

Title: 13/14 PSI - Quantum Field Theory II - Lecture 3

Date: Nov 13, 2013 09:00 AM

URL: <http://pirsa.org/13110010>

Abstract:

Scalar Free Field ; Functional Integral Quantization.

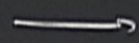
mass m $\phi(x)$ real

$$d\Omega^2 = -dt^2 + d\vec{x}^2$$

$$S[\phi] = \int d^D x \frac{1}{2} (-\partial_\nu \phi \partial^\nu \phi - m^2 \phi^2)$$

$$S_E \int d^D x_E \frac{1}{2} \partial_\nu \phi +$$

$$\int \mathcal{D}[\phi] \exp\left(-\frac{i}{\hbar} S[\phi]\right)$$

$|g\rangle$  $|\phi\rangle$

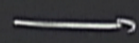
$$\phi = \{ \phi(\bar{z}) \}$$

Field configuration
on a given time slice

 QM QFT

$|q\rangle$

QM

 $|\phi\rangle$

QFT

$$\phi = \{ \phi(\bar{z}) \}$$

Field configuration
on a given time slice

$$|q\rangle \longrightarrow |\phi\rangle$$

QM QFT

$\phi = \{ \phi(\bar{z}) \}$ Field configuration
on a given time slice
subtleities \triangle

Vacuum state $|\Omega\rangle \implies$ Euclidean Time, Periodic τ_B + limit $\tau_B = \infty$

$$\langle \Omega | \text{Observable} | \Omega \rangle$$

insert

Field configuration
on a given time slice



periodic + limit
 $\tau_{\beta} = \infty$

inserting a $q(t)$ \rightarrow
in path integral
(classical variable)

$Q(t)$
quantum
operator

Field configuration
on a given time slice

inserting a $q(t) \rightarrow Q(t)$
in path integral
(classical variable)
quantum
operator

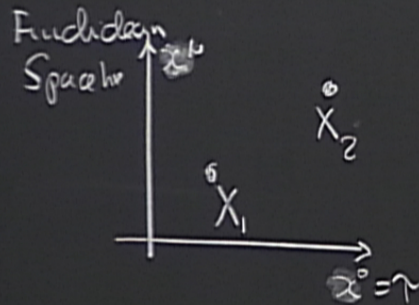
periodic + limit
 $\tau_\beta = \infty$

inserting a $\phi(x) \rightarrow \bar{\Phi}(x) ?$
Field operator

Idea: Insert $\phi(x_1) \dots \phi(x_N)$ in Euclidean Path Integral

Euclidean Functional Integral . Expectation value of the product of 2 fields
at 2 points

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{\int \mathcal{D}[\phi] \exp(-\frac{1}{\hbar} S_E[\phi]) \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] \exp(-\frac{1}{\hbar} S_E[\phi])}$$



an Functional Integral . Expectation value of the product of 2 fields = 2 point function
at 2 points

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{\int \mathcal{D}[\phi] \exp(-\frac{1}{\hbar} S_E[\phi]) \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] \exp(-\frac{1}{\hbar} S_E[\phi])}$$

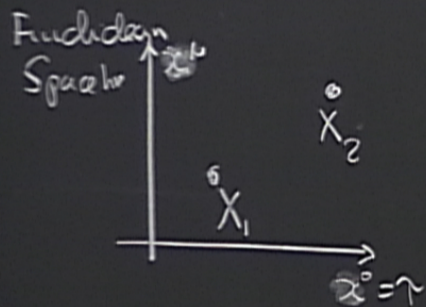
x_2

S_E is a quadratic form in ϕ
Gaussian Integral

$x_1 = \tau$

Euclidean Functional Integral . Expectation value of the product of 2 fields = 2 part at 2 points

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{\int \mathcal{D}[\phi] \exp(-\frac{1}{\hbar} S_E[\phi]) \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] \exp(-\frac{1}{\hbar} S_E[\phi])}$$



