

Title: 13/14 PSI - Quantum Field Theory II - Lecture 11

Date: Nov 25, 2013 09:00 AM

URL: <http://pirsa.org/13110022>

Abstract:

Dirac Equation: $h^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$ $d=4$

Diracology: γ^μ matrices $\mu=1 \dots 4$.
 $[2^d] \times [2^d]$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 \cdot h^{\mu\nu} \quad \text{anticommutator}$$

wave function $\Psi = (\Psi_a; a=1, [2^d])$

Dirac Equation: $h^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$ $d=4$ Dirac Action

$$S[\bar{\Psi}, \Psi] = \int d^4x \bar{\Psi}(x) \cdot (i\not{\partial} - m) \Psi(x)$$

Diracology: γ^μ matrices $\mu=1 \dots 4$ $n = \lfloor 2^{\frac{d}{2}} \rfloor$
 $\lfloor 2^{\frac{d}{2}} \rfloor \times \lfloor 2^{\frac{d}{2}} \rfloor$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 \cdot h^{\mu\nu} \text{ anticommutator}$$

wave function $\Psi = (\Psi_a; a=1, \lfloor 2^{\frac{d}{2}} \rfloor)$

$$(i\not{\partial}\Psi - m)\Psi = 0 \quad \not{\partial} = \gamma^\mu \partial_\mu$$

the Action

$$[\bar{\Psi}, \Psi] = \int d^4x \bar{\Psi}(x) \cdot (i\not{\partial} - m) \Psi(x)$$
$$= (i\not{\partial} - m) \Psi(x)$$

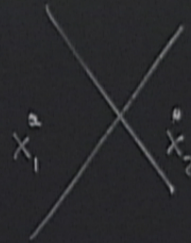
not use functional

QFT I: 2nd quantization.

$\Psi(x), \Psi^\dagger(x)$ becomes field operators

$$\{\Psi(x_1), \Psi(x_2)\} = 0 \quad \text{if } x_1, x_2 \text{ spacelike separated}$$

anticommute $\Leftrightarrow$ causality + locality



functional integration $\Leftarrow$ Grassmann variables

$$\dim(G_N) = 2^{2N}$$

Addition ∂_2

$\mathbb{G}_N \cong \Lambda_N$ Algebra over $\mathbb{C}$ (or $\mathbb{R}, \dots$)
generated by $2N$ anticommuting generators

multiplication: $\theta_i \cdot \theta_j + \theta_j \cdot \theta_i = 0$, same for $\bar{\theta}, \bar{\theta}$, $\bar{\theta}, \bar{\theta}$

anticommutation $\theta_i \bar{\theta}_j + \bar{\theta}_j \theta_i = 0$, etc. - -

nilpotent $\theta_i^2 = 0$ $\bar{\theta}_i^2 = 0$

$$+ C_{i_1 \bar{j}_1} \theta_{i_1} \bar{\theta}_{j_1} + \dots = \sum_{(i_1, \bar{j}_1), \dots, (i_k, \bar{j}_k)} C_{i_1 \bar{j}_1, \dots, i_k \bar{j}_k} \theta_{i_1} \dots \theta_{i_k} \bar{\theta}_{j_1} \dots \bar{\theta}_{j_k}$$

