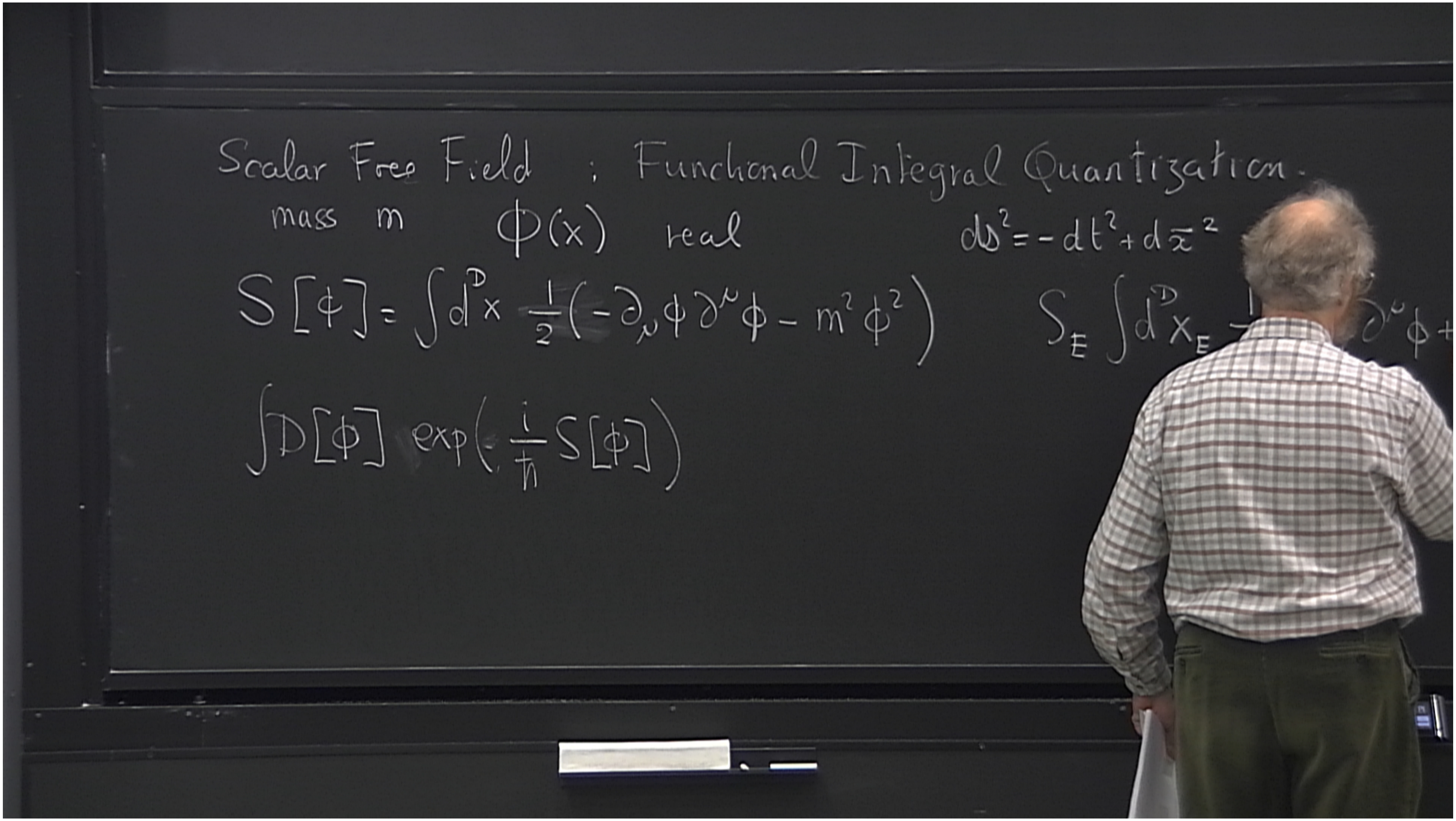
Title: 13/14 PSI - Quantum Field Theory II - Lecture 3

Date: Nov 13, 2013 09:00 AM

URL: http://pirsa.org/13110010

Abstract:


Scalar Free Field : Functional Integral Quantization.
mass m
$\phi(x)$ real
$ds^2 = -dt^2 + d\vec{x}^{\,2}$
$S[\phi] = \int d^D x \, \frac{1}{2}\left(-\partial_\nu \phi \, \partial^\nu \phi - m^2 \phi^2\right)$
$S_E \int d^D x_E$
$\int D[\phi] \, \exp\left(\frac{i}{\hbar} S[\phi]\right)$

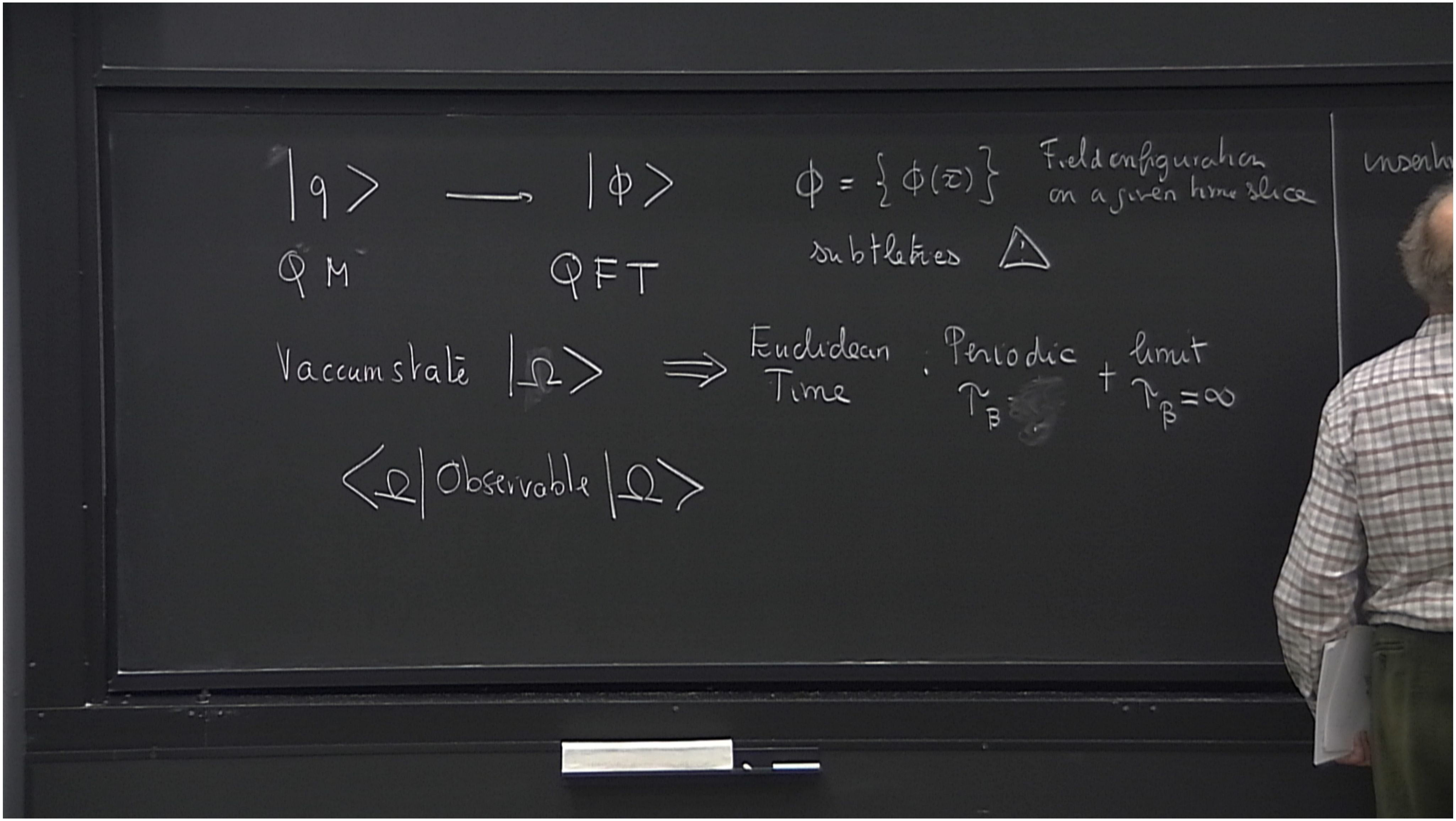
$|q\rangle \longrightarrow |\phi\rangle$ $\qquad \phi = \{\phi(\bar{x})\}$ Field configuration on a given time slice

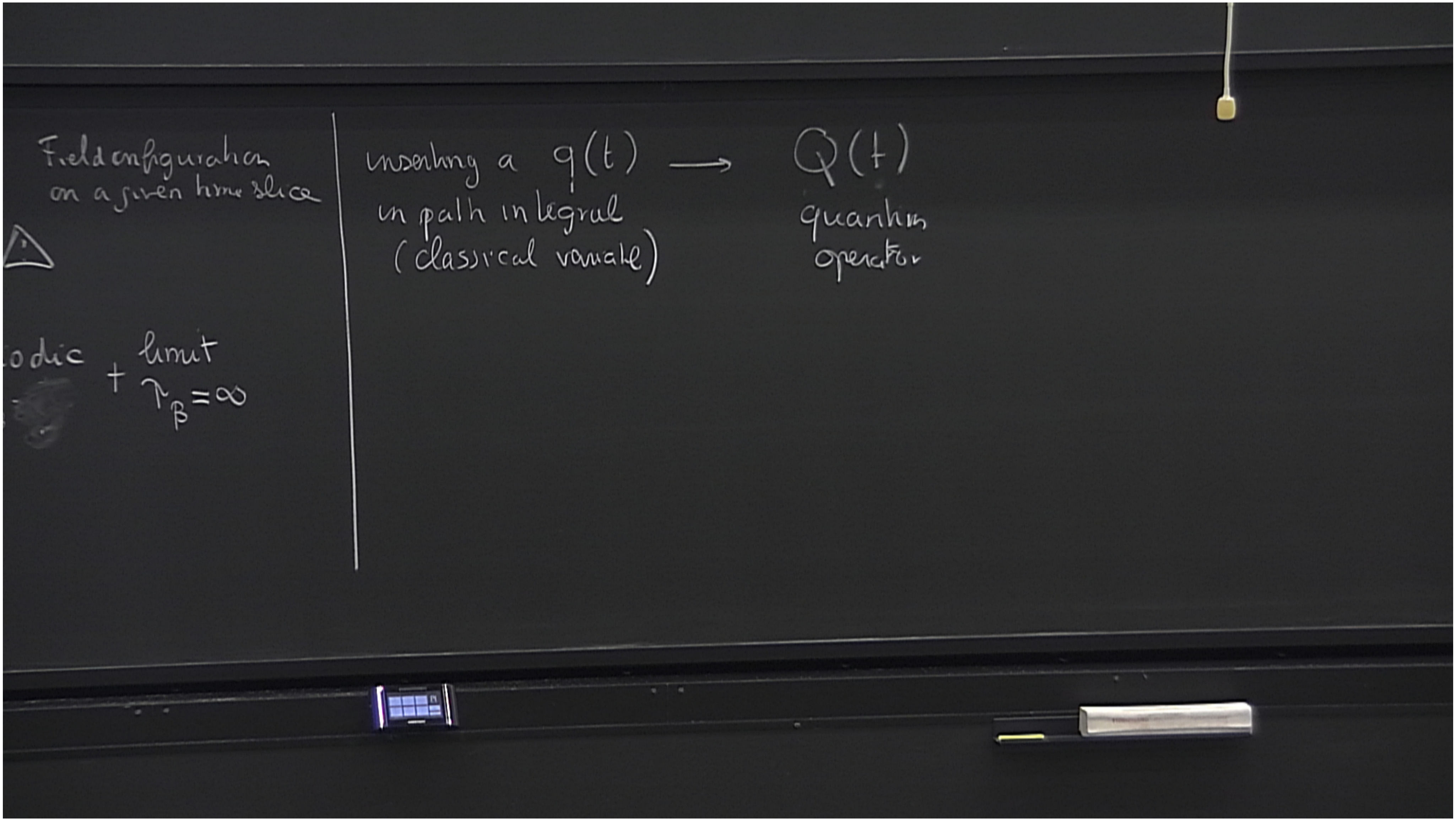
QM $\qquad$ QFT

$$|q\rangle \longrightarrow |\phi\rangle \qquad \phi = \{\phi(\bar{x})\}$$

Field configuration on a given time slice

QM $\qquad$ QFT


|q> ⟶ |φ>
QM
QFT
φ = {φ(x)}
Field configuration on a given time slice
subtleties
Vaccum state |Ω> ⟹ Euclidean Time : Periodic τ_β + limit τ_β = ∞
<Ω| Observable |Ω>


Field configuration
on a given time slice
odic
+ limit
$\tau_\beta = \infty$
inserting a $q(t)$ ⟶ $Q(t)$
in path integral
(classical variable)
quantum
operator

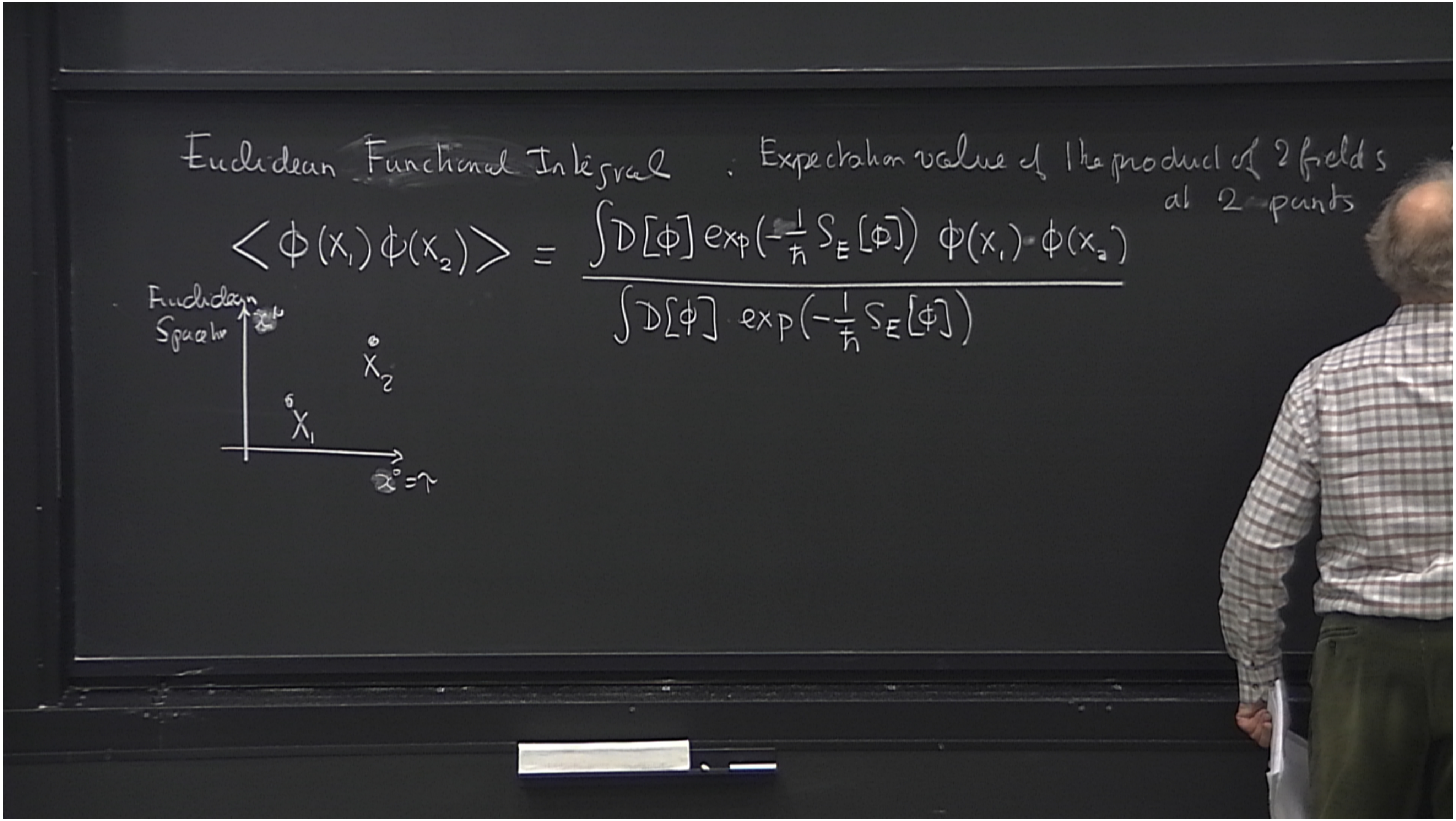
Field configuration
on a given time slice

odic + limit $\tau_\beta = \infty$

inserting a $q(t)$ $\longrightarrow$ $Q(t)$
in path integral (classical variable) — quantum operator

inserting a $\phi(x)$ $\longrightarrow$ $\Phi(x)$?
Field operator

Idea Insert $\phi(x_1)\cdots\phi(x_n)$ in Euclidean Path Integral


Euclidean Functional Integral : Expectation value of the product of 2 fields at 2 points
$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{\int D[\phi]\, \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)\, \phi(x_1)\,\phi(x_2)}{\int D[\phi]\, \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)}$$
Euclidean Spacetime
x^μ
x_2
x_1
$x^0 = \tau$

...an Functional Integral . Expectation value of the product of 2 fields = 2 point function at 2 points

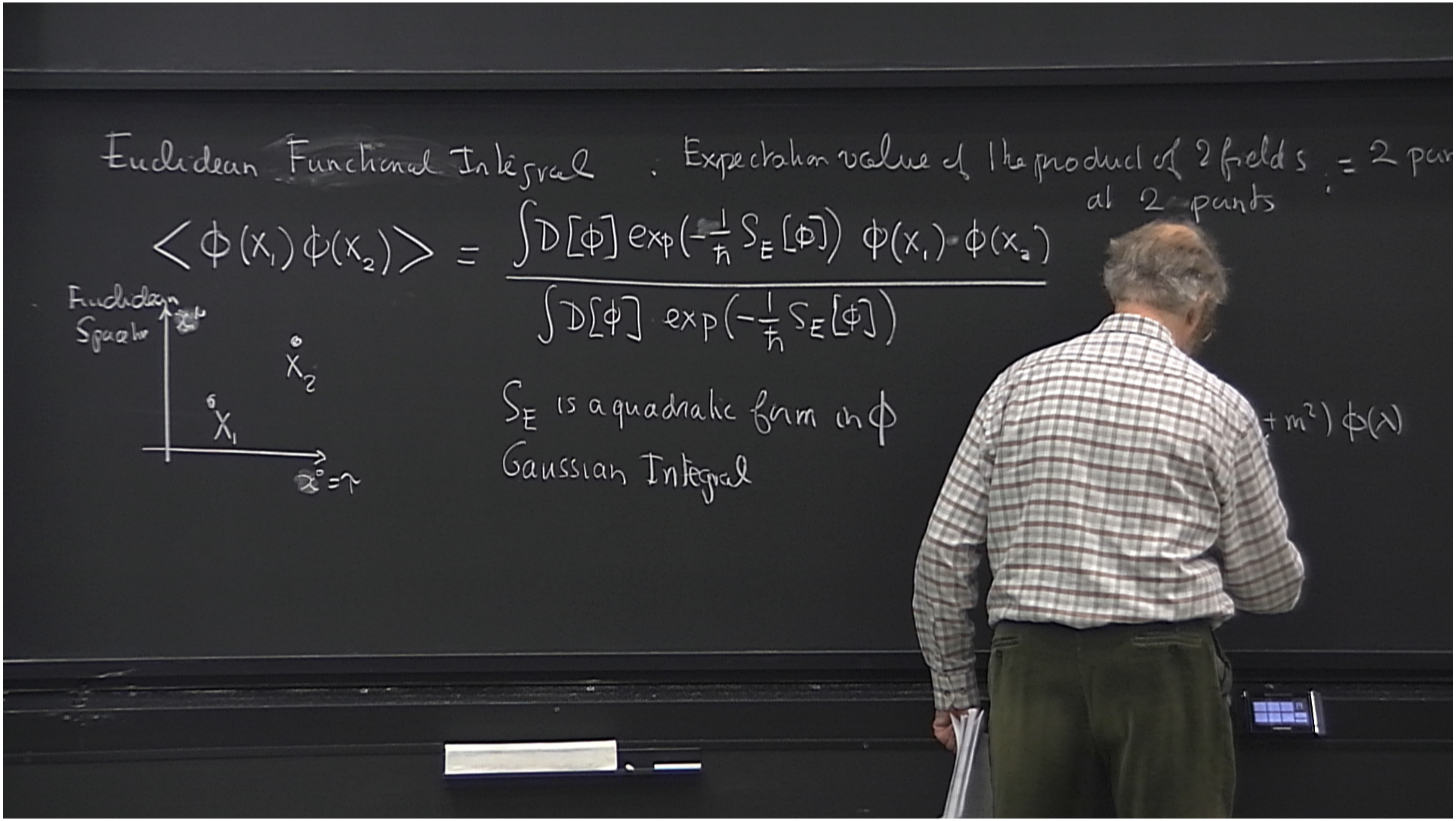
$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{\int D[\phi]\, \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)\, \phi(x_1)\,\phi(x_2)}{\int D[\phi]\, \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)}$$

