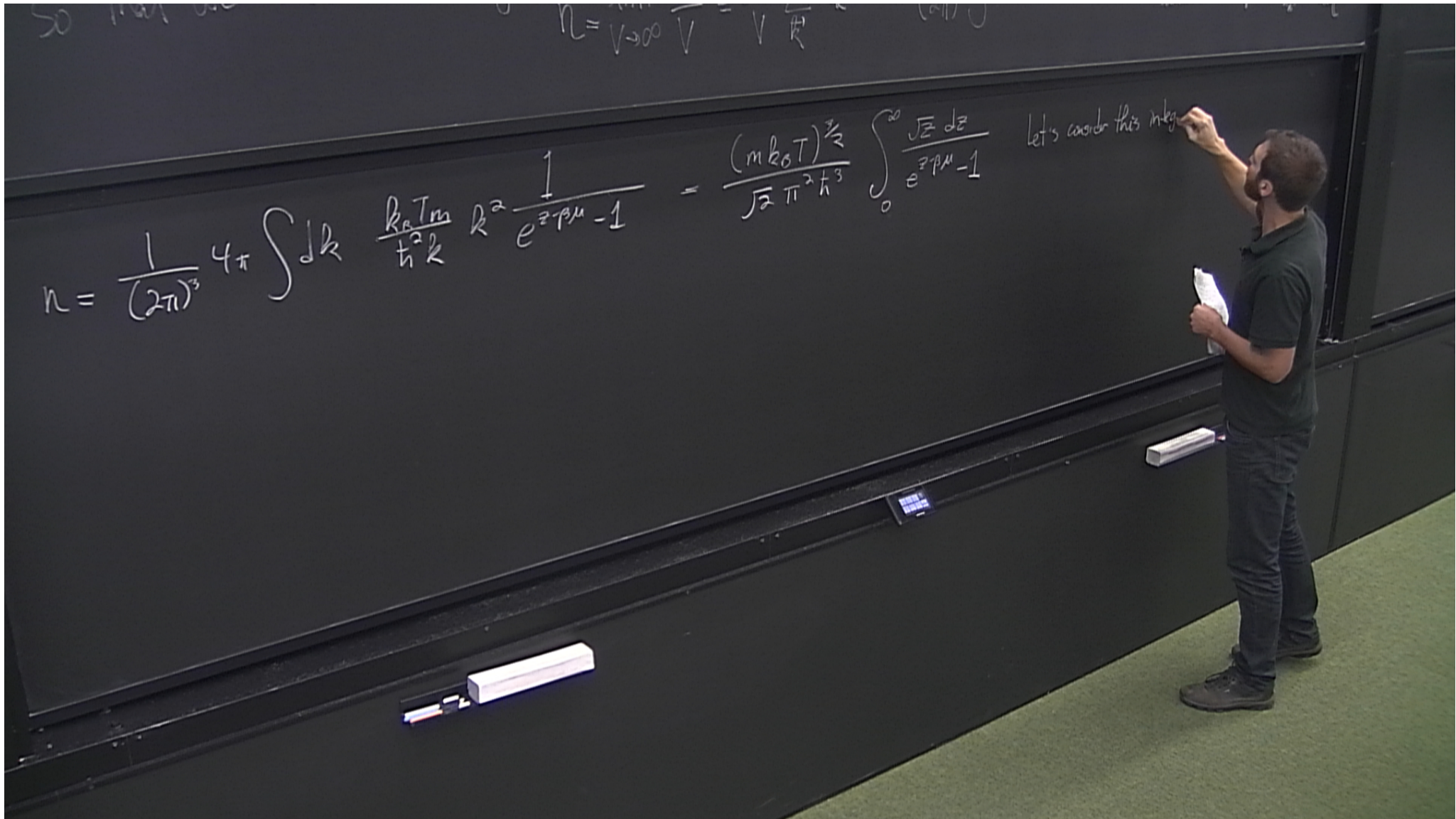
of occupied states: $N = \sum_{\vec{k}} n_{\vec{k}}$

density

$$n = \lim_{V \rightarrow \infty} \frac{N}{V} = \frac{1}{V} \sum_{\vec{k}} n_{\vec{k}} = \frac{1}{(2\pi)^3} \int d^3k$$







$$\frac{(m k_B T)^{3/2}}{\sqrt{2} \pi^2 \hbar^3}$$

$$\int_0^\infty \frac{\sqrt{z} dz}{e^{z\beta\mu} - 1}$$

let's consider this integral.



$$\frac{(m k_B T)^{3/2}}{\sqrt{2} \pi^2 \hbar^3}$$

$$\int_0^\infty \frac{\sqrt{z} dz}{e^{z-\beta\mu} - 1}$$

Let's consider this integral.
 $\Rightarrow$ constraint on μ : μ

$$n = \frac{1}{(2\pi)^3} 4\pi \int dk \frac{k_B T m}{\hbar^2 k} k^2 \frac{1}{e^{\beta \mu} - 1} = \frac{(m k_B T)^{3/2}}{\sqrt{2} \pi^2 \hbar^3}$$

To simplify this - set $\mu=0$. Then solve the integral
 This is - a given temperature there is a maximum
 of

$$n = \frac{1}{(2\pi)^3} 4\pi \int dk \frac{k_B T m}{\hbar^2 k} k^2 \frac{1}{e^{\frac{\hbar^2 k^2}{2m} - \mu} - 1} = \frac{(m k_B T)^{3/2}}{\sqrt{2} \pi^2 \hbar^3}$$

To simplify things - set $\mu=0$. Then solve the integral
 This implies that for a given temperature there is a maximum
 of particles that can be accommodated in excited level

$$N_{\max} = nV$$

$$\frac{(mk_B T)^{3/2}}{\sqrt{2} \pi^2 \hbar^3} \int_0^\infty \frac{\sqrt{z} dz}{e^{z-\beta\mu} - 1}$$

Let's consider this integral.
 $\Rightarrow$ constraint on μ : μ has to have an upper bound of zero (to keep N positive)

the integral: $n = 2.612$
 is a maximum number
 excited levels,

$$\frac{(mk_B T)^{3/2}}{(2\pi\hbar^2)^{3/2}}$$

$$\frac{3/2}{h^3} \int_0^\infty \frac{\sqrt{z} dz}{e^{z\beta\mu} - 1}$$

integral: $n = 2.612$
 maximum number
 levels.

