

Title: 13/14 PSI - Quantum Field Theory II - Lecture 9

Date: Nov 21, 2013 09:00 AM

URL: <http://pirsa.org/13110018>

Abstract:

"Wilsonian" Renormalization: K. Wilson 72

$$A = 1$$

Perturbative Renormalization ϕ^4 $d=4$ massive

$$\int \mathcal{D}[\phi] \exp(-S_R(\phi)) \quad S_R(\phi) = \int d^4x \left(\frac{A}{2} (\partial\phi)^2 + \frac{B}{2} \phi^2 + \frac{\varepsilon}{4!} \phi^4 \right)$$

renormalization scale μ
renormalized coupling g_R

UV cutoff $\Lambda \gg \mu$

$$A = 1 + O(g_R^2)$$

$$B = g_R + g_R^2 \frac{3}{2} \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right) + O(g_R^3)$$

$$B = \boxed{m_R^2} - g_R \frac{1}{2} \left(\frac{1}{(4\pi)^2} \Lambda^2 - \frac{m_R^2}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right) \right) + O(g_R^2)$$

It works

$$\Gamma_R(p) = p^2 + M_{phys}^2 + O(g_R^2)$$

exercise:

$$\boxed{M_{phys}^2 = m_R^2 + g_R \frac{m_R^2}{2} \frac{1}{(4\pi)^2} \log\left(\frac{m_R^2}{\mu^2}\right) + O(g_R^2)}$$

$$A = 1 + O(g_R^2)$$

$$B = g_R + g_R^2 \frac{3}{2} \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right) + O(g_R^3)$$

$$B = \boxed{m_R^2} - g_R \frac{1}{2} \left(\frac{1}{(4\pi)^2} \Lambda^2 - \frac{m_R^2}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right) \right) + O(g_R^2) \Rightarrow \boxed{M_{\text{phys}}^2 = m_R^2 + g_R \frac{m_R^2}{2} \frac{1}{(4\pi)^2} \log\left(\frac{m_R^2}{\mu^2}\right) + O(g_R^2)}$$

It works

RG "Beta function" for the mass m_R

g_R dimensionless

m_R dimension 1

$$\Gamma_R(p) = p^2 + M_{\text{phys}}^2 + O(g_R^2)$$

exercise:

$$\beta(g_R) = \mu \frac{\partial}{\partial \mu} g_R \Big|_{\text{Physics unchanged}} = \frac{1}{2} \frac{3}{(4\pi)^2} g_R^2$$

$$\mu \frac{\partial}{\partial x} t_R \Big| := \beta_t(g_R, t_R)$$

cc does not change
 M_{phys} " "

$$\mu \frac{\partial}{\partial \mu} t_R \Big| := \beta_t(g_R, t_R) = t_R \left(-2 + g_R \frac{1}{2} \frac{1}{(4\pi)^2} \right)$$

cc does not change
M_{phys} " "

$$\mu \frac{\partial}{\partial \mu} t_R \Big| := \beta_t(g_R, t_R) = t_R \left(-2 + g_R \frac{1}{2} \frac{1}{(4\pi)^2} \right)$$

$\mu \frac{\partial}{\partial \mu} t_R$ cc does not change
 M_{phys} " "

canonical dimension of $m_R^2 = 2$

this comes from the $\log(\mu)$

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"Beta function" for t_R

↑
 canonical dimension of
 $m_R^2 = 2$

← this comes from the $\log(\mu)$

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cc does not change M_{phys} " "

"Beta function" for t_R

canonical dimension of $m_R^2 = 2$

this comes from the $\log(\mu)$

$d=4$ $(\partial\phi)^2$, ϕ^2 , ϕ^4 "counterterms" renormalizable
 $d < 4$ ϕ^2 counterterm

$$\hbar \quad \text{---} \bigcirc \text{---} = \Lambda^2 - \underbrace{m^2 \log \Lambda^2}$$

↑



$d=2$ $\text{---} \bigcirc \text{---}$ Divergent

$d=3$ $\text{---} \bigcirc \text{---}$ $\bigcirc$ Div.

$$\mu \frac{\partial}{\partial \mu} t_R \Big| := \beta_t(g_R, t_R) = t_R \left(-2 + g_R \frac{1}{2} \frac{1}{(4\pi)^2} \right)$$

cc does not change
Mphys " "

↑
canonical dimension of
 $m_R^2 = 2$

←
this comes
from the $\log(\mu)$

"Beta function" for t_R

$d=4$ $(\partial\phi)^2$, ϕ^2 , ϕ^4 "counterterms" renormalizable

$d<4$ ϕ^2 counterterm

super-renormalizable

$d>4$

$$\mu \frac{\partial}{\partial \mu} t_R \Big| := \beta_t(g_R, t_R) = t_R \left(-2 + g_R \frac{1}{2} \frac{1}{(4\pi)^2} \right)$$

cc does not change
Mphys " " "

