

Title: 13/14 PSI - Quantum Field Theory II - Lecture 7

Date: Nov 19, 2013 09:00 AM

URL: <http://pirsa.org/13110016>

Abstract:



Renormalization of ϕ^4 QFT for $d=4$ 1

Renormalization of ϕ^4 QFT for $d=4$ 1 loop

$$S[\phi] = \int d^D x \frac{1}{2} (\partial \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4$$

2pt Function $\hat{\Gamma}(p)$

Renormalization of ϕ^4 QFT for $d=4$ 1 loop

$$S[\phi] = \int d^D x \frac{1}{2} (\partial \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4$$

Euclidean

amplitude of the
"tadpole" diagram

Irreducible 2pt Function $\Gamma(p) = (p^2 + m^2) + \frac{g}{2} F(m)$

$$= \text{---} \text{---} + g \frac{1}{2} \text{---} \bigcirc \text{---}$$

Irreducible 4pt

$$\Gamma(p_1, p_2, p_3, p_4) = \text{---} \bigcirc \text{---} =$$

Renormalization of ϕ^4 QFT for $d=4$ 1 loop

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$$= \text{---} \text{---} \text{---} + \frac{g}{2} \text{---} \bigcirc \text{---}$$

Irreducible 4pt Function $\hat{\Gamma}(p_1, p_2, p_3, p_4) = \text{---} \bigcirc \text{---} = g^2 \text{---} \times \text{---} - \frac{g^2}{2} \left(\text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} \right)$

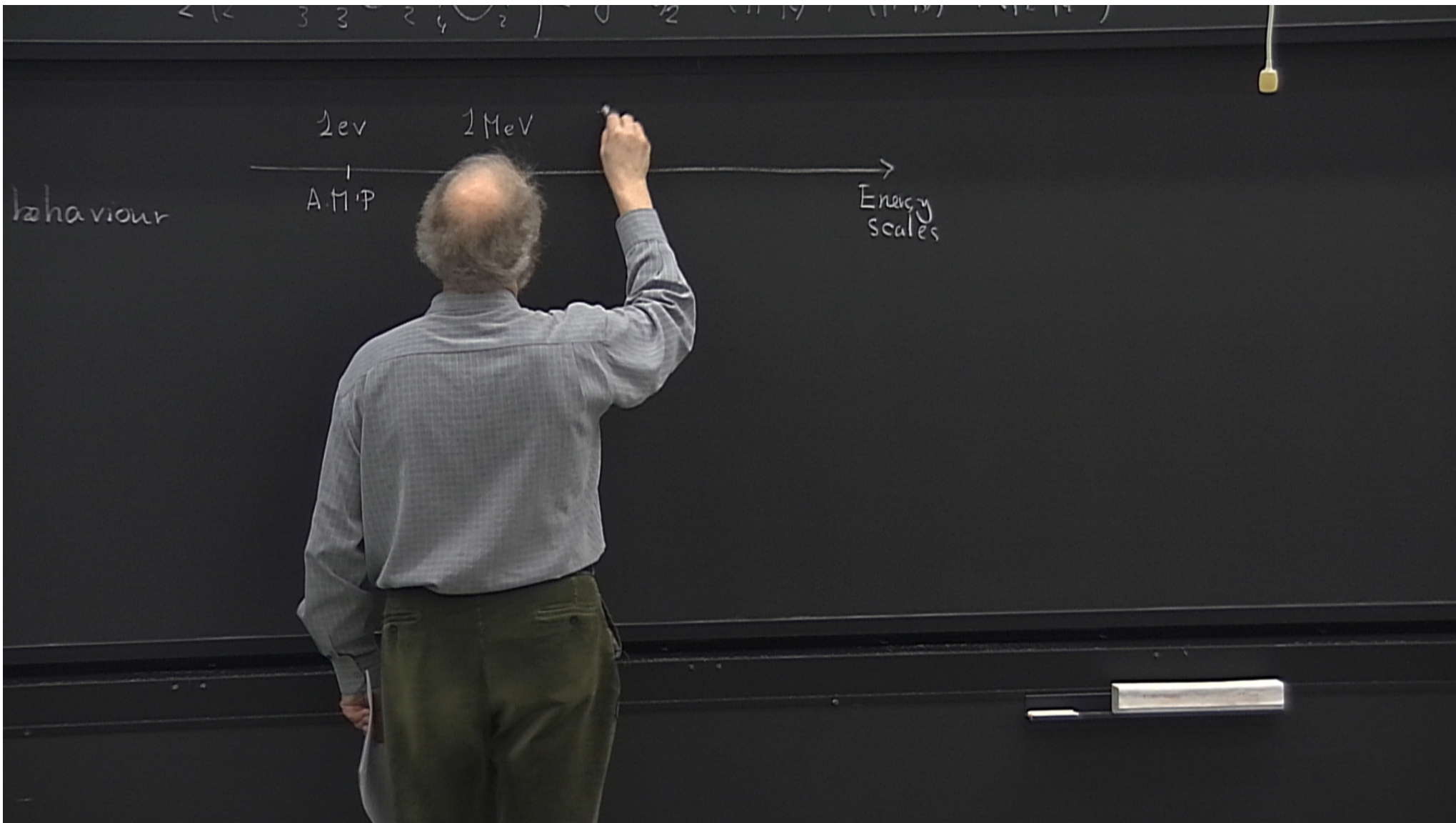
1 loop

amplitude of the
"tadpole" diagram

$$T(m) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} = \text{tadpole diagram}$$

$$= \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} \cdot \frac{1}{(p+k)^2 + m^2} = p \text{ --- } \text{bubble diagram}$$

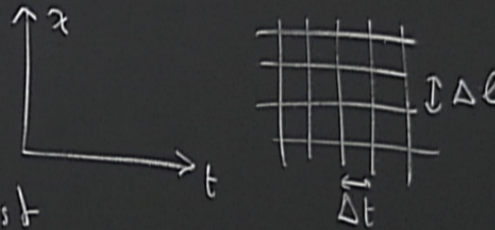
$$K = -\frac{g^2}{2} \left(\text{tadpole diagram} + \text{bubble diagram} \right) = g - \frac{g^2}{2} B(p_1 + p_2, m) + B(p_1 + p_2, m) + B(p_2 + p_1, m)$$