x_2

$x^0 = \tau$

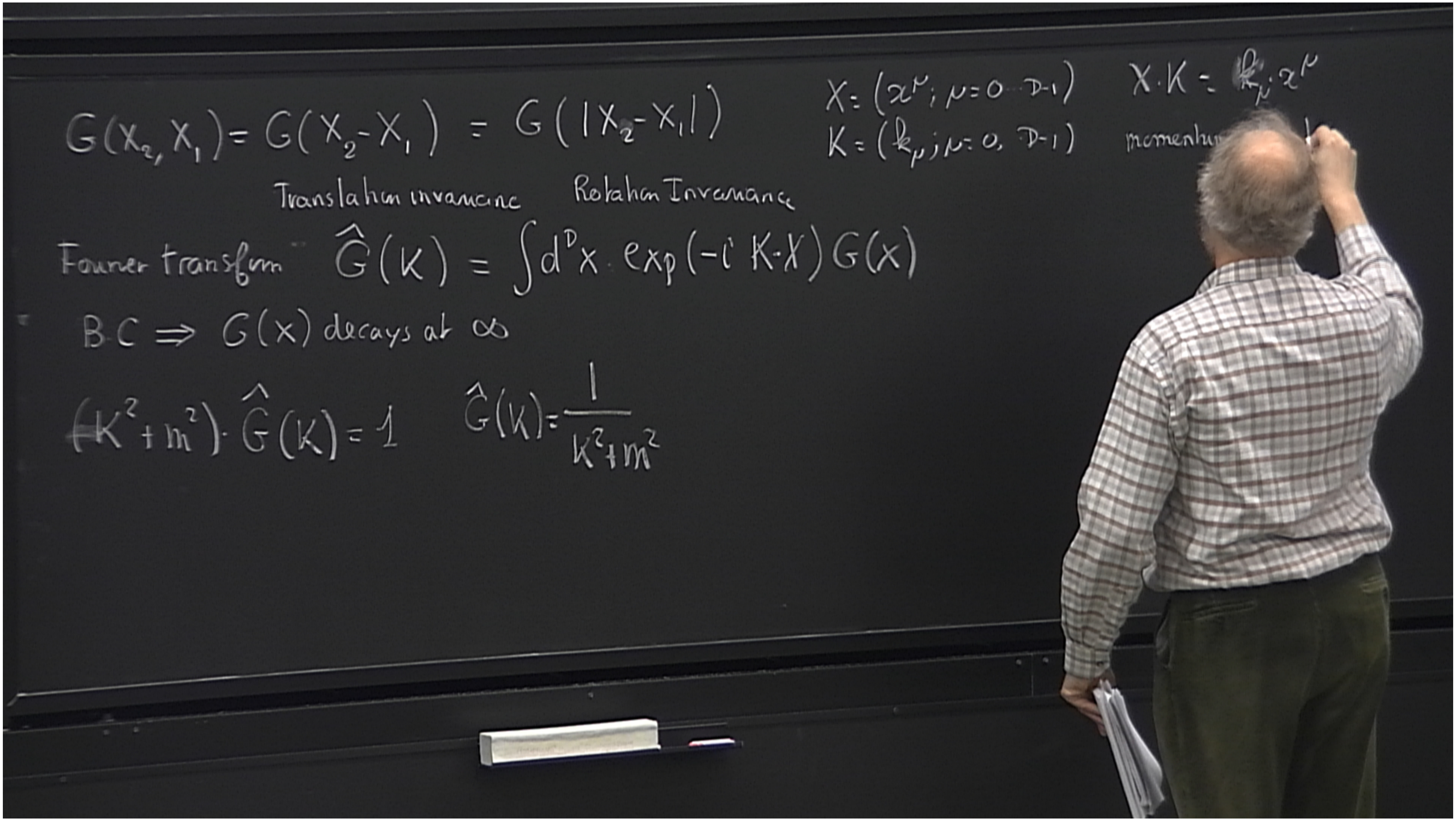
S_E is a quadratic form in ϕ

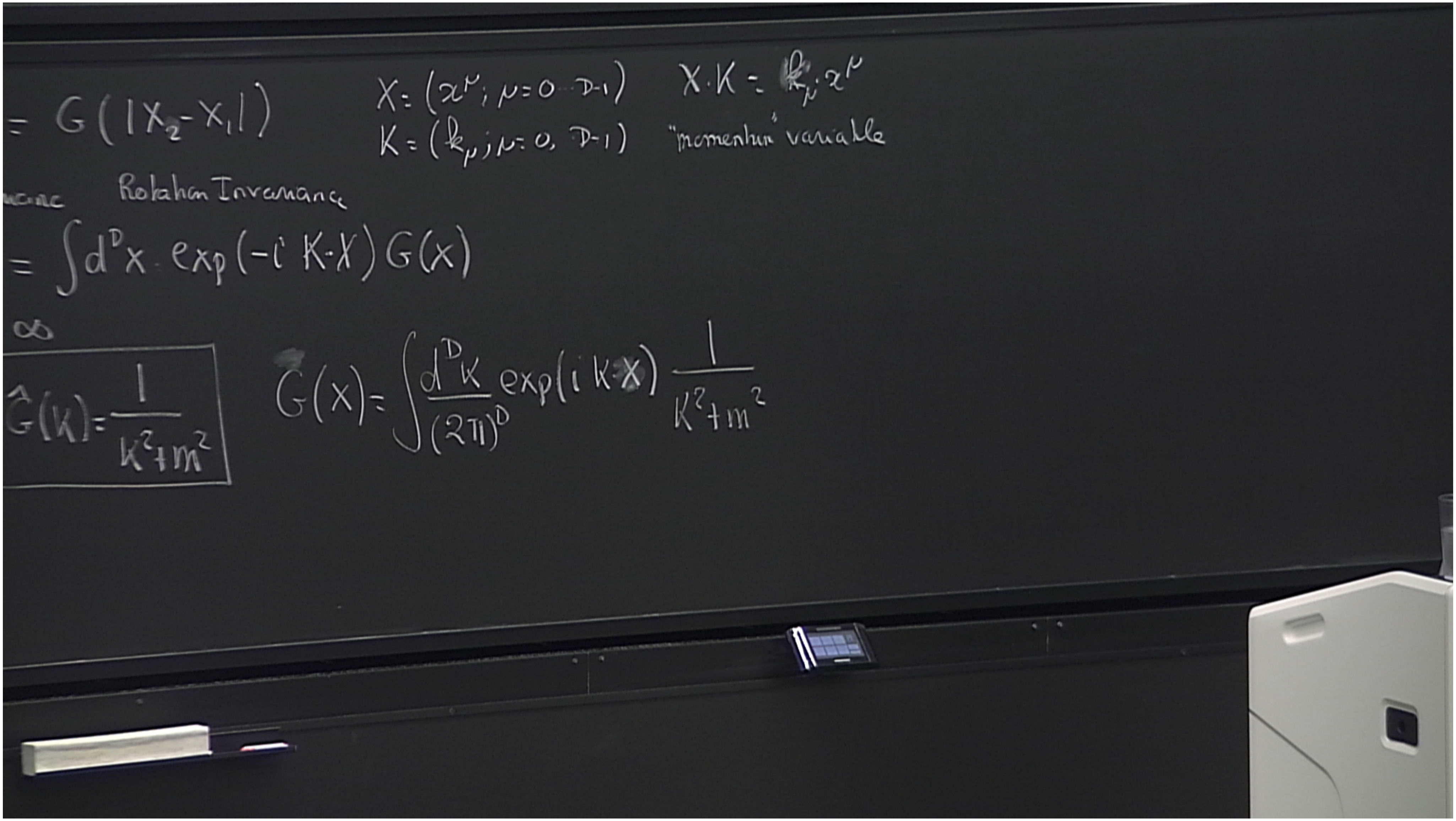
Gaussian Integral

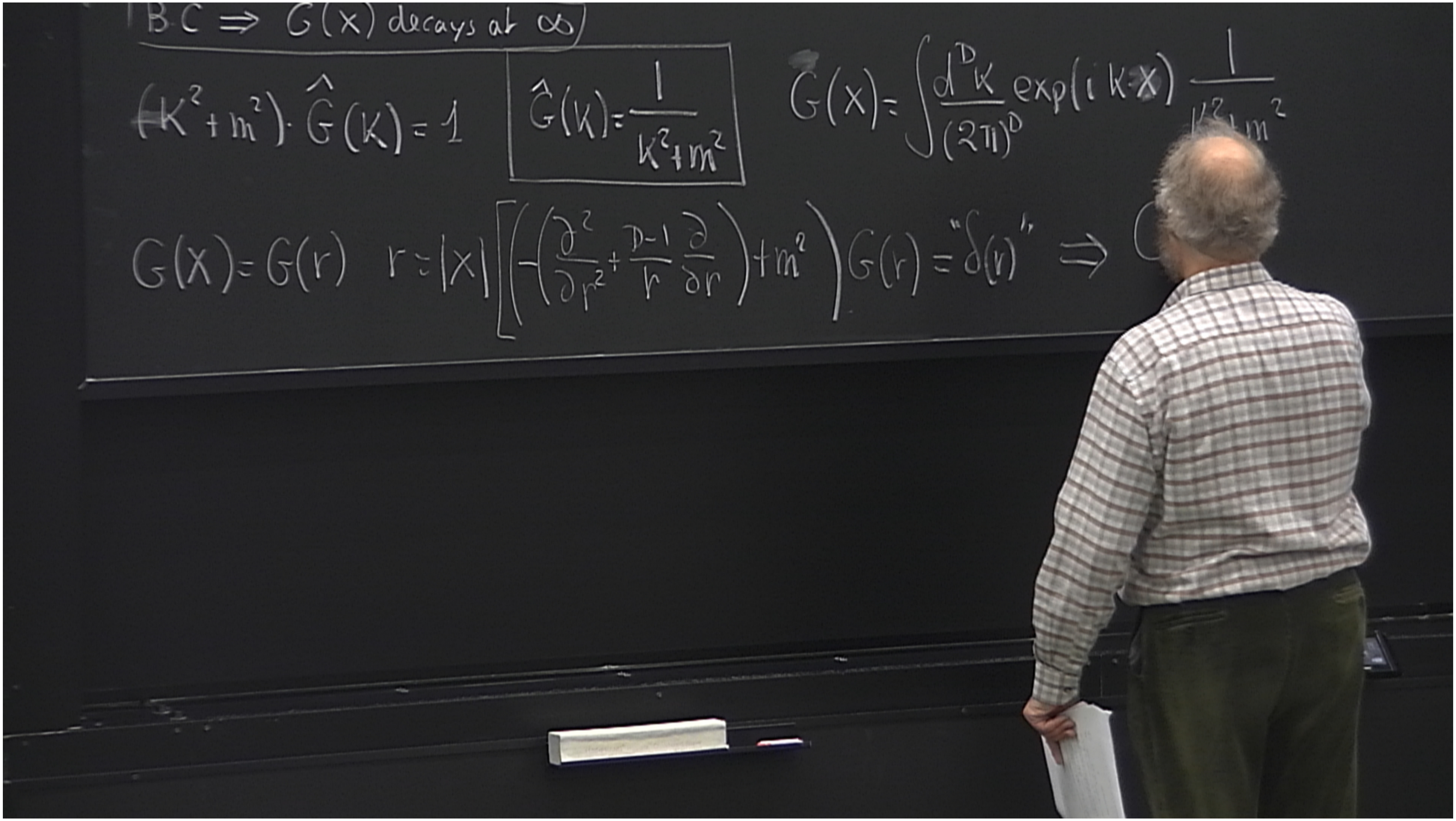

Euclidean Functional Integral
Expectation value of the product of 2 fields at 2 points
$\langle \phi(x_1)\phi(x_2)\rangle = \dfrac{\int D[\phi]\exp(-\frac{1}{\hbar}S_E[\phi])\,\phi(x_1)\phi(x_2)}{\int D[\phi]\exp(-\frac{1}{\hbar}S_E[\phi])}$
Euclidean Spacetime
x_1
x_2
$x^0 = \tau$
S_E is a quadratic form in ϕ
Gaussian Integral
$+ m^2)\,\phi(x)$

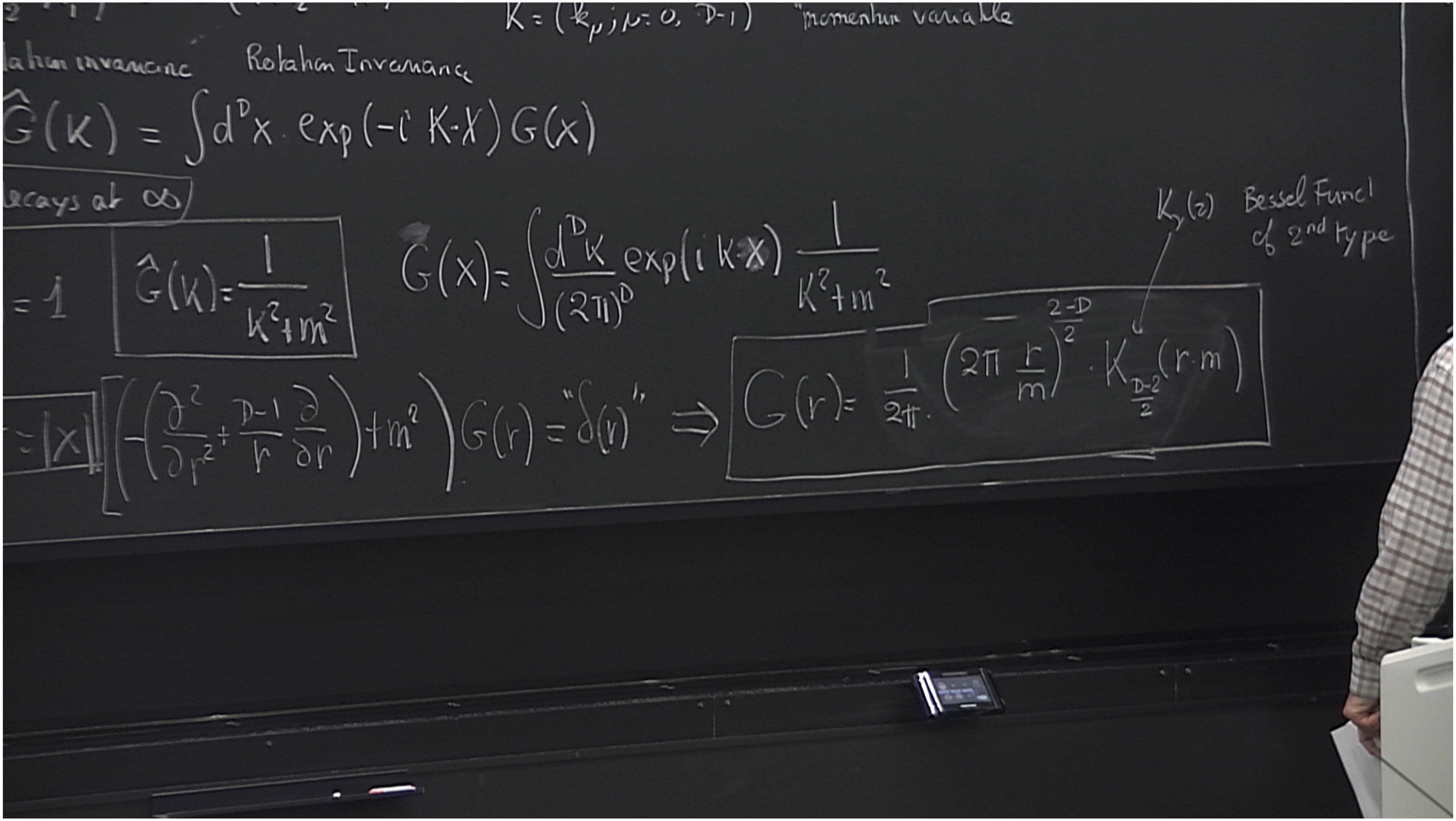
$$G(x_2, x_1) = G(x_2 - x_1) = G(|x_2 - x_1|)$$

Translation invariance   Rotation Invariance


$G(X_2, X_1) = G(X_2 - X_1) = G(|X_2 - X_1|)$
Translation invariance
Rotation Invariance
$X = (x^\mu ; \mu = 0 \cdots D-1)$
$K = (k_\mu ; \mu = 0, D-1)$
$X \cdot K = k_\mu x^\mu$
momentu
Fourier transform $\hat{G}(K) = \int d^D x \, \exp(-i K \cdot X) G(x)$
B.C $\Rightarrow$ $G(x)$ decays at ∞
$(K^2 + m^2) \cdot \hat{G}(K) = 1$
$\hat{G}(K) = \frac{1}{K^2 + m^2}$


$= G(|X_2 - X_1|)$
$X = (x^\mu ; \mu = 0 \dots D-1)$ $X \cdot K = k_\mu x^\mu$
$K = (k_\mu ; \mu = 0, D-1)$ "momentum" variable
Rotation Invariance
$= \int d^D x \, \exp(-i K \cdot X) \, G(x)$
∞
$\boxed{\hat{G}(k) = \frac{1}{k^2 + m^2}}$
$G(x) = \int \frac{d^D k}{(2\pi)^D} \exp(i k \cdot x) \frac{1}{k^2 + m^2}$


BC ⇒ G(x) decays at ∞
(k^2 + m^2) · Ĝ(k) = 1
Ĝ(k) = 1/(k^2+m^2)
G(x) = ∫ d^D k/(2π)^D exp(i k·x) 1/(k^2+m^2)
G(x) = G(r)   r = |x|
[(−(∂^2/∂r^2 + (D−1)/r ∂/∂r) + m^2) G(r) = "δ(r)" ⇒



$K = (k_\mu ; \mu = 0, \ldots D-1)$ "momentum variable"
lation invariance
Rotation Invariance
$\hat{G}(K) = \int d^D x \cdot \exp(-i K\cdot X) G(x)$
ecays at ∞
$= 1$
$\boxed{\hat{G}(K) = \frac{1}{K^2+m^2}}$
$G(x) = \int \frac{d^D K}{(2\pi)^D} \exp(i K\cdot X) \frac{1}{K^2+m^2}$
$K_\nu(z)$ Bessel Funct of 2^{nd} type
$= |x|$
$\left[-\left(\frac{\partial^2}{\partial r^2} + \frac{D-1}{r}\frac{\partial}{\partial r}\right) + m^2\right] G(r) = “\delta(r)” \Rightarrow$
$\boxed{G(r) = \frac{1}{2\pi}\cdot\left(2\pi \frac{r}{m}\right)^{\frac{2-D}{2}} \cdot K_{\frac{D-2}{2}}(r\cdot m)}$

Real time $\quad x = (t, \vec{x})$

$G(x_1, x_2)$

$$\left(+ \frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial \vec{x}}\right)^2 + m^2 \right) G(x) =$$

Real time $\quad x = (t, \vec{x})$

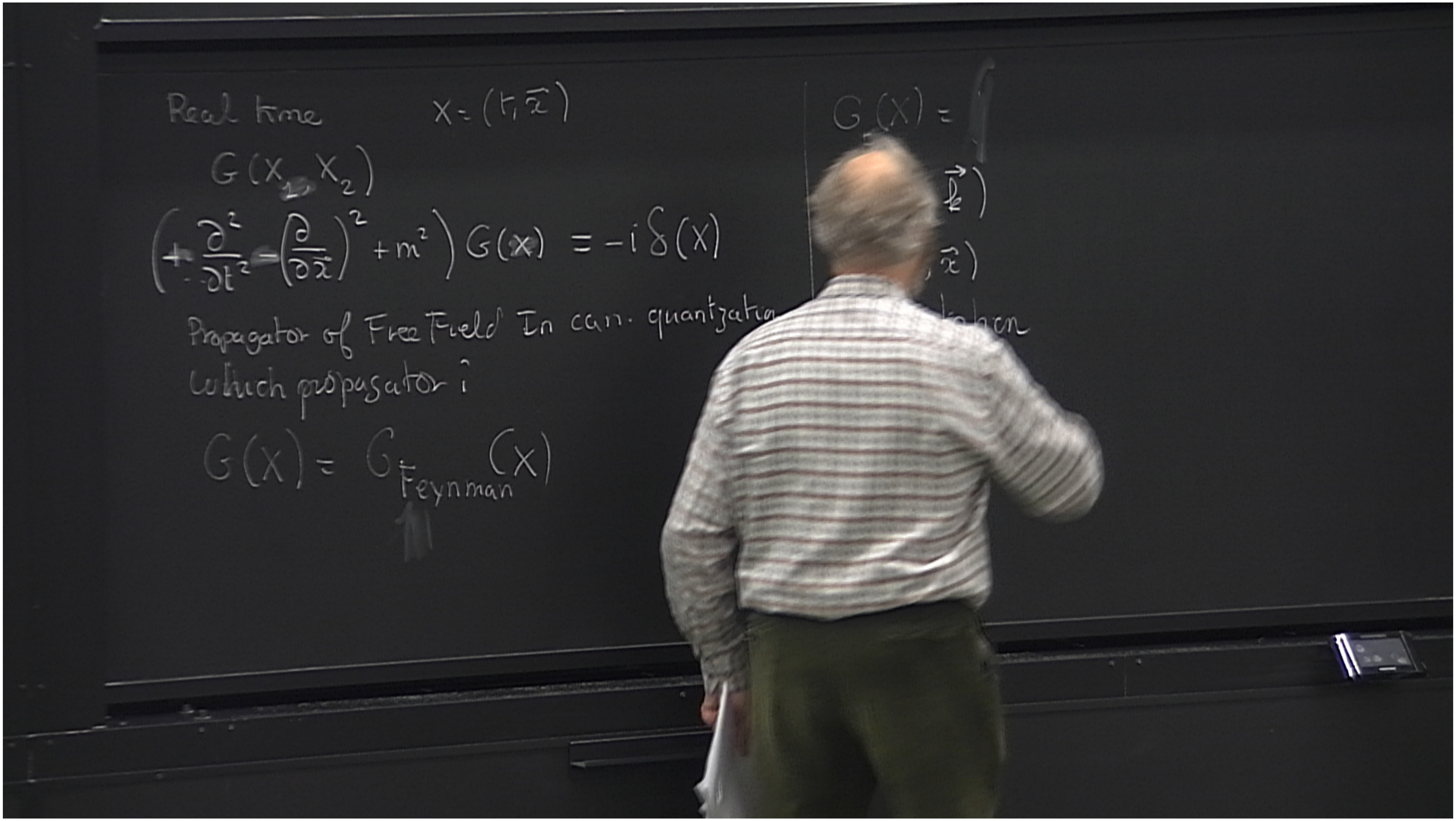
$G(x_1, x_2)$

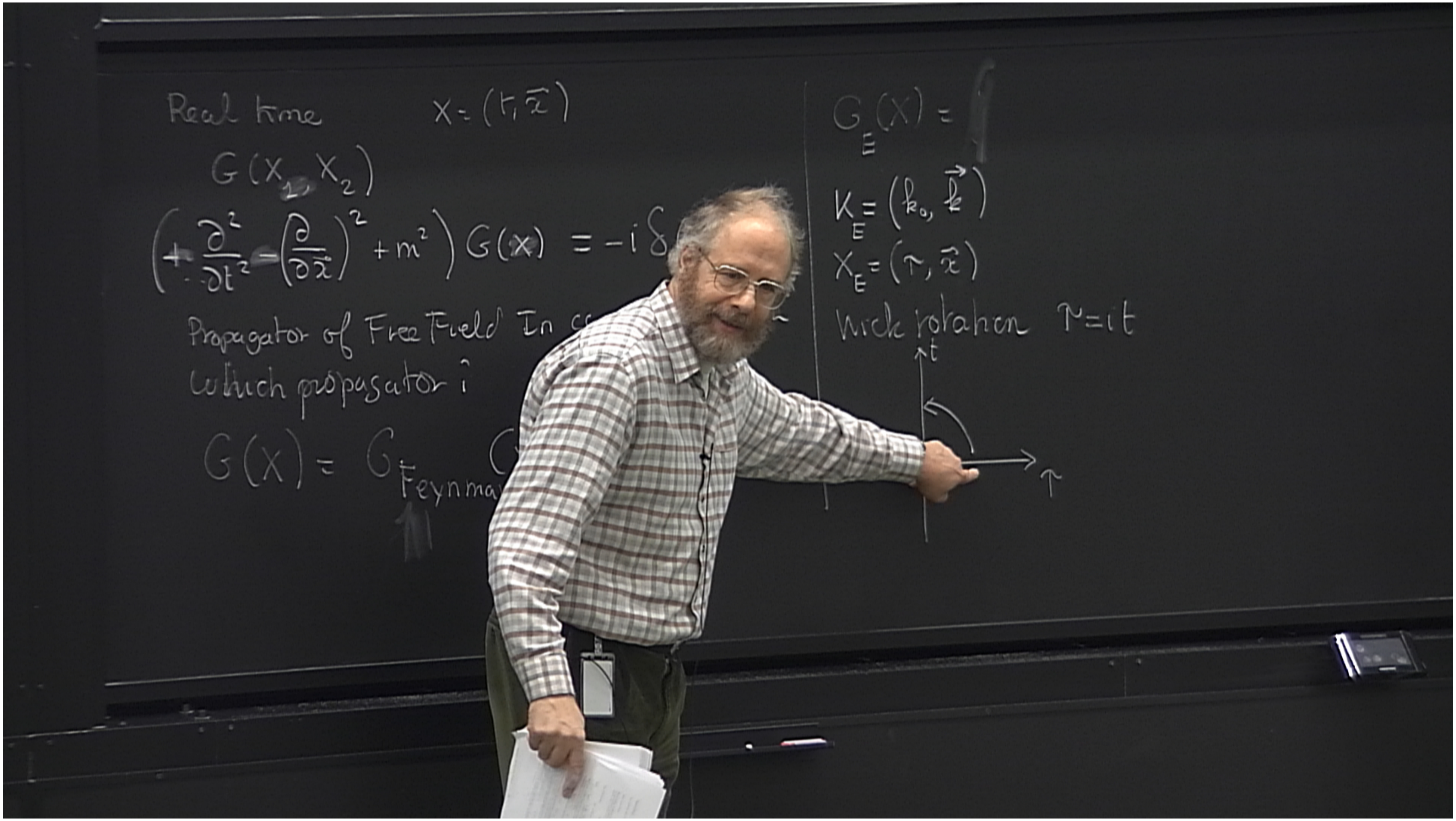
$$\left(+ \frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial \vec{x}}\right)^2 + m^2 \right) G(x) = -i\,\delta(x)$$