Let's consider this integral.
 $\Rightarrow$ constraint on μ : μ has to have an upper bound
 of zero (positive)

$$\frac{(mk_B T)^{3/2}}{(2\pi\hbar^2)^{3/2}}$$

- for sufficiently high temperatures, N_{max}
- for T "too low"

$$\frac{3/2}{h^3} \int_0^\infty \frac{\sqrt{z} dz}{e^{z\beta\mu} - 1}$$

integral: $n = 2.612$
 maximum number
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Let's consider this integral.
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$$\frac{(mk_B T)^{3/2}}{(2\pi\hbar^2)^{3/2}}$$

- for sufficiently high temperatures, $N_{\max} > N$
- for T "too low", $N_{\max} < N$
- what happens to the rest of the par

Free gas of bosons follows the following distribution $n_{\vec{k},\sigma} = \frac{1}{e^{\beta(\epsilon_{\vec{k},\sigma} - \mu)} - 1}$ eq. $\rho = \sum_{\vec{k},\sigma} \frac{n_{\vec{k},\sigma}}{\Omega} \vec{k} \otimes \vec{k}$
 Consider the total number of occupied states: $N = \sum_{\vec{k},\sigma} n_{\vec{k},\sigma}$
 So let's have a density: $n = \frac{N}{V} = \frac{1}{V} \sum_{\vec{k},\sigma} n_{\vec{k},\sigma} = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta(\epsilon_{\vec{k}} - \mu)} - 1}$, $d^3k \rightarrow 4\pi k^2 dk$
 substitute $z = \beta\mu = \frac{1}{k_B T} \frac{\hbar^2 k^2}{2m}$

$$n = \frac{1}{(2\pi)^3} 4\pi \int dk \frac{k^2}{\hbar^2 k^2} \frac{1}{e^{\beta(\epsilon_{\vec{k}} - \mu)} - 1} = \frac{(m k_B T)^{3/2}}{\sqrt{\pi} \hbar^3} \int_0^\infty \frac{z^{3/2}}{e^z - 1} dz$$

Let's consider this integral as a constraint on μ : μ has to have an upper bound of zero (to keep N positive)

To simplify things - set $\mu=0$. Then solve the integral: $n = 2.612 \frac{(m k_B T)^{3/2}}{(2\pi\hbar)^3}$
 This implies that for a given temperature there is a maximum number of particles that can be accommodated in excited levels.
 $N_{\text{max}} = nV$
 - Consider now what happens if we fix the particle number N .
 - for sufficiently high temperatures, $N_{\text{max}} > N$
 - for T too low, $N_{\text{max}} < N$
 - what happens to the rest of the particles

Our integral implied that there are zero states at zero kinetic energy - this is wrong.
 Quantum mechanically, we know there is one state at zero k .

at zero kinetic energy - this is wrong.
state at zero k. Many Bosons can pile into this state.
 $\frac{N_0}{V}$ as $V \rightarrow \infty$ (N_0 is the number of particles in your ground state.

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Free gas of bosons follows the following distribution $n_i(T) = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$ e.g. $P_i = \sum_n \frac{1}{2\pi n} \alpha_i^2 \alpha_n$

Consider the total number of occupied states: $N = \sum_i n_i$

So that we have a denser $n_i = \frac{1}{V} \sum_i n_i = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$ $d^3k = 4\pi k^2 dk$ substitute $z = \beta\epsilon_k = \frac{1}{k_B T} \frac{\hbar^2 k^2}{2m}$

$n = \frac{1}{(2\pi)^3} 4\pi \int_0^\infty dk \frac{k^2}{e^{\beta(\epsilon_k - \mu)} - 1} = \frac{1}{\sqrt{2} \pi^2 \hbar^3} \int_0^\infty \frac{(\hbar k)^2}{e^{\beta(\epsilon_k - \mu)} - 1} dz$ Let's consider this integral as constraint on μ : μ has to have an upper bound of zero (to keep N positive)

To simplify things - set $\mu=0$. Then solve the integral: $n = \frac{2.612}{(2\pi\hbar)^3} \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2}$

This implies that for a given temperature there is a maximum number of particles that can be accommodated in excited levels.

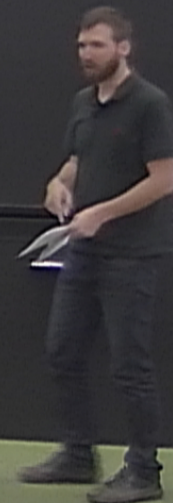
- for sufficiently high temperatures, $N_{\max} > N$
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$N_{\max} = nV$ - Consider now what happens if we fix the particle number N .

Our integral implied that there are zero states at zero kinetic energy - this is wrong.

Quantum mechanically, we know there is one state at zero k. Many Bosons can pile into this state

$n = n_0 + \frac{1}{V} \int d^3k \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$, $n_0 = \frac{N_0}{V} \approx V^{-1/3}$ (N_0 is the number of particles in your ground state.) (this is called the "condensate")



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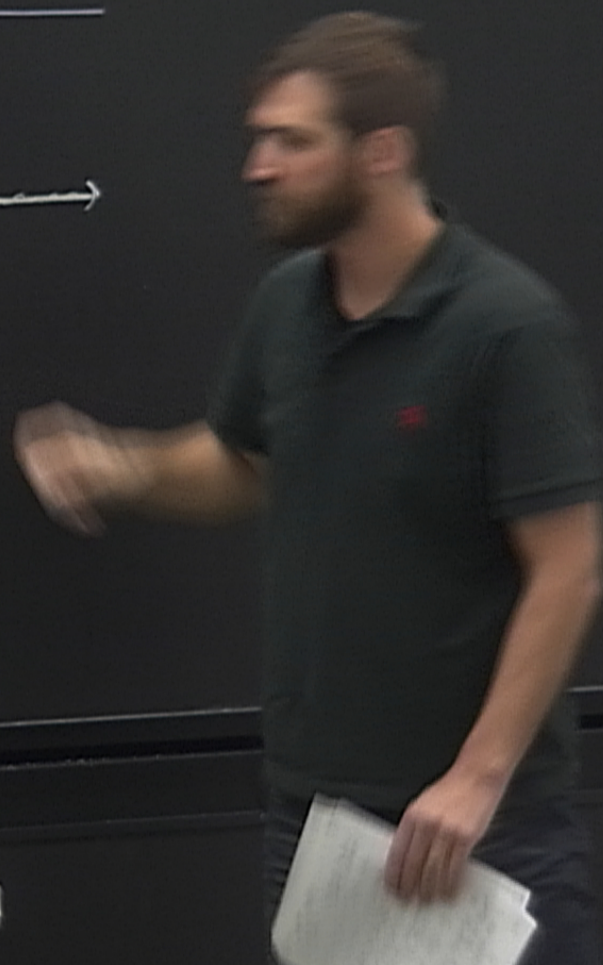
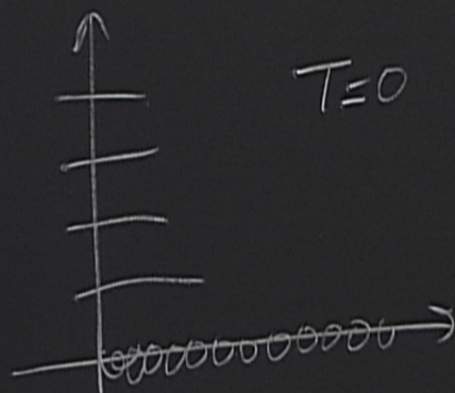
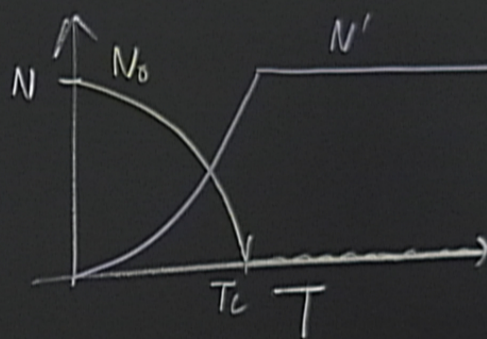
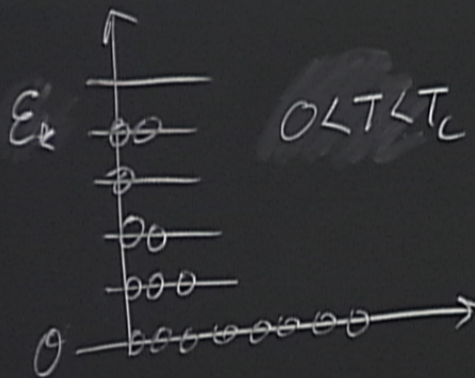
$$n = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta \hbar^2 k^2 / 2m} - 1}, \quad n_0 = \frac{N_0}{V} \text{ as } V \rightarrow \infty$$