U. V "regulator" or "cut-off"

modify the theory at high momenta $\rightarrow$ better behaviour

- 1) Discretize Space-Time
e.g. Lattice



Lorentz invariance is lost

- 2) Change the propagator in momentum space

$$= \frac{1}{k^2 + m^2} \quad |k| < \Lambda \quad \text{large momentum}$$

$$= 0 \quad |k| > \Lambda \quad \text{regulator}$$

2ev

A.M.P

do we get

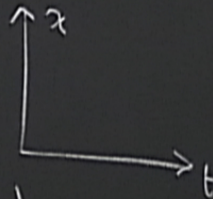
which do n

Planck Sc

U. V "regulator" or "cut-off"

modify the theory at high momenta $\rightarrow$ better behaviour

- 1) Discretize Space-Time
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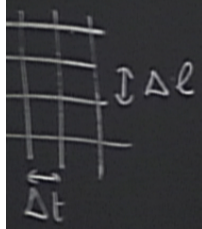
$$\begin{aligned} * &= \frac{1}{k^2 + m^2} & |k| < \Lambda & \text{large momentum} \\ &= 0 & |k| > \Lambda & \text{regulator} \end{aligned}$$

2ev

A.M.P

do we get
which do not
Planck Scale

→ better behaviour



space
imp momentum
t off

2ev	1MeV	1GeV	2TeV	???	Planch Mass
A.M.P	Nucl Phys	Had	LHC Highs Physio	Energy Scales	Q. Gravity

do we get sensible
which do not depend
Planch Scale P

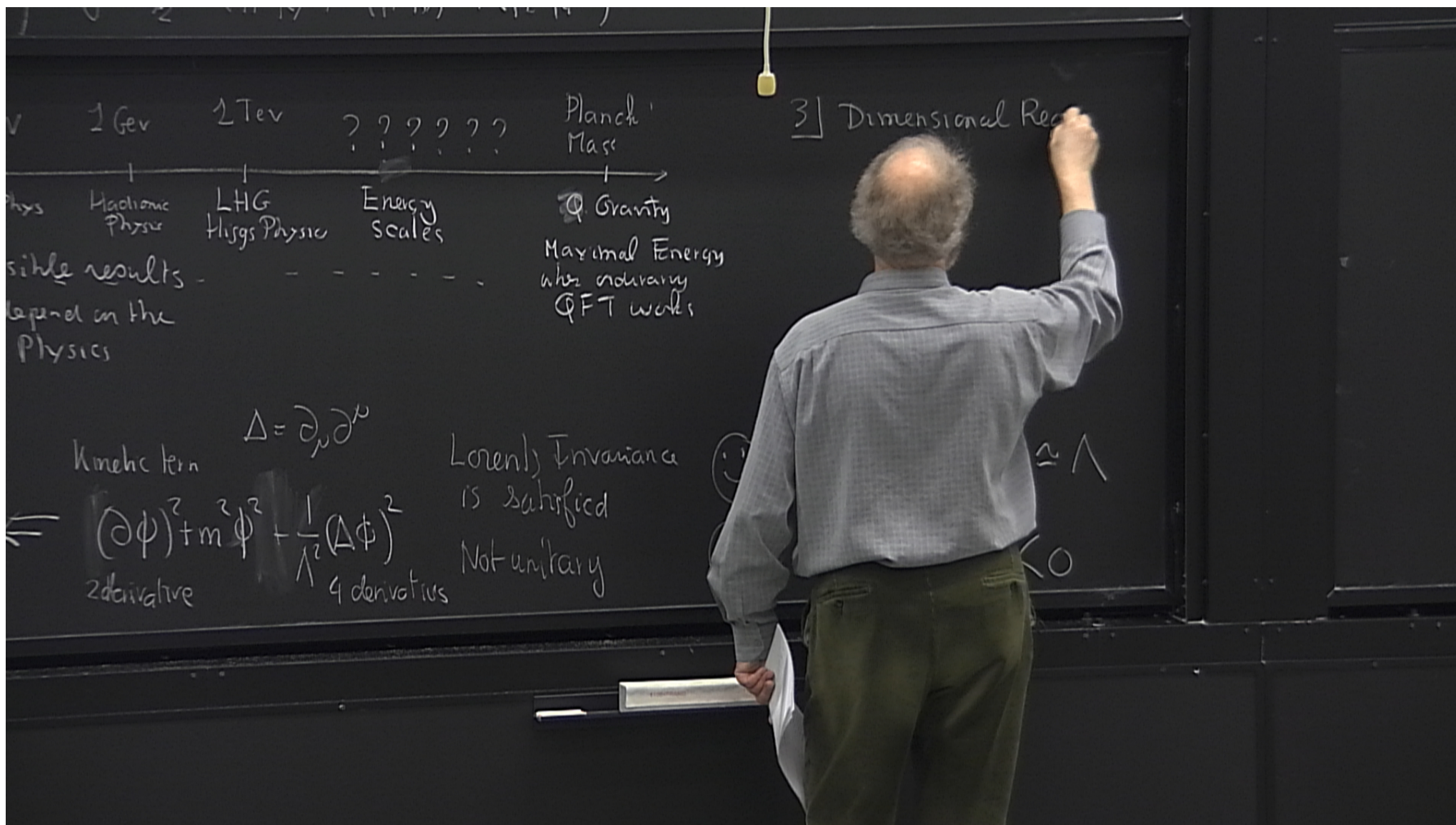
Maximal Energy
where ordinary
QFT works

2'

$$\frac{1}{k^2 + m^2 + \frac{k^4}{\Lambda^2}}$$

Pauli Villan
Regulator

$$+ m^2 \phi^2 + \frac{1}{\Lambda^2} (\Delta \phi)^2$$



Regularization

dimension is
ger. !
M. theories

Λ

< 0

$D=4$ UV divergences

$$T(m; \Lambda) = \int_{|k| < \Lambda} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2} = \cdot \int d^4 k \frac{1}{k^2 + m^2}$$

$$\vec{k} = \vec{u} \cdot k$$

$$|\vec{u}| = 1$$

Regularization

dimension is
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M-theories

Λ

< 0

$D=4$ UV-divergences

$$T(m; \Lambda) = \int_{|\vec{k}| < \Lambda} \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} = \cdot \int_0^\Lambda dk \, k^3 \frac{1}{k^2 + m^2} = \Lambda^2$$

$$\vec{k} = \vec{u} \cdot k$$

$$|\vec{u}| = 1$$

$D=4$ UV divergences

$$T(m; \Lambda) = \int_{|\vec{k}| < \Lambda} \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} = \int_0^\Lambda dk \, k^3 \frac{1}{k^2 + m^2} = \frac{1}{(4\pi)^2} \Lambda^2$$

$$\vec{k} = \vec{u} \cdot k$$

$$|\vec{u}| = 1$$

$$m^2 \text{Log}(\Lambda^2)$$

$D=4$ UV divergences

$$T(m; \Lambda) = \int \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} = \cdot \int_0^\Lambda d\vec{k} \frac{1}{\vec{k}^2 + m^2} = \frac{1}{(4\pi)^2} \Lambda^2 - \frac{1}{(4\pi)^2} m^2 \text{Log}(\Lambda^2) + \text{Finite terms}$$

$$\vec{k} = \vec{u} \cdot k$$

$$|\vec{u}| = 1$$

$$\int d\vec{k} \cdot k \quad \int d\vec{k} \frac{m^2}{k}$$

$D=4$ UV divergences

$$T(m; \Lambda) = \int \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} = \int_0^\Lambda d\vec{k} \cdot \vec{k}^3 \frac{1}{\vec{k}^2 + m^2} = \frac{1}{(4\pi)^2} \Lambda^2 - \frac{1}{(4\pi)^2} m^2 \text{Log}(\Lambda^2) + \text{Finite terms}$$

$$\vec{k} = \vec{u} \cdot k$$

$$|\vec{u}| = 1$$

$$B(p) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} \cdot \frac{1}{(p+k)^2 + m^2}$$

