

Title: 13/14 PSI - Quantum Field Theory II - Lecture 8

Date: Nov 20, 2013 09:00 AM

URL: <http://pirsa.org/13110017>

Abstract:

Renormalization Group:

massless ϕ^4 theory
1 loop - perturbation theory

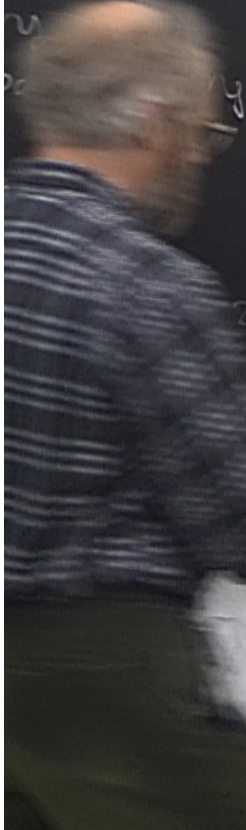
$$\int \mathcal{D}[\phi] \exp(-S_R[\phi])$$

$$S_R[\phi] = \int d^4x \left[\frac{A}{2} (\partial\phi)^2 + \frac{B}{2} \phi^2 + \frac{C}{4!} \phi^4 \right]$$

$$\langle \phi(x_1) \dots \phi(x_n) \rangle_R = \int \mathcal{D}[\phi] \exp(-S_R[\phi])$$

Correlation function
for the renormal

our p:



$$\langle \phi(x_1) \dots \phi(x_N) \rangle_R = \frac{\int \mathcal{D}[\phi] \exp(-S_R(\phi)) \phi(x_1) \dots \phi(x_N)}{\int \mathcal{D}[\phi] \exp(-S_R(\phi))}$$

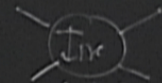
Correlation function
for the renormalized theory

one well defined when $\Lambda \rightarrow \infty$

when expressed in term of the

renormalized coupling g_R (at 1st order)

$g_R =$



4pt func

for $\{p_1, p_4\} \simeq \mu$

$$+ \frac{g}{4!} \phi^4$$



Renormalization Group:

- massless ϕ^4 theory
- 1 loop - perturbation theory

$$\int \mathcal{D}[\phi] \exp(-S_R[\phi])$$

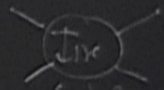
$$S_R[\phi] = \int d^4x \left[\frac{A}{2} (\partial\phi)^2 + \frac{B}{2} \phi^2 + \frac{G}{4!} \phi^4 \right]$$

$$\langle \phi(x_1) \dots \phi(x_N) \rangle_R = \frac{\int \mathcal{D}[\phi] \exp(-S_R[\phi]) \phi(x_1) \dots \phi(x_N)}{\int \mathcal{D}[\phi] \exp(-S_R[\phi])}$$

Correlation function
for the renormalized theory

are well defined when $\Lambda \rightarrow \infty$

when expressed in terms of the
renormalized coupling g_R (at 1st order)

$g_R =$ 
for $\{p_1, p_4\} \simeq \mu$

$$A = 1 + \dots + O(g_R^2)$$

$$B = 0 - g_R \frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2 + O(g_R^2)$$

$$G = g_R + g_R^2 \frac{3}{2} \frac{1}{(4\pi)^2} \log(\Lambda^2/\mu^2) + O(g_R^2)$$

01 02 03 04 05

How does a correlation function of the ϕ^4 theory
depends on the energy scale ?

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depend on the energy scale?

$\langle \phi(x)$

How does a correlation function of the ϕ^4 theory depends on the energy scale?

$$\langle \phi(x_1) \dots \phi(x_N) \rangle_R = G_R(x_1, x_N; g_R, \mu)$$

compare it to

$$G_R(Sx_1, Sx_N; g_R, \mu)$$

S scale factor

N -point function
in position space

(same for momentum space)

$$S > 1$$

large distances / low energy / m

$$S < 1$$

small distances / high E / m

How does a correlation function of the ϕ^4 theory depends on the energy scale?