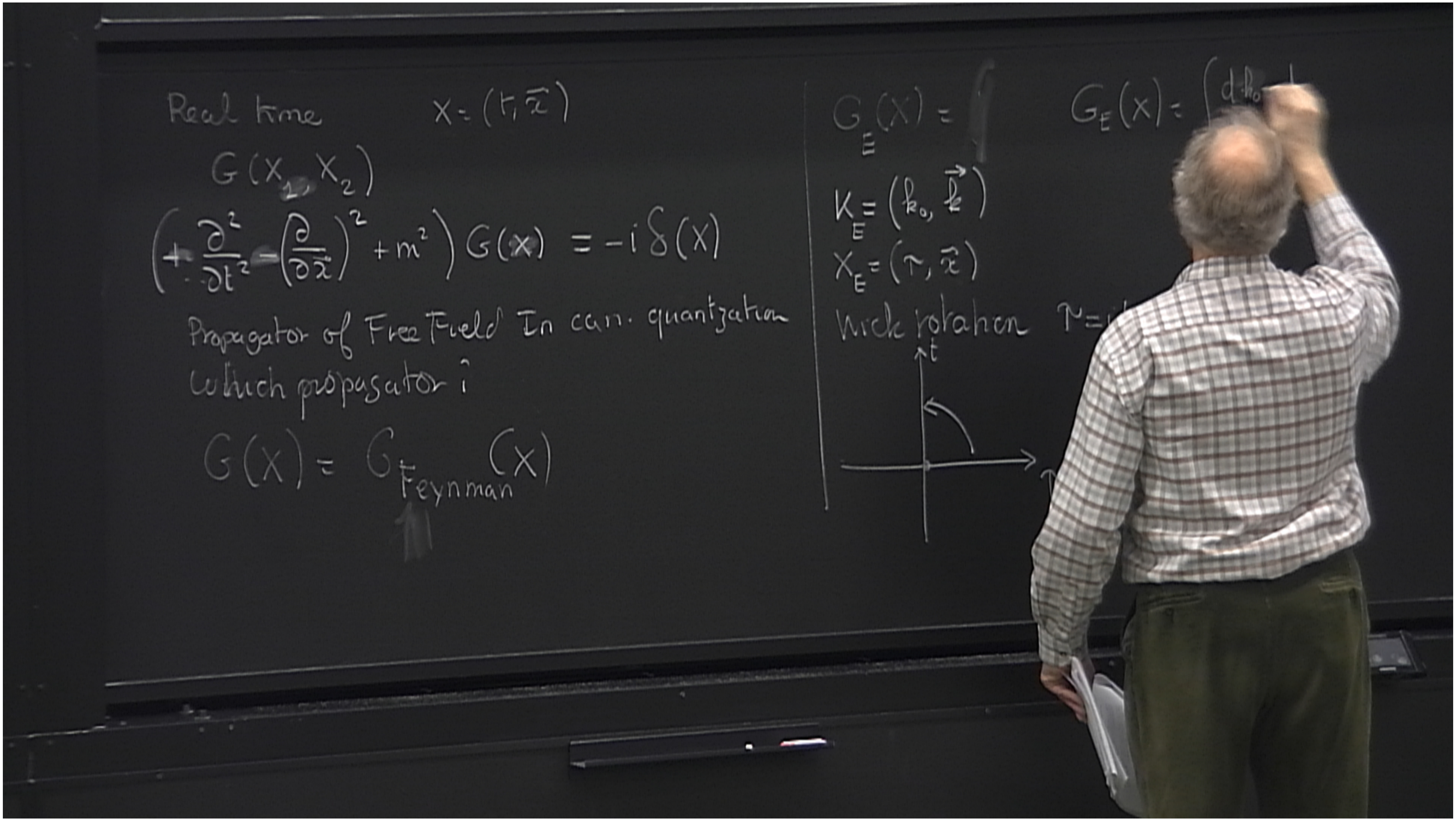
Propagator of Free Field In can. quantization

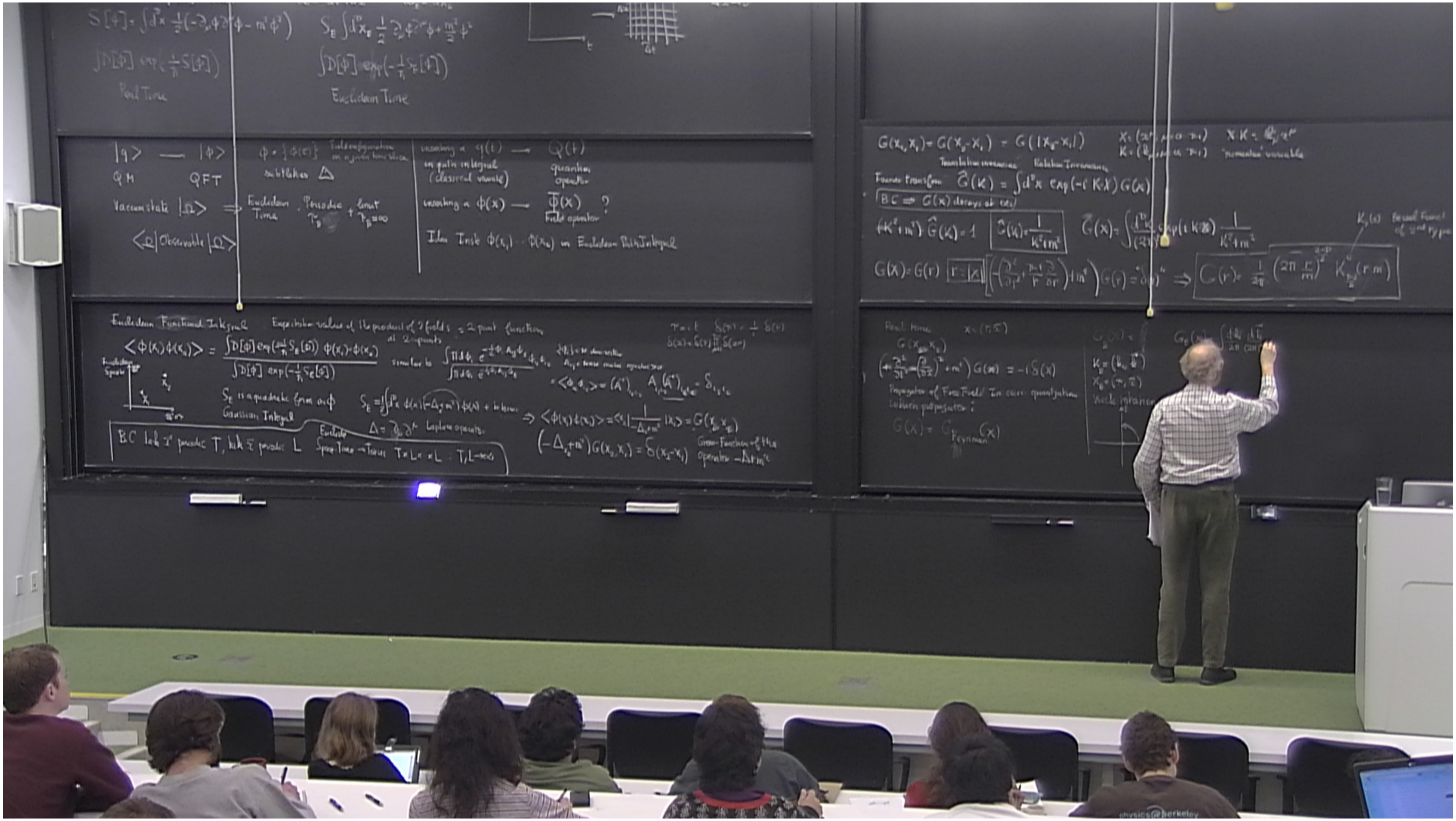
Which propagator ?

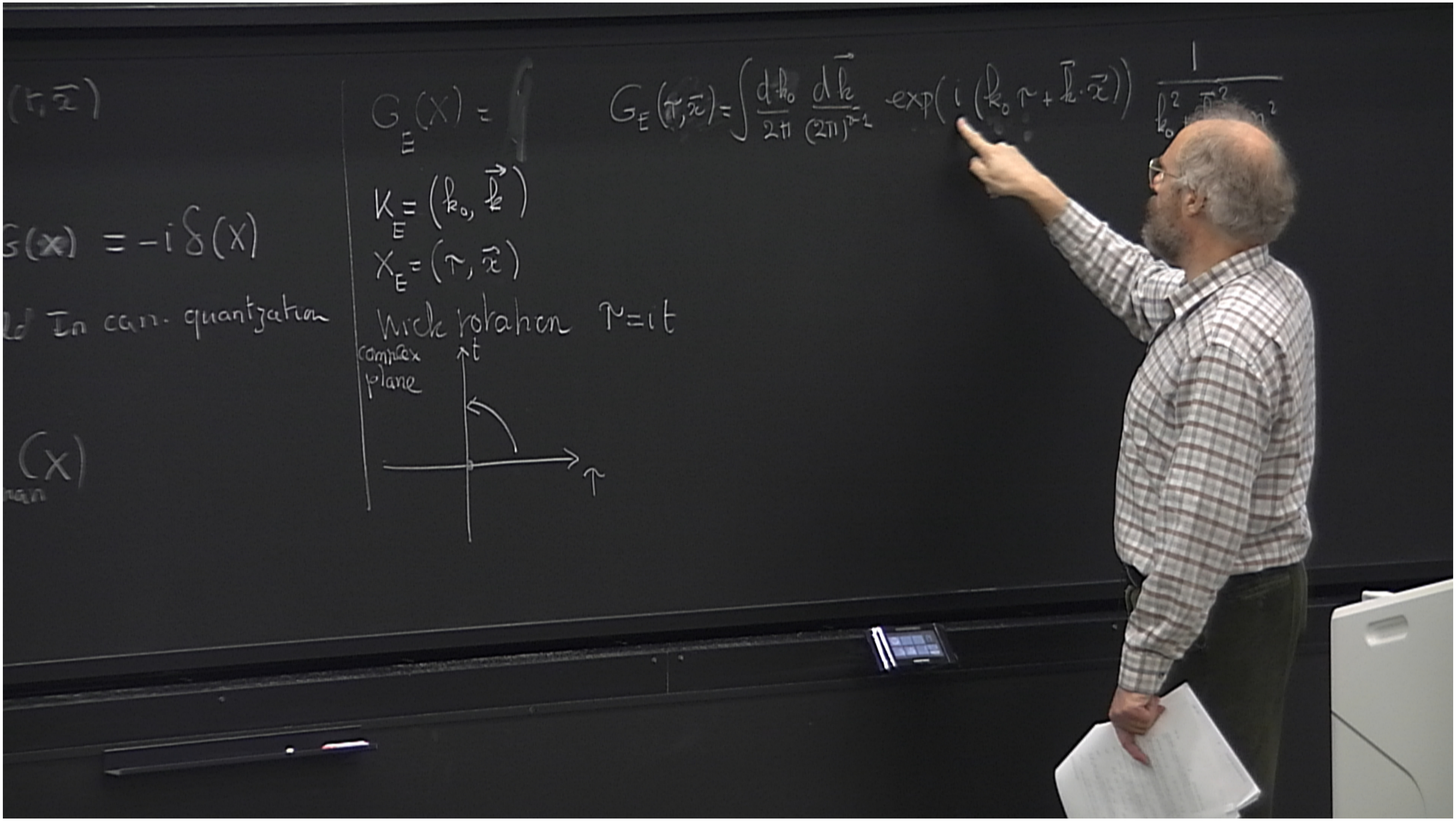
$$G(x) = G_{\text{Feynman}}(x)$$

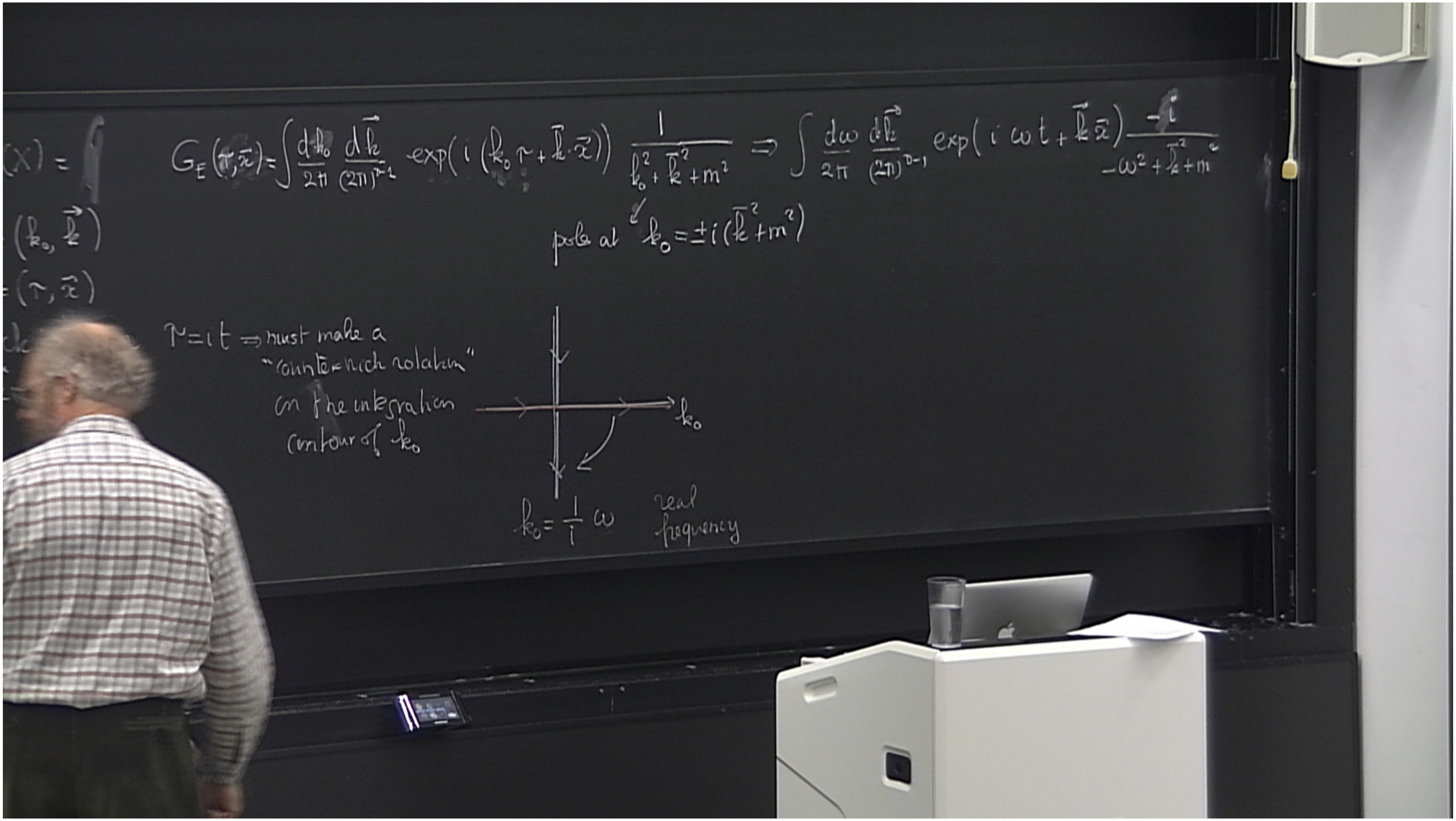

Real time X = (t, x⃗)
G(X1, X2)
(∂²/∂t² − (∂/∂x⃗)² + m²) G(x) = −i δ(X)
Propagator of Free Field In can. quantization
which propagator ?
G(X) = G Feynman (X)
G(X) =

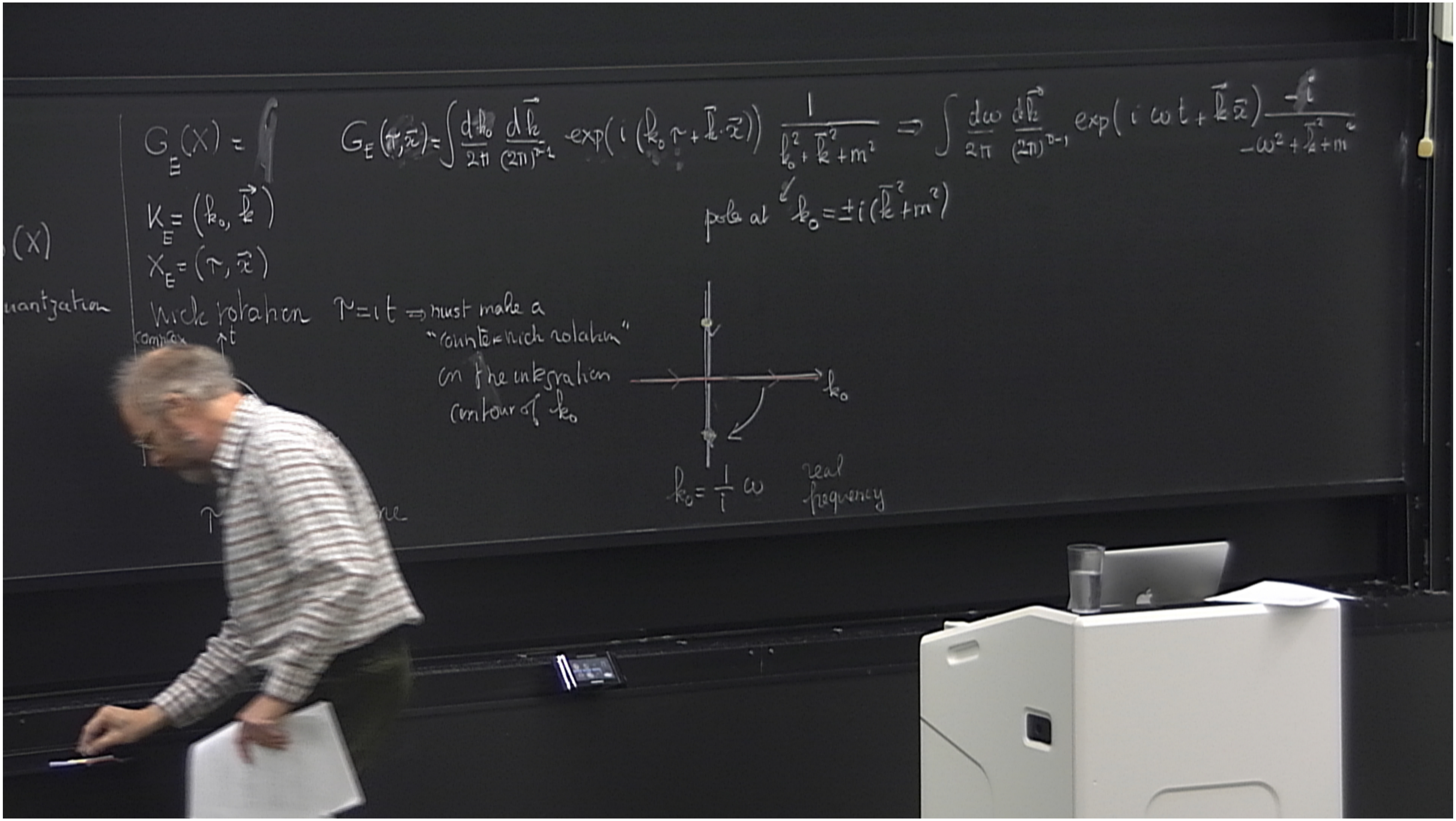

Real time
X = (t, x)
G(X1, X2)
Propagator of Free Field
which propagator ?
G(X) = G Feynman
G_E(X) =
K_E = (k_0, k)
X_E = (τ, x)
Wick rotation τ = it