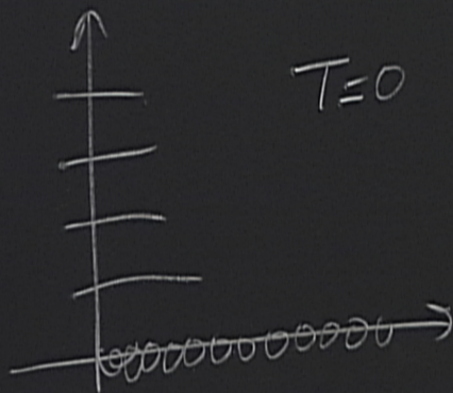
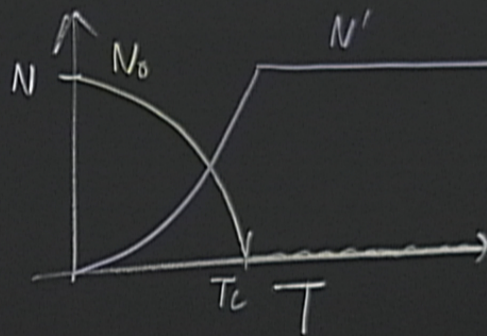
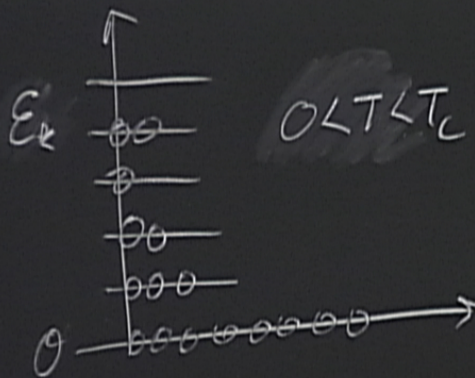
If we in

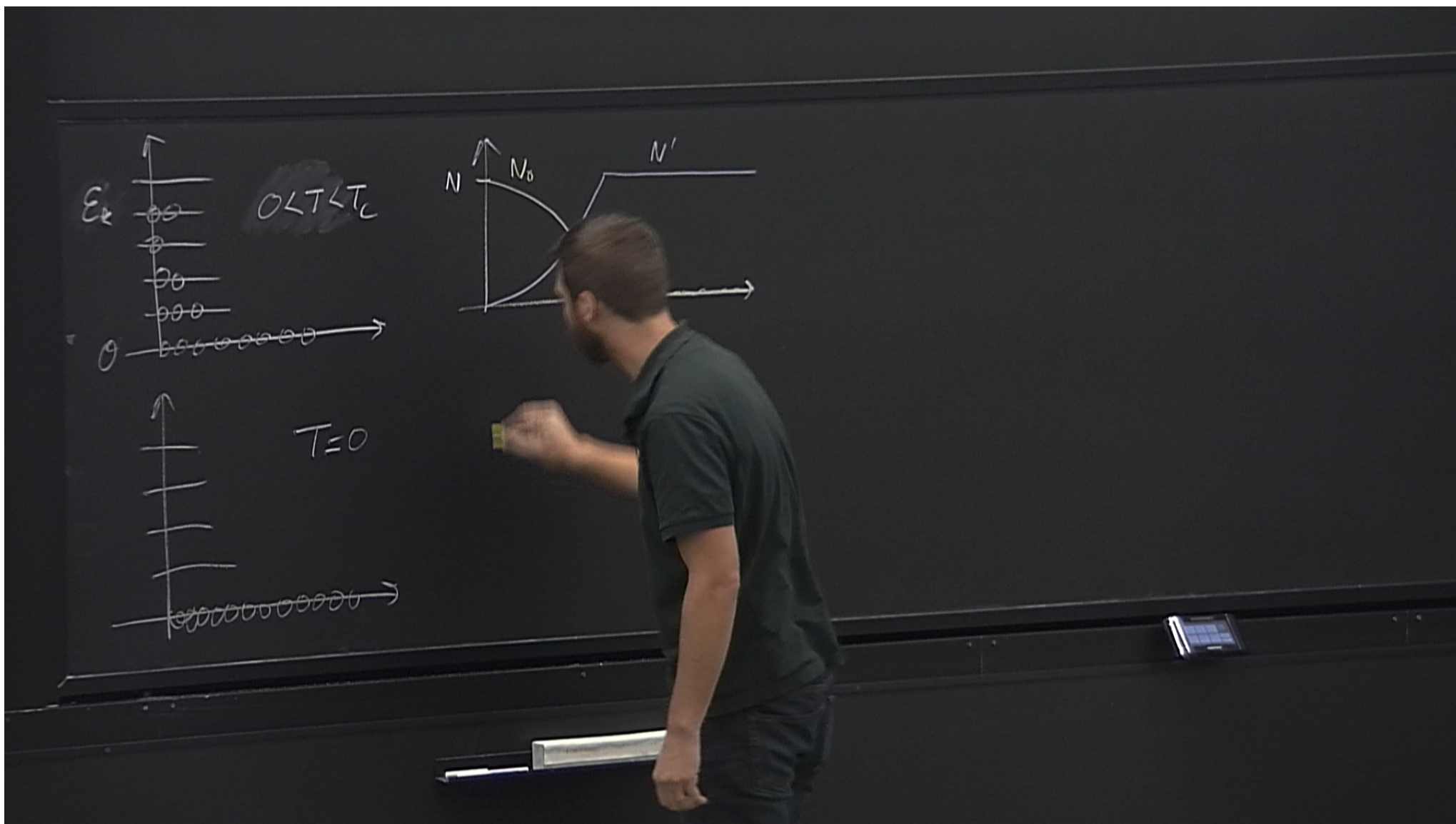
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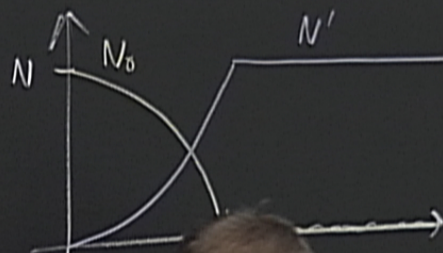
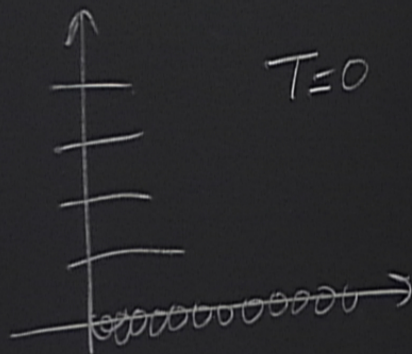
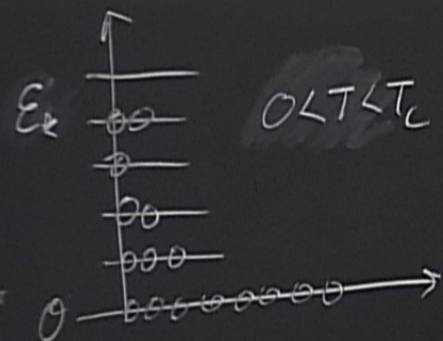
$$n = n_0 + \frac{1}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta \hbar^2 k^2 / 2m} - 1}, \quad n_0 = \frac{1}{V} \rightarrow V \rightarrow \infty$$