$D=4$ UV divergences

$$T(m; \Lambda) = \int_{|\vec{k}| < \Lambda} \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} = \int_0^\Lambda d\vec{k} \cdot \vec{k}^3 \frac{1}{\vec{k}^2 + m^2} = \frac{1}{(4\pi)^2} \Lambda^2 - \frac{1}{(4\pi)^2} m^2 \text{Log}(\Lambda^2) + \text{Finite terms}$$

$\int d\vec{k} \cdot \vec{k} \quad \int d\vec{k} \frac{m^2}{\vec{k}^2}$

$$\vec{k} = \vec{u} \cdot k$$

$$|\vec{u}| = 1$$

$$B(p, m; \Lambda) = \int_{\substack{|\vec{k}| < \Lambda \\ |p + \vec{k}| < \Lambda}} \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} \cdot \frac{1}{(p + \vec{k})^2 + m^2}$$

$D=4$ UV divergences

$$T(m; \Lambda) = \int \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} = \int_0^\Lambda d\vec{k} \cdot \vec{k}^3 \frac{1}{\vec{k}^2 + m^2} = \frac{1}{(4\pi)^2} \Lambda^2 - \frac{1}{(4\pi)^2} m^2 \text{Log}(\Lambda^2) + \text{Finite terms}$$

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$D=4$ UV divergences

$$T(m; \Lambda) = \int_{|\vec{k}| < \Lambda} \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} = \int_0^\Lambda d\vec{k} \cdot \vec{k} \frac{1}{k^2 + m^2} = \frac{1}{(4\pi)^2} \Lambda^2 - \frac{1}{(4\pi)^2} m^2 \text{Log}(\Lambda^2) + \text{Finite terms}$$

$$\vec{k} = \vec{u} \cdot k$$

$$|\vec{u}| = 1$$

$$B(p, m; \Lambda) = \int_{|\vec{k}| < \Lambda} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2} \cdot \frac{1}{(p+k)^2 + m^2} \approx \int_{|\vec{k}| < \Lambda} \frac{d^4 k}{(2\pi)^4} \frac{1}{(\vec{k}^2)^2}$$

$$|p| \ll \Lambda$$

$$|\vec{k}| < \Lambda$$

$$|p+k| < \Lambda$$

$D=4$ UV divergences

$$T(m; \Lambda) = \int_{|\vec{k}| < \Lambda} \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} = \int_0^\Lambda d\vec{k} \cdot \vec{k}^3 \frac{1}{\vec{k}^2 + m^2} = \frac{1}{(4\pi)^2} \Lambda^2 - \frac{1}{(4\pi)^2} m^2 \text{Log}(\Lambda^2) + \text{Finite terms}$$

$\vec{k} = \vec{u} \cdot k$
 $|\vec{u}| = 1$

$\int d\vec{k} \cdot \vec{k} \quad \int d\vec{k} \frac{m^2}{\vec{k}^2}$

$$B(p, m; \Lambda) = \int_{|\vec{k}| < \Lambda} \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{\vec{k}^2 + m^2} \cdot \frac{1}{(p + \vec{k})^2 + m^2} \approx \int_{|\vec{k}| < \Lambda} \frac{d^4 \vec{k}}{(2\pi)^4} \frac{1}{(\vec{k}^2)^2} = \frac{1}{(4\pi)^2} \text{Log}(\Lambda^2) + \text{Finite terms}$$

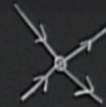
$|p| < \Lambda$
 $|p + \vec{k}| < \Lambda$

$(p, m, \dots)$
 independent of p & m

$$= 0 \quad |k| > \Lambda \quad \text{cut off}$$

$$\Lambda^2$$

massive spin 0 particles + contact repulsive
(K.G.)



$$= 0 \quad |k| > \Lambda \quad \text{cut off} \quad \Lambda^2$$

massive spin 0 particles + contact repulsive interactions
(K.G.)



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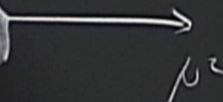
$$\Lambda^2$$

massive spin 0 particles + contact repulsive interactions (local)



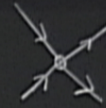
spectrum of the theory: particle + n-particle states

spectral representation,



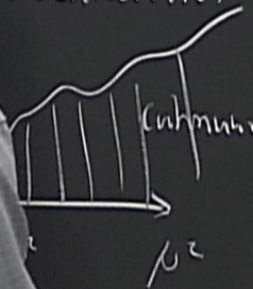
$$= 0 \quad |k| > \Lambda \quad \text{cut off} \quad \Lambda^2$$

massive spin 0 particles + contact repulsive interactions (local)



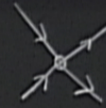
Spectrum of the theory: particle + n-particle states

Sp presentation:



$$= 0 \quad |k| > \Lambda \quad \text{cut off}$$

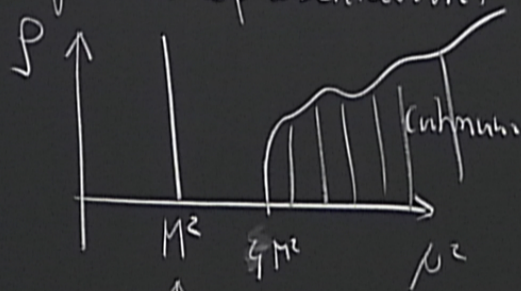
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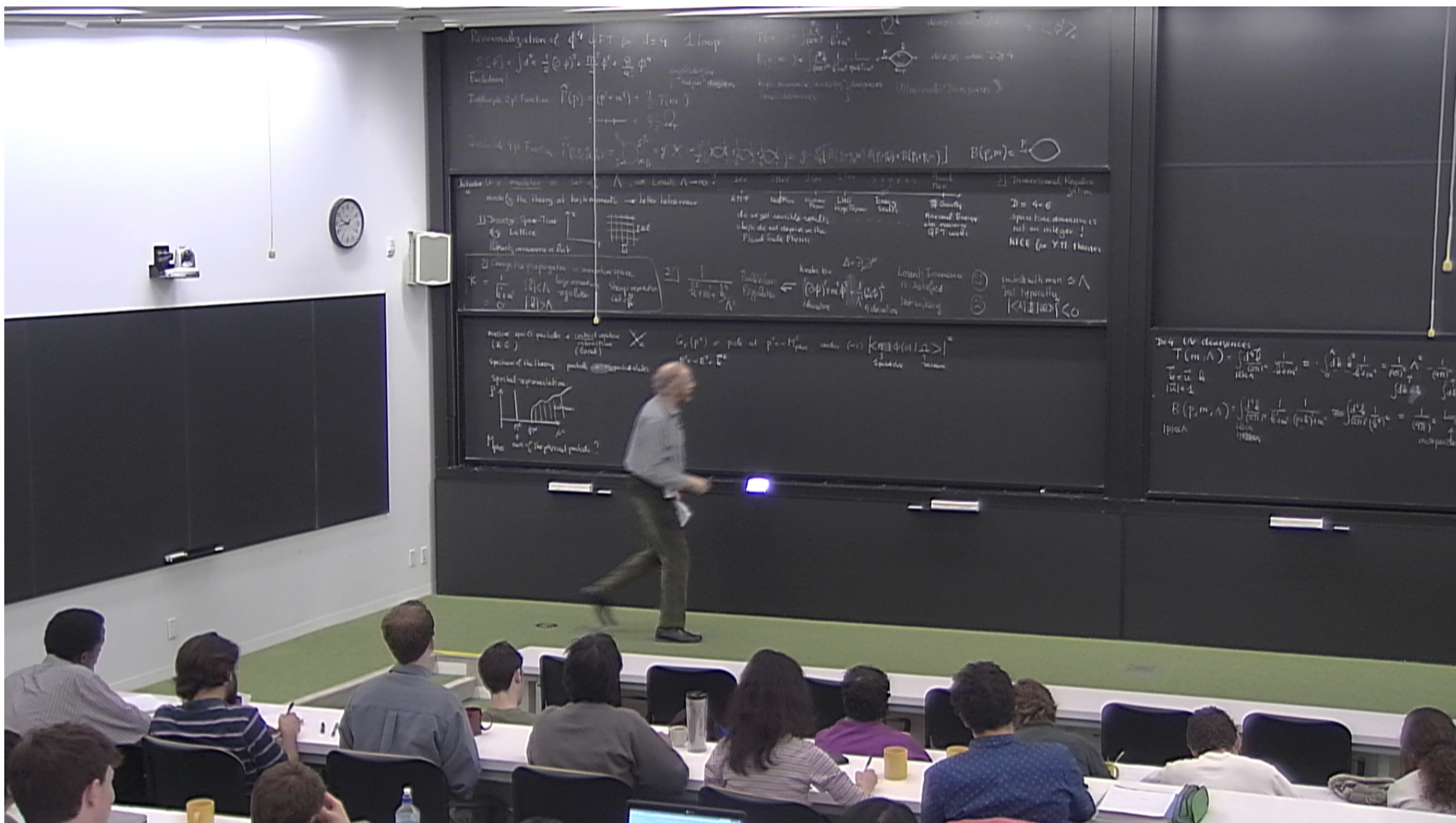
$$G_F(p^2) = \frac{1}{p^2 + M^2}$$

Spectrum of the theory: particle + n-particle states

Spectral representation



mass of the physical particle?



cut off

Λ^2

2 derivative

4 derivatives

attract repulsive
interactions
(local)



de + n-particle states

$G_F(p)$ pole at $p^2 = -M_{\text{phys}}^2$, residue $(-i) \left\langle \underbrace{\Omega}_{\substack{\uparrow \\ \text{2 particle state}}} \mid \underbrace{\phi(0)}_{\substack{\uparrow \\ \text{field operator}}} \mid \underbrace{\Omega}_{\substack{\uparrow \\ \text{vacuum}}} \right\rangle$

$p^2 = -E^2 + \vec{p}^2$

$$\Gamma(p) = p^2 + \underbrace{m^2 + g \frac{1}{2} T(m)}_{\text{Mass}} + O(g^2)$$

$$M_{\text{phys}}^2 = m^2 + g \frac{1}{2} T(m) + O(g^2) \neq m$$

Cut off

Λ^2

2 derivative

4 derivatives

attract repulsive
interactions
(local)



de + n-particle states

Mass

$G_F(p)$ pole at $p^2 = -M_{phys}^2$, residue $(-i) \langle \Omega | \phi(0) | \Omega \rangle$

$$p^2 = -E^2 + \vec{p}^2$$

$\uparrow$ particle state
 $\uparrow$ vacuum
field operator

$$G(p)^{-1} = \Gamma(p) \left(p^2 + m^2 + g \frac{1}{2} T(m) + O(g^2) \right)$$

Physical Mass

$$p^2 + m^2 + g \frac{1}{2} T(m) + O(g^2)$$

Renormal

Renormalization: Express physical correlation functions in terms of measurable parameters

Quantization

$$\int_{\Lambda}^{\text{cut-off}} \mathcal{D}[\phi] \exp(-S_{\text{REN}}(\phi)) \quad \text{renormalized action}$$

$$S_{\text{REN}}(\phi) = \int d^4x \left[\frac{A}{2} (\partial\phi)^2 + \frac{B}{2} \phi^2 + \frac{E}{4!} \phi^4 \right]$$

which A, B, E should take to get a sensible physical theory when $\Lambda \rightarrow \infty$

Massless ϕ^4 theory (First)

$$S(\phi) = \int \left[\frac{1}{2} (\partial\phi)^2 + \frac{g}{4} \phi^4 \right] d^4x$$

Scale Invariance in $d=4$

$$\phi(x) \rightarrow \phi_\lambda(x) = \lambda \phi(\lambda x) \quad \text{Scale Transformation}$$

$$S[\phi_\lambda] = S[\phi]$$

Renormalization. Express physical correlation functions in terms of measurable parameters