"Beta function" for t_R

canonical dimension of $m_R^2 = 2$

this comes from the $\log(\mu)$

$d=4$ $(\partial\phi)^2, \phi^2, \phi^4$ "counterterms" renormalizable

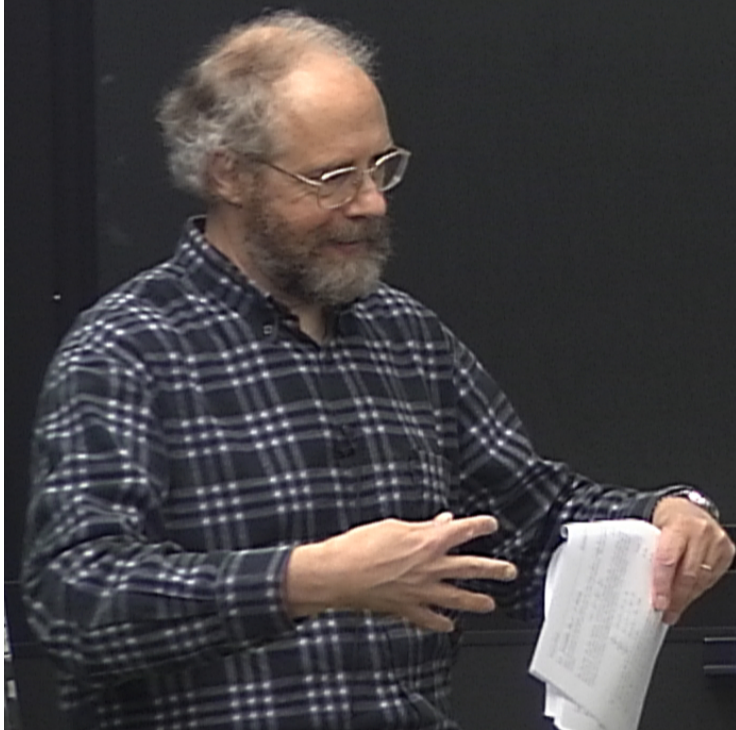
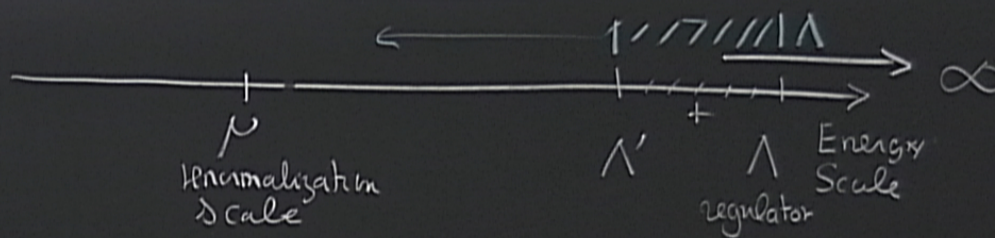
$d<4$ ϕ^2 counterterm super-renormalizable

$d>4$ " + $\phi^6 + (\partial\phi)^2 \phi^2 + (\Delta\phi)^2$ "not under" non-renormalizable
control.

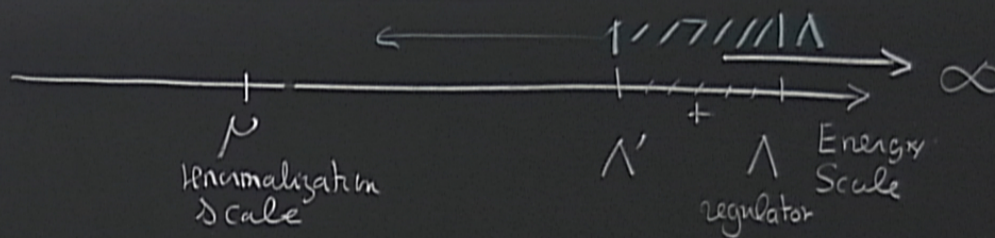
Wilson.