If we invert our formula for n , we can define the critical temperature for the appearance of the condensate









No particle
zero mo.
→ no pressure
viscosity

Free gas of Bosons follows the following distribution $n_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$ eq. $P_h = \sum_i \frac{n_i \epsilon_i}{\sum_i n_i}$
 Consider the total number of occupied states: $N = \sum_i n_i$
 In 3D we have a box V $n_i = \frac{1}{V} \sum_i n_i = \frac{1}{V} \int \frac{1}{e^{\beta(\epsilon - \mu)} - 1} dN$ $d^3k \rightarrow 4\pi k^2 dk$
 substitute $z = \beta\epsilon = \frac{h^2 k^2}{2m}$

$$n = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta(\epsilon - \mu)} - 1} = \frac{(m k_B T)^{3/2}}{\sqrt{\pi} \lambda^3} \int_0^\infty \frac{x^2}{e^x - 1} dx$$

Let's consider this integral as a function of μ : μ has to have an upper bound of zero (to keep N positive)

To simplify things - set $\mu = 0$. Then solve the integral: $n = 2.612 \frac{(m k_B T)^{3/2}}{(2\pi h)^3}$
 This implies that for a given temperature there is a maximum number of particles that can be accommodated in excited levels
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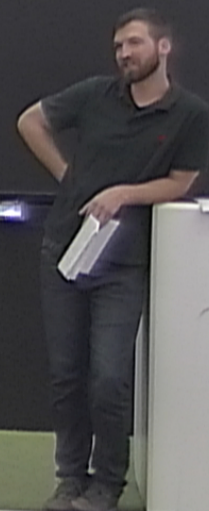
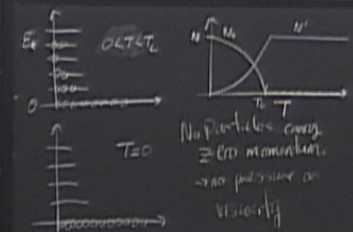
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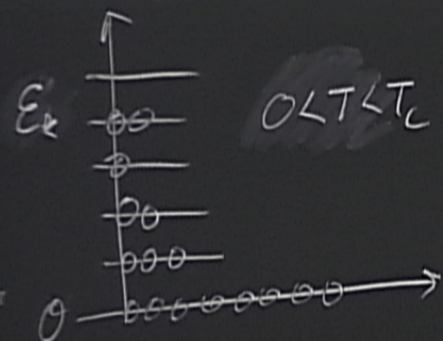
Quantum mechanically, we know there is one state at zero k. Many Bosons can pile into this state

$$n = n_0 + \frac{1}{V} \int \frac{1}{e^{\beta(\epsilon - \mu)} - 1} dN, \quad n_0 = \frac{N_0}{V} \approx V \rightarrow \infty \quad (N_0 \text{ is the number of particles in your ground state})$$

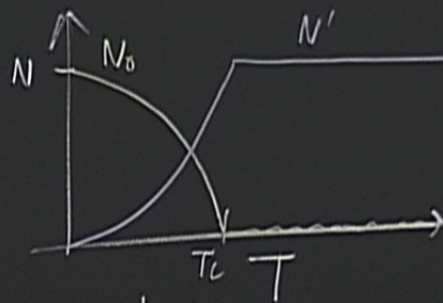
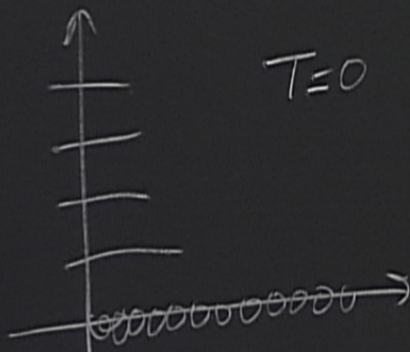
this is called the "condensate"

If we insert our formula for n , we can define the critical temperature for the appearance of the condensate
 $T_c = \frac{2\pi h^2}{m k_B} \left(\frac{N/V}{2.612} \right)^{2/3}$ below T_c particles start moving into the lowest energy level.
 occupation $N' = N \left(\frac{T}{T_c} \right)^{3/2}$ (excited)
 $N_0 = N \left(1 - \left(\frac{T}{T_c} \right)^{3/2} \right)$



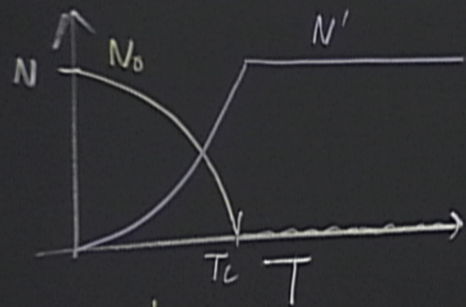


$T=0$



No particles carry zero momentum.
 $\rightarrow$ no pressure or viscosity

Let's write T_c in terms of the de Broglie



No particles carry
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→ no pressure or
viscosity

Let's write T_c in terms
of the de Broglie λ



$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} - 1} \quad \text{eg. } \rho(\epsilon) = \sum \frac{1}{2m} \text{ at } \epsilon_0$$

$$N = \sum n(\epsilon)$$

$$N = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

$$\epsilon_k = \frac{\hbar^2 k^2}{2m}$$

$$\text{substitute } z = e^{\beta(\mu - \epsilon_k)} = \frac{1}{e^{\beta(\epsilon_k - \mu)}} = \frac{\hbar^2 k^2}{2m}$$