Quantization

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Scale Invariance in $d=4$

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$$S[\phi_\lambda] = S[\phi]$$

theory (Fust)

$$\frac{1}{2}(\partial\phi)^2 + \frac{g}{4}\phi^4 d^4x$$

in $d=4$: g is dimensionless

$\phi(x) = \lambda \phi(\lambda x)$ Scale Transformation

$S[\phi]$ g being fixed

$$1, \dim[g]=0$$

Quantum massless theory

take

$$A=1$$

$$= 0$$

$$g \frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2$$

$$M_{\text{phys}}^2 = 0$$

theory (Fierz)

$$\frac{1}{2}(\partial\phi)^2 + \frac{g}{4}\phi^4 \int d^4x$$

in $d=4$: g is dimensionless

$\phi(\lambda x) = \lambda \phi(x)$ Scale Transformation

$S[\phi]$ g being fixed

$$1, \dim[g]=0$$

Quantum massless theory

take

$$M_{\text{phys}}^2 = 0$$

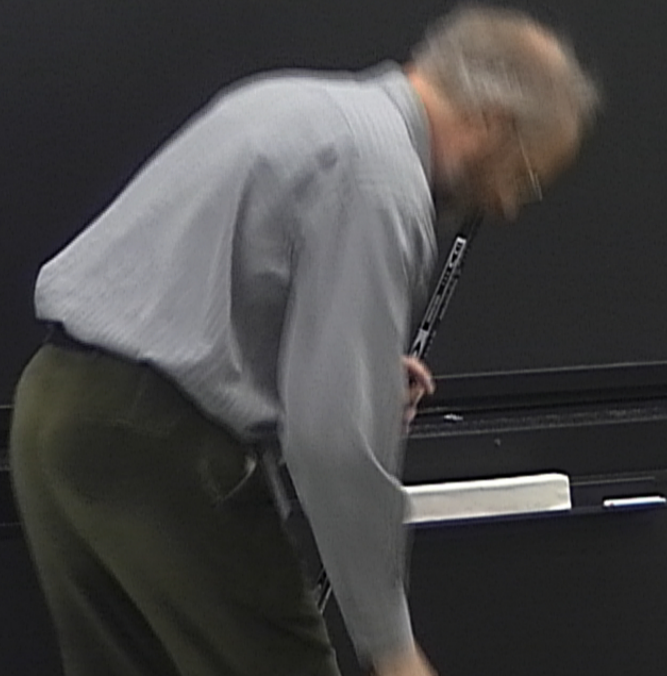
$$A = 1 + \phi \cdot g + O(g^2)$$

$$B = 0 - g \frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2 + O(g^2)$$

$$C = g + \dots$$

when $\Lambda \rightarrow \infty$

What is a "physical" coupling?



coupling ?

4 pt function
for the massless
theory

$$B(p, m=0) \approx \left[\frac{1}{(4\pi)^2} \log(\Lambda^2/p^2) + \text{const.} \right]$$

depends on the
regularization
(not important)

renormalization scale $\mu \ll \Lambda$
(mass or energy)

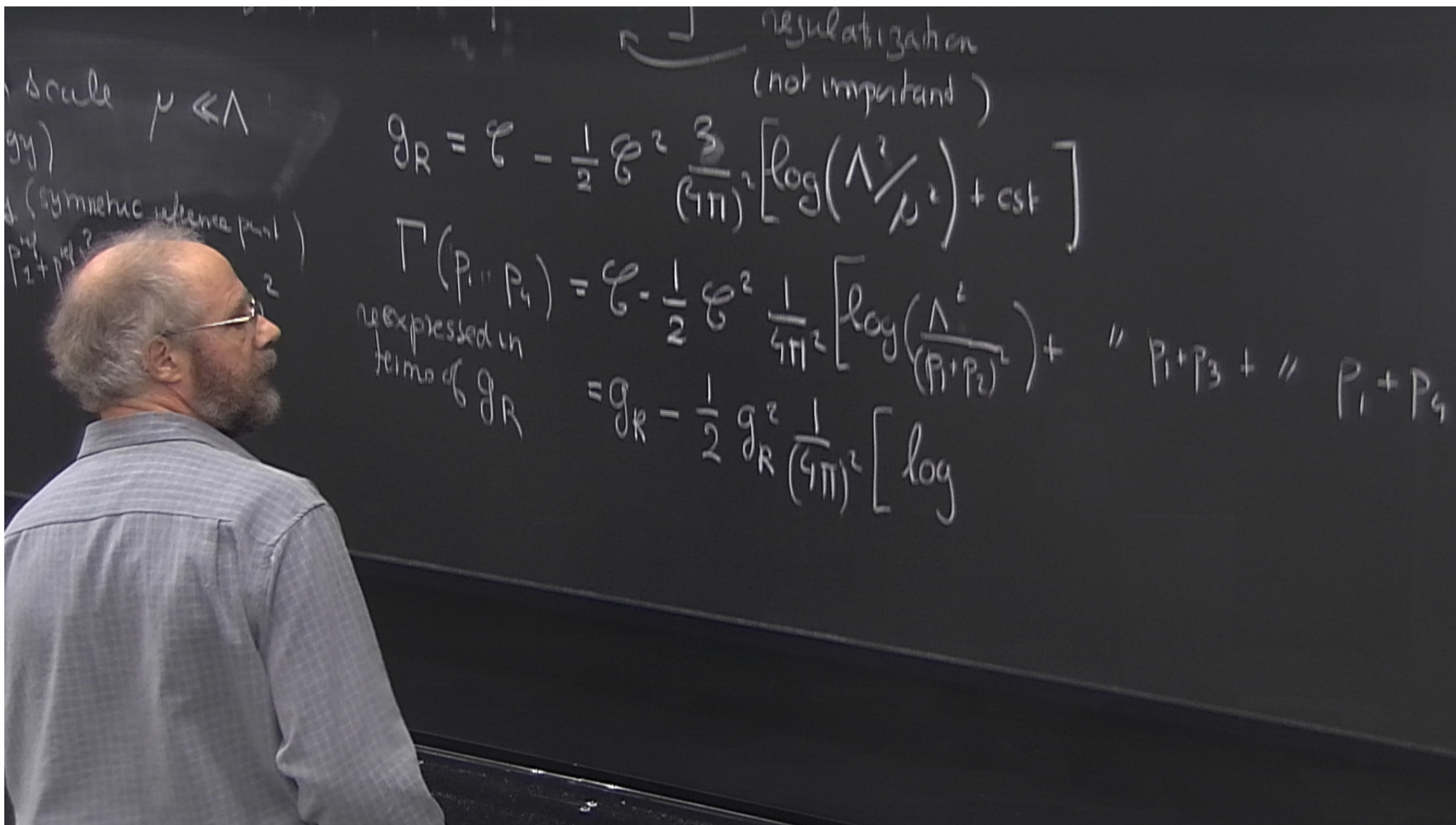
For simplicity (symmetric reference point)