Energy
Scale

Wilson:



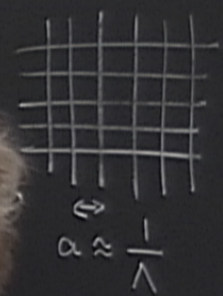
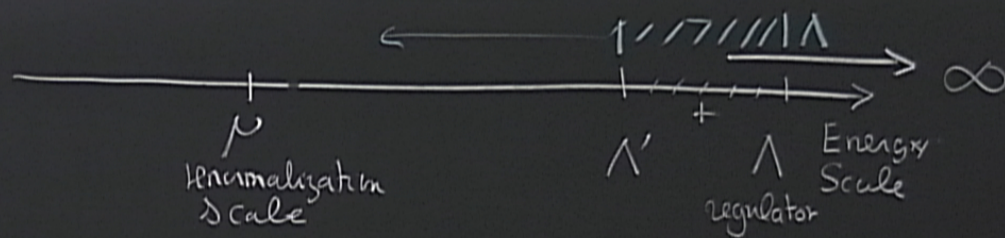
Wilson:



$$\phi(k) \quad |k| < \Lambda'$$

$$\Lambda' < |k| < \Lambda \quad \Lambda > \Lambda'$$

Wilson.



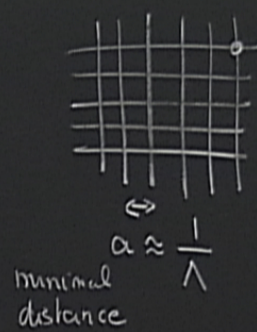
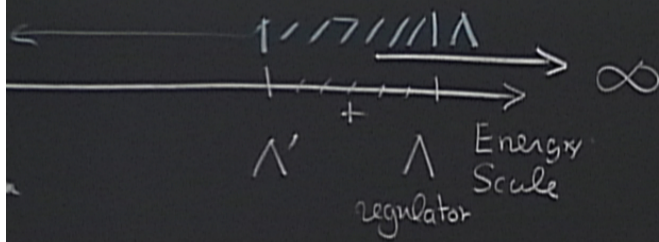
$$\phi(k) \quad |k| < \Lambda'$$

$$\Lambda' < |k| < \Lambda \quad \Lambda > \Lambda'$$

Start from the Theory at Fixed cut-off Λ (High Energy)
with a given action "Bare action"

Integrate out the High momenta $\Lambda' < |k| < \Lambda$

→ "Renormalized"



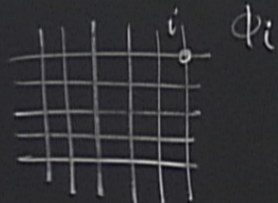
$$\int D[\Phi] \exp(-$$

cut off a

off Λ (High Energy)

ion' $\longrightarrow$ "Renormalized effective action"

$\Lambda' < |k| < \Lambda$



$\Leftrightarrow$
 $a \approx \frac{1}{\Lambda}$
 minimal distance

$$\int d^D x = \sum_{\text{sites } i} a^D \phi_i \quad \text{Sums}$$

$$\partial = \frac{\phi_{i+1} - \phi_i}{a} \text{ etc. } \quad \text{Finite differences}$$

etc...

$$\int_{\text{all } \phi} \mathcal{D}[\phi] \exp(-\mathcal{A}[\phi])$$

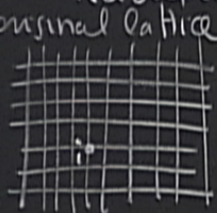
$$\mathcal{A}[\phi] = \int d^D x \left[\underbrace{\frac{1}{2}(\nabla\phi)^2 + \frac{t}{2}\phi^2 + \frac{g}{4!}\phi^4}_{\text{usual terms}} + \underbrace{\frac{h}{6!}\phi^6 + \dots + (\Delta\phi)^2}_{\text{"non-renormalizable couplings"}} + \dots \right]$$

$$\text{Action} = \mathcal{A}[\phi]$$

Scaling factors S

"Bloc Spin Transformation" = Renormalization transformation
new lattice

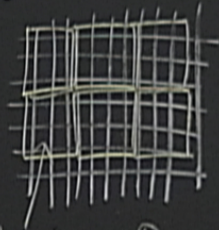
Kadanoff ≈ 60
original lattice



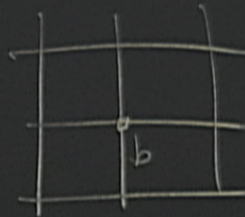
i

$$\phi_i \in \mathbb{Z}^D$$

size of the original
lattice



Block size $B \times a$
here $B=3$
Block b

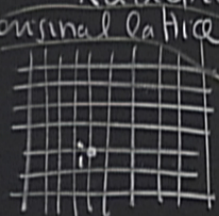


Block Field variable = average in a
block

$$\tilde{\phi}_b = \frac{1}{B} \sum_{i \in b} \phi_i$$

"Bloc Spin Transformation" = Renormalization transformation
new lattice $\tilde{\Lambda}$