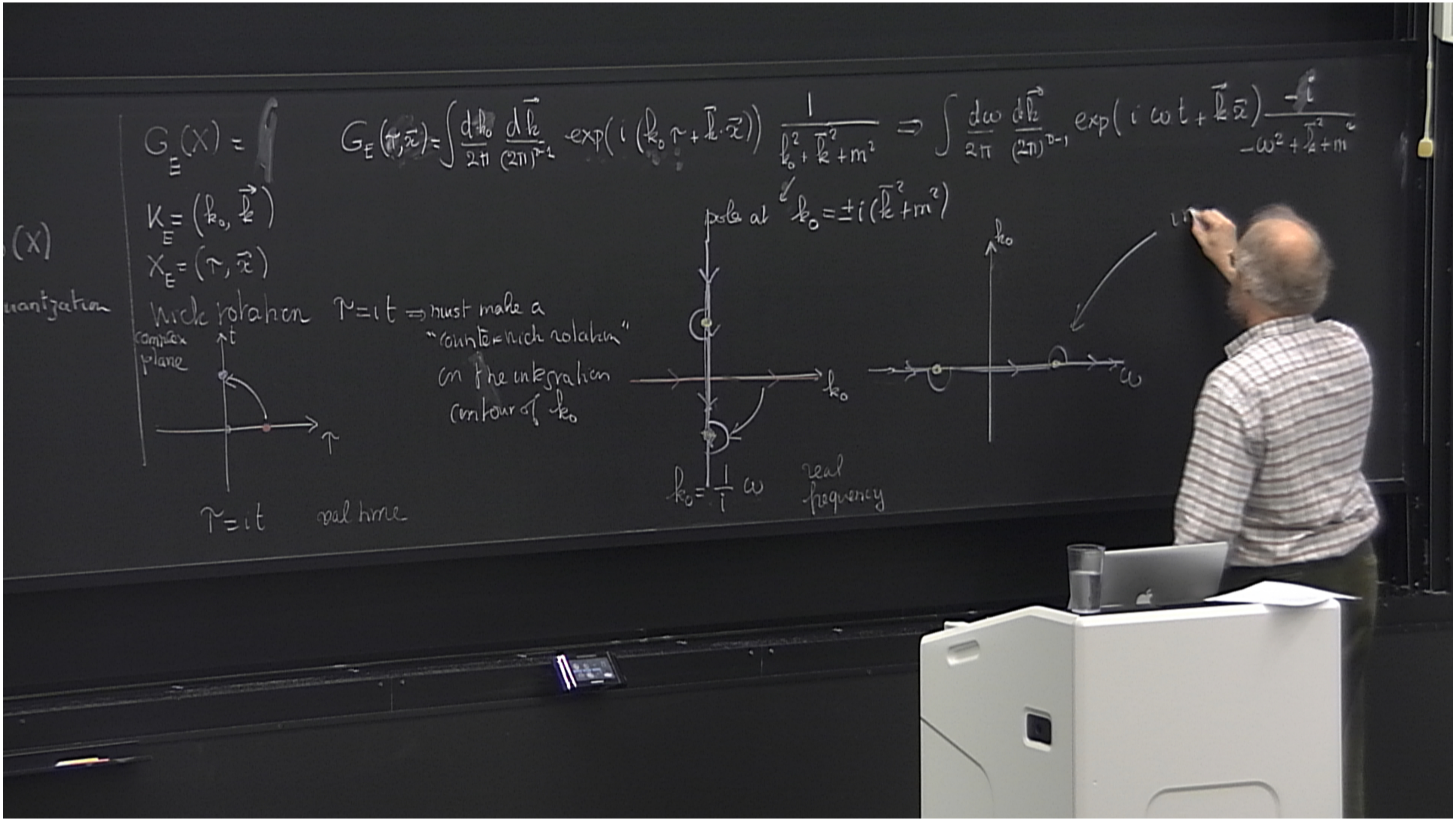

Real time $x = (t, \vec{x})$
$G(x_1, x_2)$
$\left(+\frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial \vec{x}}\right)^2 + m^2\right) G(x) = -i\delta(x)$
Propagator of Free Field In can. quantization
which propagator ?
$G(x) = G_{\text{Feynman}}(x)$
$G_E(x) =$
$G_E(x) = \int dk_0$
$K_E = (k_0, \vec{k})$
$X_E = (\tau, \vec{x})$
Wick rotation
t

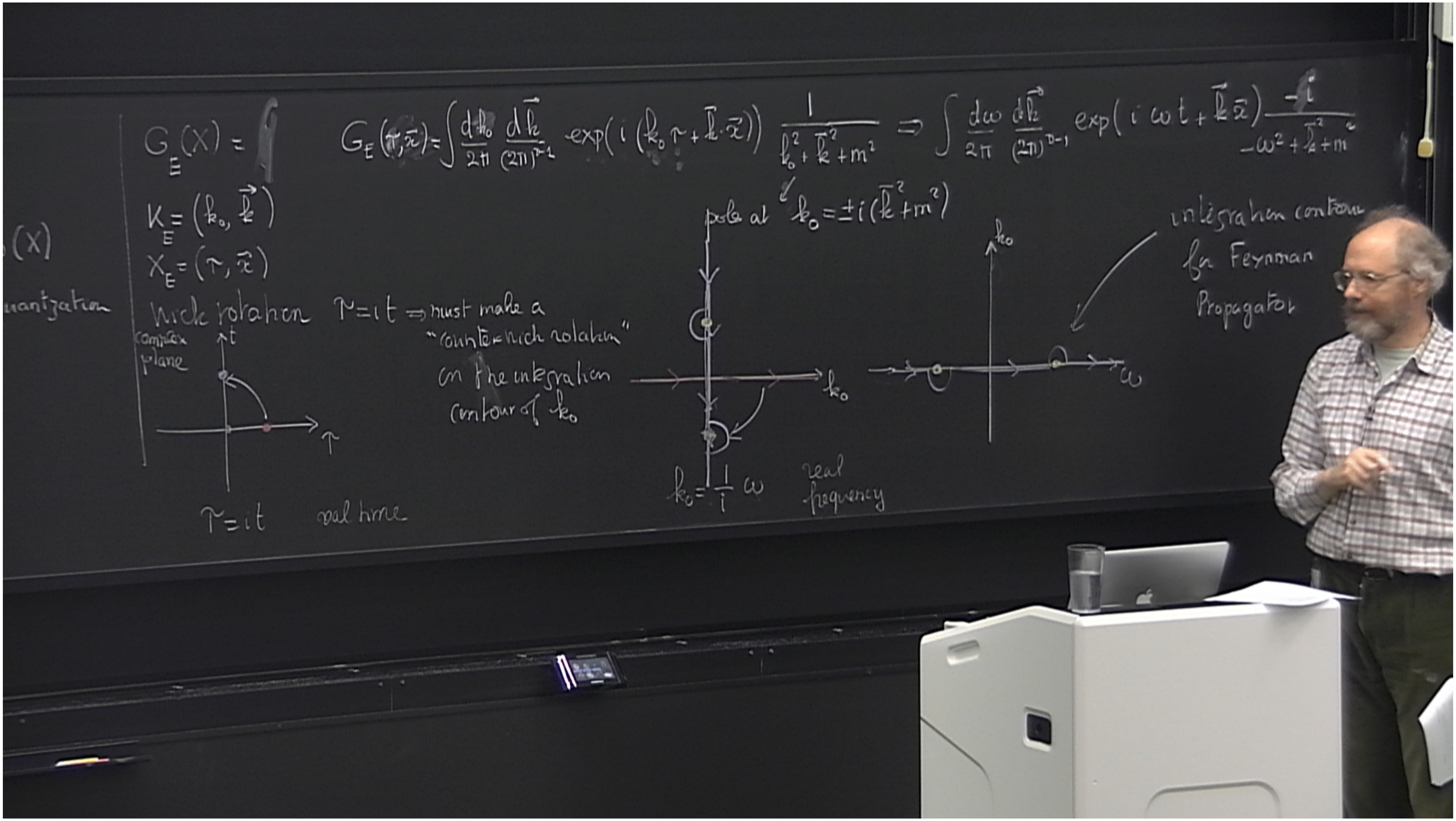



G_E(X) =
K_E = (k_0, k)
X_E = (τ, x)
Wick rotation τ = it
complex
plane
In can. quantization
G_E(τ, x) = ∫ dk_0/2π dk/(2π)^{d-1} exp(i(k_0 τ + k·x))

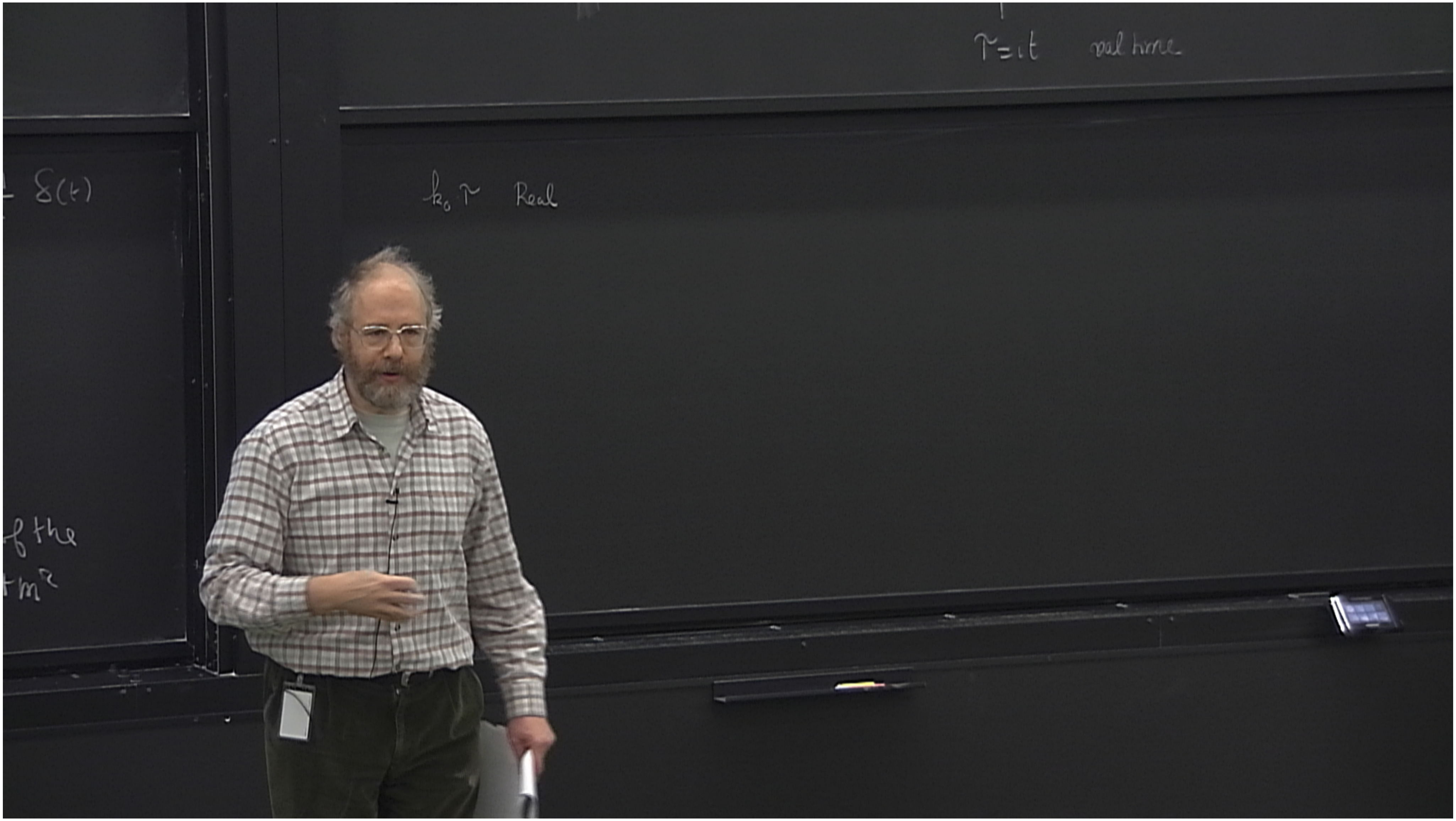

(x) =
$(k_0, \vec{k})$
$(\tau, \vec{x})$
$G_E(\tau,\vec{x}) = \int \frac{dk_0}{2\pi} \frac{d\vec{k}}{(2\pi)^{D-1}} \exp(i(k_0\tau + \vec{k}\cdot\vec{x})) \frac{1}{k_0^2 + \vec{k}^2 + m^2} \Rightarrow \int \frac{d\omega}{2\pi} \frac{d\vec{k}}{(2\pi)^{D-1}} \exp(i\omega t + \vec{k}\vec{x}) \frac{-i}{-\omega^2 + \vec{k}^2 + m^2}$
pole at $k_0 = \pm i(\vec{k}^2 + m^2)$
$\tau = it \Rightarrow$ must make a "counterclock rotation" on the integration contour of k_0
k_0
$k_0 = \frac{1}{i}\omega$
real frequency

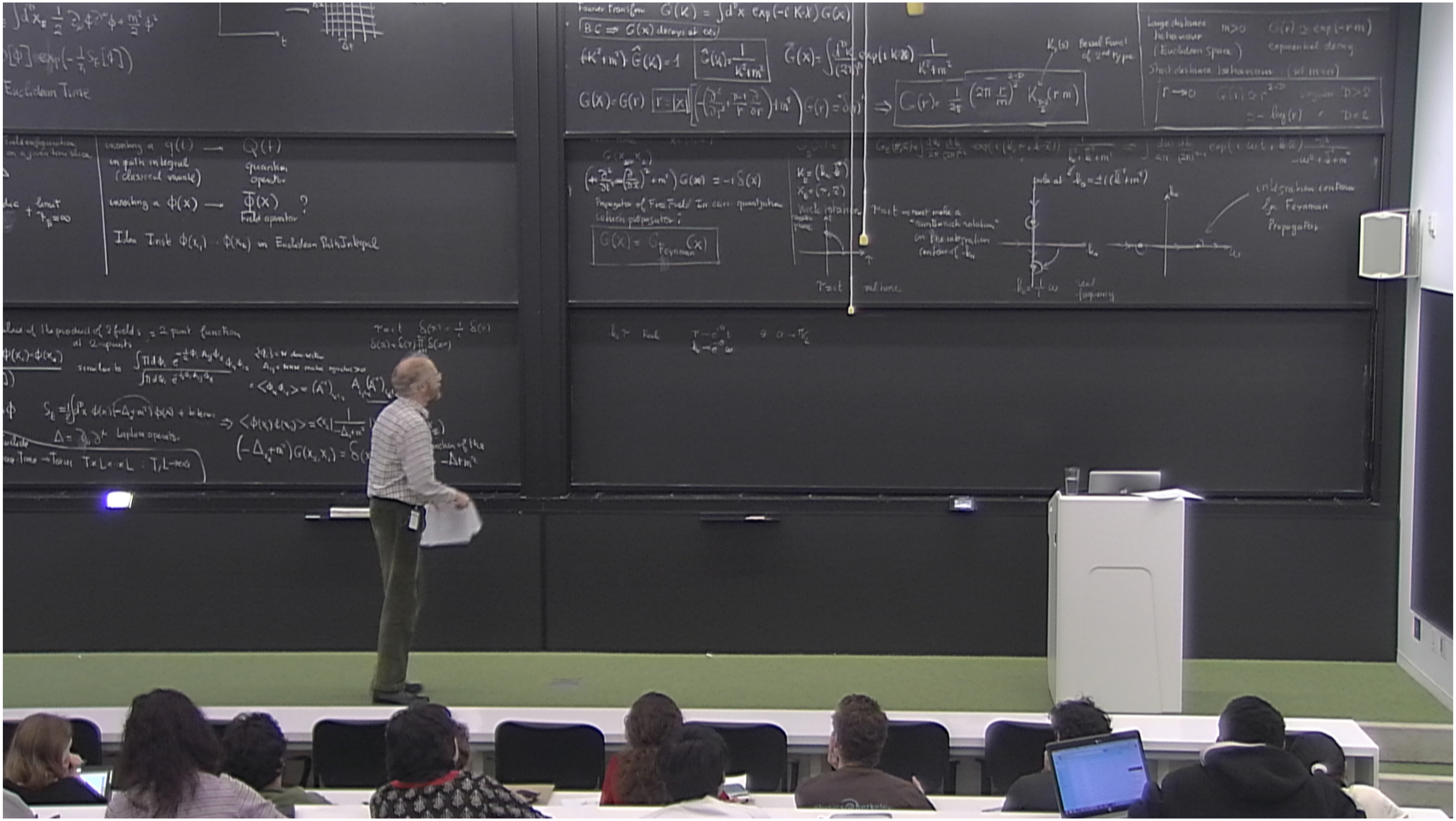

G_E(X) =
G_E(τ, x) = ∫ dk_0/2π dk/(2π)^{D-1} exp(i(k_0 τ + k·x)) 1/(k_0^2 + k^2 + m^2) ⇒ ∫ dω/2π dk/(2π)^{D-1} exp(iωt + k x) -i/(-ω^2 + k^2 + m^2)
K_E = (k_0, k)
X_E = (τ, x)
pole at k_0 = ±i(k^2 + m^2)
Wick rotation τ = it ⇒ must make a
"counter-wick rotation"
on the integration
contour of k_0
k_0
k_0 = (1/i) ω
real
frequency

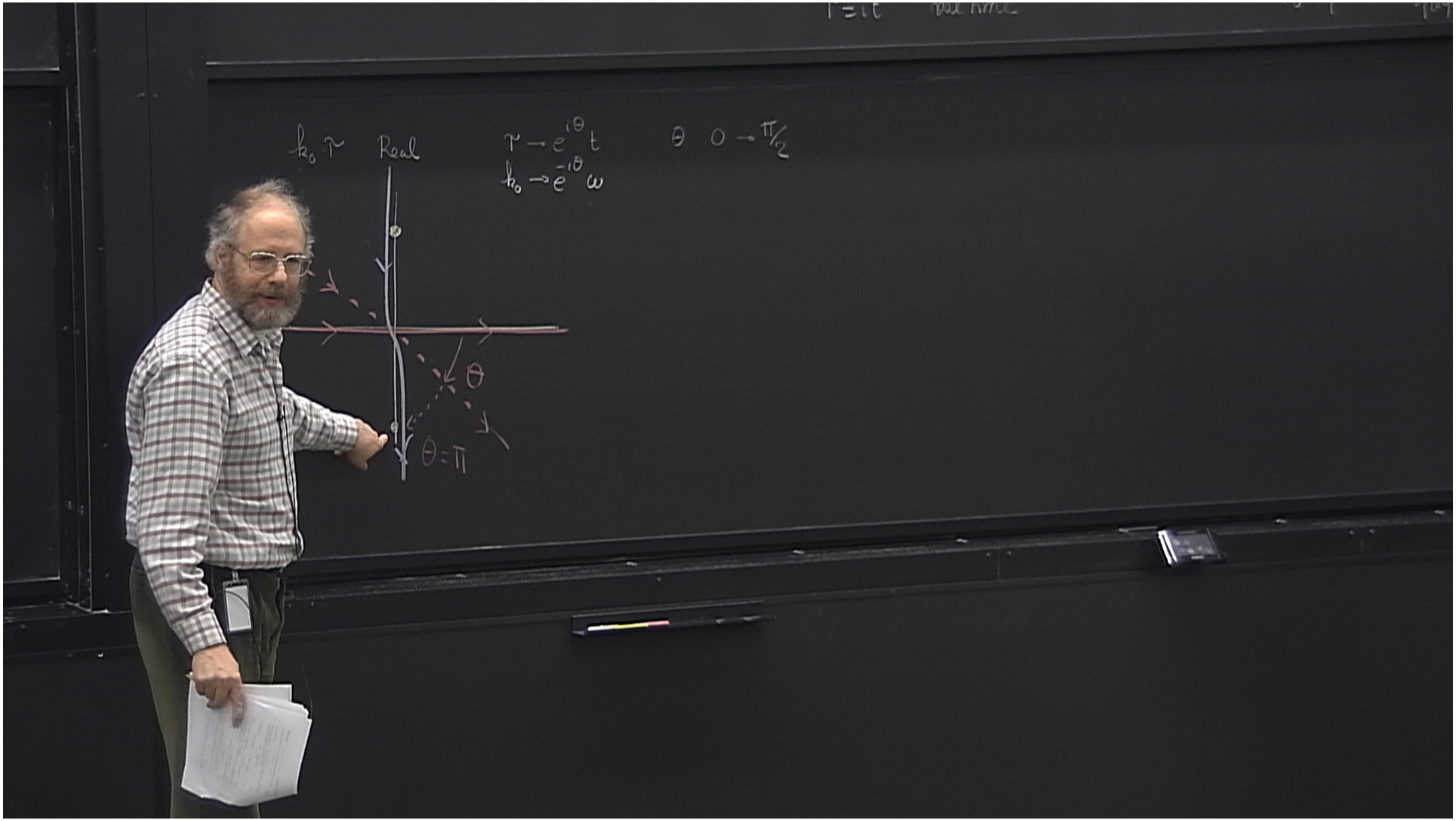

$G_E(X) =$
$G_E(\tau,\vec{x}) = \int \frac{dk_0}{2\pi} \frac{d\vec{k}}{(2\pi)^{D-1}} \exp(i(k_0\tau + \vec{k}\cdot\vec{x})) \frac{1}{k_0^2 + \vec{k}^2 + m^2} \Rightarrow \int \frac{d\omega}{2\pi} \frac{d\vec{k}}{(2\pi)^{D-1}} \exp(i\omega t + \vec{k}\vec{x}) \frac{-i}{-\omega^2 + \vec{k}^2 + m^2}$
$K_E = (k_0, \vec{k})$
$X_E = (\tau, \vec{x})$
Wick rotation $\tau = it \Rightarrow$ must make a "counter-Wick rotation" on the integration contour of k_0
complex plane
t
τ
$\tau = it$ real time
pole at $k_0 = \pm i(\vec{k}^2 + m^2)$
k_0
$k_0 = \frac{1}{i}\omega$ real frequency
ω
(x)
quantization

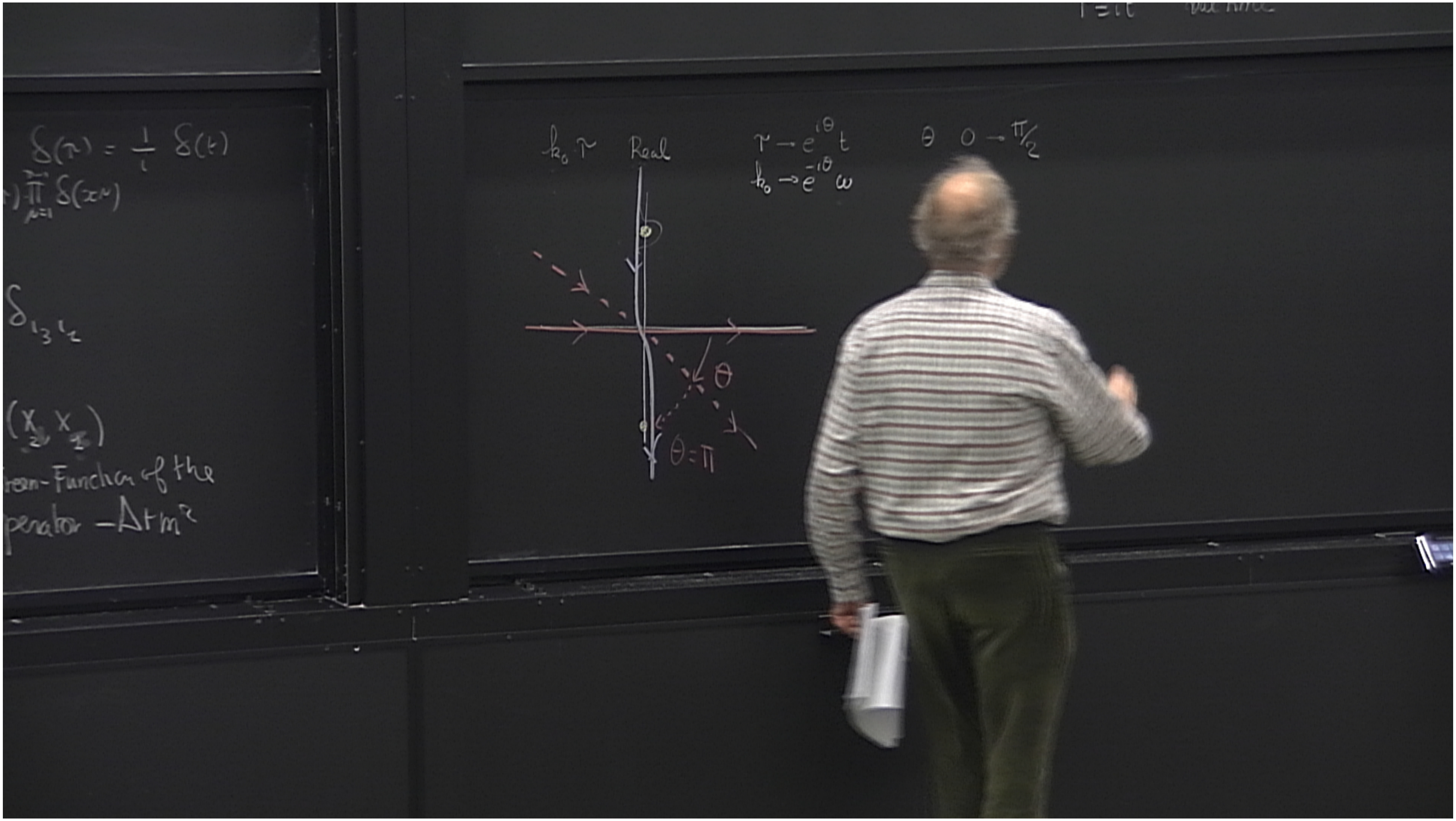


G_E(X) =
K_E = (k_0, k)
X_E = (τ, x)
Wick rotation τ = i t ⇒ must make a "counter-Wick rotation" on the integration contour of k_0
complex plane
τ = i t
real time
pole at k_0 = ± i (k² + m²)
real frequency
integration contour for Feynman Propagator