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$S > 1$ large distances / low energy / momentum

$S < 1$ small distances / high E / momentum

does a correlation function of the ϕ^4 theory
depend on the energy scale?

$$X \rightarrow SX \Leftrightarrow P \rightarrow S^{-1}P$$

$$\langle \phi(x_1) \dots \phi(x_N) \rangle_R = G_R(x_1, \dots, x_N; g_R, \mu)$$

map it to

$$(SX_1, \dots, SX_N; g_R, \mu)$$

$$\parallel$$

$$(X_1, \dots, X_N; g_R, \mu/S)$$

N -point function
in position space
(same for momentum space)

S scale

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compare it to

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S scale factor $S > 1$ large

$S < 1$ small

$$\bar{S}^{-N} \cdot G_R(x_1, \dots, x_N; g_R, \mu S) = \bar{S}^{-N} G_R(x_1, x_N; g_R(S), \mu)$$

Fourier Transform

just dimensional analysis

Renormalization

$$X \rightarrow SX \Leftrightarrow P$$

the X

N -point function
in position space

(same for momentum space)

ation of the ϕ^4 theory
le ?

$X_N; g_R, \mu$) N-point function
in position space
(same for momentum space)

S scale factor $S > 1$ large distances / low energy / momentum
 $S < 1$ small distances / high E / momentum

$$= \bar{S}^{-N} G_R(X_1, \dots, X_N; g_R(S), \mu)$$

Renormalization $\xrightarrow{\quad}$

$$X \rightarrow SX \Leftrightarrow P \rightarrow \bar{S}' P \Leftrightarrow P \rightarrow SP$$

Rescaling the X's (or the P's)
amounts to a change in the coupling

$$\rightarrow SX \Leftrightarrow P \rightarrow \bar{S}' P \Leftrightarrow \mu \rightarrow S\mu$$

Rescaling the X 's (or the P 's)
amounts to a change in the coupling

low energy/momentum
vs. high E /momentum

R.G. equation for
 $g_R(s)$

↑
running coupling
large when $s \rightarrow 0$
small when $s \rightarrow \infty$

R.G. Flow equation for the coupling

$$S \frac{\partial}{\partial S} g_R(s) = - \beta(g_R(s))$$

↑
minus sign rescaling on X 's
instead of $\mu = P$'s

Massless theory : no ϕ renormalization
 $A = 1$

mass + field renormalization

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$$G_R(sX_1, \dots, sX_N; g_R, \mu) = G_R(X_1, \dots, X_N; g_R(s), \mu)$$

$$X \Leftrightarrow P \rightarrow \bar{S}' P \Leftrightarrow p \rightarrow S p$$

scaling the X 's (or the P 's)
leads to a change in the coupling

/low energy/momentum
(e)/high E /momentum

R.G equation for
 $g_R(s)$

↑
running coupling
large when $s \rightarrow 0$
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function of $\left(\frac{\mu}{p}\right)$ $\mu/p \rightarrow \mu/p/s = \frac{S\mu}{p}$

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minus sign rescaling on X 's
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Massless theory, no ϕ renormalization
 $A = 1$

mass + field renormalization

$$G_R(sX_1, \dots, sX_N; g_R, \mu) = Z(s)^{-N} G_R(X_1, \dots, X_N; g_R(s), \mu)$$

$$S \frac{d}{ds} \log Z(s) = -\Delta_\phi(g_R(s)) \quad \left| \begin{array}{l} \uparrow \\ S^{-N} \text{ at 1 loop} \end{array} \right.$$

$\uparrow$
scaling dimension of
the field ϕ

$$\Delta_\phi = 1 + O(g_R^2)$$

Massless theory : no ϕ renormalization
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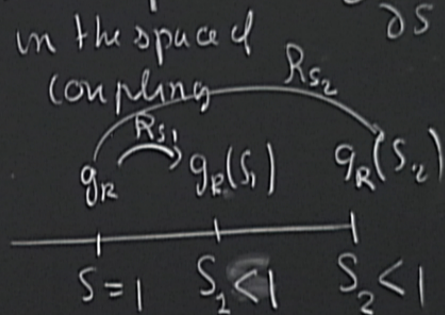
Flow equation $s \frac{d}{ds} g$

ren

$X_\mu, g_R(s), \mu$

oop

Flow equation $S \frac{\partial}{\partial S} g_R(s) = \beta(g_R(s))$



$$g_R(s_1) = R_{S_1}(g_R)$$

$$R_{S_1 \cdot S_2} = R_{S_1} \circ R_{S_2}$$

Group : Renormalization group

Scale Invariance

Classical theory

$$\mathbb{S}[\phi] = \int d^d x \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4!} \phi^4$$