Let's consider this integral
 as constraint on μ : μ has to have an upper bound
 of zero (to keep N positive)

the integral: $N = 2.612 \frac{(mk_B T)^{3/2}}{(2\pi\hbar)^3}$

is a maximum number
 of particles

happens if
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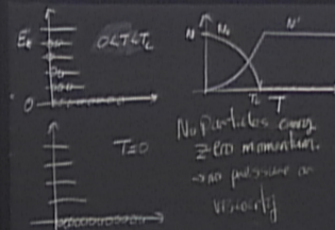
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 state at zero k. Many Bosons can pile into this state
 $n_0 = \frac{N_0}{V} \approx V \rightarrow \infty$ (this is the number of particles in your ground state)
 this is called the "condensate"

$T_c = \frac{2\pi\hbar^2}{mk_B} \left(\frac{N/V}{2.612} \right)^{2/3}$

below T_c particles
 start moving into
 the lowest energy level.

occupation
 $N' = N \left(\frac{T}{T_c} \right)^{3/2}$ (excited)
 $N_0 = N \left(1 - \left(\frac{T}{T_c} \right)^{3/2} \right)$

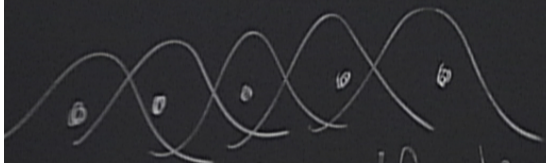


Let's write T_c in terms
 of the de Broglie wavelength
 $k_B T_c = \frac{\hbar^2}{2m} \Rightarrow \lambda_D = 2.44 n^{-1/3}$

wavelength close to the interparticle spacing.

Let's write T_c in terms
of the de Broglie wavelength

$$k_B T = \frac{p^2}{2m} \Rightarrow \lambda_D = 2.44 n^{-1/3}$$



wavelength close to the interparticle spacing.

Note: the creation/annihilation operators

$$\text{obey } [a_0, a_0^\dagger] = 1$$

$$\langle a_0^\dagger a_0 \rangle = N_0$$

$$\Rightarrow a_0^\dagger = \sqrt{N_0} \quad a_0 = \sqrt{N_0}$$

in terms
of the wavelength

$$\lambda_D = 2.44 n^{-1/3}$$

the interparticle spacing.

Note: the creation/annihilation operators

obey $[a, a^\dagger] = \frac{1}{\sqrt{V}}$

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$$\langle a_0^\dagger a_0 \rangle = N_0$$

$$\Rightarrow a_0^\dagger = \sqrt{N_0} e^{-i\theta} \quad a_0 = \sqrt{N_0} e^{i\theta}$$

in terms
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Note: the creation/annihilation operators

obey $[a_0, a_0^\dagger] = \frac{1}{\Omega}$ $\rightarrow$ obey

$$\langle a_0^\dagger a_0 \rangle = N_0$$

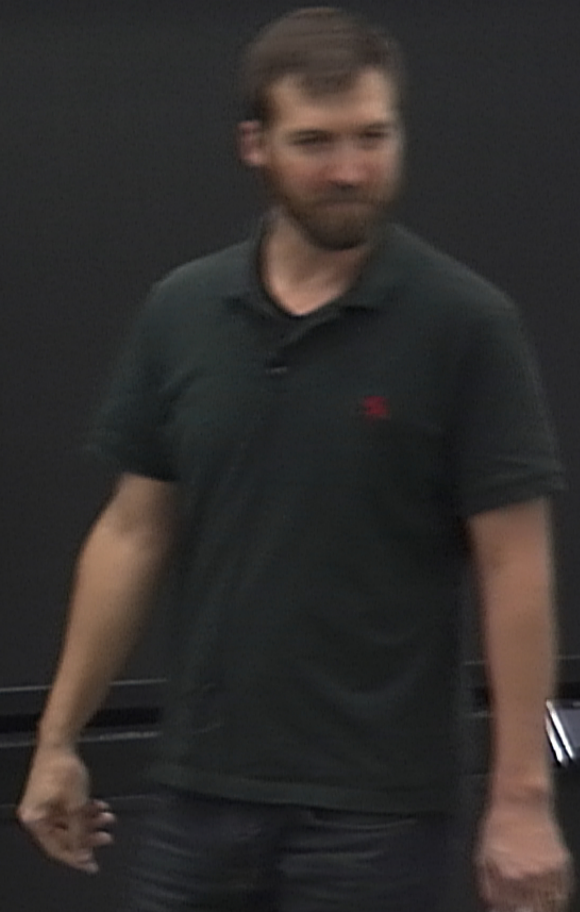
$$\Rightarrow \langle a_0^\dagger \rangle = \sqrt{N_0} e^{-i\theta} \quad \langle a_0 \rangle = \sqrt{N_0} e^{i\theta}$$

$$\langle a_0^\dagger \rangle = \sqrt{N_0}$$



Magnons and Spin Waves

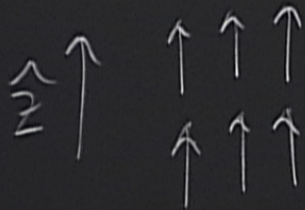
Imagine a magnetic system
with broken time-reversal symmetry.