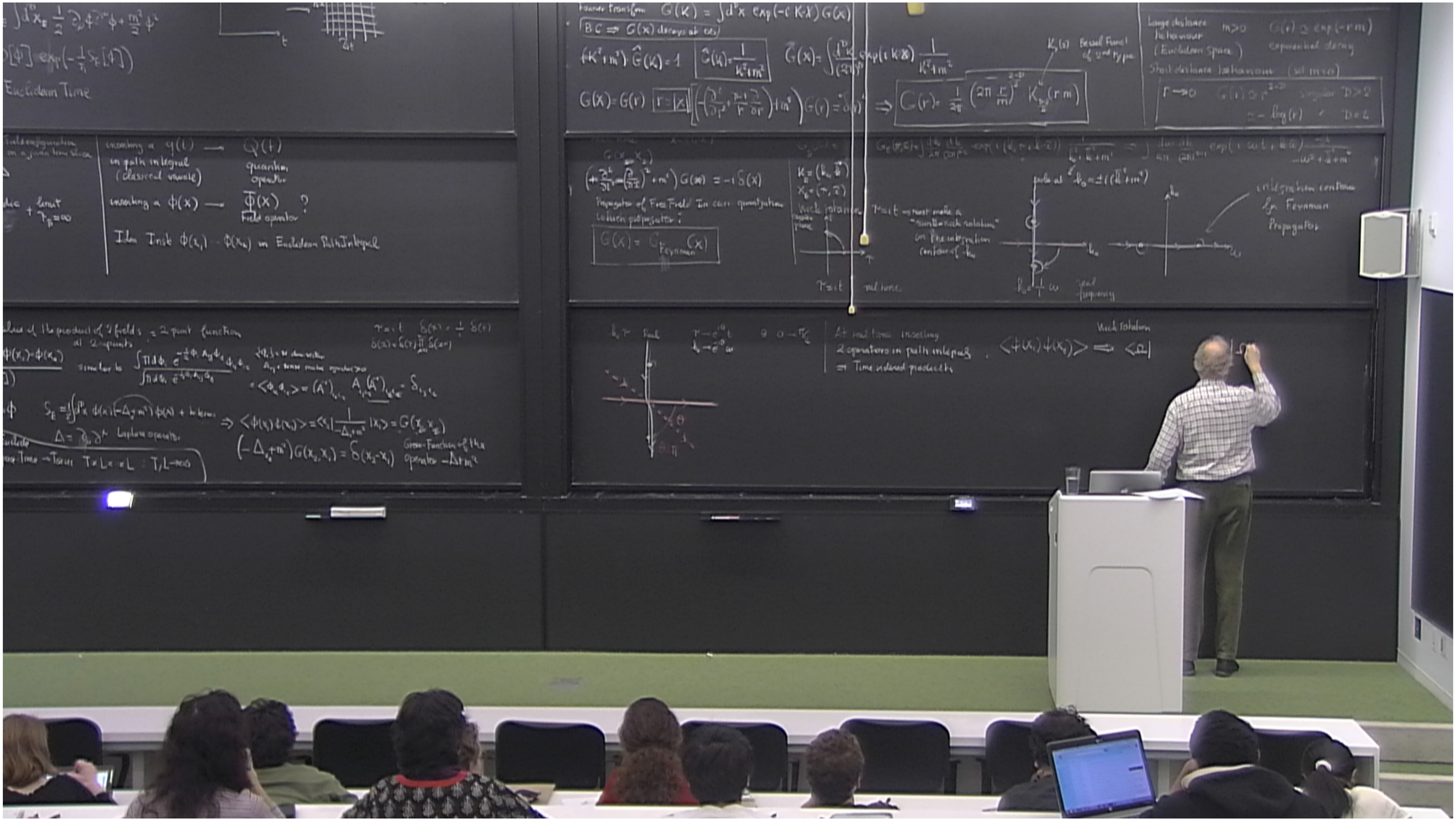

$\tau = i t$ real time
$k_0 \tau$ Real
$\delta(t)$
f the
m^2

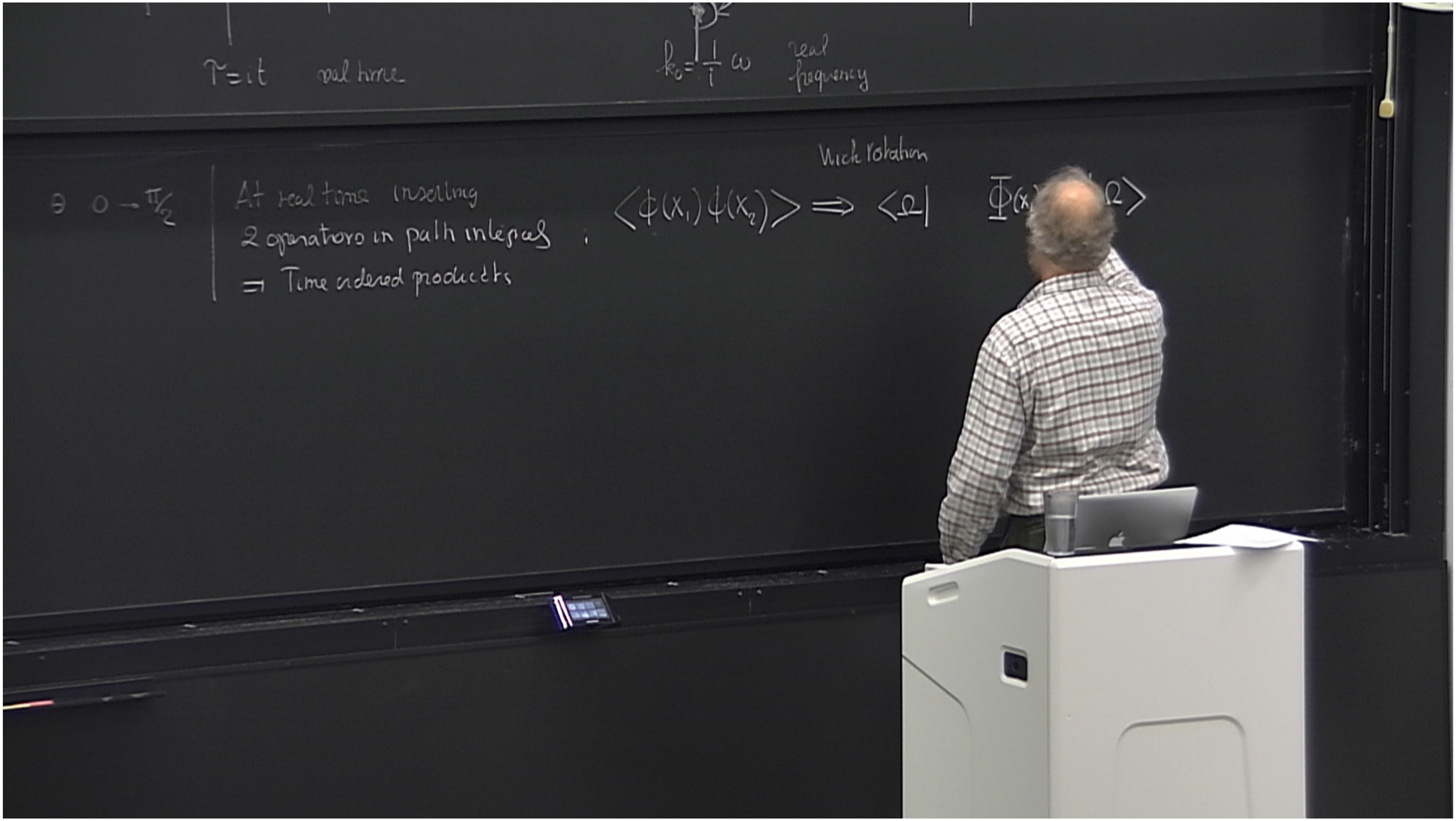

Euclidean Time
Large distance behaviour
(Euclidean Space)
Exponential decay
Short distance behaviour
Wick rotation
integration contour
for Feynman
Propagator
Propagator of Free Field / In can. quantization
quantum operator
Field operator
Idea Insert Φ(x₁) · Φ(x₂) in Euclidean Path Integral
Laplace operator
Bessel Funct of 2nd type
real frequency
real time

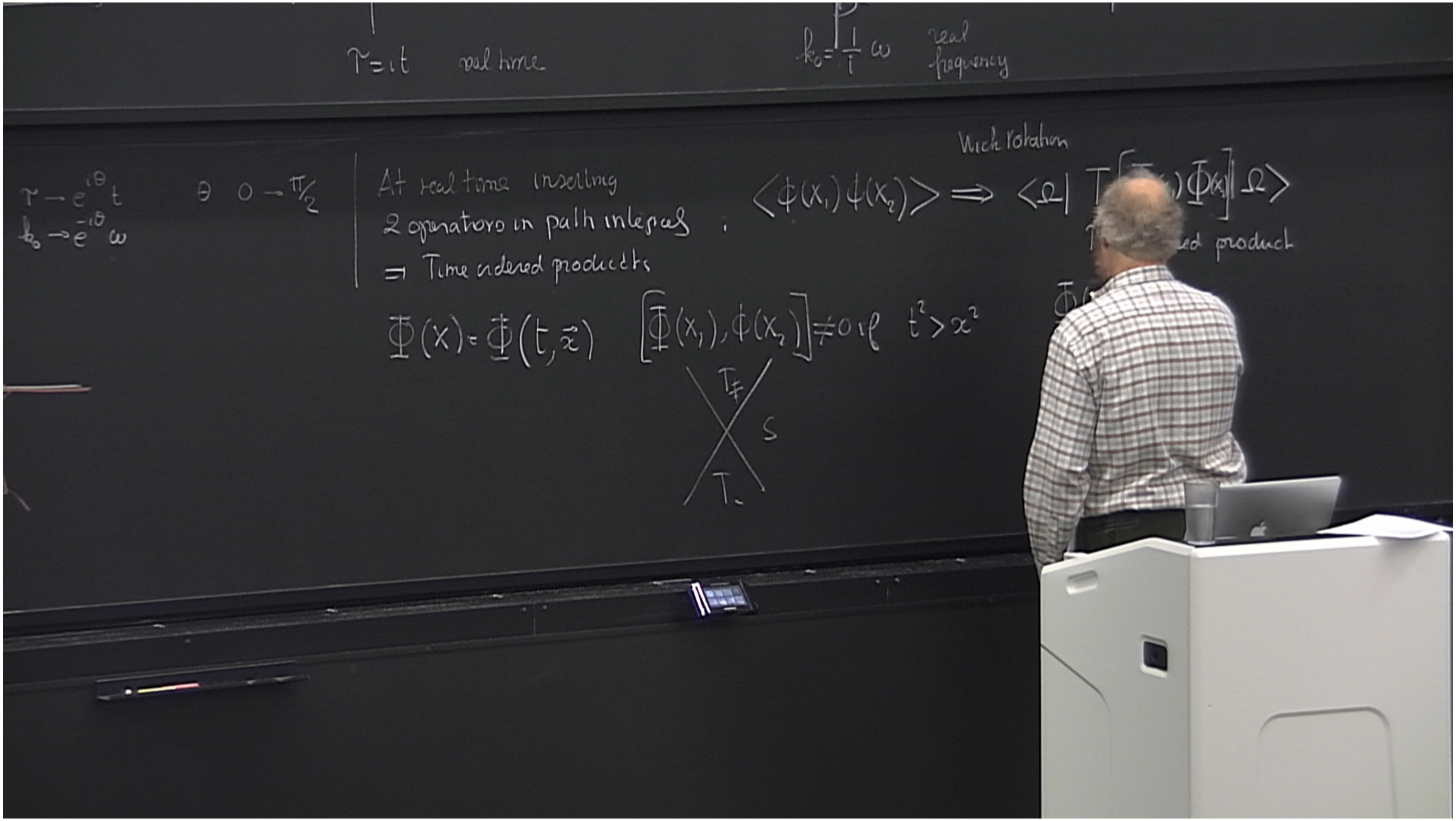

$k_0 \sim$ Real
$\tau \to e^{i\theta} t$
$k_0 \to e^{-i\theta} \omega$
$\theta \quad 0 \to \pi/2$
θ
$\theta = \pi$

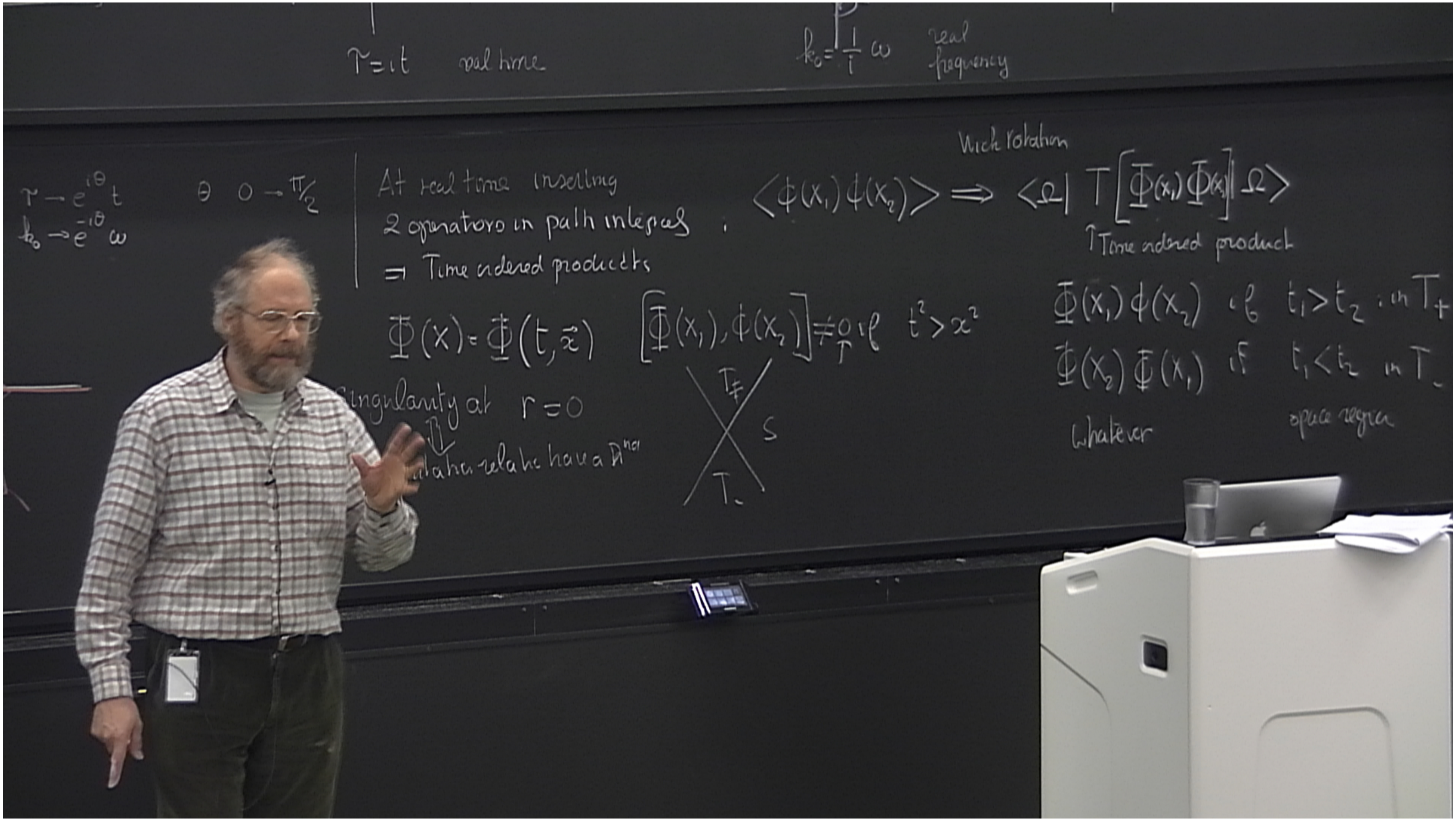


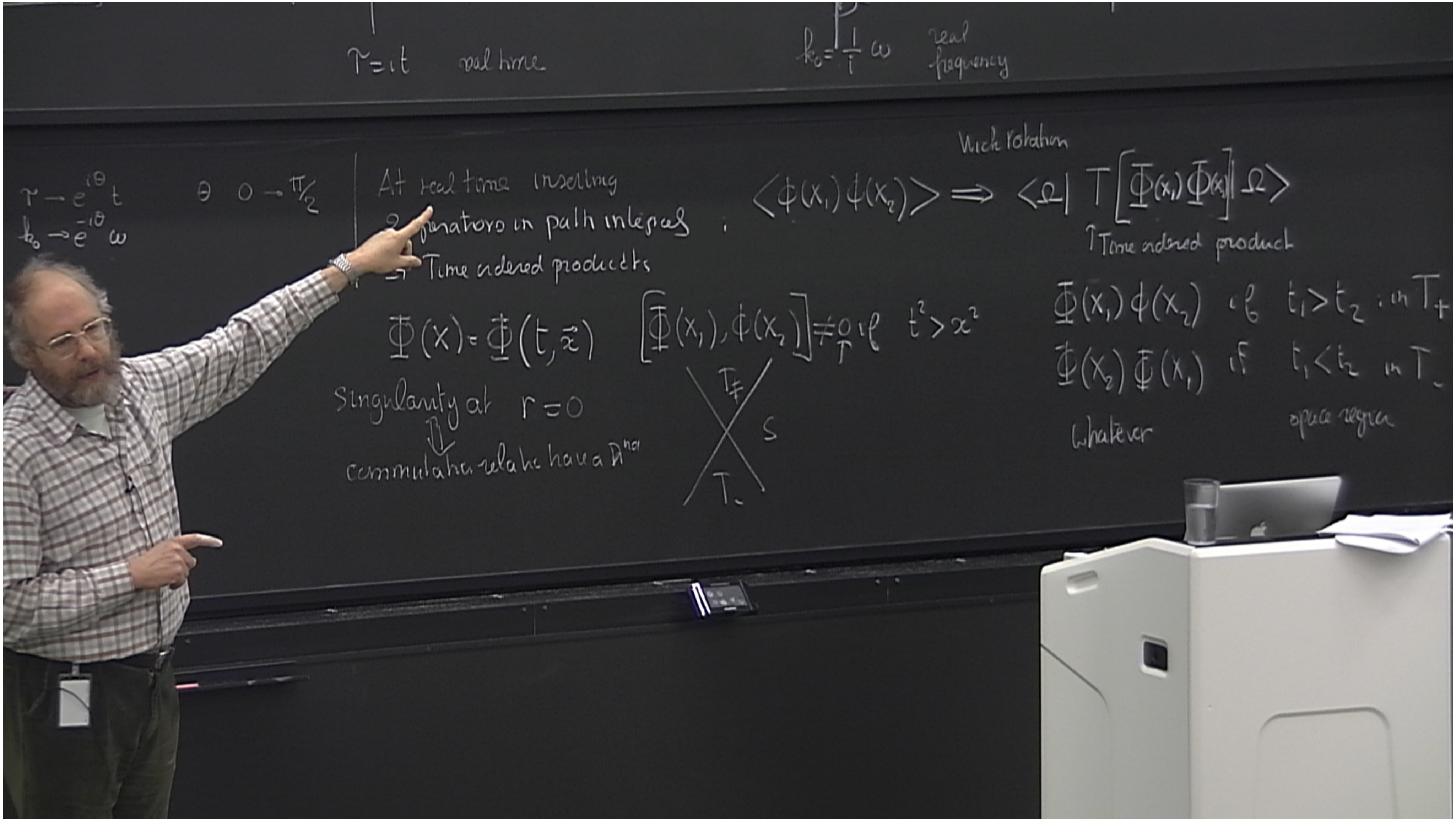
$\delta(\tau) = \frac{1}{i}\,\delta(t)$
$\prod_{\mu=1} \delta(x^\mu)$
reen-Function of the
perator $-\Delta + m^2$
$k_0\ \tau$ Real
$\tau \to e^{i\theta} t$
$k_0 \to e^{-i\theta}\omega$
$\theta\ \ 0 \to \pi/2$
θ
$\theta = \pi$

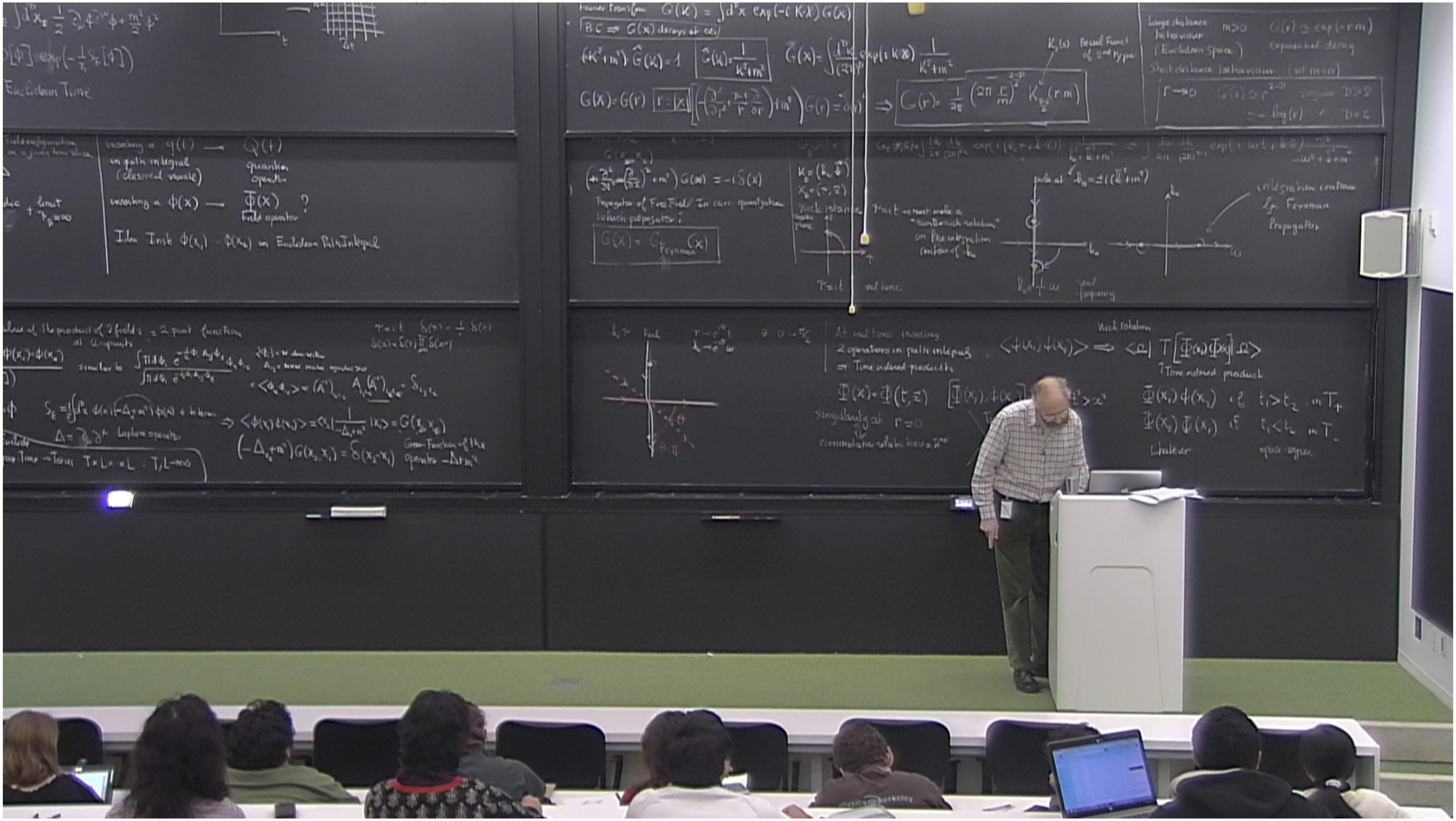

Euclidean Time
Fourier Transform
Large distance behaviour
(Euclidean Space)
Short distance behaviour
Bessel Funct of 2nd type
Propagator of Free Field in canonical quantization
Feynman propagator
Wick rotation
pole at
integration contour for Feynman Propagator
real frequency
real time
At real time inserting
2 operators in path integral
⇒ Time ordered products
Green Function of the operator