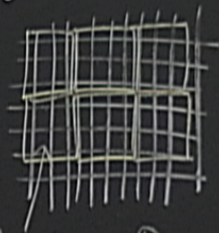
Kadanoff ≈ 60
original lattice Λ



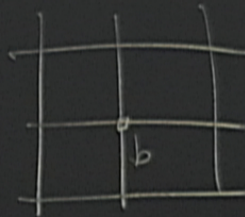
Λ

$\phi_i \in \mathbb{Z}^D$

1 site of the original
lattice



Block of size $B \times a$
here $B=3$
Block b



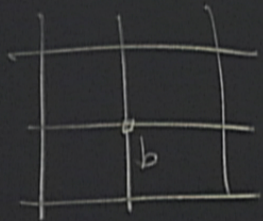
Block Field variable = average on a
block

$$\tilde{\phi}_b = \vec{B} \sum_{i \in b} \phi_i$$

$$e^{-A_{\text{eff}}(\tilde{\phi}_b)} = \int d\phi_i e^{-A[\phi_i]} \delta[\tilde{\phi}_b - \vec{B} \sum_{i \in b} \phi_i]$$

$$\int \prod_{i \in \Lambda} d\phi_i e^{-A[\phi_i]} = \int \prod_{b \in \tilde{\Lambda}} d\tilde{\phi}_b \delta[\tilde{\phi}_b - \vec{B} \sum_{i \in b} \phi_i] e^{-A[\phi_i]} = \int \prod_{b \in \tilde{\Lambda}} d\tilde{\phi}_b e^{-A_{\text{eff}}[\tilde{\phi}_b]}$$

" = Renormalization transformation
new lattice $\tilde{\Lambda}$



Bloc Field variable = average in a bloc

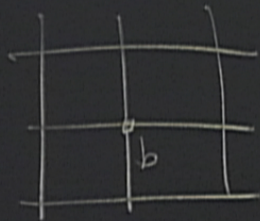
$$\tilde{\phi}_b = \vec{B} \sum_{i \in b} \phi_i$$

$$e^{-A_{\text{eff}}(\tilde{\phi}_b)} = \int \prod_{i \in \Lambda} d\phi_i \prod_b \delta[\tilde{\phi}_b - \frac{1}{\vec{B}} \sum_{i \in b} \phi_i] e^{-A[\phi_i]}$$

Effective action for the bloc variables $\tilde{\phi}_b$

$$\prod_{b \in \tilde{\Lambda}} d\tilde{\phi}_b \delta[\tilde{\phi}_b - \vec{B} \sum_{i \in b} \phi_i] e^{-A[\phi_i]} = \int \prod_{b \in \tilde{\Lambda}} d\tilde{\phi}_b e^{-A_{\text{eff}}[\tilde{\phi}_b]}$$

" = Renormalization transformation
new lattice $\tilde{\Lambda}$



Bloc Field variable = average in a bloc

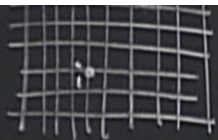
$$\tilde{\phi}_b = B^D \sum_{i \in b} \phi_i$$

$$\prod_{b \in \tilde{\Lambda}} d\tilde{\phi}_b \delta[\tilde{\phi}_b - B^D \sum_{i \in b} \phi_i] e^{-A[\phi_i]} = \int \prod_{b \in \tilde{\Lambda}} d\tilde{\phi}_b e^{-A_{\text{eff}}[\tilde{\phi}_b]}$$

$$e^{-A_{\text{eff}}(\tilde{\phi}_b)} = \int \prod_{i \in \Lambda} d\phi_i \prod_b \delta[\tilde{\phi}_b - \frac{1}{B^D} \sum_{i \in b} \phi_i] e^{-A[\phi_i]}$$

Effective action for the bloc variables $\tilde{\phi}_b$

Step 2: compare $A[\phi_i]$ with $A_{\text{eff}}[\tilde{\phi}_b]$



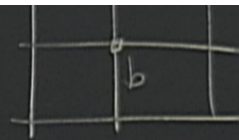
i

$\phi_i \in \mathbb{Z}^D$

1 site of the original lattice



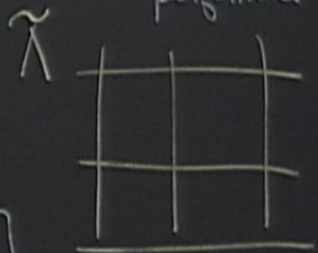
Block of size $B \times a$
here $B=3$
Block b