$$i_1 < i_2 < \dots < i_k, \bar{j}_1 < \bar{j}_2 < \dots < \bar{j}_k$$

variables

$$\dim(G_N) = 2^{2N}$$

Addition $g_1 + g_2$

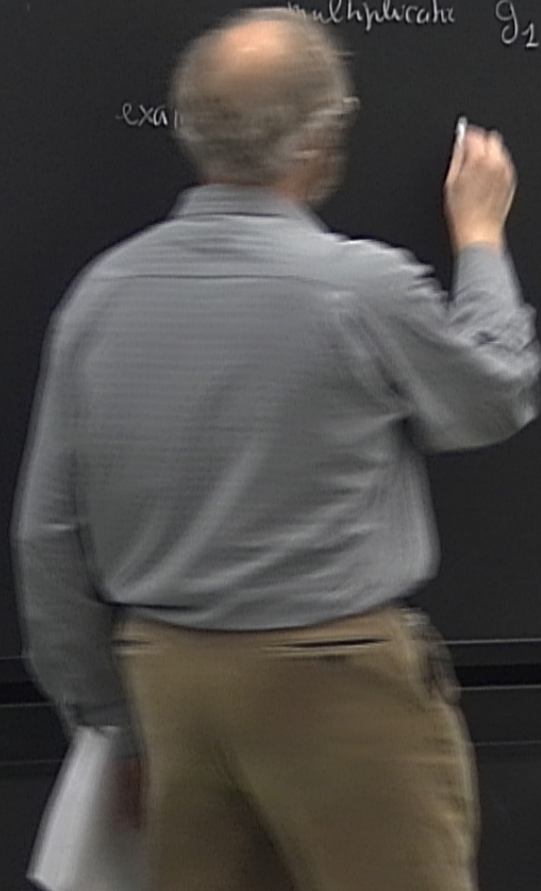
multiplication $g_1 \cdot g_2$

use the anticommutation rule

generators

for $\theta, \bar{\theta}, \bar{\theta}, \theta$

exa



variables

$$\dim(G_N) = 2^{2N}$$

Addition $g_1 + g_2$

multiplication $g_1 \cdot g_2$

use the anticommutation rule

generators

for θ

example: $N=1$

$$g_1 = a_1 + b_1 \theta + c_1 \bar{\theta} + d_1 \theta \bar{\theta}$$

$$g_2 = a_2 + b_2 \theta + c_2 \bar{\theta} + d_2 \theta \bar{\theta}$$

$$g_1 g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1) \theta + a_1 (c_2 + a_2 c_1) \bar{\theta} +$$

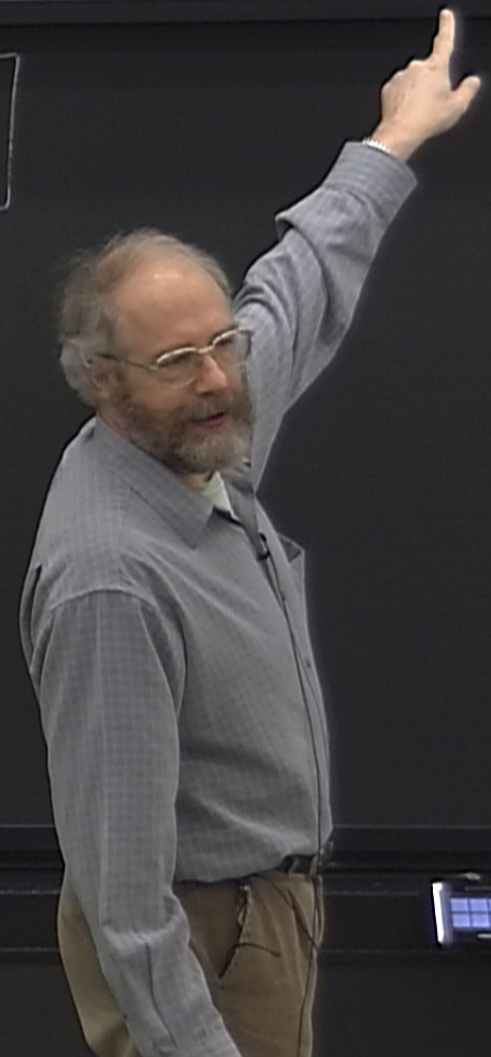
$$\mathbb{Z}_2 \text{ graded algebra: super-algebra } (a_2 d_1 + a_1 d_2 + b_1 c_2 - b_2 c_1) \theta \bar{\theta}$$

$$G = G_{\text{even}} \oplus G_{\text{odd}}$$

Integration: $g \rightarrow \int d\theta_i g$?

take $\int d\theta_i := \frac{\partial}{\partial \theta_i}$

$$\int d\theta_i \frac{\partial}{\partial \theta_i} g = 0$$



$$g = 0$$

Berezin integrals over Gaussians &

$$G_N \quad N \theta_i, \quad N \bar{\theta}_i$$

$$\theta \in G_N \quad e = \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

Fermionic version of the Wick Theorem

$$A_{ij} \text{ some } N \times N \text{ matrix } A = A^\dagger$$

$$\exp(g) = 1 + g + \frac{1}{2} g^2 + \dots$$

Berezin integrals over Gaussians & Fermionic version of the Wick Theorem

$$G_N \quad N \theta_i, \quad N \bar{\theta}_i$$

$$e \in G_N \quad e = \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

$$A_{ij} \text{ some } N \times N \text{ matrix } A = A^\dagger \quad g = \bar{\theta}_i A_{ij} \theta_j$$

$$\exp(g) = 1 + g + \frac{1}{2} g^2 + \dots \quad \text{ends at order } N$$

Berezin integrals over Gaussians &

Fermionic version of the Wick Theorem

$$G_N \quad N \theta_i, \quad N \bar{\theta}_i$$

$$A_{ij} \text{ some } N \times N \text{ matrix } A = A^\dagger \quad g = \bar{\theta}_i A_{ij} \theta_j$$