S_E is a quadratic form in φ
Gaussian Integral

+ m²) φ(x)

$$G(x_2, x_1) = G(x_2 - x_1) = G(|x_2 - x_1|)$$

Translasi invariance Rotasi Invariance

$$G(x_2, x_1) = G(x_2 - x_1) = G(|x_2 - x_1|)$$

Translasi invariance

Rotasi Invariance

Fourier transform $\hat{G}(k) = \int d^D x \cdot \exp(-i k \cdot x) G(x)$

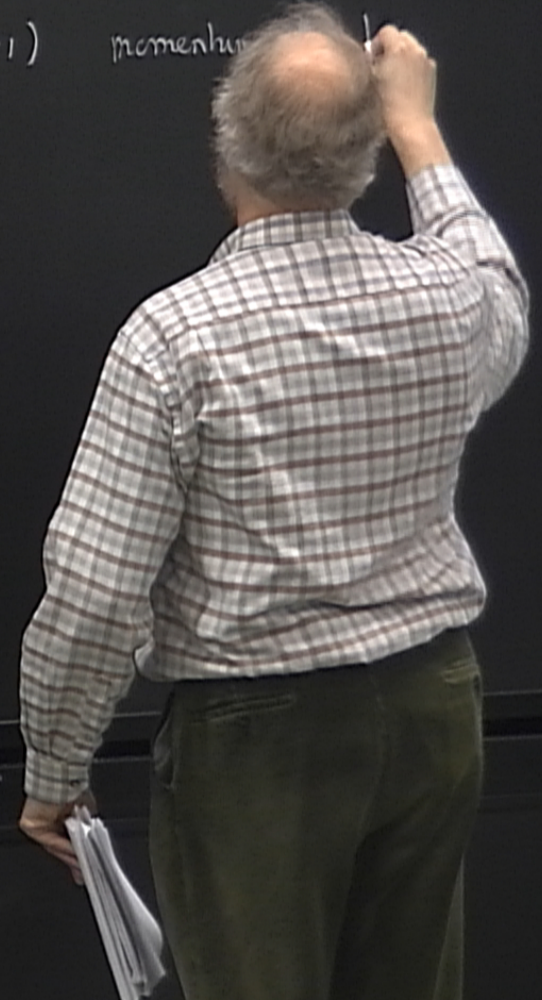
B.C $\Rightarrow G(x)$ decays at ∞

$$(k^2 + m^2) \cdot \hat{G}(k) = 1 \quad \hat{G}(k) = \frac{1}{k^2 + m^2}$$

$$x = (x^\mu; \mu = 0, \dots, D-1)$$
$$k = (k_\mu; \mu = 0, \dots, D-1)$$

$$x \cdot k = k_\mu \cdot x^\mu$$

menentukan



$$= G(|x_2 - x_1|) \quad X = (x^\mu; \mu = 0, \dots, D-1) \quad X \cdot K = k_\mu \cdot x^\mu$$

$$K = (k_\mu; \mu = 0, \dots, D-1) \quad \text{"momentum" variable}$$

Rotations Invariance

$$= \int d^D x \cdot \exp(-i K \cdot X) G(x)$$

$$\hat{G}(k) = \frac{1}{k^2 + m^2}$$

$$G(x) = \int \frac{d^D k}{(2\pi)^D} \exp(i k \cdot X) \frac{1}{k^2 + m^2}$$

B.C $\Rightarrow G(x)$ decays at ∞

$$(k^2 + m^2) \cdot \hat{G}(k) = 1$$

$$\hat{G}(k) = \frac{1}{k^2 + m^2}$$

$$G(x) = \int \frac{d^D k}{(2\pi)^D} \exp(i k \cdot x) \frac{1}{k^2 + m^2}$$

$$G(x) = G(r) \quad r = |x| \left[- \left(\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right) + m^2 \right] G(r) = \delta(r) \Rightarrow$$

$K = (k_\mu; \mu = 0, D-1)$ "momentum variable"

Rotational Invariance

$$\hat{G}(K) = \int d^D x \cdot \exp(-i K \cdot X) G(X)$$

decays at ∞

$$\hat{G}(K) = \frac{1}{k^2 + m^2}$$

$$G(X) = \int \frac{d^D k}{(2\pi)^D} \exp(i k \cdot X) \frac{1}{k^2 + m^2}$$

$K_\nu(z)$ Bessel Funct
of 2nd type

$$\left[-\left(\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right) + m^2 \right] G(r) = \delta(r) \Rightarrow G(r) = \frac{1}{2\pi} \left(2\pi \frac{r}{m} \right)^{\frac{2-D}{2}} K_{\frac{D-2}{2}}(r \cdot m)$$

Real time $X = (t, \vec{x})$

$G(X_1, X_2)$

$$\left(+ \frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial \vec{x}} \right)^2 + m^2 \right) G(x) =$$