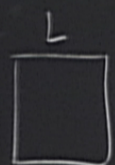
Block Field variable = average in a block

$$\tilde{\phi}_b = \vec{B} \sum_{i \in b} \phi_i$$

Step 2: compare
perform a rescaling



$$\vec{a} = B \vec{a}$$



$$\int \prod_{i \in \Lambda} d\phi_i e^{-\mathcal{A}[\phi_i]} = \int \prod_{b \in \tilde{\Lambda}} d\tilde{\phi}_b \delta[\tilde{\phi}_b - \vec{B} \sum_{i \in b} \phi_i] e^{-\mathcal{A}[\phi_i]}$$

$\mathcal{A} = (L/a)^D$ variables

$$= \int \prod_{b \in \tilde{\Lambda}} d\tilde{\phi}_b e^{-\mathcal{A}_{eff}[\tilde{\phi}_b]}$$

variable $\mathcal{A}_{eff} = (L/Ba)^D = \mathcal{A}/B^D$

$A = (L/a)^D$ variables

$i \in \Lambda$ $\prod_{i \in \Lambda} \phi_i$

variable $(\Lambda' = (L/a'))^D$

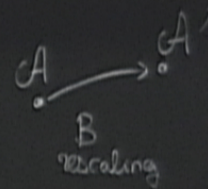
Renormalized action $A'[\phi'] = A_{\text{eff}}[\tilde{\phi}]$

$$\text{original sum} = \int \prod_{i \in \Lambda'} d\phi'_i e^{-A'[\phi']}$$

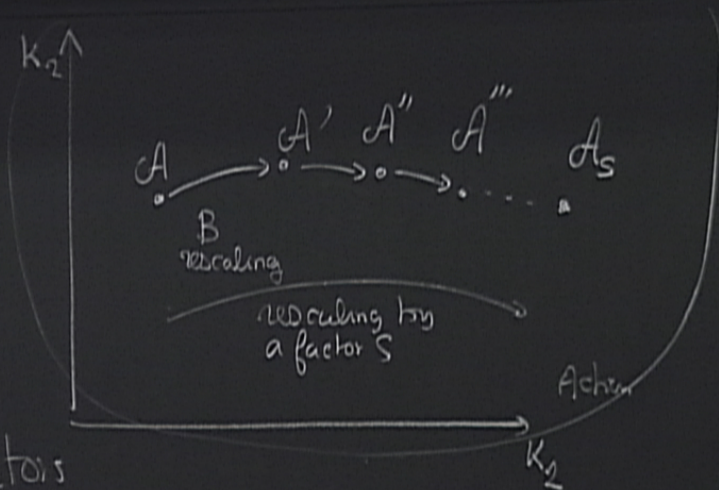
Transformation in the space of Actions $A[\phi]$

$$A[\phi] = \int \sum_{\alpha} \kappa_{\alpha} O_{\alpha}(\phi)$$

$O_{\alpha}(\phi)$ are "local" operators
function of $\phi, \nabla\phi, \nabla^2\phi$, etc
at a given point



Variable $(A) = (1/Ba)^{1/B}$



$$\{k_\alpha\} \xrightarrow{\text{B rescaling}} \{k'_\alpha\}$$

Perform this N times so that
you get a rescaling by a factor

$$S = B^N$$

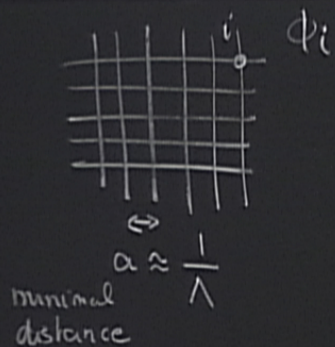
$\tilde{\phi}$

ϕ

local operators

$\nabla\phi, \nabla^2\phi$, etc

(ϕ is
primary)



$$\int d^D x = \sum_{\text{sites } i} a^D \phi_i$$

$$\partial = \frac{\phi_{i+1} - \phi_i}{a} \text{ etc.} \quad \text{Finite differences}$$

etc..

$$\int_{\text{cut off } a} D[\phi] \exp(-\mathcal{A}[\phi])$$

$$\mathcal{A}[\phi] = \int d^D x \left[\underbrace{\frac{1}{2}(\nabla\phi)^2 + \frac{t}{2}\phi^2 + \frac{g}{4!}\phi^4}_{\text{usual terms}} + \underbrace{\frac{h}{6!}\phi^6 + \dots}_{\text{"non-renormalizable couplings"}} + (\Delta\phi)^2 + \phi^2(\nabla\phi)^2 \right]$$