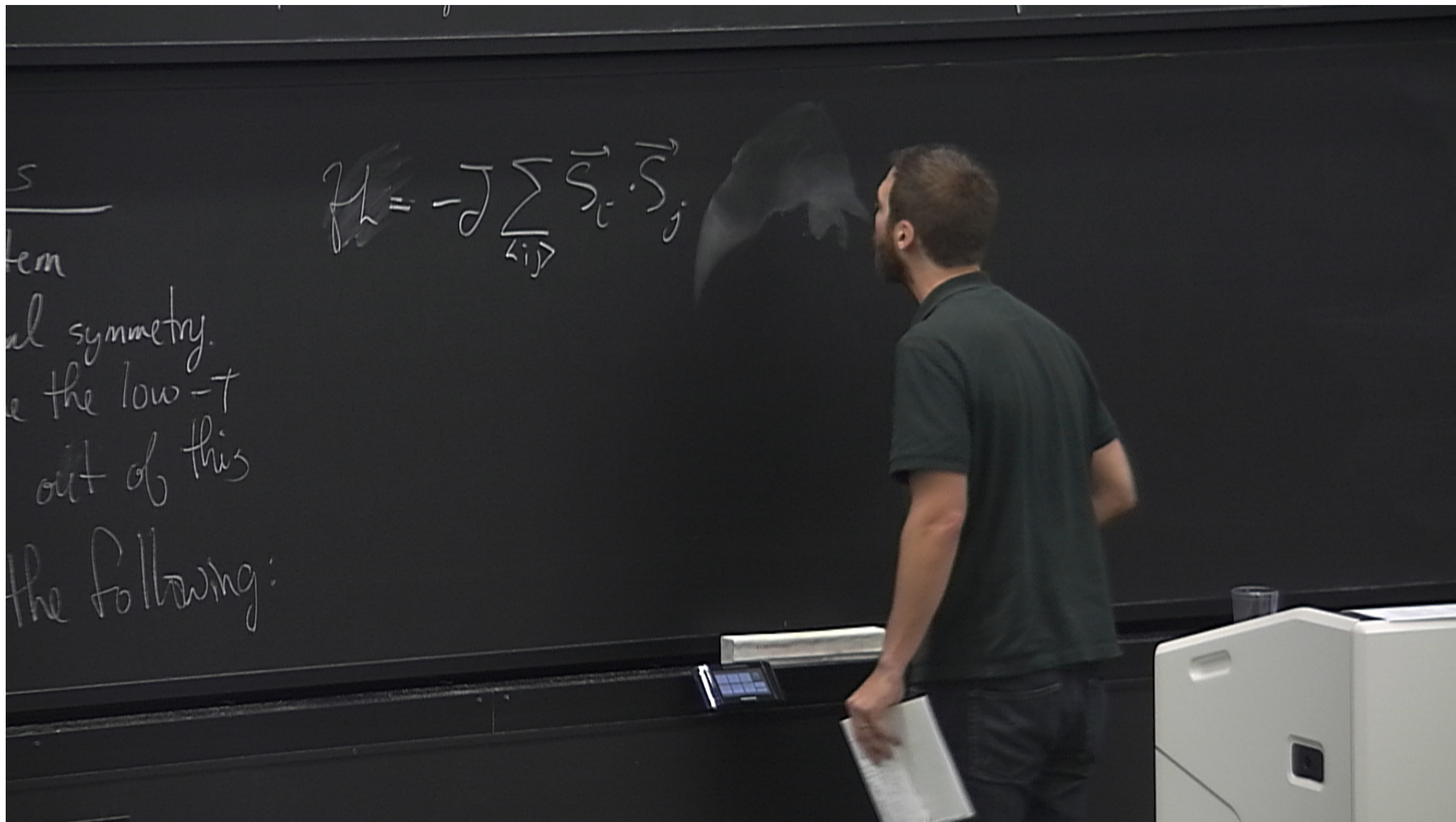


Magnons and Spin Waves

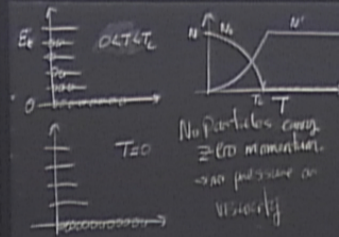
Imagine a magnetic system
with broken time-reversal symmetry.

Let's examine the low- T
excitations out of this





$\rho(\mathbf{r}, \mathbf{r}') = e^{\beta(\mu - \epsilon(\mathbf{r}, \mathbf{r}'))} - 1$ eq. $\rho = \sum_{\mathbf{p}} \frac{1}{2m} \mathbf{p}^2 \rho$
 $\rho = \sum_{\mathbf{p}} \frac{1}{2m} \mathbf{p}^2 \rho$
 $\rho = \frac{1}{(2\pi)^3} \int d^3p \frac{1}{\beta \epsilon(\mathbf{p}) - 1}$ $\int d^3p \rightarrow 4\pi p^2 dp$
 substitute $z = \beta \mu = \frac{1}{k_B T} \frac{\hbar^2 k^2}{2m}$
 $\rho = \frac{1}{(2\pi)^3} \int d^3p \frac{1}{\beta \epsilon(\mathbf{p}) - 1}$
 $\rho = \frac{1}{(2\pi)^3} \int d^3p \frac{1}{\beta \epsilon(\mathbf{p}) - 1}$
 Let's consider this integral
 as constraint on μ : μ has to have an upper bound
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 is a maximum number
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 state at zero k. Many Bosons can pile into this state
 $n = \frac{N_0}{V} \approx N_0/V \rightarrow \infty$ (this is the number of particles in your ground state)
 this is called the "condensate"
 $T_c = \frac{2\pi\hbar^2}{mk_B} \left(\frac{N/V}{2.612} \right)^{2/3}$ below T_c particles
 start moving into
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 occupation
 $N' = N - (N_0/V)^{3/2}$ (excited)
 $N_0 = N(1 - (T/T_c)^3)$



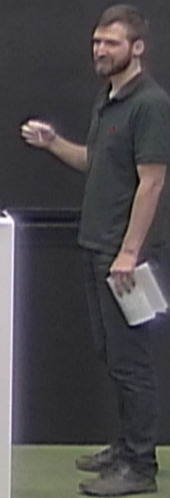
Let's write T_c in terms
 of the de Broglie wavelength
 $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$
 $\lambda_D = 2.44 n^{-1/3}$
 wavelength close to the interparticle spacing.