Real time $X = (t, \vec{x})$

$G(X_1, X_2)$

$$\left(+ \frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial \vec{x}} \right)^2 + m^2 \right) G(X) = -i \delta(X)$$

Propagator of Free Field In can. quantization
Which propagator?

$$G(X) = G_{\text{Feynman}}(X)$$

Real time

$$X = (t, \vec{x})$$

$$G(X_1, X_2)$$

$$\left(+ \frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial \vec{x}} \right)^2 + m^2 \right) G(X) = -i \delta(X)$$

Propagator of Free Field In can. quantization
Which propagator?

$$G(X) = G_{\text{Feynman}}(X)$$

$$G(X) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikX}}{k^2 - m^2 + i\epsilon}$$

$$\vec{k}$$

$$\vec{x}$$

haben

Real time

$$X = (t, \vec{x})$$

$$G(X_1, X_2)$$

$$\left(\frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial \vec{x}} \right)^2 + m^2 \right) G(X) = -i \delta$$

Propagator of Free Field In c...

Which propagator i

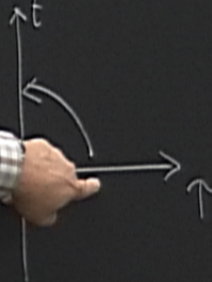
$$G(X) = G_{\text{Feynman}}$$

$$G_E(X) = \int$$

$$K_E = (k_0, \vec{k})$$

$$X_E = (\tau, \vec{x})$$

Wick rotation $\tau = it$



Real time $X = (t, \vec{x})$

$$G(X_1, X_2)$$

$$\left(+ \frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial \vec{x}} \right)^2 + m^2 \right) G(X) = -i \delta(X)$$

Propagator of Free Field In can. quantization
which propagator?

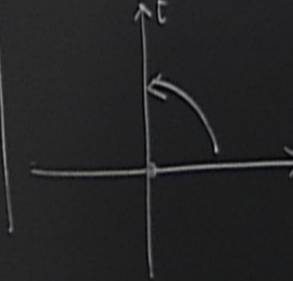
$$G(X) = G_{\text{Feynman}}(X)$$

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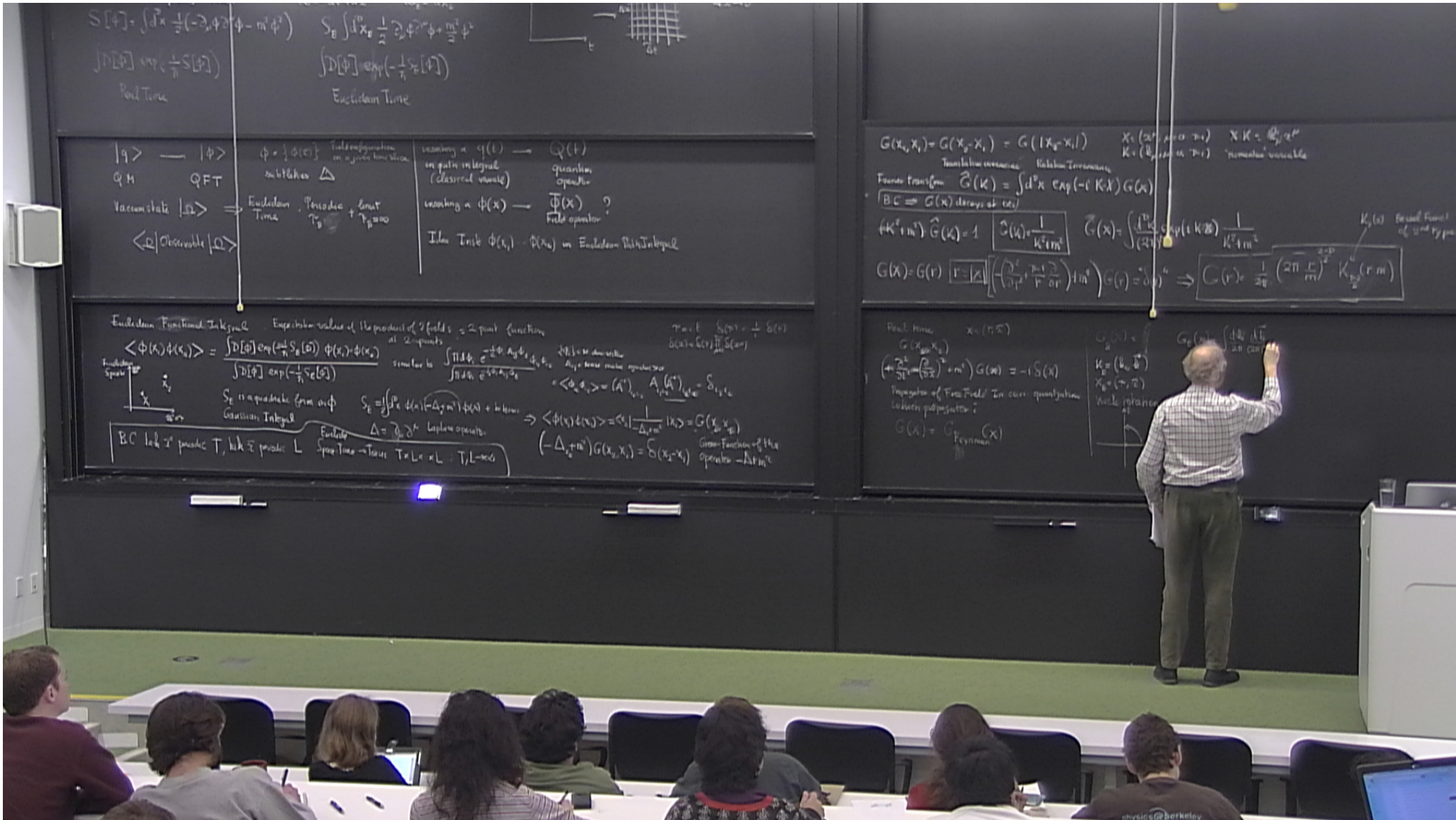
$$K_E = (k_0, \vec{k})$$