discretized
no UV divergences

sums

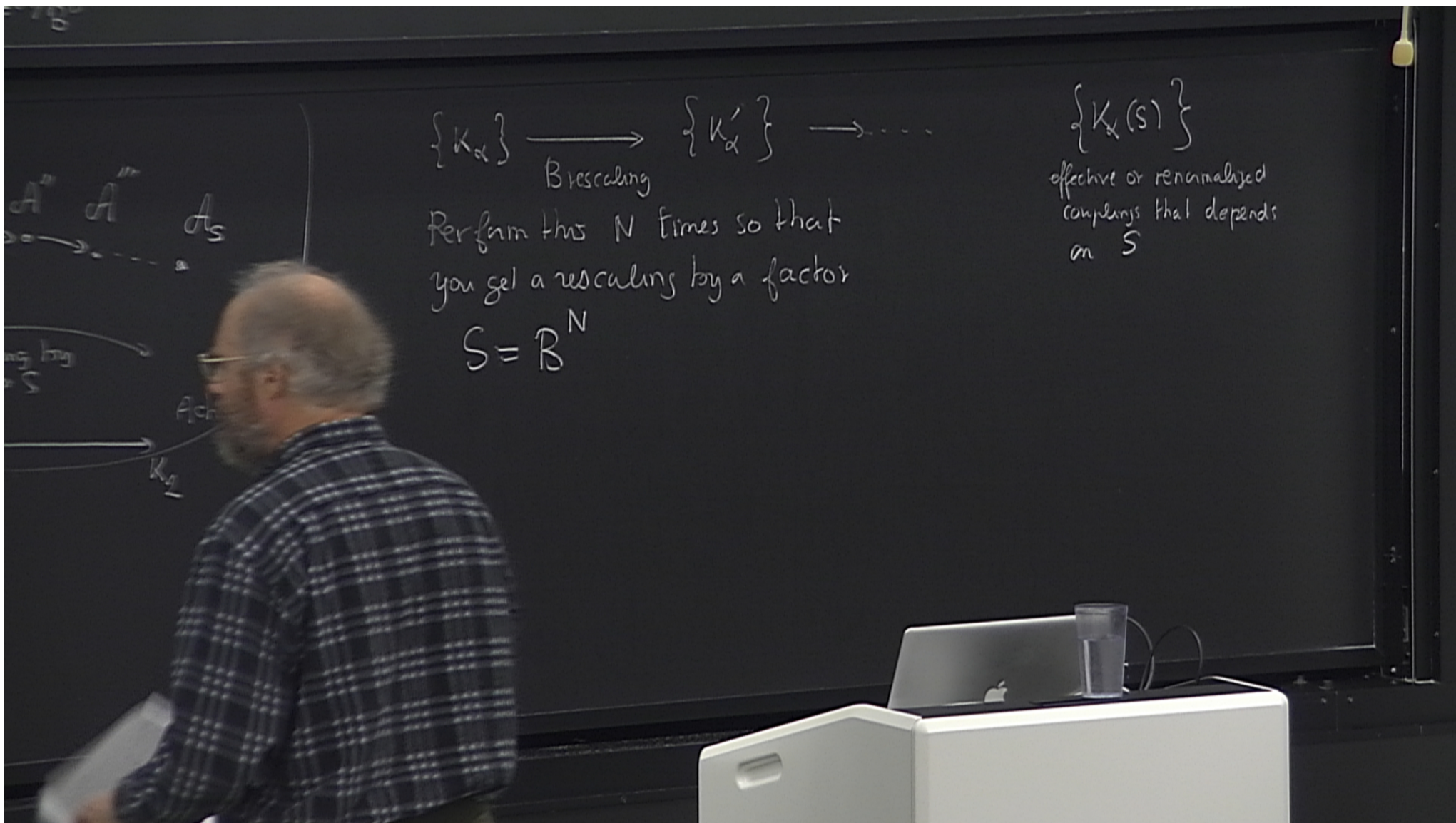
original

$$\text{Action} = \mathcal{A}[\phi]$$

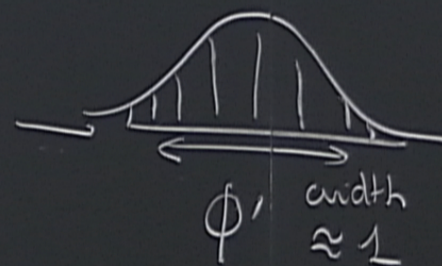
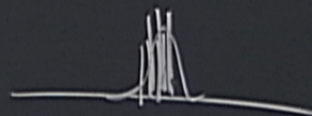
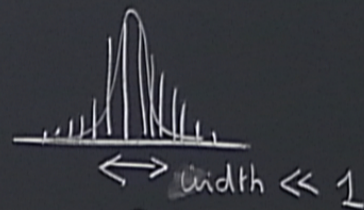
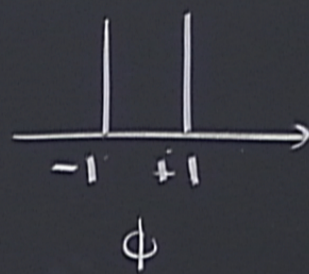
Scaling
factors