$$e \in G_N \quad e = \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

$$\exp(g) = 1 + g + \frac{1}{2} g^2 + \dots$$

ends at order N order

$= 0$

Berezin integrals over Gaussians & Fermionic version of the Wick Theorem

G_N $N \theta_i, N \bar{\theta}_i$

$$e = \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

ex

$$1 + A \theta \bar{\theta}$$

A_{ij} some $N \times N$ matrix $A = A^\dagger$ $g = \bar{\theta}_i A_{ij} \theta_j$

$$\exp(g) = 1 + g + \frac{1}{2} g^2 + \dots$$

ends at order N

order $M >$

Integrals over Gaussians & Fermionic version of the Wick Theorem

$$\theta_i, N \bar{\theta}_i$$

A_{ij} some $N \times N$ matrix $A = A^\dagger$

$$g = \bar{\theta}_i A_{ij} \theta_j$$

complex matrix

$$e = \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

$$\exp(g) = 1 + g + \frac{1}{2} g^2 + \dots$$

ends at order N

order $M > N$ necessarily θ_i^2

$N=1$

$$e^{A \bar{\theta} \theta} = 1 + A \bar{\theta} \theta$$

$$\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2$$

$$e^{A_{ij} \bar{\theta}_i \theta_j} = 1 + \dots (A_{12} A_{21} - A_{11} A_{22}) \bar{\theta}_1 \theta_2 \bar{\theta}_1 \theta_2$$

Gaussian integration

$$\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i = \int d\bar{\theta}_1 d\theta_1 \int d\bar{\theta}_2 d\theta_2 \dots \exp(-\bar{\theta}_i A_{ij} \theta_j) = \text{term of highest degrees with signs}$$

$$\Delta_{\text{ordering}} = \det(A)$$

erezin integrals over Gaussians &

Fermionic version of the Wick Theorem

$$\theta_i, \bar{\theta}_i \in G_N$$

$$e = \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

$$A_{ij} \text{ some } N \times N \text{ matrix } A = A^\dagger \quad g = \bar{\theta}_i A_{ij} \theta_j$$

complex matrix

$$\exp(g) = 1 + g + \frac{1}{2} g^2 + \dots$$

ends at order N

order M > N necessarily $\theta_i^2 = 0$

example N=1

$$\exp(-\bar{\theta} A \theta) = 1 + A \bar{\theta} \theta$$

$$\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2$$

$$1 + (A_{12} A_{21} - A_{11} A_{22}) \theta_1 \theta_2 \bar{\theta}_1 \bar{\theta}_2$$

$$\left(\theta_1 A_{11} \theta_1 + \bar{\theta}_1 A_{21} \theta_1 + \bar{\theta}_2 A_{22} \theta_2 \right)^2$$

Gaussian integrals

$$\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i = \int d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 \dots \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

$$\Delta_{\text{ordering}} = \det(A)$$

term of highest degrees with signs

Berezin integrals over Gaussians &

Fermionic version of the Wick Theorem

$$G_N \quad N \theta_i, \quad N \bar{\theta}_i$$

$$A_{ij} \text{ some } N \times N \text{ matrix } A = A^\dagger \quad g = \bar{\theta}_i A_{ij} \theta_j \quad \text{complex}$$

$$e \in G_N \quad e = \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

$$\exp(g) = 1 + g + \frac{1}{2} g^2 + \dots$$

example $N=1$

$$\exp(-\bar{\theta} A \theta) = 1 + A \bar{\theta} \theta$$

$$N=2 \quad \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2$$

$$\exp(-\bar{\theta}_i A_{ij} \theta_j) = 1 + \dots + (A_{12} A_{21} - A_{11} A_{22}) \bar{\theta}_1 \theta_2 \bar{\theta}_1 \bar{\theta}_2$$

$$\frac{1}{2} \left(\bar{\theta}_1 A_{11} \bar{\theta}_1 + \bar{\theta}_1 A_{12} \bar{\theta}_2 + \bar{\theta}_2 A_{21} \bar{\theta}_1 + \bar{\theta}_2 A_{22} \bar{\theta}_2 \right)^2$$

12

ends at order N

order $M > N$ necessary

Gaussian integrals over exponentials in complex Grassmann Algebra

$$\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i = \int d\bar{\theta}_1 d\theta_1 \int d\bar{\theta}_2 d\theta_2 \dots \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

term of highest order with signs

$$\Delta_{\text{ordering}} = \det(A)$$

Integration: $g \rightarrow \int d\theta_i g$?