Scale transformation

$$\phi \rightarrow \phi_s$$

$$\phi_s(x) = s^\Delta \phi(sx)$$

$$\Delta = \frac{d-2}{2} \quad \text{canonical dimension of } \phi$$

$$\text{In } d=4, \Delta=1$$

$$\mathbb{S}[\phi_s] = \mathbb{S}[\phi]$$

↑ action

Scale Invariance

Noether theorem $\Rightarrow$ conserved current

Classical theory

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$$\mathcal{S}[\phi_s] = \mathcal{S}[\phi] \Rightarrow \text{scale symmetry}$$

$\uparrow$ action

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$\uparrow$
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Noether theorem $\Rightarrow$ conserved current

Noether's theorem $\Rightarrow$ conserved current

$$\mathcal{S}[\phi_s] = \int d^4x \frac{1}{2} S^2 \frac{\partial}{\partial x}$$

ed current

$$\begin{aligned}
 \mathcal{S}[\phi_s] &= \int d^4x \left[\frac{1}{2} S^2 \left[\frac{\partial}{\partial x^\mu} (\phi(sx)) \right]^2 + \frac{g}{4} S^4 \phi^4(sx) \right] \\
 &= \int d^4x \left[\frac{1}{2} S^4 \left[\left(\frac{\partial}{\partial x} \phi \right)(sx) \right]^2 + \frac{g}{4} S^4 \phi^4(sx) \right] \\
 &\quad X = S^{-1} Y \\
 &= \int d^4Y \ S^{-4} \left[\frac{1}{2} \left(\frac{\partial}{\partial Y} \phi(Y) \right)^2 + \frac{g}{4} \phi^4(Y) \right] = \mathcal{S}[\phi]
 \end{aligned}$$

Scale Invariance

Classical theory

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Scale transformation

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↑ action

Noether theorem $\Rightarrow$ conserved current

$$J^\mu(x) = T^\mu_\nu X^\nu + \frac{d-2}{2} \phi \partial^\mu \phi$$

stress-energy tensor

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \left(\frac{1}{2} (\partial\phi)^2 + \frac{g}{4!} \phi^4 \right)$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{Equation of motion } -\Delta\phi + \frac{g}{6}\phi^3 = 0$$

translation invariance

Scale Invariance

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translation invariance

$$\partial_\mu T^\mu = X^\nu (\partial_\mu T^{\mu\nu}) + T^\mu_\mu + \frac{d-2}{2} \partial_\mu (\phi \partial^\mu \phi) = \frac{d-2}{2} \phi (\Delta \phi) - d \frac{g}{2} \phi$$

T^μ conserved? $\parallel$
 0

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$$\text{Tr}(\hat{g}^{\mu\nu}) = g^\mu_\mu = d$$

T^μ conserved? $\overset{||}{0}$

$\overset{=}{\text{E.O.M}}$

$$\partial_\mu T^\mu \underset{\text{E.O.M}}{=} (d-4) \frac{g}{4!} \phi^4 = \frac{d-4}{4} \phi (\Delta \phi)$$

$$d=4 \text{ classical } \partial_\mu T^\mu = 0$$

$$\partial_\mu T^\mu = X^\nu (\partial_\nu T^{\mu\nu}) + T^\mu_\mu + \frac{d-2}{2} \partial_\nu (\phi \partial^\nu \phi) = \frac{d-2}{2} \phi (\Delta \phi) - d \frac{g}{4!} \phi^4$$

$$\text{Tr}(\delta^{\mu\nu}) = \delta$$

T^μ conserved? $\parallel$
0

=
E.O.M

$$\partial_\mu T^\mu \underset{\text{E.O.M}}{=} (d-4) \frac{g}{4!} \phi^4 = \frac{d-4}{4} \phi (\Delta \phi)$$

$d=4$ classical $\partial_\mu T^\mu = 0$ scale invariance; $\dim[g] = 0$

Quantum theory? ?