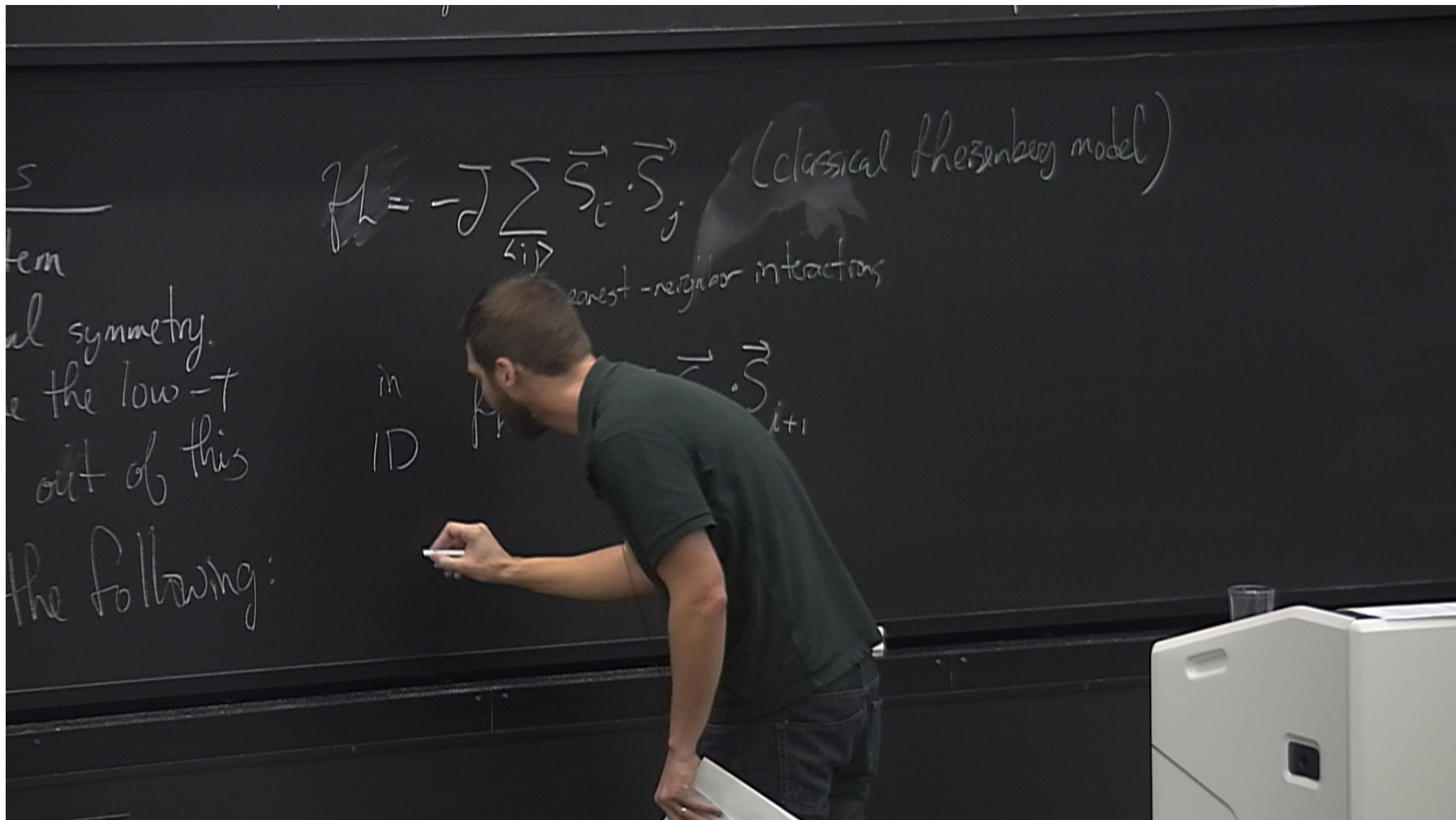
Note: the creation/annihilation operators
 obey $[a_i, a_j^\dagger] = \delta_{ij}$ (bosons)
 $[a_i, a_j] = 0$
 $\langle a_i^\dagger \rangle = \sqrt{N_0} e^{i\theta}$
 $\langle a_i \rangle = \sqrt{N_0} e^{-i\theta}$
 $\Psi = 1/4 e^{i\theta} \sim e^{i\theta N}$

Magnons and Spin Waves

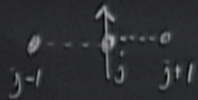
Imagine a magnetic system
 with broken time-reversal symmetry
 $\uparrow \uparrow \uparrow$ Let's examine the low-T
 $\uparrow \uparrow \uparrow$ excitations out of this
 state under the following:

$H = -J \sum_{i,j} \vec{S}_i \cdot \vec{S}_j$ (classical Heisenberg model)
 in 1D $H = -J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$





classical Heisenberg model)



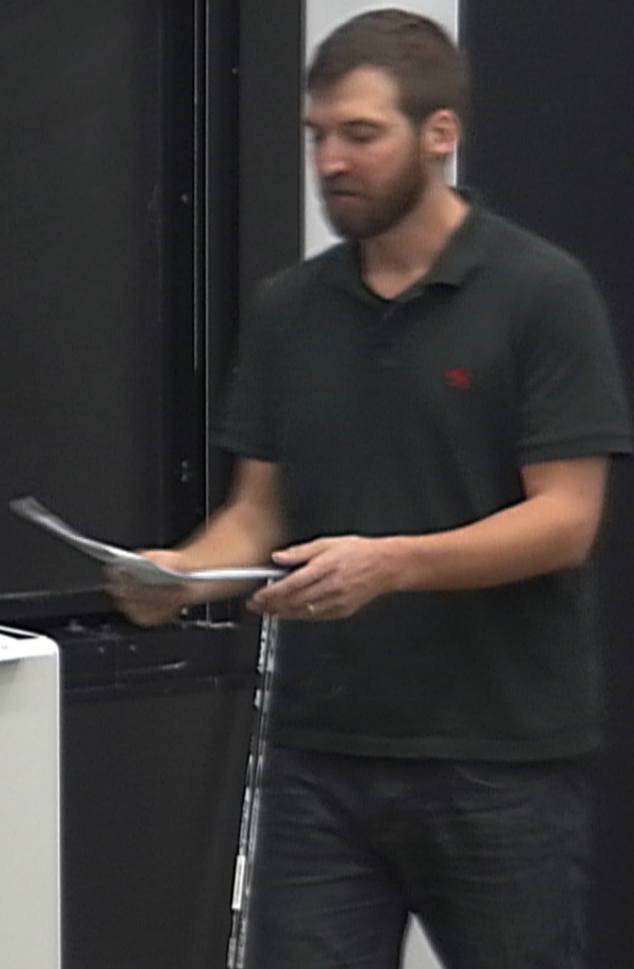
interactions

$$-\vec{S}_j \cdot (\vec{S}_{j-1} + \vec{S}_{j+1}) = -\vec{S}_j \cdot \vec{H}_j$$

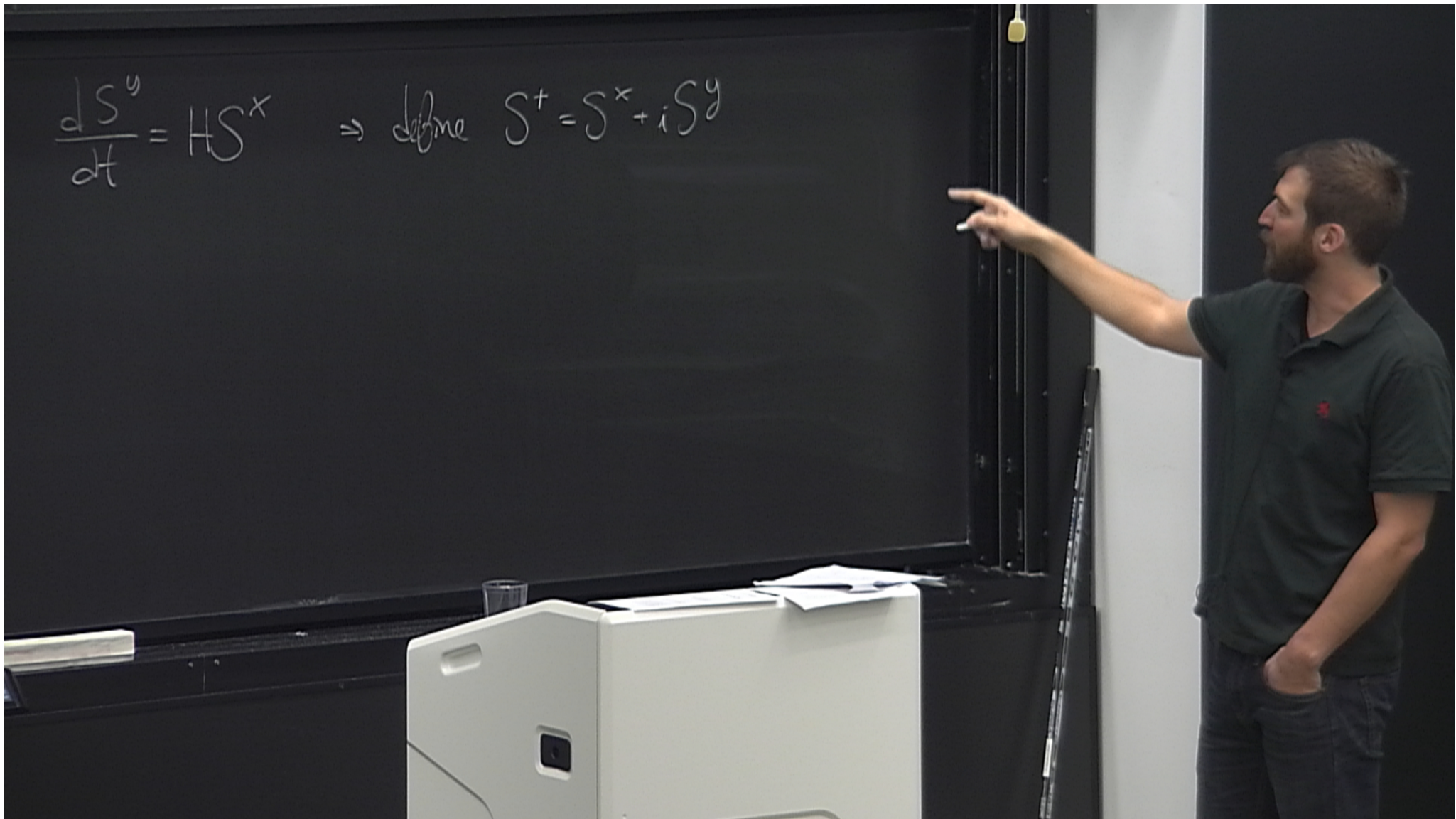
The classical equations of motion are.