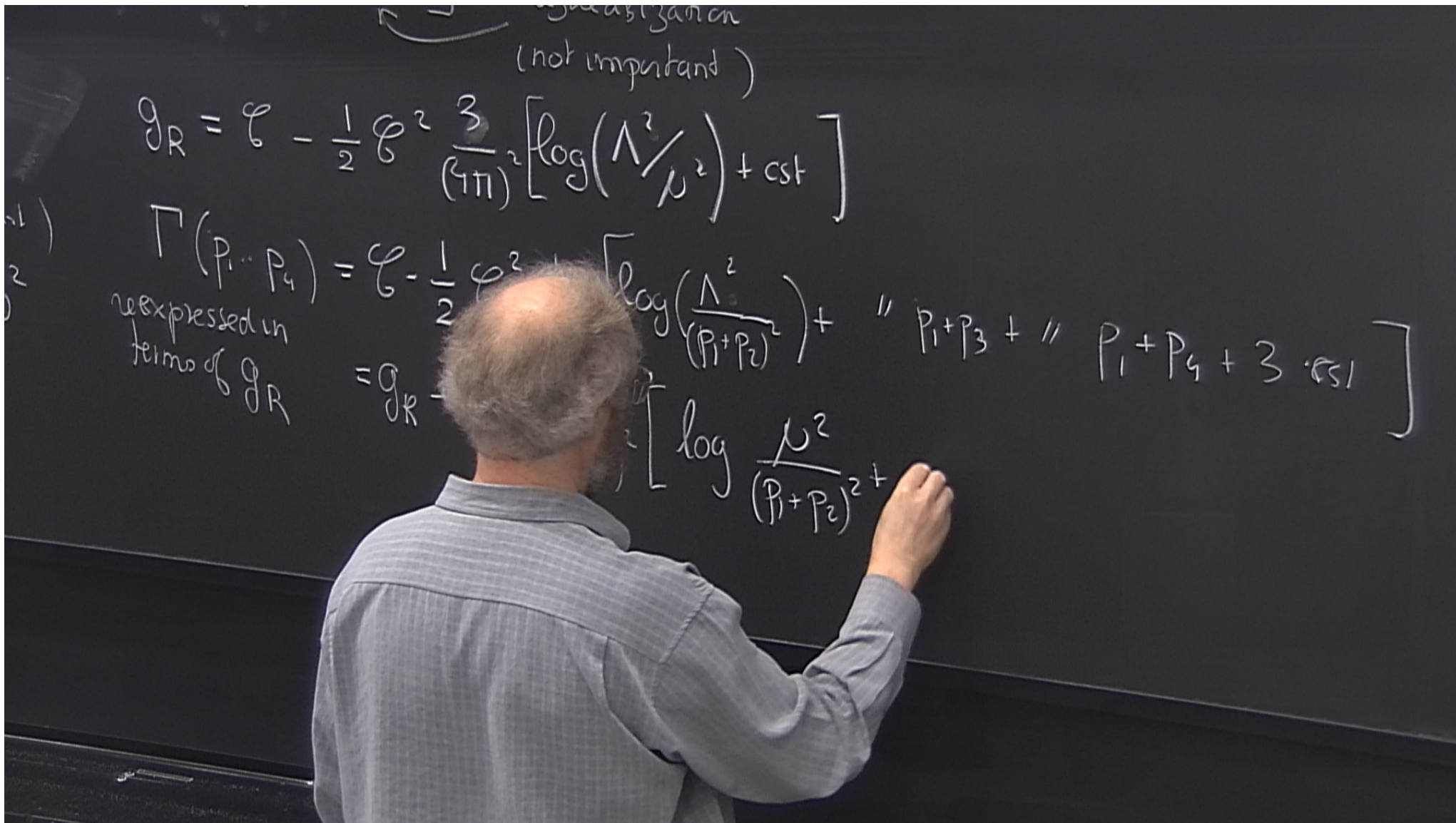
$$(p_1^{\text{rel}} + p_2^{\text{rel}})^2 = (p_1^{\text{rel}} + p_3^{\text{rel}})^2 = (p_1^{\text{rel}} + p_4^{\text{rel}})^2$$