$$X_E = (\tau, \vec{x})$$

Wick rotation $\tau = it$



$$G_E(X) = \int d^4 k_0$$



$$S[\phi] = \int d^4x \frac{1}{2} (\partial_\mu \phi)^2 - m^2 \phi^2$$

$$S_E[\phi] = \int d^4x_E \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2$$

Real Time $\int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$

Euclidean Time $\int \mathcal{D}[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$



$|q\rangle \rightarrow |\phi\rangle$
 QM QFT
 Vacuum state $|\Omega\rangle$
 $\langle \Omega | \text{Observable} | \Omega \rangle$

$\phi = \{ \phi(x) \}$ configuration on a given time slice
 substitution Δ

Euclidean Time $\tau, \tau_0, \tau_1, \tau_2, \tau_3, \tau_4$

considering a $\phi(t) \rightarrow Q(t)$
 on path integral (classical variable) quantum operator

considering a $\phi(x) \rightarrow \hat{\Phi}(x)$?
 field operator

Idea: Insert $\phi(x_1) \dots \phi(x_n)$ in Euclidean Path Integral

$$G(x_2, x_1) = G(x_2, x_1) = G(x_2 - x_1) \quad x_i = (x_i^\mu, t_i) \quad x_i K = \frac{d^4x_i}{(2\pi)^4} \quad K = (k_i^\mu, \omega_i) \quad \text{"momentum" variable}$$

Translation invariance: Kaldeus-Trennung

Fourier transform $\hat{G}(K) = \int d^4x \exp(-i Kx) G(x)$

BC $\Rightarrow G(x)$ decays at ∞

$$(K^2 + m^2) \hat{G}(K) = 1 \quad \hat{G}(K) = \frac{1}{K^2 + m^2} \quad \hat{G}(x) = \int \frac{d^4K}{(2\pi)^4} \exp(i Kx) \frac{1}{K^2 + m^2}$$

$$G(x) = G(r) \quad r = |x| \quad \left[\left(\frac{\partial}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial}{\partial r} \right] G(r) = \delta(r) \Rightarrow G(r) = \frac{1}{2\pi} \int_0^\infty \frac{dk}{k} K_0(kr, m)$$

$K_0(x)$ Bessel function of order 0

Euclidean Functional Integral: Expectation value of the product of n fields = n -point function at 2-points

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{\int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S_E[\phi]\right) \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)}$$

similar to $\int \mathcal{D}\phi \exp\left(-\frac{1}{2} \phi A \phi\right)$ $A_{ij} = \text{matrix}$ A_{ij}^{-1} inverse matrix operator