τ = it real time
k0 = 1/i ω real frequency
Wick rotation
θ 0 → π/2
At real time inserting
2 operators in path integral :
⟹ Time ordered products
⟨φ(x1) φ(x2)⟩ ⟹ ⟨Ω|

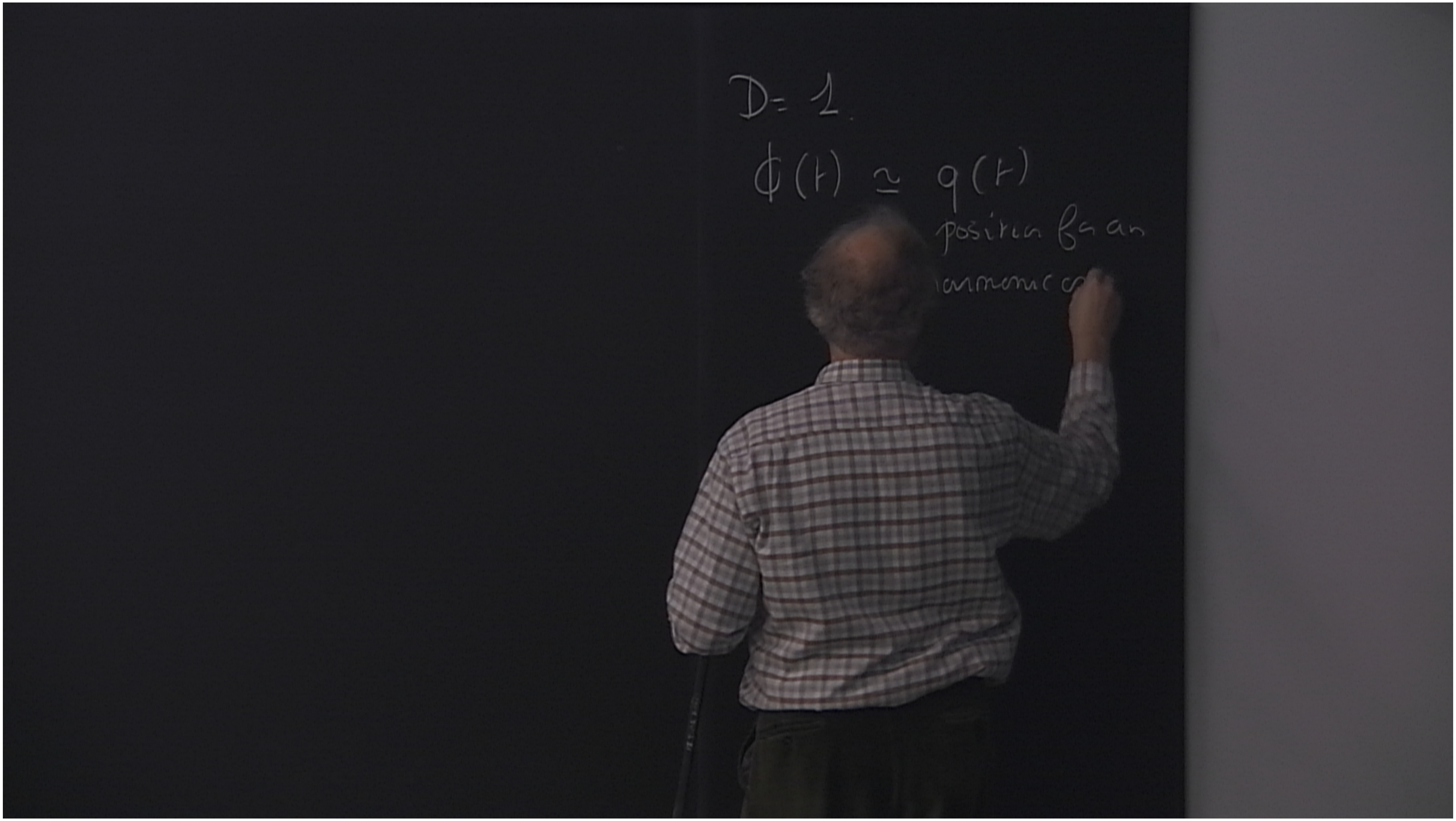

τ = it   real time
k0 = 1/i ω   real frequency
Wick rotation
τ → e^iθ t   θ 0 → π/2
k0 → e^-iθ ω
At real time inserting
2 operators in path integral
⇒ Time ordered products
⟨φ(x1) φ(x2)⟩ ⟹ ⟨Ω| T
Φ(x) = Φ(t, x⃗)
[Φ(x1), φ(x2)] ≠ 0 if t² > x²
product


τ = it real time
k₀ = 1/i ω real frequency
Wick rotation
τ → e^{iθ} t θ 0 → π/2
k₀ → e^{-iθ} ω
At real time inselling
2 operators in path integral
⇒ Time ordered products
⟨φ(x₁) φ(x₂)⟩ ⇒ ⟨Ω| T[Φ(x₁) Φ(x₂)] |Ω⟩
↑ Time ordered product
Φ(x₁) φ(x₂) if t₁ > t₂ in T₊
Φ(x₂) Φ(x₁) if t₁ < t₂ in T₋
Whatever
space region
Φ(x) = Φ(t, x⃗)
[Φ(x₁), φ(x₂)] ≠ 0 if t² > x²
T₊
S
T₋
Singularity at r = 0

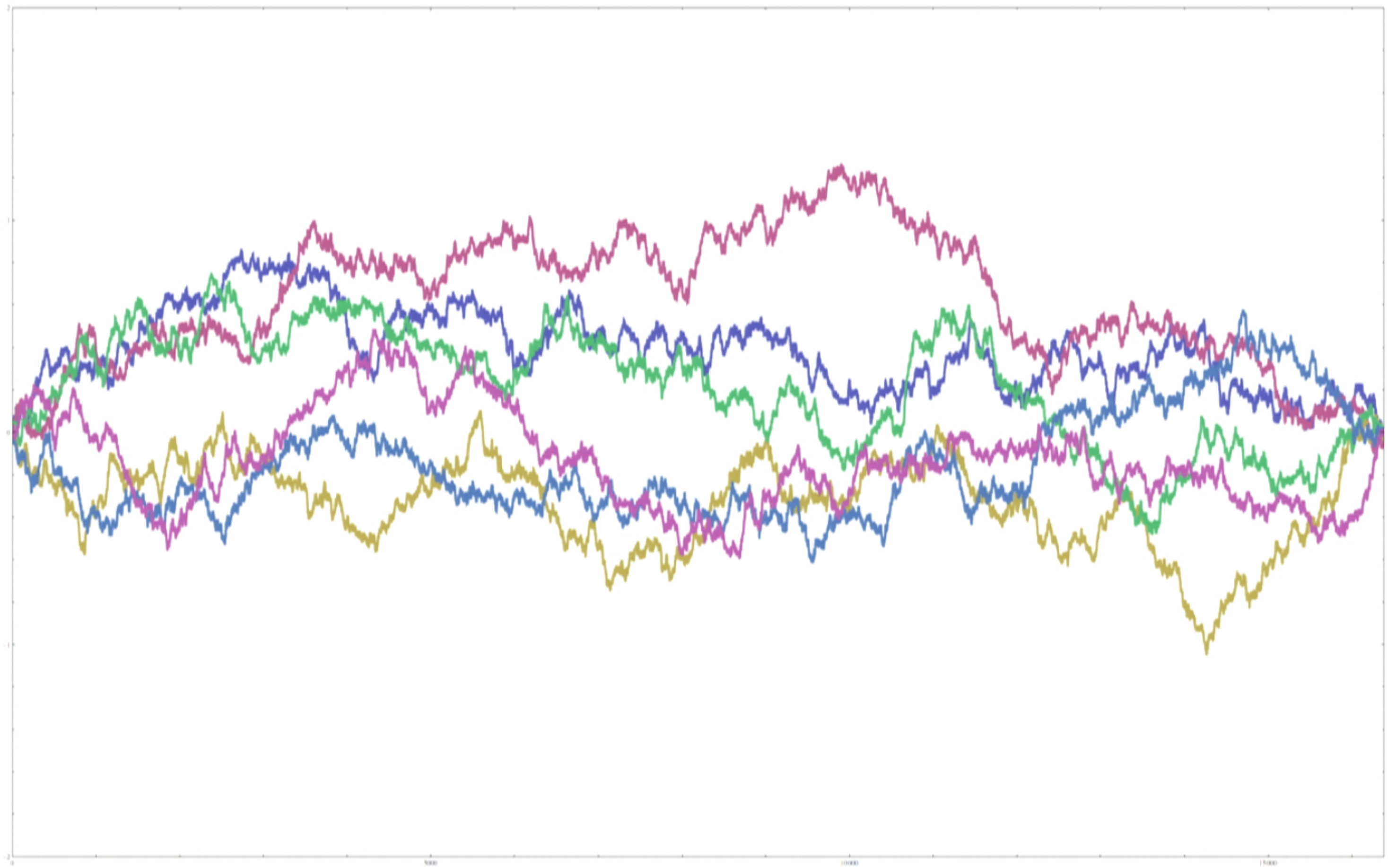

τ = it real time
k₀ = (1/i) ω real frequency
τ → e^{iθ} t
k₀ → e^{-iθ} ω
θ 0 → π/2
At real time insertion
of operators in path integral
⇒ Time ordered products
Φ(x) = Φ(t, x⃗)
[Φ(x₁), φ(x₂)] ≠ 0 if t² > x²
Singularity at r = 0
commutation relations have a ħ
T₊
S
T₋
Wick rotation
⟨φ(x₁) φ(x₂)⟩ ⟹ ⟨Ω| T [Φ(x₁) Φ(x₂)] |Ω⟩
Time ordered product
Φ(x₁) φ(x₂) if t₁ > t₂ in T₊
Φ(x₂) Φ(x₁) if t₁ < t₂ in T₋
Whatever
space region


Euclidean Time
Wick rotation
Time ordered product
Propagator

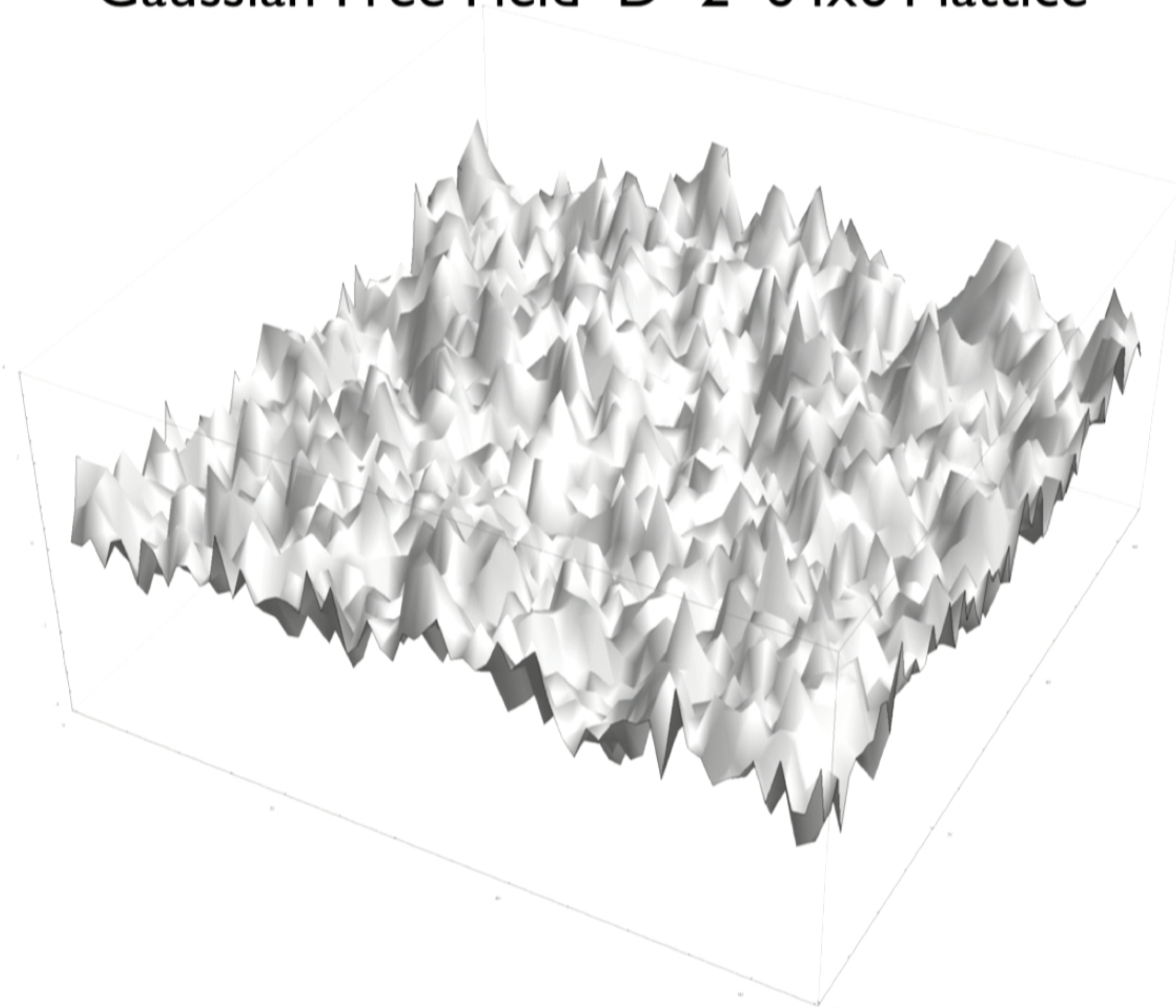
Typical free field configurations

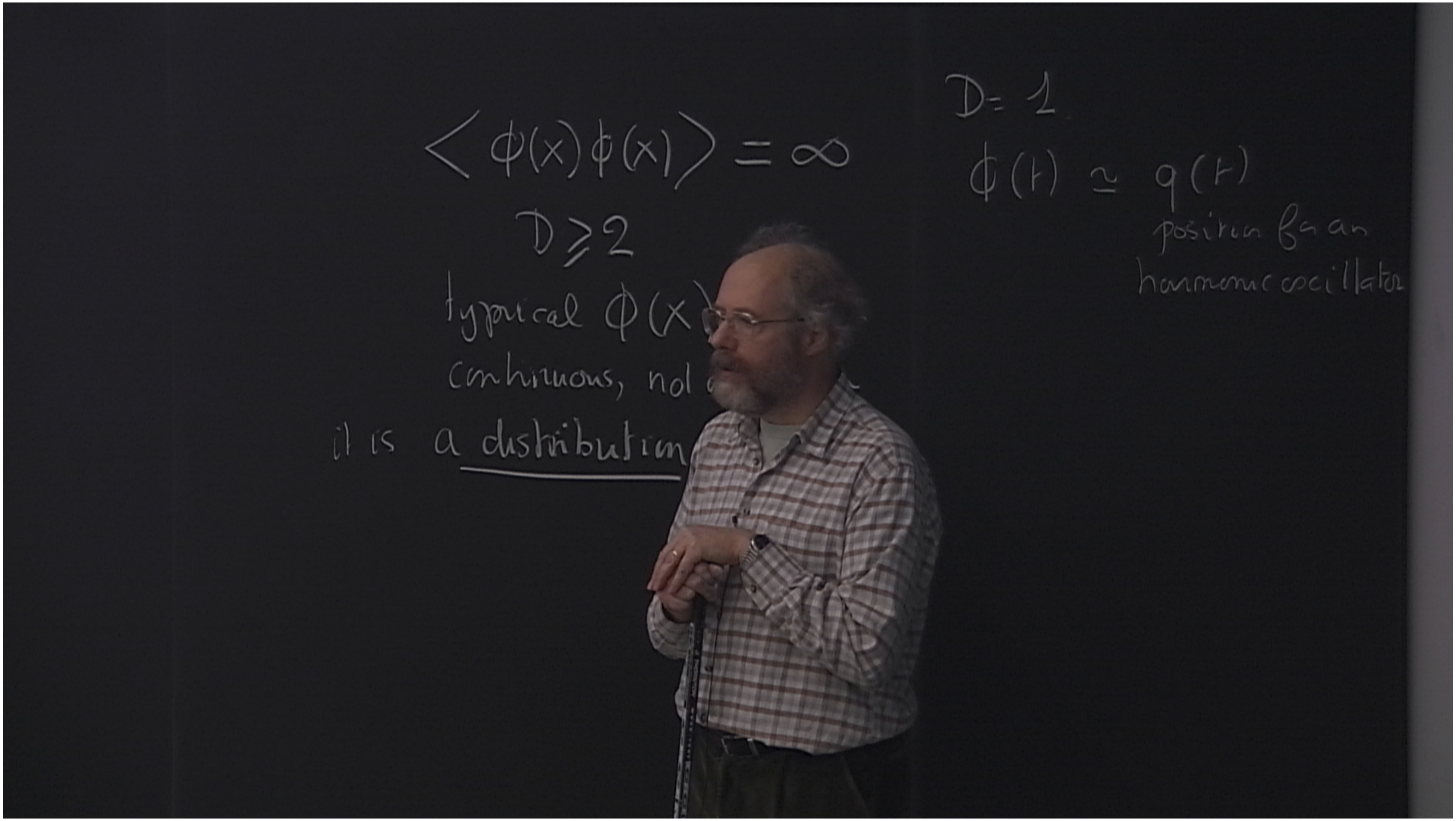

D= 1.
$\phi(t) \simeq q(t)$
position for an

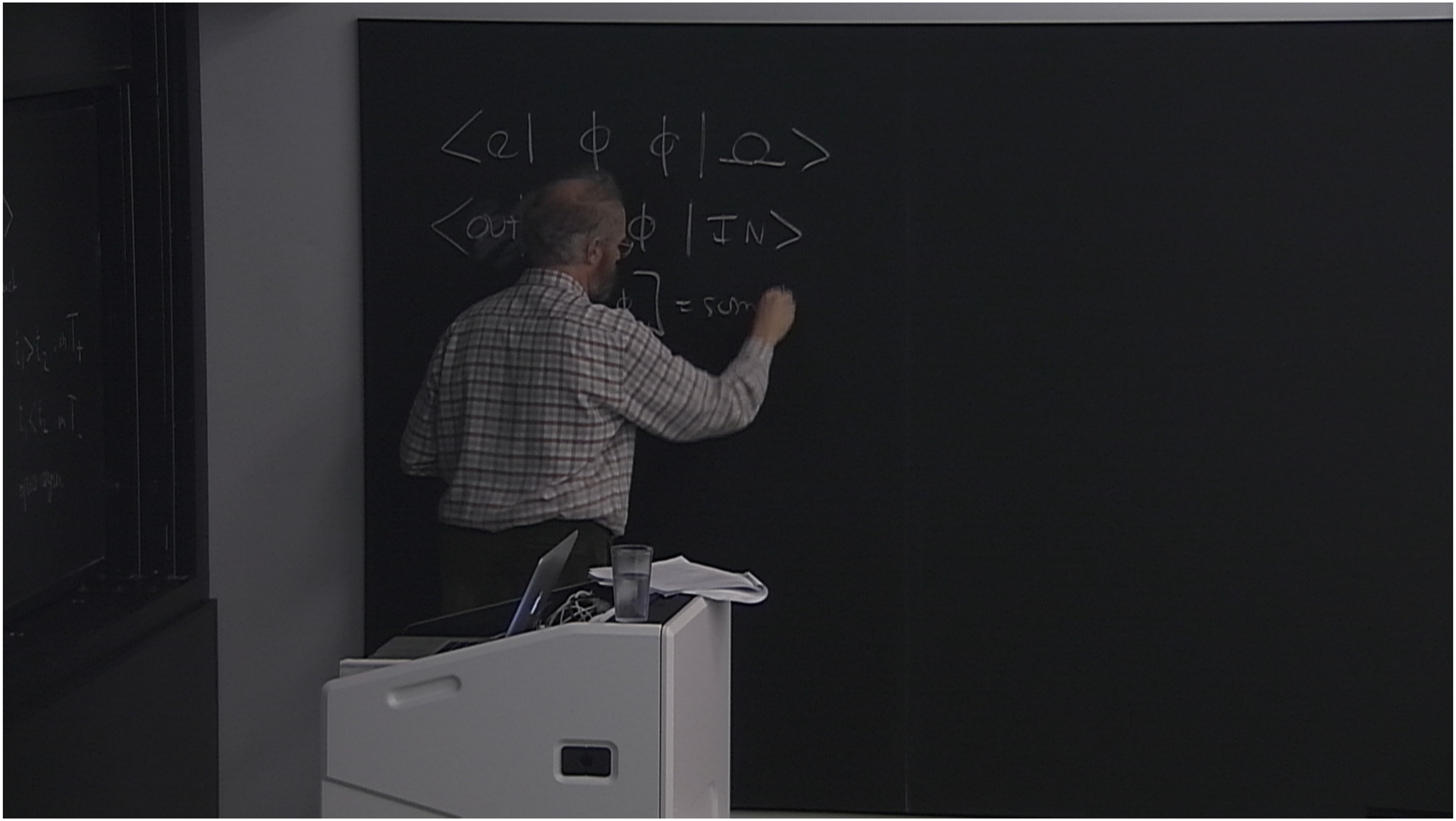
D=1 : Gaussian Free Fields = Random Walk
(i.e. Brownian or Wiener Process)