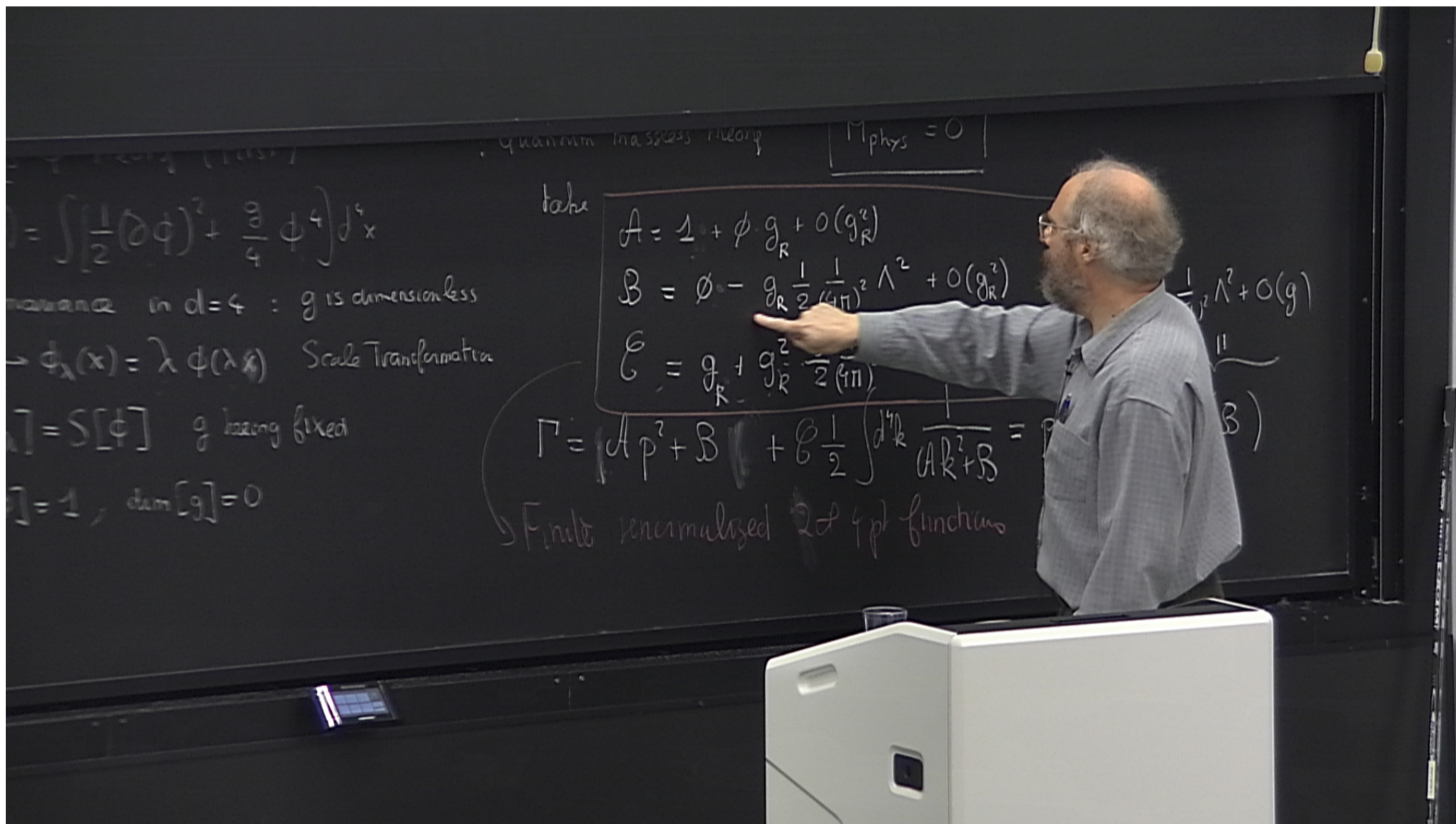
coupling

At a reference point
in momentum space

$$g - \frac{1}{2} g^2 \frac{1}{\epsilon}$$

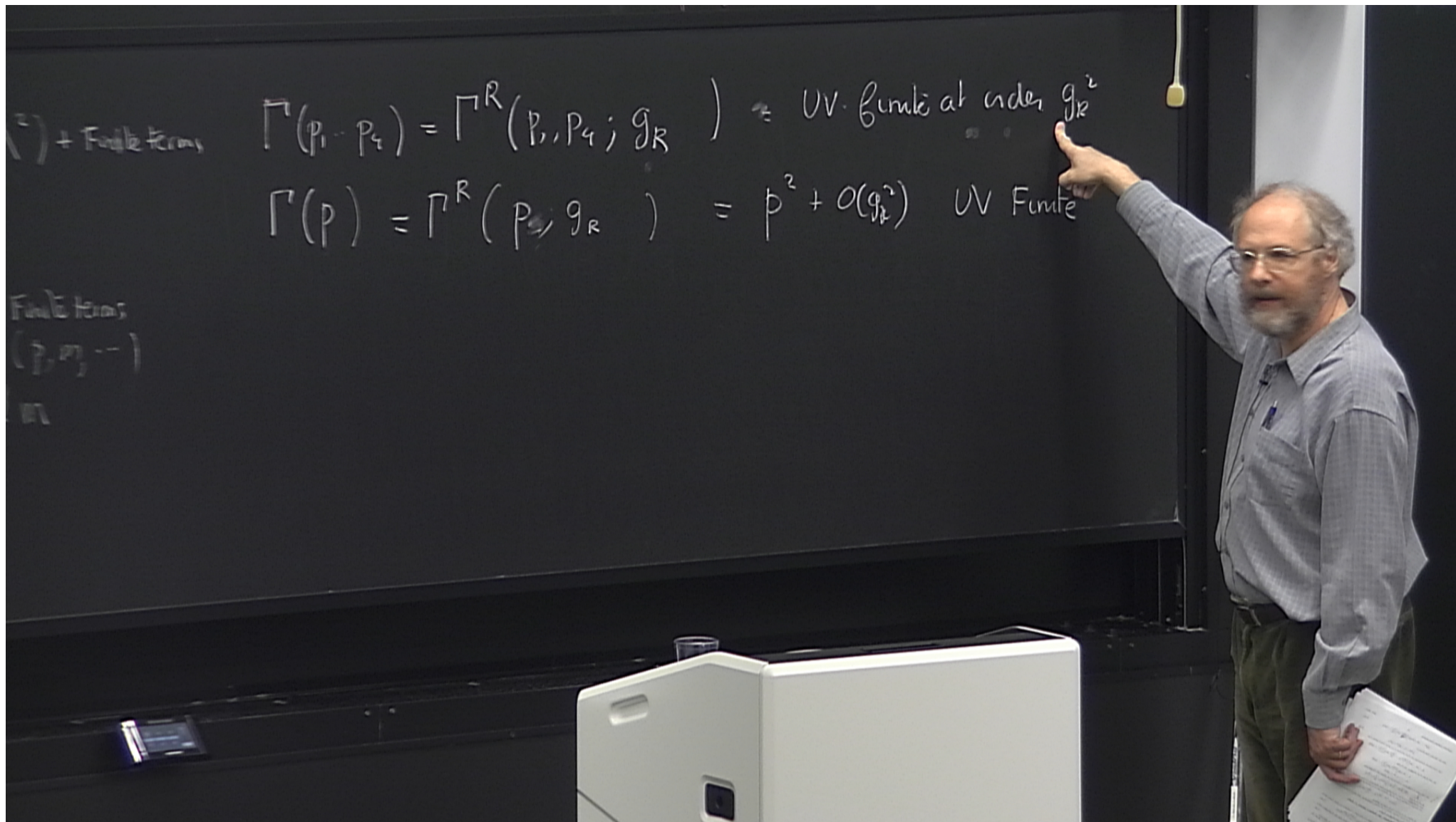


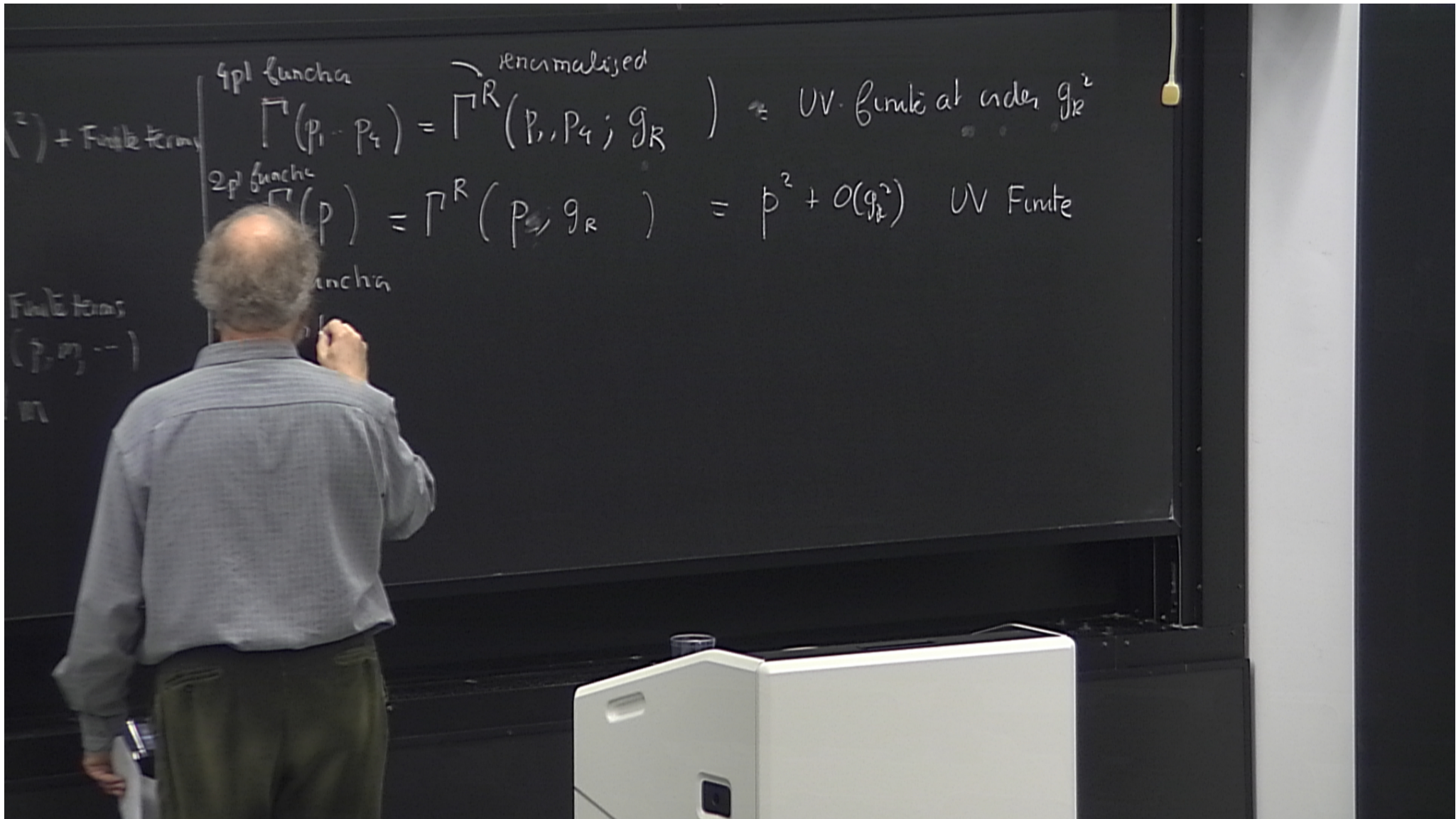




the terms $\Gamma(p_1, p_4) = \Gamma^R(p_1, p_4; g_R)$ = UV. finite at

$$\Gamma(p) = \Gamma^R(p; g_R) =$$





4pt function $\Gamma(p_1, p_4) = \overset{\text{renormalized}}{\Gamma^R}(p_1, p_4; g_R)$ UV. finite at order g_R^2

2pt function $\Gamma(p) = \Gamma^R(p, g_R) = p^2 + O(g_R^2)$ UV Finite

Finite terms
(p, p, ...)

Renormalized 4 pt function
depends both on g_k and on μ ?