$$\int d\theta_i \frac{\partial}{\partial \theta_i} g = 0$$

take $\int d\theta_i := \frac{\partial}{\partial \theta_i}$

$$\int d\bar{\theta}_i := \frac{\partial}{\partial \bar{\theta}_i}$$

$$\int d\theta_i 1 = 0$$

$$\int d\theta_i \theta_i = 1$$

$$\int \prod_i d z_i d \bar{z}_i \exp(-\bar{z}_i A_{ij} z_j) = * \frac{1}{(\det A)}$$

Berezin integrals over Grass

$$G_N \quad N \theta_i, N \bar{\theta}_i$$

$$e \in G_N \quad e = \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

example $N=1$

$$\exp(-\bar{\theta} A \theta) = 1 + A \bar{\theta} \theta$$

$$N=2 \quad \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2$$

$$\exp(-\bar{\theta}_i A_{ij} \theta_j) = 1 + \dots$$

$$\frac{1}{2} (\bar{\theta}_1 A_{11} \bar{\theta}_1 + \bar{\theta}_1 A_{12} \bar{\theta}_2 + \bar{\theta}_2 A_{21} \bar{\theta}_1 + \bar{\theta}_2 A_{22} \bar{\theta}_2)$$

12

Integration: $g \rightarrow \int d\theta_i g$?

$$\int d\theta_i \frac{\partial}{\partial \theta_i} g = 0$$

take $\int d\theta_i := \frac{\partial}{\partial \theta_i}$

$$\int d\bar{\theta}_i := \frac{\partial}{\partial \bar{\theta}_i}$$

$$\int d\theta_i 1 = 0$$

$$\int d\theta_i \theta_i = 1$$

for complex Gaussian integration

$$\int \prod_i dz_i d\bar{z}_i \exp(-\bar{z}_i A_{ij} z_j) = \frac{1}{(\det A)}$$

Berezin integrals over Grass

$$G_N \quad N \theta_i, N \bar{\theta}_i$$

$$e \in G_N \quad e = \exp(-\bar{\theta}_i A_{ij} \theta_j)$$

example $N=1$

$$\exp(-\bar{\theta} A \theta) = 1 + A \bar{\theta} \theta$$

$$N=2 \quad \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2$$

$$\exp(-\bar{\theta}_i A_{ij} \theta_j) = 1 + \dots$$

$$\frac{1}{2} (\bar{\theta}_1 A_{11} \bar{\theta}_1 + \bar{\theta}_1 A_{12} \theta_2 + \bar{\theta}_2 A_{21} \theta_1 + \bar{\theta}_2 A_{22} \bar{\theta}_2)$$

12

Fermionic "expectation values"

$$\begin{array}{c} \langle g \rangle \\ \uparrow \\ \in G_N \end{array} := \frac{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j) \cdot g}{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j)} = \frac{\text{complex number}}{\text{complex number}}$$

Fermionic "expectation values"

$$\langle g \rangle := \frac{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j) \cdot g}{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j)} = \frac{\text{complex number}}{\text{complex number}}$$

$\uparrow$
 $e \in G_N$

$$\langle \theta_i \dots \rangle = 0 \quad \text{if } i \text{ is odd} \quad \langle \theta_i \bar{\theta}_j \rangle$$

$$\frac{N-1}{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta}) \theta \bar{\theta}} = \frac{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta})}{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta})}$$

Fermionic "expectation values"

$$\langle g \rangle := \frac{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j) \cdot g}{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j)} = \frac{\text{complex number}}{\text{complex number}}$$

$$\langle \theta_i \bar{\theta}_j \rangle = \frac{\sum_{k=1 \text{ pairs}} A \cdots A (-1)^{\#} \text{Minor}(A)_{ij}}{1} = \text{Minor}(A)_{ij}$$

$$\frac{N-1}{N} \frac{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta}) \theta \bar{\theta}}{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta})} = \frac{1}{A}$$

Fermionic "expectation values"

$$\langle g \rangle := \frac{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j) \cdot g}{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j)} = \frac{\text{complex number}}{\text{complex number}}$$