$\langle \phi_i \phi_j \rangle = (A^{-1})_{ij} = A_{ij}^{-1} \delta_{ij} = \delta_{ij} \delta_{x_i x_j}$

S_E is quadratic form in ϕ
 Gaussian Integral $S_E = \int d^4x \phi(x) (-\Delta + m^2) \phi(x) + \text{linear}$

Euclidean $\Delta = \partial_\mu^2$ Laplace operator

Space-Time \rightarrow Time $T \times L^3 \times L$ $T, L \rightarrow \infty$

BC: $\text{left } x^0 \text{ periodic } T, \text{ left } x^i \text{ periodic } L$

$\langle \phi(x_1) \phi(x_2) \rangle = \langle \phi | \frac{1}{-\Delta + m^2} | \phi \rangle = G(x_2, x_1)$
 Green-Function of this operator $-\Delta + m^2$

Pauli time $x_0 = (t, \vec{x})$

$$G(x_2, x_1) = \int \frac{d^4K}{(2\pi)^4} \exp(i Kx) \frac{1}{K^2 + m^2}$$

$K = (k, \omega)$
 $x_0 = (t, \vec{x})$
 Wick rotation at $t \rightarrow i\tau$

Propagator of Free Field in canon. quantization
 Leibniz rule

$G(x) = G_{\text{Feynman}}(x)$

(t, \vec{x})

$$G(x) = -i \delta(x)$$

In can. quantization

(X)
can

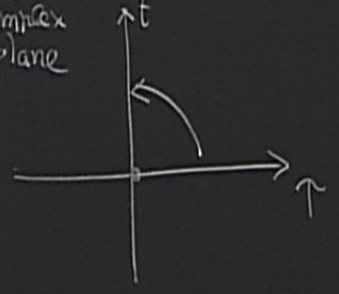
$$G_E(X) = \int$$

$$K_E = (k_0, \vec{k})$$

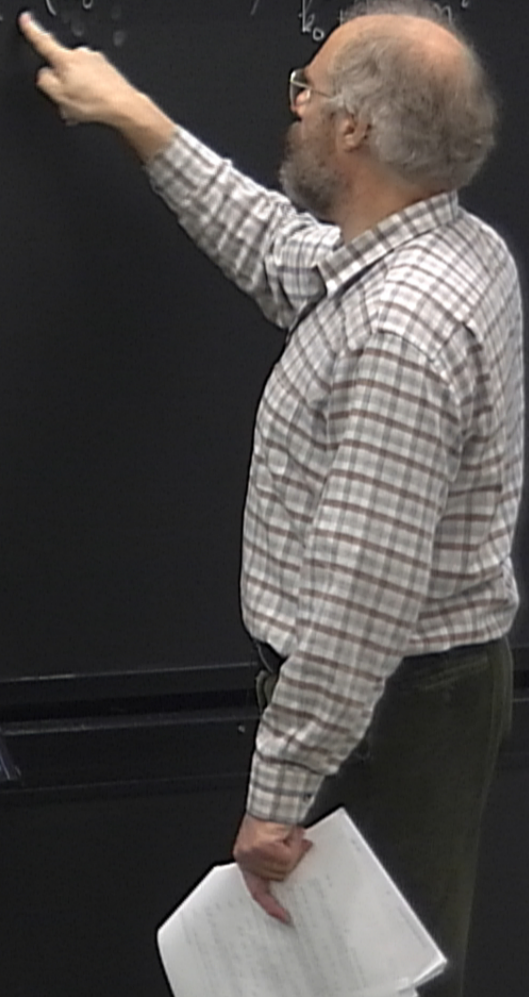
$$X_E = (\tau, \vec{x})$$

Wick rotation $\tau = it$

complex plane



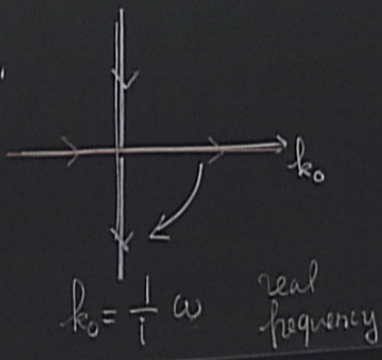
$$G_E(\tau, \vec{x}) = \int \frac{d k_0}{2\pi} \frac{d \vec{k}}{(2\pi)^{d-1}} \exp(i(k_0 \tau + \vec{k} \cdot \vec{x})) \frac{1}{k_0^2 + \vec{k}^2 + m^2}$$