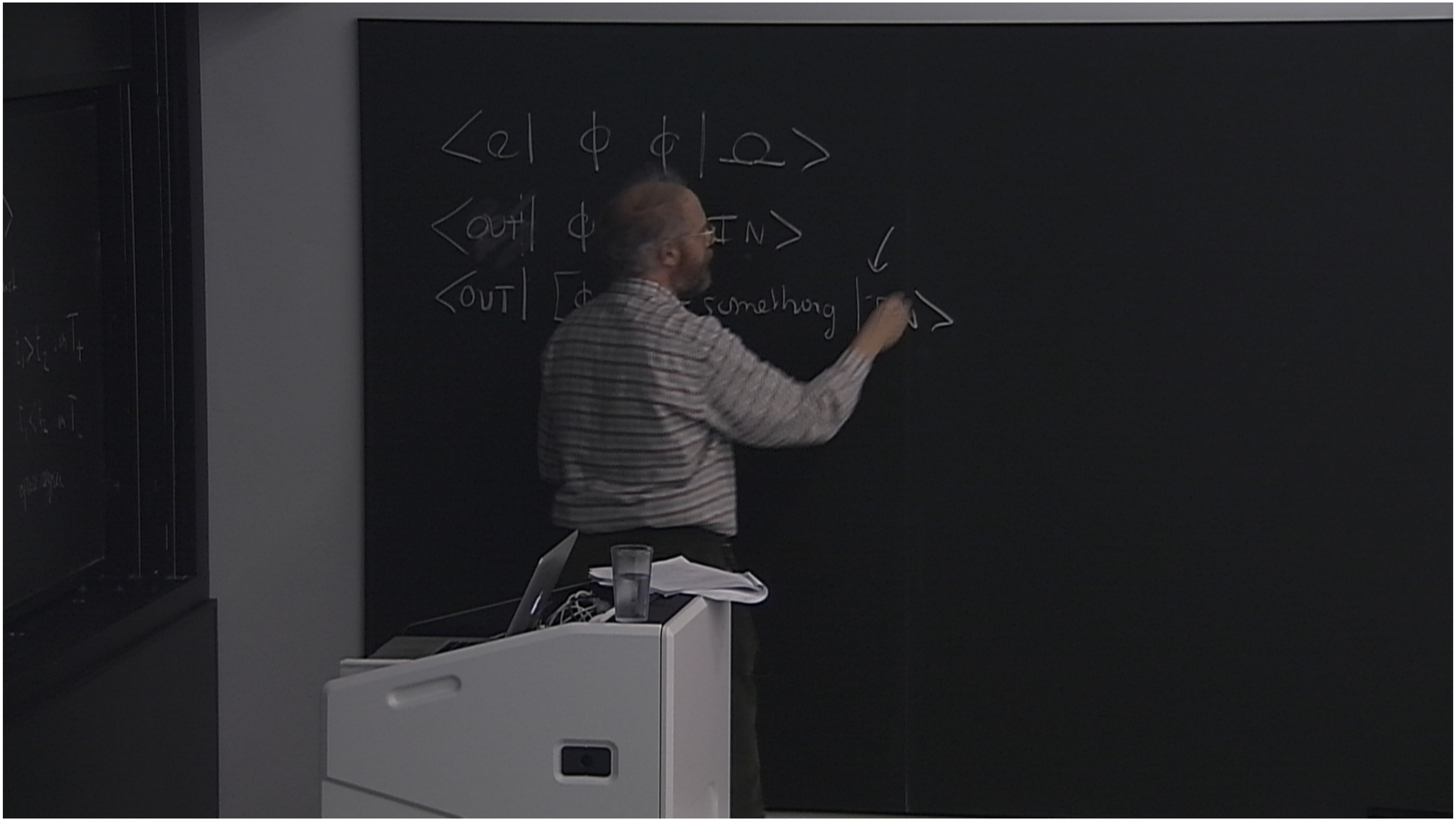
Gaussian Free Field D=2 64x64 lattice

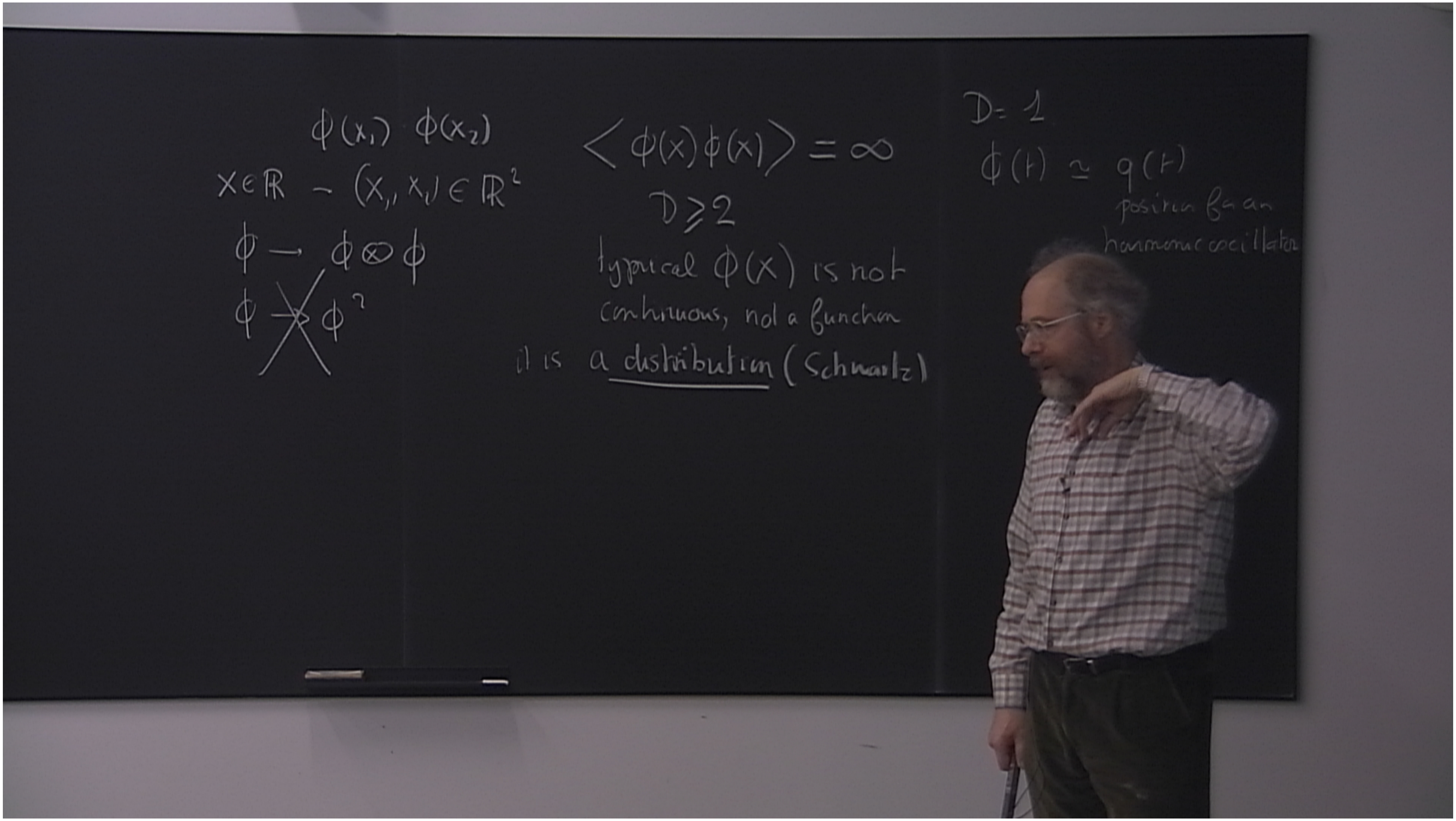



D= 1
$\langle \phi(x)\phi(x) \rangle = \infty$
D ≥ 2
typical $\phi(x)$
continuous, not
it is a distribution
$\phi(t) \simeq q(t)$
position for an
harmonic oscillator

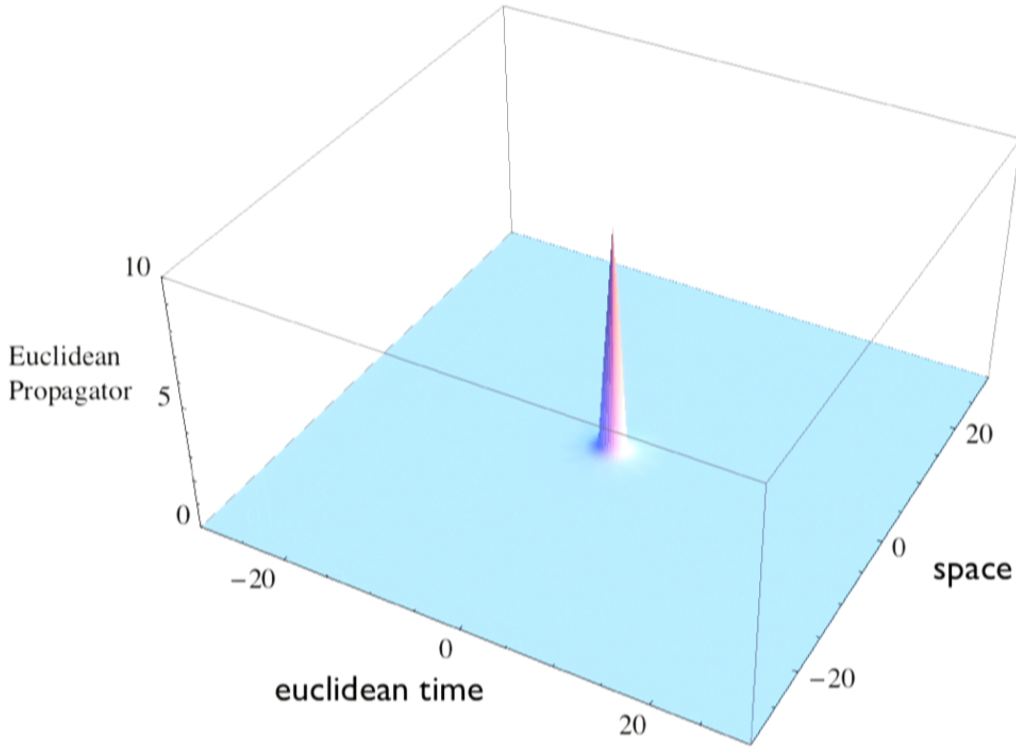


⟨Ω| φ φ |Ω⟩
⟨OUT| φ φ |IN⟩
φ] = sum

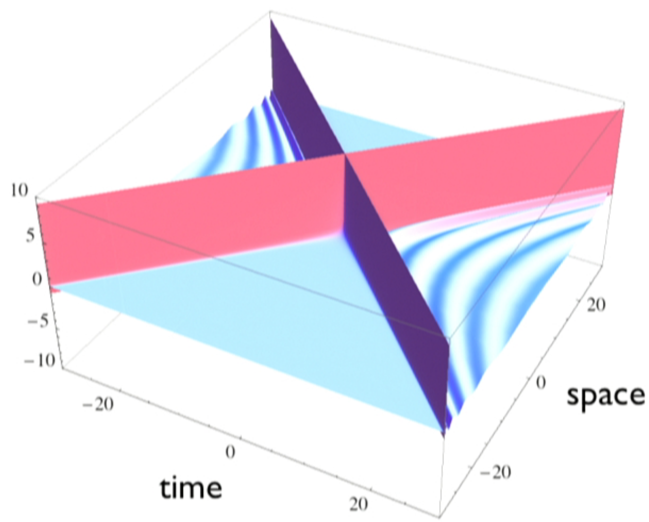

⟨e| φ φ |Ω⟩
⟨OUT| φ
IN⟩
⟨OUT| [φ
something |


φ(x₁) φ(x₂)
x ∈ ℝ → (x₁, x₁) ∈ ℝ²
φ → φ ⊗ φ
φ → φ²
< φ(x) φ(x) > = ∞
D ≥ 2
typical φ(x) is not
continuous, not a function
it is a distribution (Schwartz)
D = 1
φ(t) ≃ q(t)
position for an
harmonic oscillator

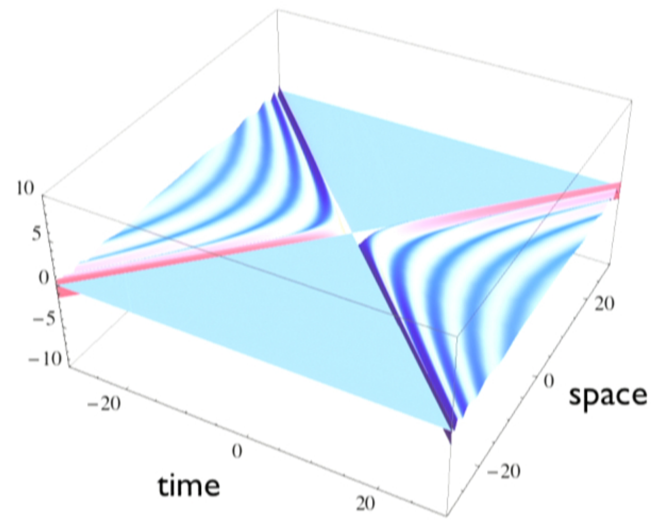
Free field mass m , dimension D=2 propagator in Euclidean space



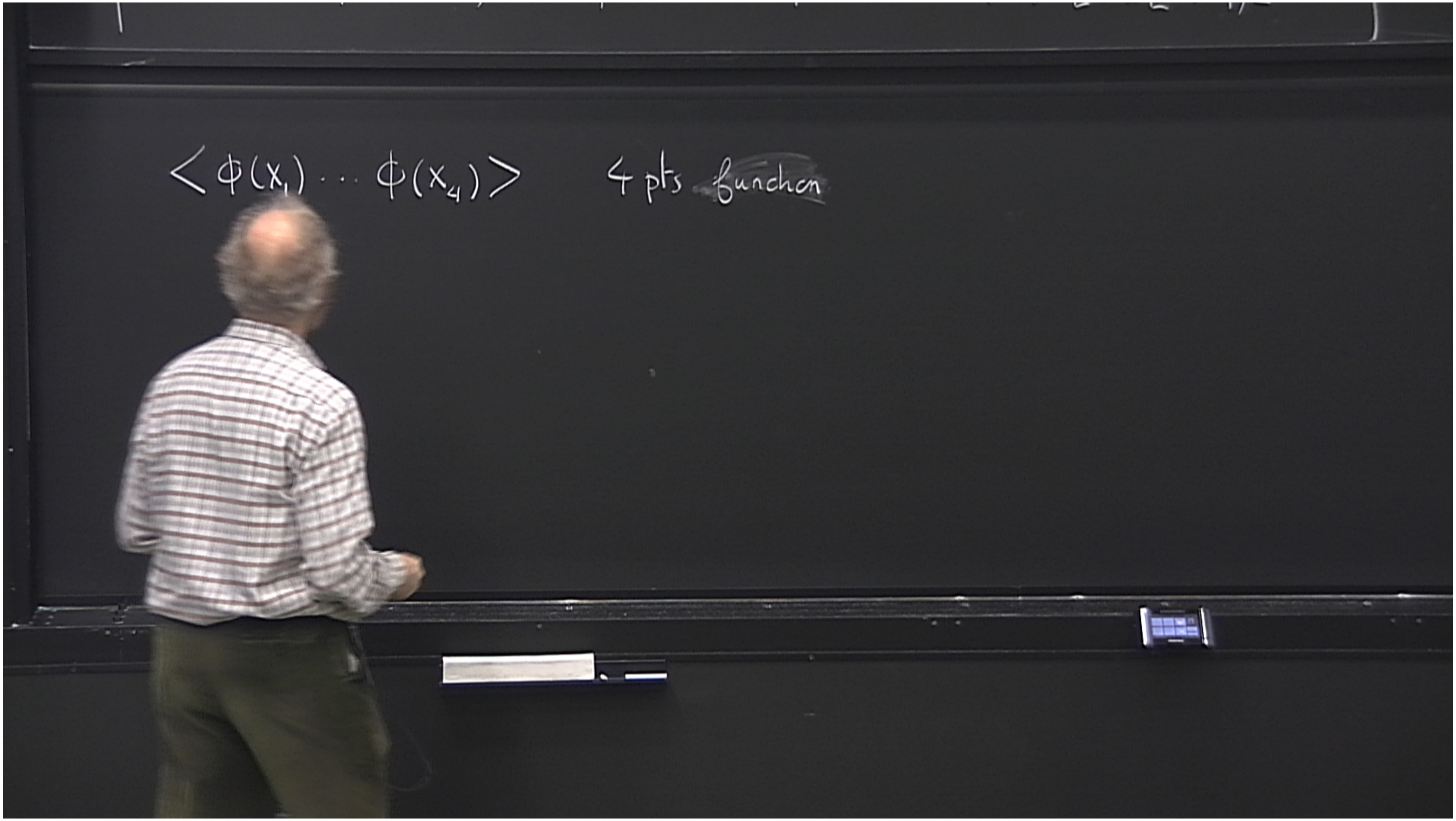
Free field mass m , dimension D=2 propagator in Minkowski space = Feynman propagator