$\in G_N$

$$\langle \theta_i \dots \rangle = 0 \quad \text{if } i \text{ is odd}$$

$$\langle \theta_i \bar{\theta}_j \rangle = \frac{\sum_{k=1}^N A_{ik} A_{kj} (-1)^{\#}}{\det A} = \frac{\text{Minor}(A)_{ji}}{\det A}$$

$$= (\bar{A}^{-1})_{ij}$$

$$\frac{N-1}{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta}) \theta \bar{\theta}} = \frac{1}{A}$$

$$\underline{N=1} \quad \frac{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta}) \theta \bar{\theta}}{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta})} = \frac{1}{A}$$

$$\theta_i \rightarrow \theta'_i = M_{ij} \theta_j$$

idem for $\bar{\theta}$

$$\int d\bar{\theta}'_i d\theta'_i = |\det M|^{-2} \int d\bar{\theta}_i d\theta_i$$

Fermionic "expectation values"

$$\langle g \rangle := \frac{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j) \cdot g}{\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j)} = \frac{\text{complex number}}{\text{complex number}}$$

$\uparrow$
 e^{G_N}

$$\langle \theta_i \dots \rangle = 0$$

odd



$$\langle \theta_i \bar{\theta}_j \rangle = \frac{\sum_{k=1, k \neq i}^N A_{ik} \dots A_{kj} (-1)^{\#}}{\det A} = \frac{\text{Minor}(A)_{ij}}{\det A}$$

$$= (\bar{A}')_{ij}$$

$$\langle \bar{\theta}_i \theta_j \rangle = -(\bar{A}')_{ij}$$

$$\frac{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta}) \theta \bar{\theta}}{\int d\bar{\theta} d\theta (1 + A \theta \bar{\theta})} =$$

Back to the Dirac Field

$$\Psi = \left\{ \Psi^a(x) ; \begin{array}{l} x \in M^{1,3} \\ \text{space time} \end{array} ; a=1, \dots, 2^{\lfloor \frac{4}{2} \rfloor} \right\}$$

c Dirac indices

$$\overline{\Psi} = \left\{ \overline{\Psi}^a(x), \dots, \dots, \dots \right\}$$

↑
independent generators of a
very big Grassmann algebra

$$\theta_i \quad i=1 \dots N, \Rightarrow \Psi^a(x) \quad \begin{array}{l} a=1 \dots 4 \\ x \in M_1 \end{array}$$

$$\dim[G] = 2^{\times [4 \times \# \text{points in space-time}]}$$

Back to the Dirac Field

$$\Psi = \left\{ \Psi^a(x) ; \begin{array}{l} x \in M^{1,3} \\ \text{spacetime} \end{array} ; a=1, \dots, \binom{4}{2} \right\}$$

Dirac indices

$$\overline{\Psi} = \left\{ \overline{\Psi}^a(x), \dots \right\}$$

↑
independent generators of a
very big Grassmann algebra

$$\theta_i \quad i=1 \dots N, \Rightarrow \Psi^a(x) \quad \begin{array}{l} a=1 \dots 4 \\ x \in M_1 \end{array}$$

$$\dim[G] = 2^{\times [4 \times \# \text{points in spacetime}]}$$

Back to the Dirac Field

$$\Psi = \left\{ \Psi^a(x) ; \begin{array}{l} x \in M^{1,3} \\ \text{spacetime} \end{array} ; a=1, \dots, \binom{4}{2} \right\}$$

Dirac indices

$$\bar{\Psi} = \left\{ \bar{\Psi}^a(x), \dots \right\}$$

↑ independent generators of a
very big Grassmann algebra

$$\theta_i \quad i=1 \dots N, \Rightarrow \Psi^a(x) \quad \begin{array}{l} a=1 \dots 4 \\ x \in M_1 \end{array}$$

$$\dim[G] = 2^{\times [4 \times \# \text{points in spacetime}]}$$

Path Integral = Grassmann Gaussian Integral

Path Integral = Grassmann Gaussian Integral

$$\langle \psi(x) \bar{\psi}(y) \rangle = \langle 0 |$$

$$\int \underbrace{D[\bar{\psi}, \psi]}_{\text{measure}} \exp[i \underbrace{S[\bar{\psi}, \psi]}_{\text{Dirac}}] = \text{Det} [i \not{D} - m]$$

Functional determinant
(it has to be regularized)

$$\prod_{a=1}^4 \prod_x d\bar{\psi}^a(x) d\psi^a(x)$$

$$S[\bar{\psi}, \psi] = \int d^4x \bar{\psi}^a(x) [i(\gamma^\mu)_{ab} \partial_\mu - m \delta_{ab}] \psi^b(x)$$

Dirac action

$$(-\cancel{p} + m + i\varepsilon_+)^{-1}_{ab}$$

$$\left(\frac{1}{-\cancel{p} + m + i\varepsilon_+} \right)_{ab} = \left(\begin{array}{c} -1 \\ \end{array} \right)_{ab}$$