$$\langle \phi(x_1) \phi(x_2) \partial_\mu T^\mu(x) \rangle = 0$$

$x_1 \neq x_2 \neq x$

$$\frac{d-2}{2} \phi (\Delta \phi) - d \frac{g}{4!} \phi^4 \quad \left| \quad \text{Q.E.O.M} \quad \langle \phi(x_1) \phi(x_2) [-\Delta \phi(x) + \frac{g}{6} \phi^3(x)] \rangle = 0 \right.$$

$x_1 \neq x_2 \neq x$

$\uparrow$
 $\text{Tr}(g^{\mu\nu}) = g^{\mu}_{\mu} = d$

$$]=0$$

$$\partial_\mu T^\mu = X^\nu (\partial_\nu T^{\mu\nu}) + T^\mu_\mu + \frac{d-2}{2} \partial_\nu (\phi \partial^\nu \phi) = \frac{d-2}{2} \phi (\Delta \phi) - d \frac{g}{4!} \phi^4 \quad \left| \begin{array}{l} \text{Q.E.D.} \\ \uparrow \\ T^\mu_\mu = \delta^\mu_\mu = d \end{array} \right.$$

T^μ conserved? $\parallel$
0

$$\partial_\mu T^\mu \underset{\text{E.O.M}}{=} (d-4) \frac{g}{4!} \phi^4 = \frac{d-4}{4} \phi (\Delta \phi)$$

$\underset{\text{E.O.M}}{=}$

Δ

forget the UV regul

$d=4$ classical $\partial_\mu T^\mu = 0$ scale invariance; $\dim[g]=0$

Quantum theory? ?

$$\langle \phi(x_1) \phi(x_2) \partial_\mu T^\mu(x) \rangle = 0$$

$x_1 \neq x_2, \dots \neq x$

$$\langle \phi(x_1) \dots \phi(x_n) (d-4) \frac{g}{4!} \phi^4(x) \rangle$$

$$\begin{aligned}
 \delta^2 \phi &= \frac{d-2}{2} \phi (\Delta \phi) - d \frac{g}{4!} \phi^4 & \text{Q.E.O.M} & \quad \langle \phi(x_1) \cdot \phi(x_2) [-\Delta \phi(x) + \frac{g}{6} \phi^3(x)] \rangle = 0 \\
 & & \text{Tr}(\delta^{\mu\nu}) = \delta^\mu_\mu = d & \quad x_1 \neq x_2 \neq x \\
 & & & \\
 & \stackrel{\text{E.O.M}}{=} & & \\
 & \triangle \text{ do not forget the regularization} & \partial_\nu J^\mu \approx \dots + \text{regularization terms} & \xrightarrow{\Lambda \rightarrow \infty} -0 ???
 \end{aligned}$$

$\dim[g] = 0$
 $(d-4) \frac{g}{4!} \phi^4(x) >$

$$\text{Tr}(g^{\mu\nu}) = g^{\mu}_{\mu} = d$$

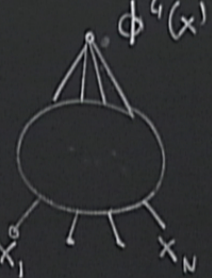
$$x_i \neq x_N \neq x$$

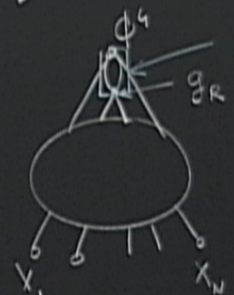
$$\Delta \text{ do not forget the UV regularization}$$

$$\partial_\mu J^\mu \approx \dots + \text{regularization terms} \xrightarrow{\Lambda \rightarrow \infty} -0 ???$$

$$\text{Dimensional regularization}$$

$$\log \Lambda \rightarrow \frac{1}{d-4} \text{ pole}$$

$$(d-4) \frac{g}{4!} \phi^4(x) > = \sum$$




$$\Rightarrow \frac{1}{d-4}$$

$$\text{E.O.M}$$

$$\dim[g] = 0$$

$\text{Tr}(g^{\mu\nu}) = g^{\mu}_{\mu} = d$

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Δ do not forget the UV regularization