S

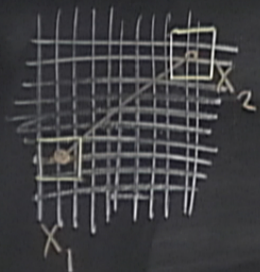
new ones



$\phi_i = \pm 1$ Ising Model



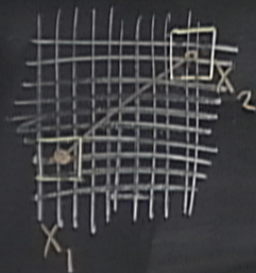
correlation functions



$$\langle \phi(x_1) \phi(x_2) \rangle_A$$

$$\frac{\int \prod_{i \in A} e^{-A[\phi]} \phi(x_1) \phi(x_2)}{\int \prod_{i \in A} e^{-A[\phi]}}$$

correlation functions



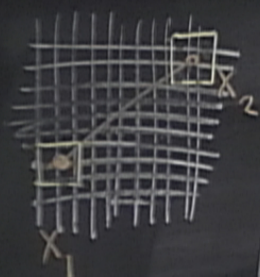
$$\langle \phi(x_1) \phi(x_2) \rangle_A \simeq \langle \tilde{\phi}(x_1) \tilde{\phi}(x_2) \rangle_{A_{\text{eff}}} =$$

if $|x_1 - x_2| \gg Ba$

$$\frac{\int \prod_{i \in A} d\phi_i e^{-A[\phi]} \phi(x_1) \phi(x_2)}{\int \prod_{i \in A} d\phi_i e^{-A[\phi]}}$$

$$\langle x_2 \rangle_{d_{\text{eff}}} = B^{-2\Delta} \langle \phi'(x_1/B) \phi'(x_2/B) \rangle_{\mathcal{A}'} \simeq \dots \simeq Z(s)^{-2} \langle \phi_s(x_1/s) \phi_s(x_2/s) \rangle$$

correlation functions



$$\langle \phi(x_1) \phi(x_2) \rangle_A$$

$$\frac{\int \prod_{i \in A} e^{-a[\phi]} \phi(x_1) \phi(x_2)}{\int \prod_{i \in A} e^{-a[\phi]}}$$

$$\simeq \langle \tilde{\phi}(x_1) \tilde{\phi}(x_2) \rangle_{d_{\text{eff}}} = B^{-2\Delta} \langle \phi'(x_1/B) \phi'(x_2/B) \rangle_{A'} \simeq \dots$$

if $|x_1 - x_2| \gg Ba$

$$G(x_1, x_2; k_a) \simeq Z(s)^{-N} G(x_1/s, x_2/s; k_a(s))$$

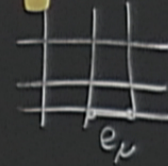
$$\underset{\sim}{\langle} \tilde{\Phi}(x_1) \tilde{\Phi}(x_2) \underset{\sim}{\rangle} = B^{-2\Delta} \underset{\sim}{\langle} \Phi'(x_1/B) \Phi'(x_2/B) \underset{\sim}{\rangle}_{A'} \simeq \dots \simeq Z(s)^{-2} \underset{\sim}{\langle} \phi_s(x_1/s) \phi_s(x_2/s) \underset{\sim}{\rangle}_{A_s}$$

if $|x_1 - x_2| \gg Ba$

this approximation is valid
as long as the A_s has only
short range interactions

$$Z(s)^{-N} G(x_1/s, x_2/s; K_d(s))$$

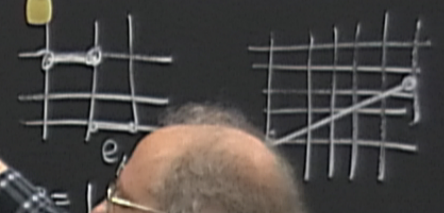
Free Field $\mathcal{A} = \int (\partial \phi)^2 = \sum_i a^{D-2} \sum_{\mu=1}^D (\phi_{i+e_\mu} - \phi_i)^2$



log(μ)

Free Field $\mathcal{A} = \int (\partial \phi)^2 = \sum_i \sum_{\mu=1}^D (\phi_{i+\hat{\mu}} - \phi_i)^2$

$\downarrow$
 $\mathcal{A}'[\phi'] = \sum_{(i,j)} K'_{ij} (\phi_i - \phi_j)^2$
 $\uparrow$ some kernel