what does it means ?

Physical Theory with $\Rightarrow$ 4pt function
well defined N -pt functions

Renormalized 4 pt function

depends both g_R and on μ ?

what does μ mean?

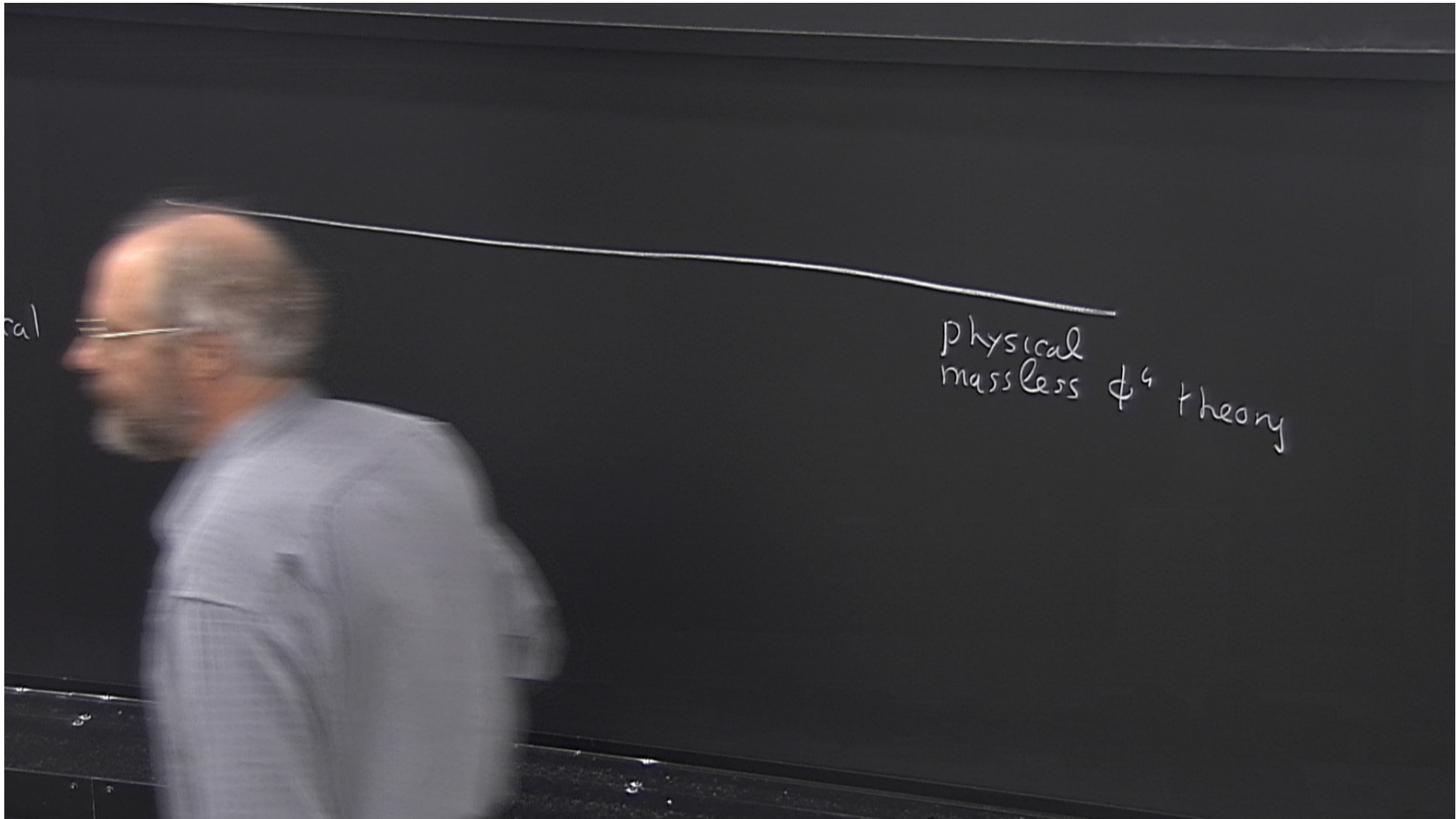
Physical quantities with $\Rightarrow$ 4pt function
well defined functions

choose a
renormalization scale

μ

choose a different
 μ'

renormalized
coupling
 g_R



First $g \rightarrow 0$, then $\Lambda \rightarrow \infty$

Today
in pert theory

non perturbative
QFT

First $g \rightarrow 0$, then $\Lambda \rightarrow \infty$

$\Lambda \rightarrow \infty$, then g_R per