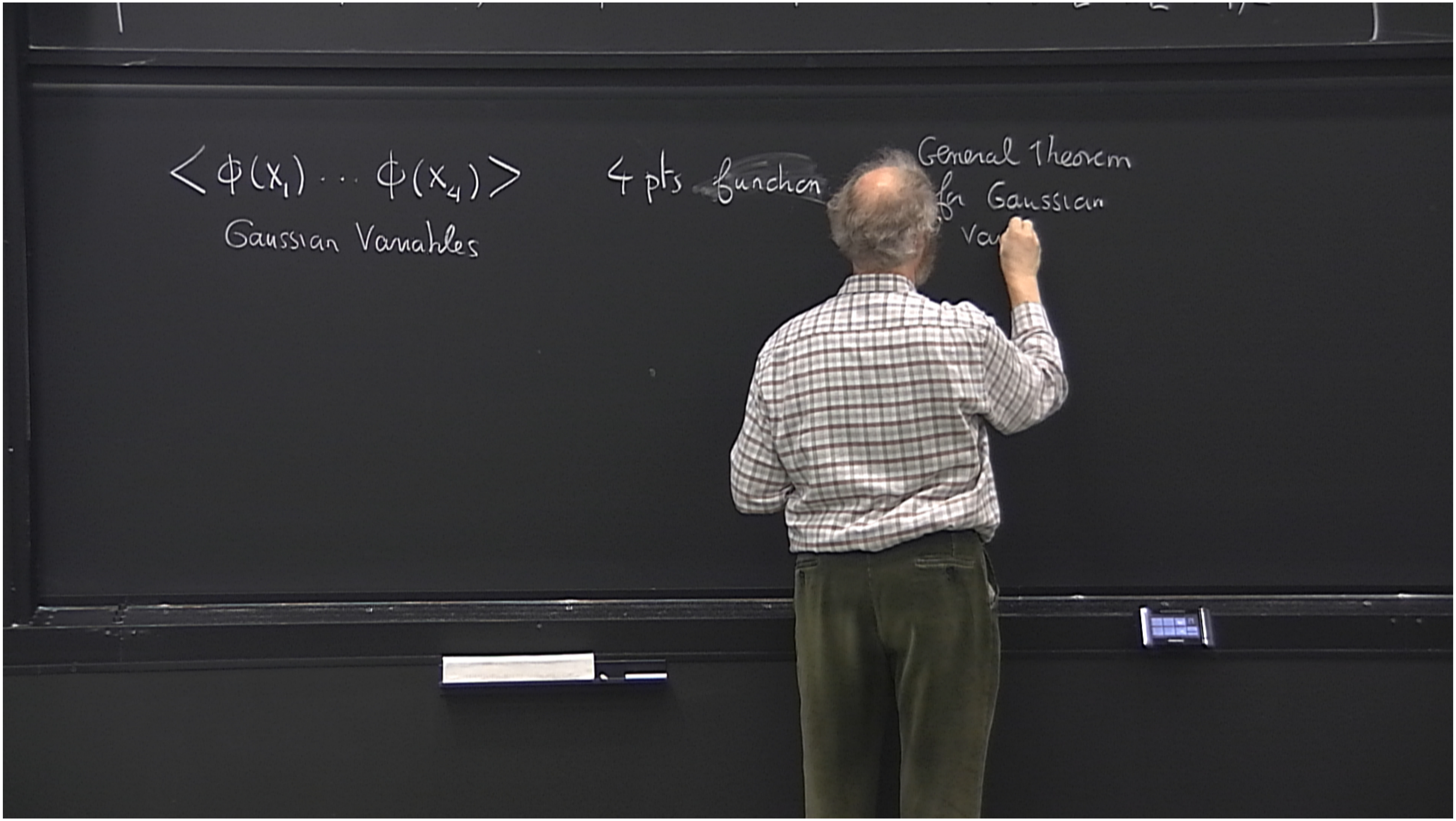


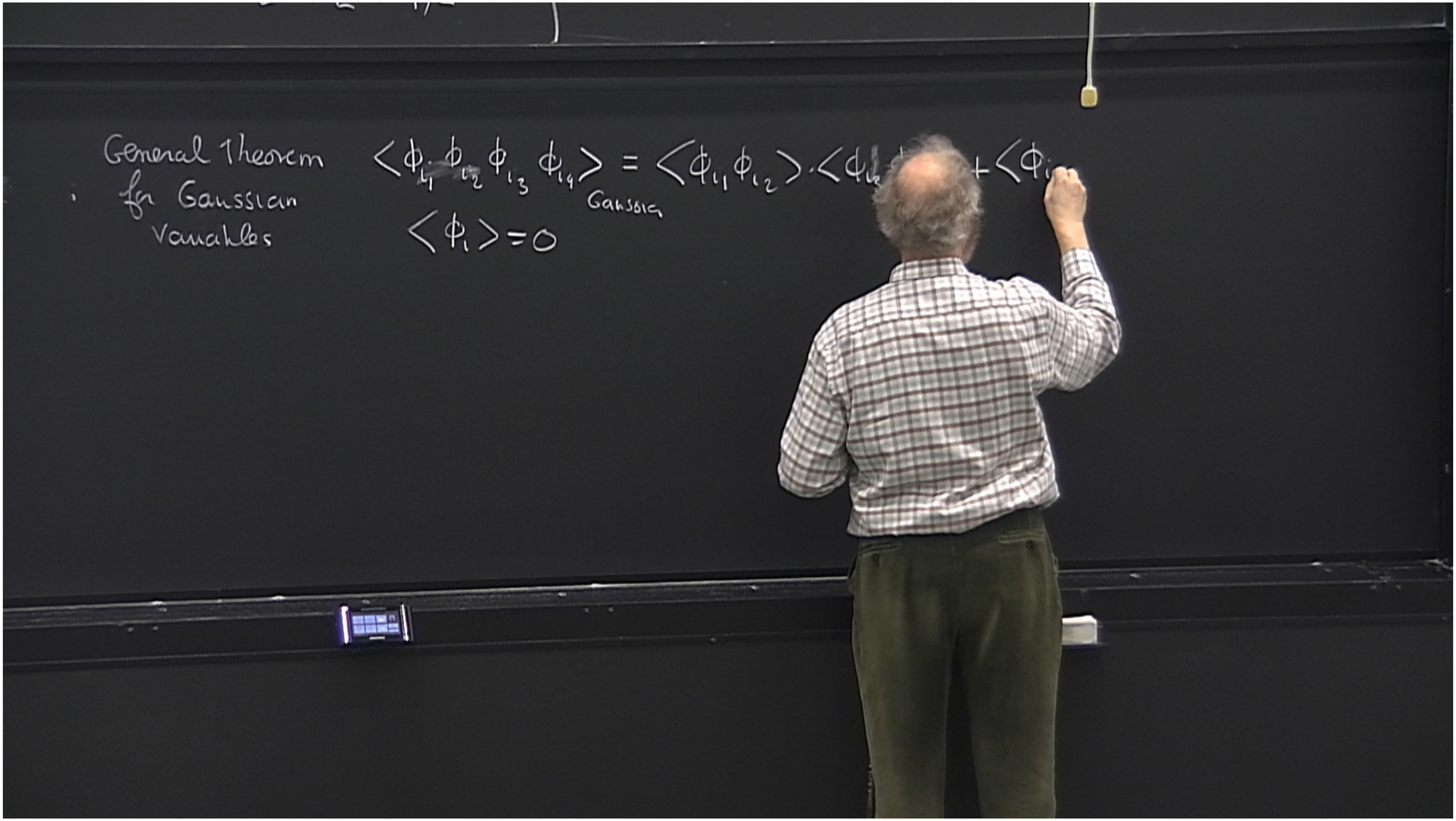
Real part

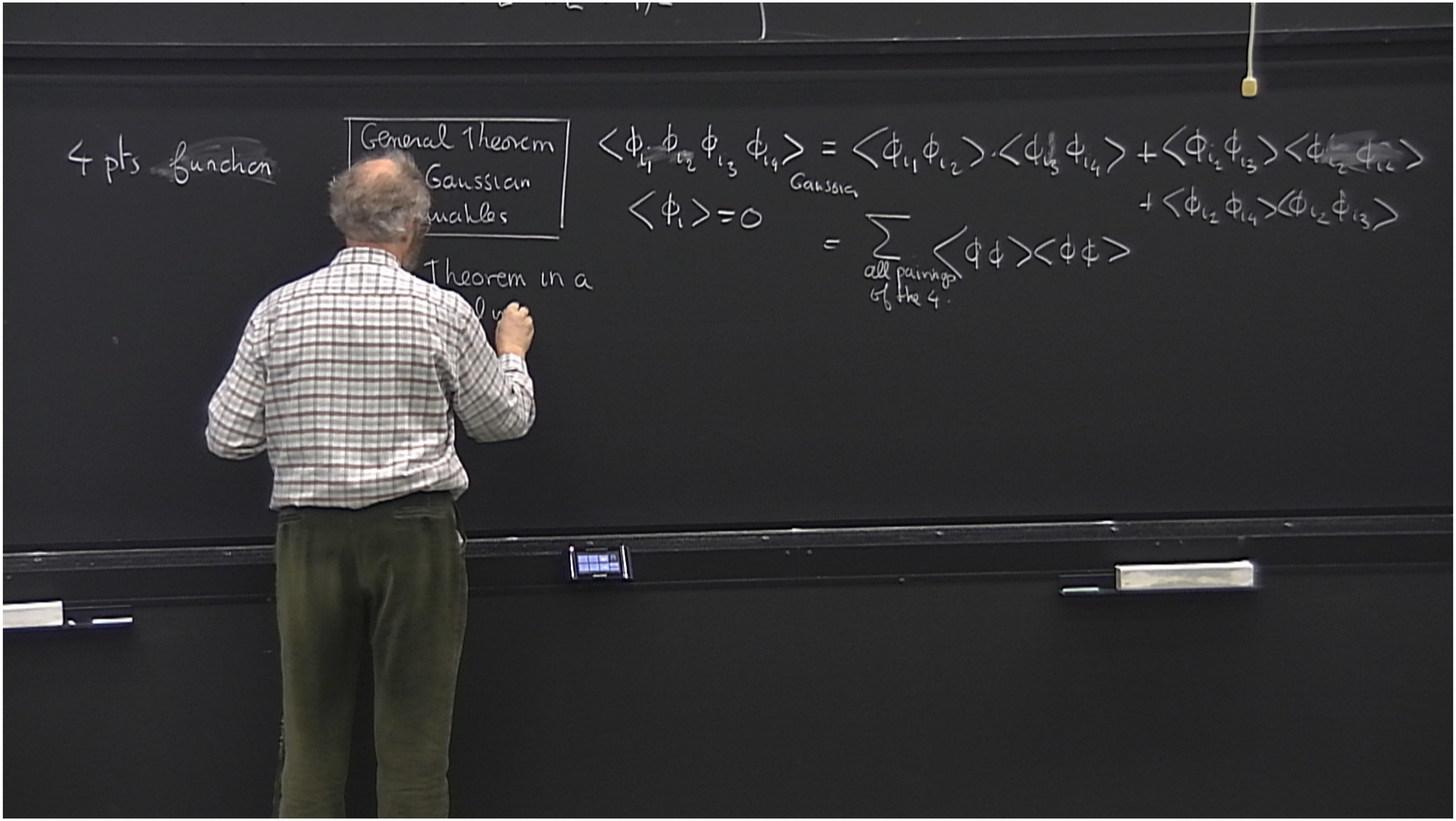


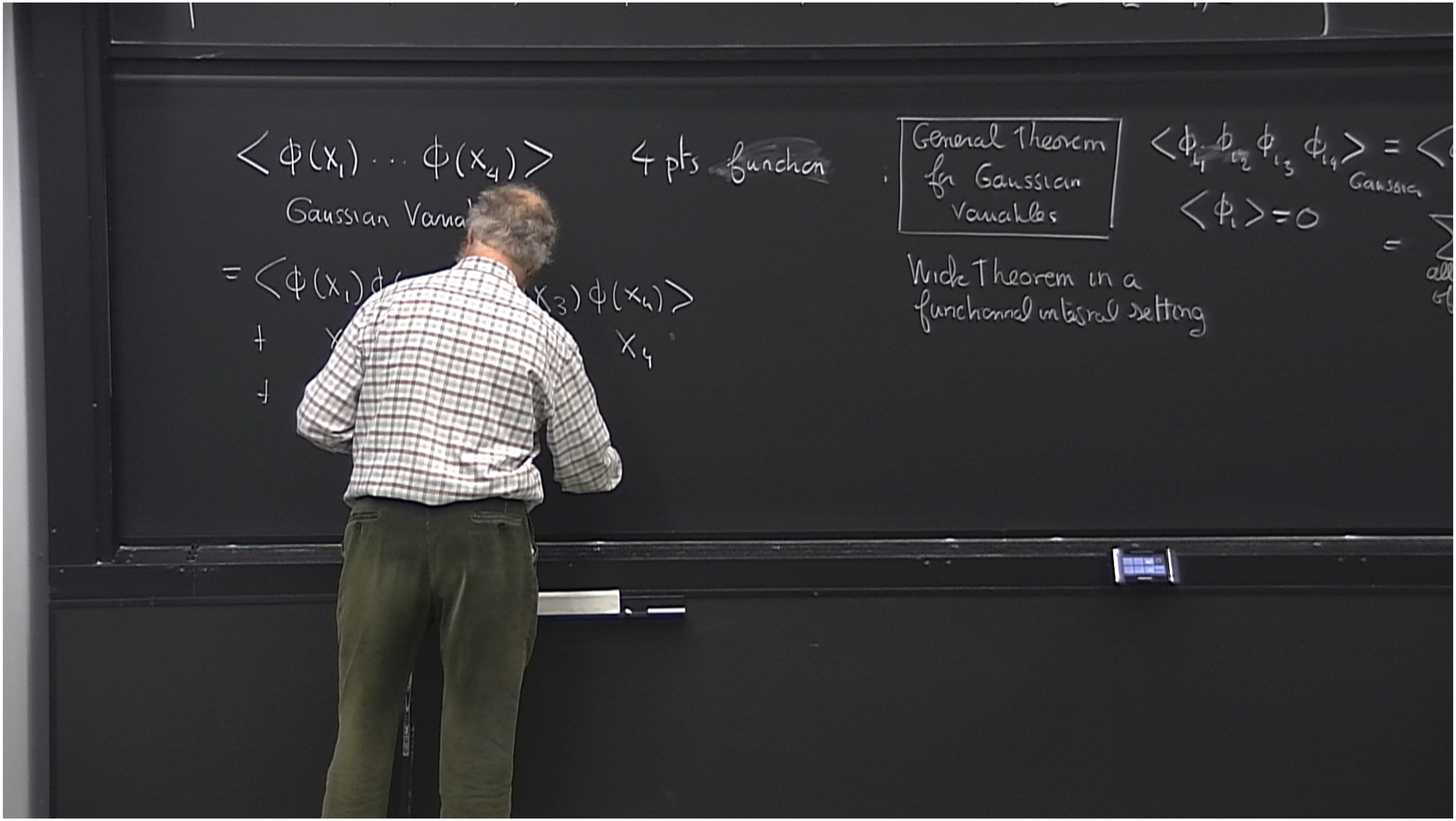
Imaginary part

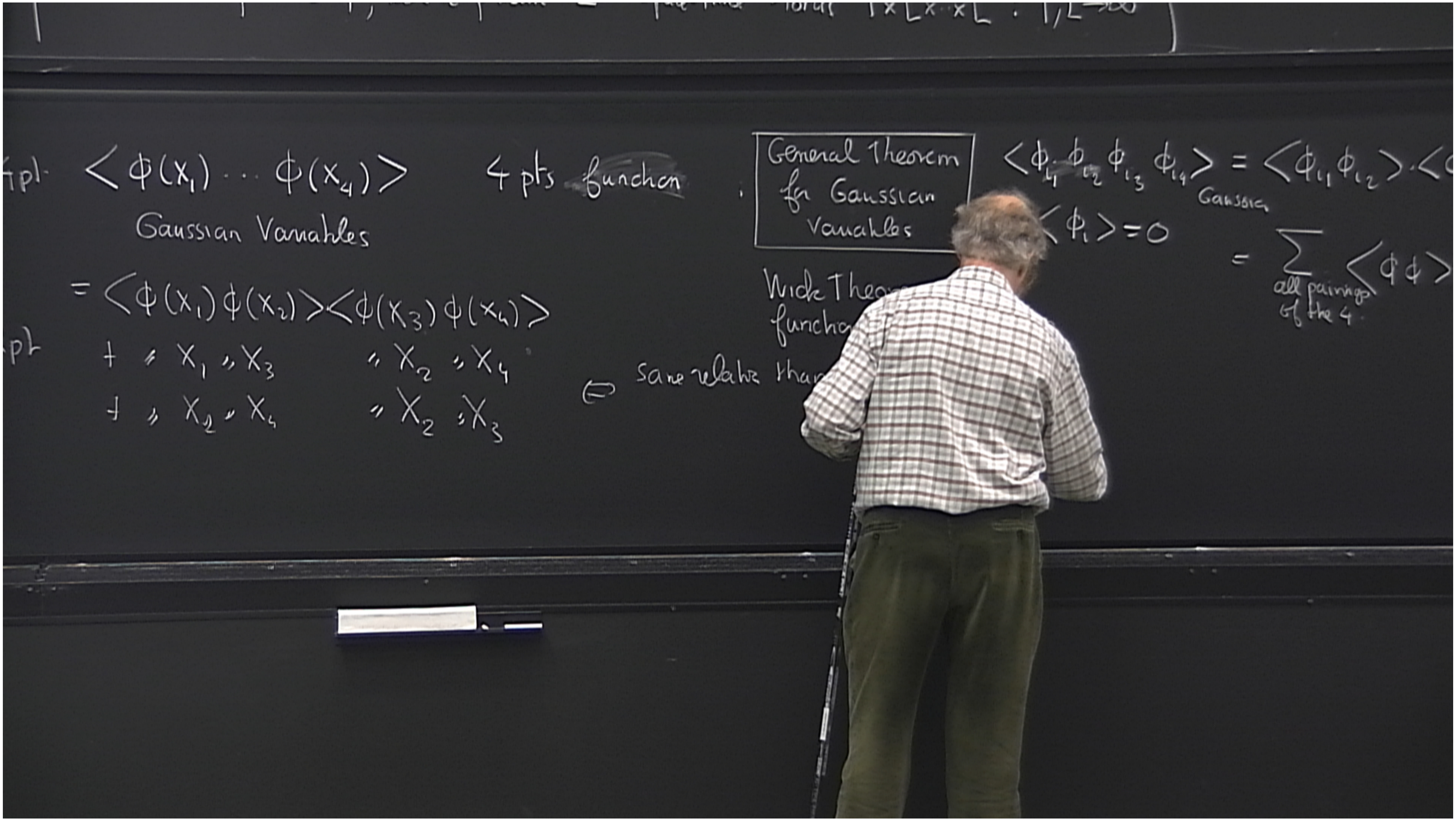

$\langle \phi(x_1) \cdots \phi(x_4) \rangle$ 4 pts function


$\langle \phi(x_1) \cdots \phi(x_4) \rangle$
4 pts function
Gaussian Variables
General Theorem
for Gaussian

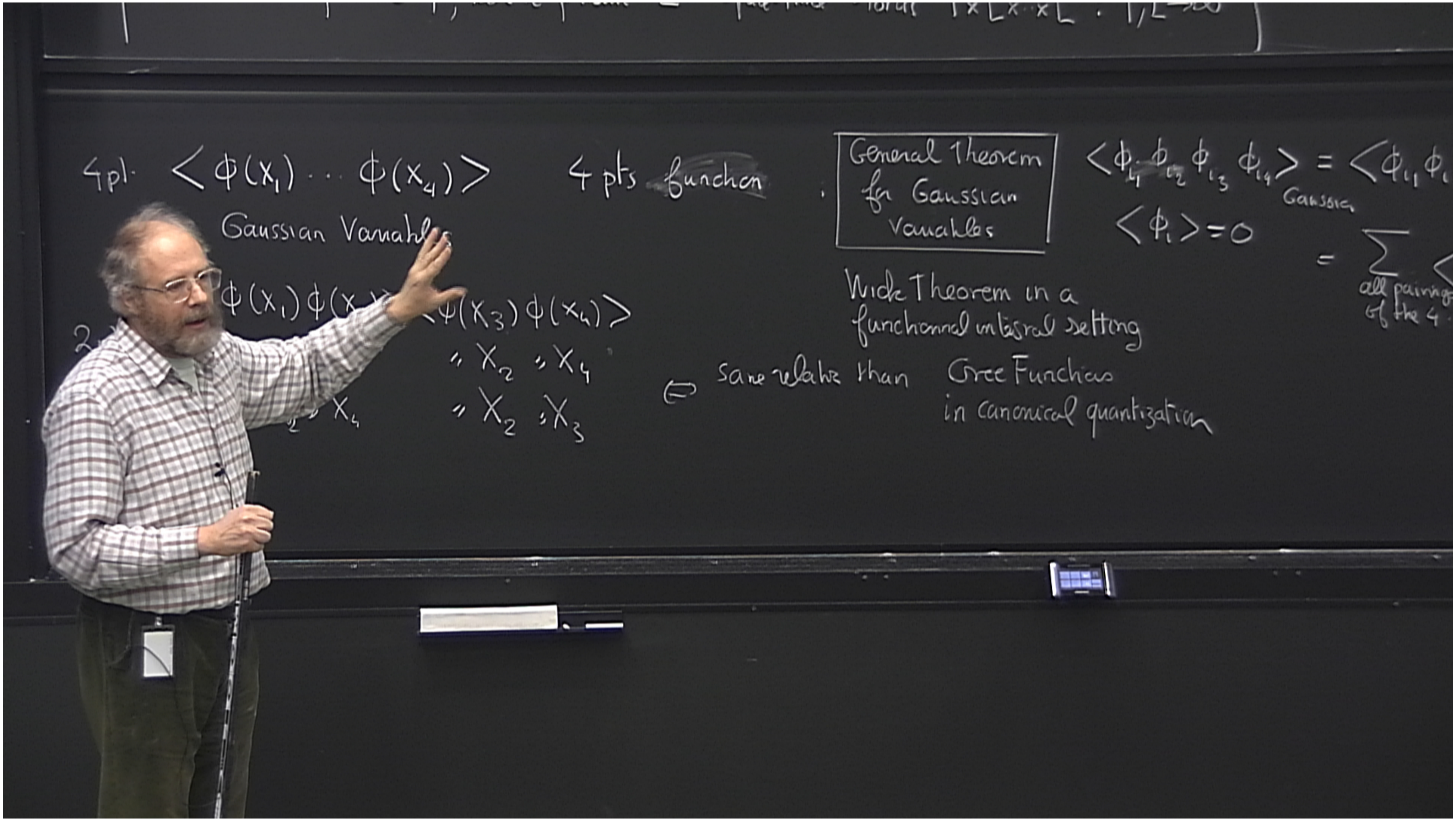

General Theorem
for Gaussian
Variables
$\langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \rangle = \langle \phi_{i_1} \phi_{i_2} \rangle \cdot \langle \phi_{i}$
Gaussian
$\langle \phi_i \rangle = 0$

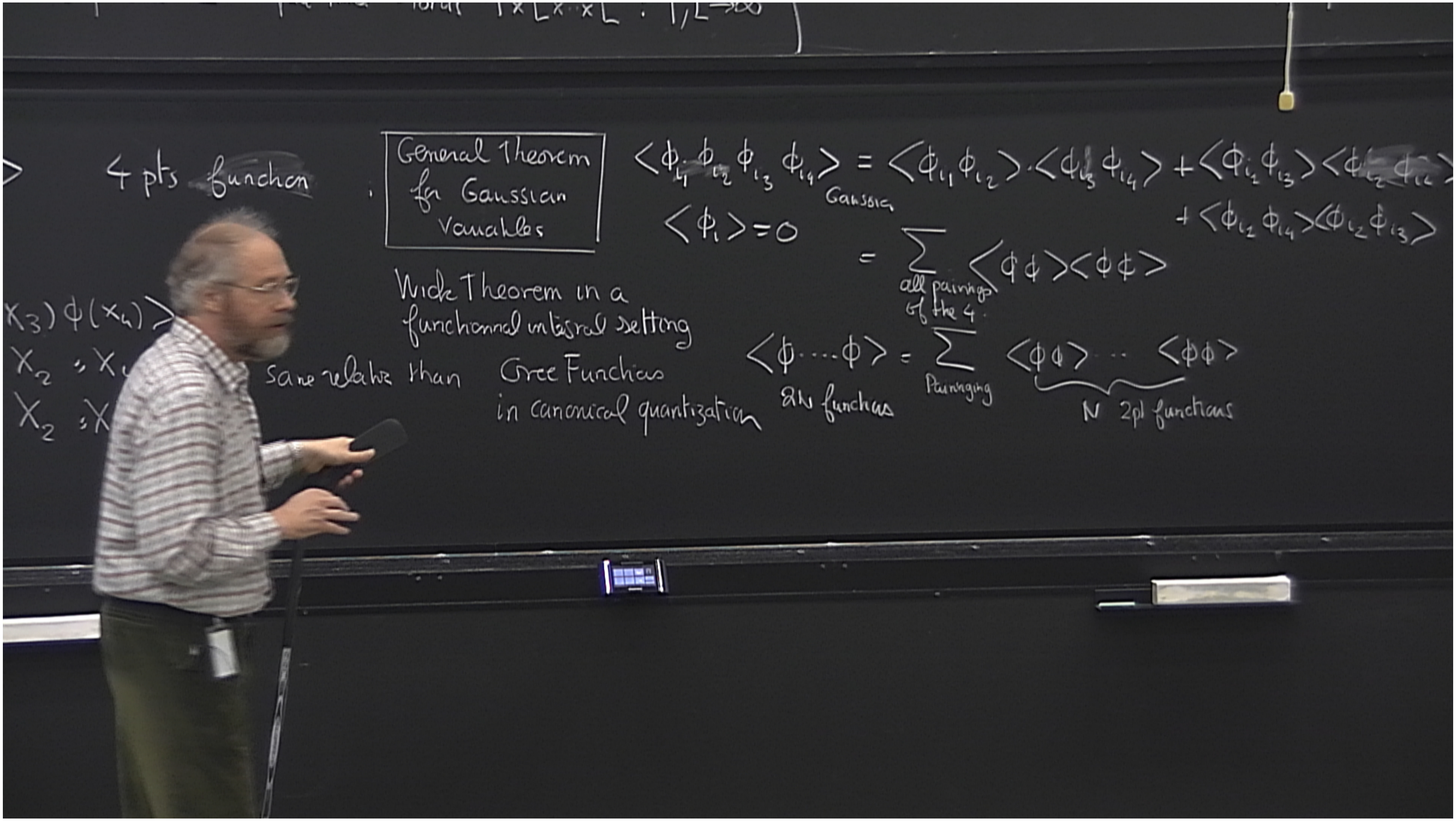

4 pts function
General Theorem
Gaussian
$\langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \rangle = \langle \phi_{i_1} \phi_{i_2} \rangle \cdot \langle \phi_{i_3} \phi_{i_4} \rangle + \langle \phi_{i_1} \phi_{i_3} \rangle \langle \phi_{i_2} \phi_{i_4} \rangle$
$+ \langle \phi_{i_1} \phi_{i_4} \rangle \langle \phi_{i_2} \phi_{i_3} \rangle$
Gaussian
$\langle \phi_i \rangle = 0$
$= \sum_{\text{all pairings of the 4}} \langle \phi \phi \rangle \langle \phi \phi \rangle$
Theorem in a


⟨φ(x1) ... φ(x4)⟩ 4 pts function
General Theorem for Gaussian Variables
⟨φi1 φi2 φi3 φi4⟩ =
Gaussian
⟨φi⟩ = 0
Wick Theorem in a functional integral setting



$\langle \phi(x_1) \cdots \phi(x_4) \rangle$ 4 pts function
Gaussian Variables
$= \langle \phi(x_1)\phi(x_2)\rangle \langle \phi(x_3)\phi(x_4)\rangle$
$+ \; x_1 \; x_3 \quad x_2 \; x_4$
$+ \; x_2 \; x_4 \quad x_2 \; x_3$
same relative
General Theorem for Gaussian Variables
Wick Theo
$\langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\phi_{i_4}\rangle = \langle \phi_{i_1}\phi_{i_2}\rangle$
Gaussian
$\langle \phi_i \rangle = 0$
$= \sum \langle \phi \phi \rangle$
all pairings of the 4


4pt· < φ(x1) ··· φ(x4) >
4 pts function
Gaussian Variables
General Theorem for Gaussian Variables
<φi1 φi2 φi3 φi4> =
Gaussian
<φi> = 0
Wick Theorem in a functional integral setting
same relation than Green Functions in canonical quantization


4 pts function
General Theorem for Gaussian Variables
Gaussian
⟨φ_i⟩ = 0
= Σ ⟨φφ⟩⟨φφ⟩
all pairings of the 4
Wick Theorem in a functional integral setting
Same relation than Green Functions in canonical quantization
2N functions
Pairings
N 2pt functions