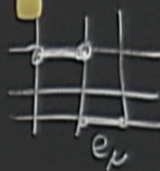


Free Field $A = \int (\partial \phi)^2 = \sum_i \sum_{\mu=1}^D (\phi_{i+e_\mu} - \phi_i)^2$

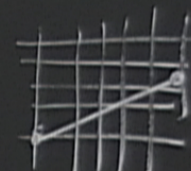
$\downarrow$
 $A'[\phi'] = \sum_{(i,j)} K'_{ij} (\phi_i - \phi_j)^2$
 $\uparrow$ some kernel

$\phi \rightarrow \phi + csl$
 massless theory
 at a critical point

$\xi = \infty$



$a=1$



Free Field $A = \int (\partial \phi)^2 = \sum_i \sum_{\mu=1}^D (\phi_{i+\vec{e}_\mu} - \phi_i)^2$

$\downarrow$
 $A'[\phi'] = \sum_{\langle i,j \rangle} K_{ij} (\phi'_i - \phi'_j)^2$
 $\uparrow$ some kernel

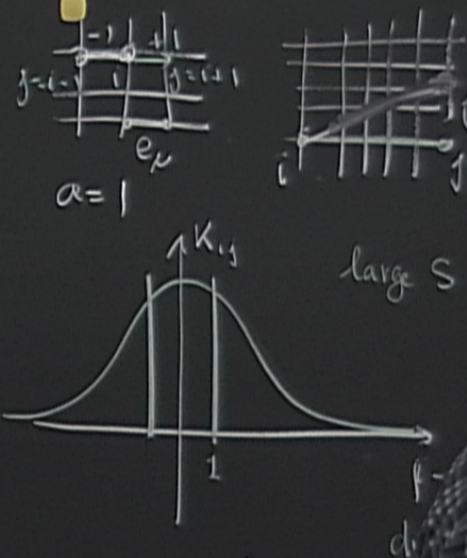
$\phi \rightarrow \phi + c s$
 massless theory
 at a critical point

$\xi = \infty$

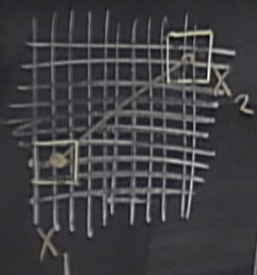
$K_{ij} = K(\vec{i} - \vec{j})$ Translation invariance

$A_S[\phi_S] = \sum_{\langle i,j \rangle} \underline{K_{ij}(s)} (\phi_i - \phi_j)^2$

$K_{ij}(s) \rightarrow K(l)$ decays exponentially at $l \rightarrow \infty$



correlation functions



$$\langle \phi(x_1) \phi(x_2) \rangle_A = \frac{\int \prod_{i \in A} e^{-\mathcal{A}[\phi]} \phi(x_1) \phi(x_2)}{\int \prod_{i \in A} e^{-\mathcal{A}[\phi]}}$$

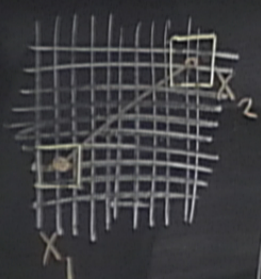
$$\simeq \langle \tilde{\phi}(x_1) \tilde{\phi}(x_2) \rangle_{\mathcal{A}_{\text{eff}}} = B^{-2\Delta} \langle \phi'(x_2/B) \rangle_{A'} \simeq \dots$$

if $|x_1 - x_2| \gg Ba$

approximation is valid if the $\mathcal{A}_S$ has only interactions

$$G(x_1, x_2; k_q) \simeq Z(s)^{-2} G(x_1/s, x_2/s; k_q)$$

correlation functions



$$\langle \phi(x_1) \phi(x_2) \rangle_{\mathcal{A}}$$

$$\frac{\int \prod_{i \in \mathcal{A}} e^{-d[\phi]} \phi(x_1) \phi(x_2)}{\int \prod_{i \in \mathcal{A}} e^{-d[\phi]}}$$

$$\simeq \langle \tilde{\phi}(x_1) \tilde{\phi}(x_2) \rangle_{\mathcal{A}'} = B^{-2\Delta} \langle \phi'(x_1/B) \phi'(x_2/B) \rangle_{\mathcal{A}'} \simeq \dots$$

if $|x_1 - x_2| \gg Ba$ d_{eff}

this approximation is valid as long as the $\mathcal{A}_S$ has only short range interactions

$$G(x_1, x_2; k_\alpha) \simeq Z(s)^{-2} G(x_1/s, x_2/s; k_\alpha(s))$$

Large distances

Small distances

$$S \gg 1$$

Fixed