$$G_E(\vec{r}, \vec{x}) = \int \frac{d^d k_0}{2\pi} \frac{d^d \vec{k}}{(2\pi)^{d-1}} \exp(i(k_0 r + \vec{k} \cdot \vec{x})) \frac{1}{k_0^2 + \vec{k}^2 + m^2} \Rightarrow \int \frac{d\omega}{2\pi} \frac{d^d \vec{k}}{(2\pi)^{d-1}} \exp(i\omega t + \vec{k} \cdot \vec{x}) \frac{-i}{-\omega^2 + \vec{k}^2 + m^2}$$

pole at $k_0 = \pm i(\vec{k}^2 + m^2)$

$\tau = it \Rightarrow$ must make a
 "counter-clockwise rotation"
 on the integration
 contour of k_0



$$G_E(X) = \int \frac{d^4 k}{(2\pi)^4} \exp(i(k_0 \tau + \vec{k} \cdot \vec{x})) \frac{1}{k_0^2 + \vec{k}^2 + m^2} \Rightarrow \int \frac{d\omega}{2\pi} \frac{d^3 \vec{k}}{(2\pi)^{D-1}} \exp(i\omega t + \vec{k} \cdot \vec{x}) \frac{-i}{-\omega^2 + \vec{k}^2 + m^2}$$

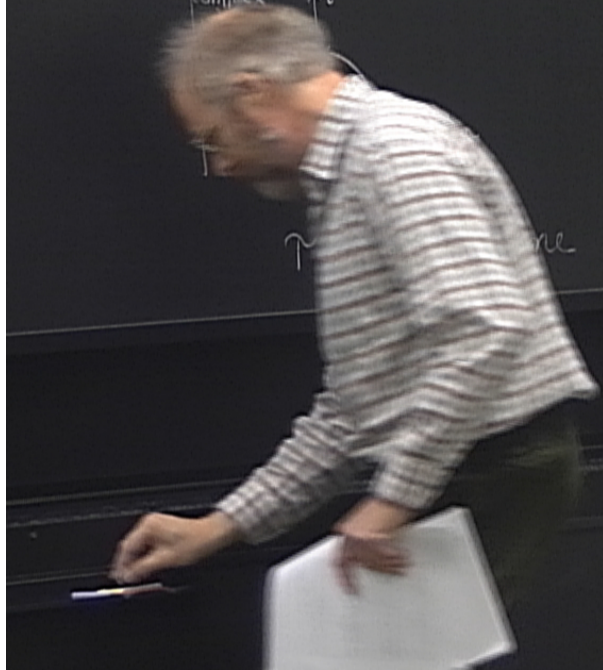
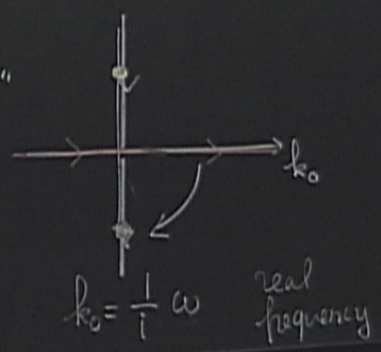
$$K_E = (k_0, \vec{k})$$

$$X_E = (\tau, \vec{x})$$

poles at $k_0 = \pm i(\vec{k}^2 + m^2)$

(X)
quantization

Wick rotation $\tau = it \Rightarrow$ must make a "counterclockwise rotation" in the integration contour of k_0