Dimensional regularization $\log \Lambda \rightarrow \frac{1}{d-4}$ pole

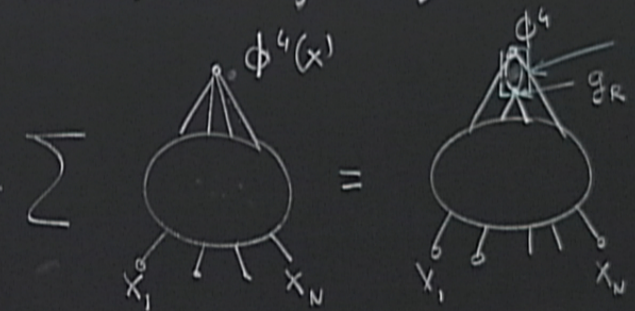
$\dim[g] = 0$

$(d-4) \frac{g}{4!} \phi^4(x) > = \sum$

$\partial_\mu J^\mu \approx \dots + \text{regularization terms} \xrightarrow{\Lambda \rightarrow \infty} -0 ???$

$\Rightarrow \frac{1}{d-4}$ pole

$(d-4) \frac{g}{4!} \phi^4 \Rightarrow \frac{d}{4!}$



$$\text{Tr}(g^{\mu\nu}) = g^{\mu}_{\mu} = d$$

$$X_1 \neq X_N \neq X$$

do not forget the UV regularization

$$\partial_\mu J^\mu \approx \dots + \text{regularization terms} \xrightarrow{\Lambda \rightarrow \infty} \dots$$

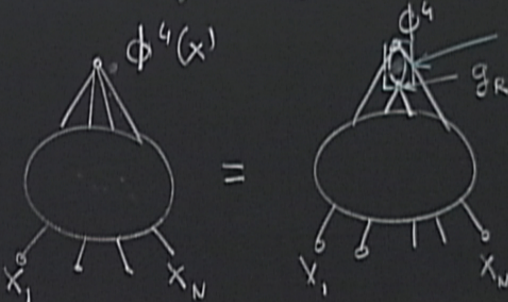
Dimensional regularization

$$\log \Lambda \rightarrow \frac{1}{d-4} \text{ pole}$$

at high momenta

$$= 0$$

$$\phi^4(x) = \sum$$



$$\Rightarrow \frac{1}{d-4}$$

$$(d-4) \frac{g}{4!} \phi^4 \Rightarrow \left(\frac{d-4}{d-4} \right) \frac{g}{4!} g \phi^4$$



$$\phi(x_1) \phi(x_2) \phi(x_3) \dots \phi(x_n) = 0$$

$$x_1 \neq x_2 \neq x_3 \neq \dots \neq x_n$$

$$\phi(x_1) \phi(x_2) \phi(x_3) \dots \phi(x_n)$$

$$4! = 24$$



Final result at 1 loop

$$\langle \phi(x_1) \phi(x_2) \partial_\mu \phi(x) \rangle = 0$$

$$x_1 \neq x_2, \dots \neq x$$

$$\langle \phi(x_1) \phi(x_2) \rangle$$

$$4! \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle$$



Final result at 1 loop

$$\partial_\mu T^\mu(x) = \beta(g) \frac{1}{4!} \phi^4 \neq 0$$

Scale invariance is "broken" by quantum effects

$$\langle \phi(x_1) \phi(x_2) \partial_\mu \phi(x) \rangle = 0$$

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Final result at 1 loop

$$\partial_\mu T^\mu(x) = \beta(g) \frac{1}{4!} \phi^4 \neq 0$$

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$$\beta(g) \neq 0$$

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$$x_1 \neq x_2 \neq x$$

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \rangle = 0$$

$$4! \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = 0$$



Final result at 1 loop

Scale Anomaly

QED

$$\partial_\mu T^\mu(x) = \beta(g) \frac{1}{4!} \phi^4 \neq 0$$

scale invariance is "broken" by quantum effects

$$\beta(g) \neq 0$$

$\phi^4 \neq 0$
quantum effects

Scale Anomaly

QED Axial Anomaly $\Rightarrow$ important consequences
 $\psi_{\text{Dirac}} \rightarrow e^{i\alpha \gamma_5} \psi_{\text{Dirac}}$ massless Dirac Fermions