$$\frac{d\vec{S}}{dt} = \vec{S} \times \vec{H}$$

field:

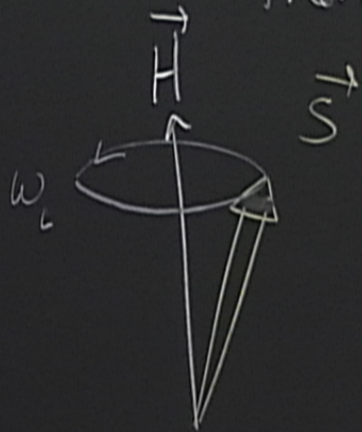


$$\frac{dS^y}{dt} = HS^x \Rightarrow \text{define } S^+ = S^x + iS^y$$



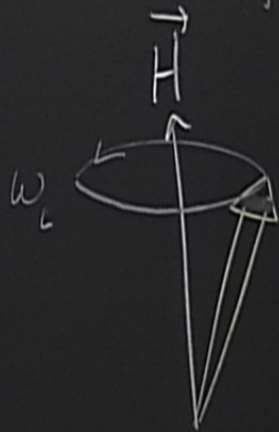
For example: $\vec{H}$ in the $\hat{z}$ direction, $\frac{dS^z}{dt}=0$, $\frac{dS^+}{dt}=$

then $\frac{dS^+}{dt} = i\omega_L S^+$, $S^+(t) = S^+(0)e^{i\omega_L t}$



For example: $\vec{H}$ in the $\hat{z}$ direction, $\frac{dS^z}{dt}=0$, $\frac{dS^{\pm}}{dt}=$

then $\frac{dS^{\pm}}{dt} = i\omega_L S^{\pm}$, $S^{\pm}(t) = S^{\pm}(0)e^{i\omega_L t}$



$$N_{\max} = nV$$

- Consider now what happens to the free particle number

$$\frac{dS^z}{dt} = 0$$

$$\frac{dS^x}{dt} = -\omega_0 \left(S^y + S_{j+1}^y - 2S_j^y \right)$$

$$\frac{dS^y}{dt} = \omega_0 \left(S^x + S_{j+1}^x - 2S_j^x \right)$$

define $S_j^+ = S_j^x + iS_j^y \Rightarrow$

$$\frac{dS_j^+}{dt} = -i\omega_0 \left(S_{j+1}^+ + S_{j-1}^+ - 2S_j^+ \right)$$

$$S_j^+(t) = S_j^+(0) e^{-i\omega_0 t}$$

- Consider now what happens to the fix the particle number N .

- what happens

$$\ddot{x} = -\omega_0 (S_{j-1}^y + S_{j+1}^y - 2S_j^y) \quad \frac{dS_j^y}{dt} = \omega_0 (S_{j-1}^x + S_{j+1}^x - 2S_j^x)$$

$$\Rightarrow \frac{dS_j^{\pm}}{dt} = i\omega_0 (S_{j+1}^{\pm} + S_{j-1}^{\pm} - 2S_j^{\pm}) \quad \text{very similar to lattice vibrat}$$

$$e^{i[kx_j - \omega(k)t]} \quad \begin{array}{c} \nearrow \\ \text{substitute} \\ \text{into this} \end{array} \Rightarrow$$

- Consider now what happens to the fix the particle number N .

- what happens

$$\ddot{x} = -\omega_0 (S_{j-1}^y + S_{j+1}^y - 2S_j^y), \quad \frac{dS_j^y}{dt} = \omega_0 (S_j^x + S_{j+1}^x - 2S_j^x)$$

$$\Rightarrow \frac{dS_j^+}{dt} = i\omega_0 (S_{j+1}^+ + S_{j-1}^+ - 2S_j^+) \quad \text{very similar to the vibrat}$$

$$e^{i[kx_j - \omega(k)t]} \xrightarrow{\text{substitute into this}} \Rightarrow \omega(k) = 2\omega_0$$

Quantum mechanically: Spins are represented by operators.

Introduce a bosonic operator $\hat{n}_i = S_i^z - n_i$, $S_i^+ = \sqrt{2n_i} \hat{a}_i$, $S_i^- = \sqrt{2(n_i+1)} \hat{a}_i^\dagger$

Consider a lattice of N
we expect a running
of spin "deviation" -
non

normal mode) are on y .

and to linear approximation

$$\hat{S}_j^+ = \sqrt{\frac{2s}{N}} \sum_{\vec{k}} e^{-i\vec{k} \cdot \vec{r}_j} \hat{a}_{\vec{k}}$$

$$\hat{S}_j^- = \sqrt{\frac{2s}{N}} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{r}_j} \hat{a}_{\vec{k}}^{\dagger}$$

Consider the Heisenberg FM: $H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$

substitute in these operators.

$$H = -JNzS^2 + 2J \sum_{\langle i,j \rangle} (1 - \gamma_k) \hat{a}_i^\dagger \hat{a}_j$$

$$\text{and } \gamma_k = \frac{1}{z} \sum_{\delta} e^{i\vec{k} \cdot \vec{\delta}}$$