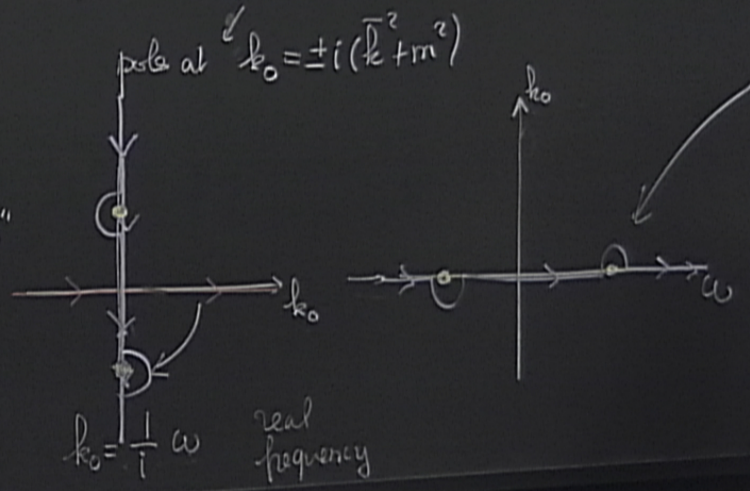
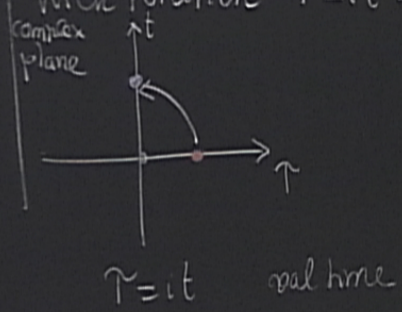
$$G_E(X) = \int \frac{d^d k_0}{2\pi} \frac{d^d \vec{k}}{(2\pi)^{D-1}} \exp(i(k_0 \tau + \vec{k} \cdot \vec{x})) \frac{1}{k_0^2 + \vec{k}^2 + m^2} \Rightarrow \int \frac{d\omega}{2\pi} \frac{d^d \vec{k}}{(2\pi)^{D-1}} \exp(i\omega t + \vec{k} \cdot \vec{x}) \frac{-i}{-\omega^2 + \vec{k}^2 + m^2}$$

$$K_E = (k_0, \vec{k})$$

$$X_E = (\tau, \vec{x})$$

(X)
quantization

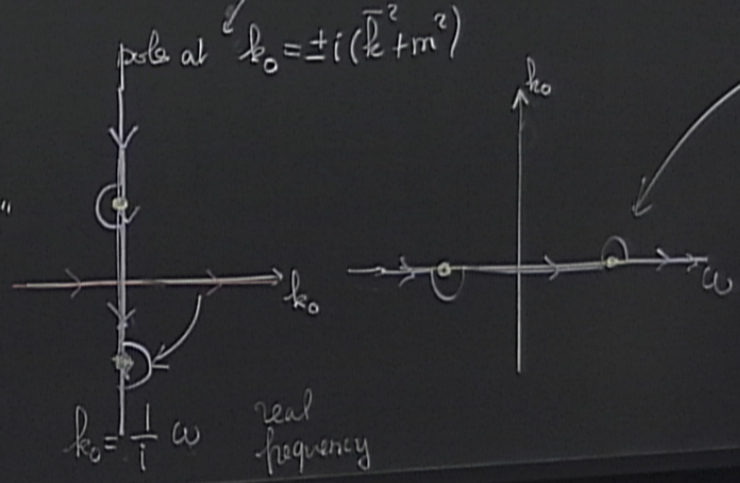
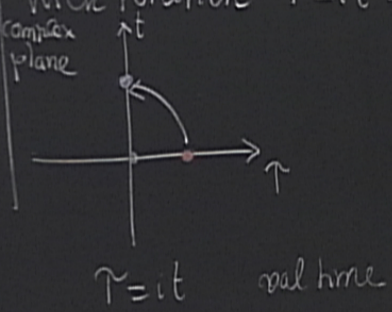
Wick rotation $\tau = it \Rightarrow$ must make a "counterclockwise rotation" in the integration contour of k_0



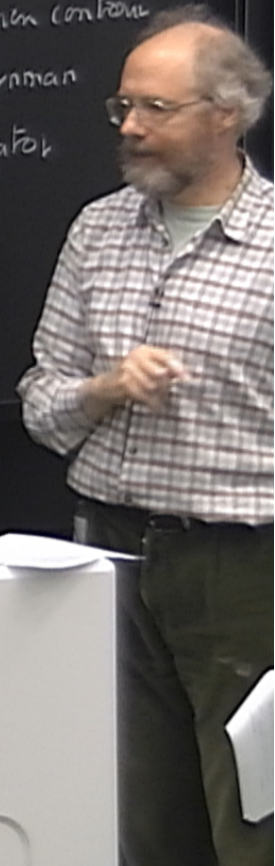
$$G_E(X) = \int \frac{d^4 k}{(2\pi)^4} \exp(i(k_0 \tau + \vec{k} \cdot \vec{x})) \frac{1}{k_0^2 + \vec{k}^2 + m^2} \Rightarrow \int \frac{d\omega}{2\pi} \frac{d^3 \vec{k}}{(2\pi)^{D-1}} \exp(i\omega t + \vec{k} \cdot \vec{x}) \frac{-i}{-\omega^2 + \vec{k}^2 + m^2}$$

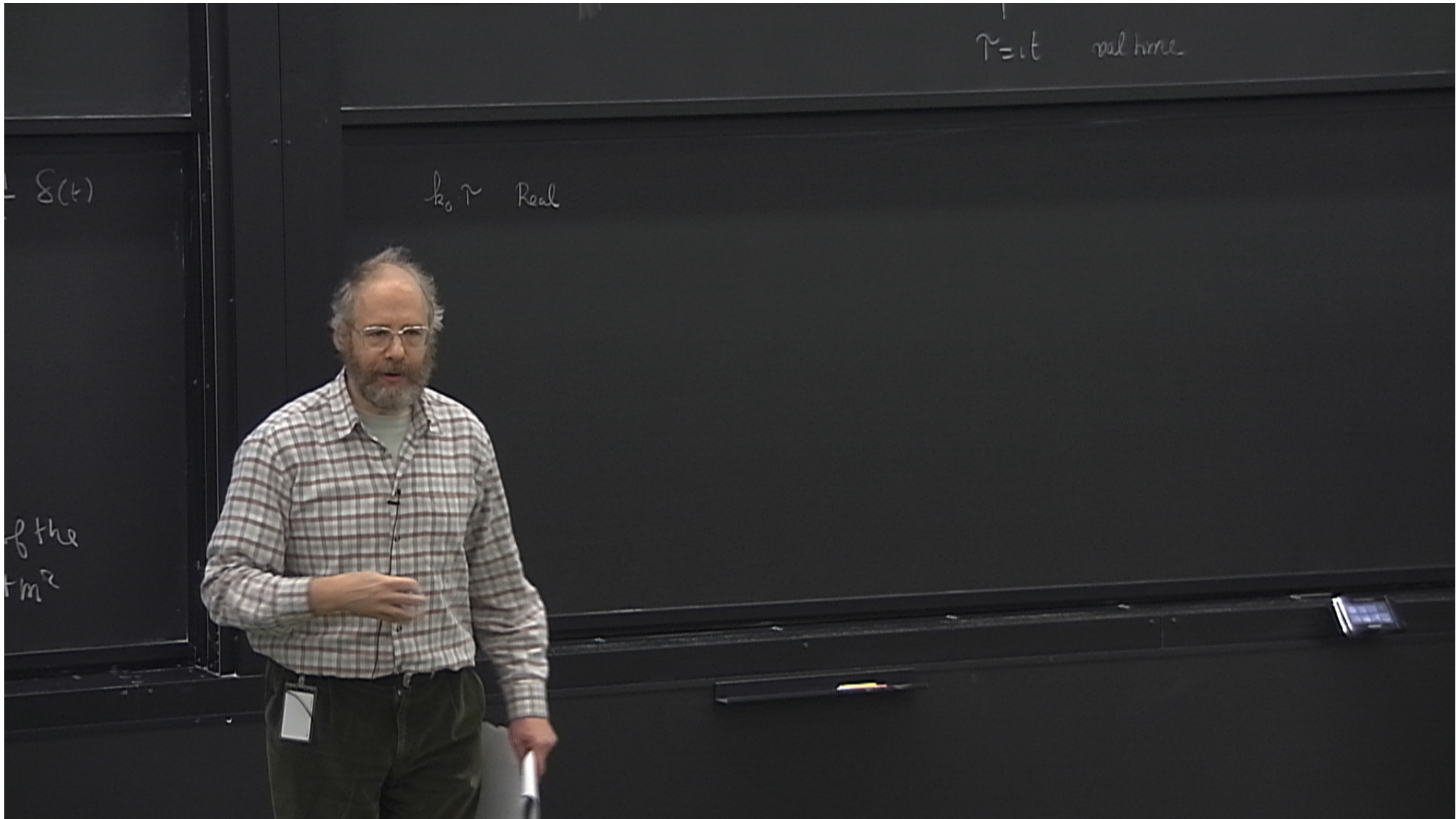
$K_E = (k_0, \vec{k})$
 $X_E = (\tau, \vec{x})$

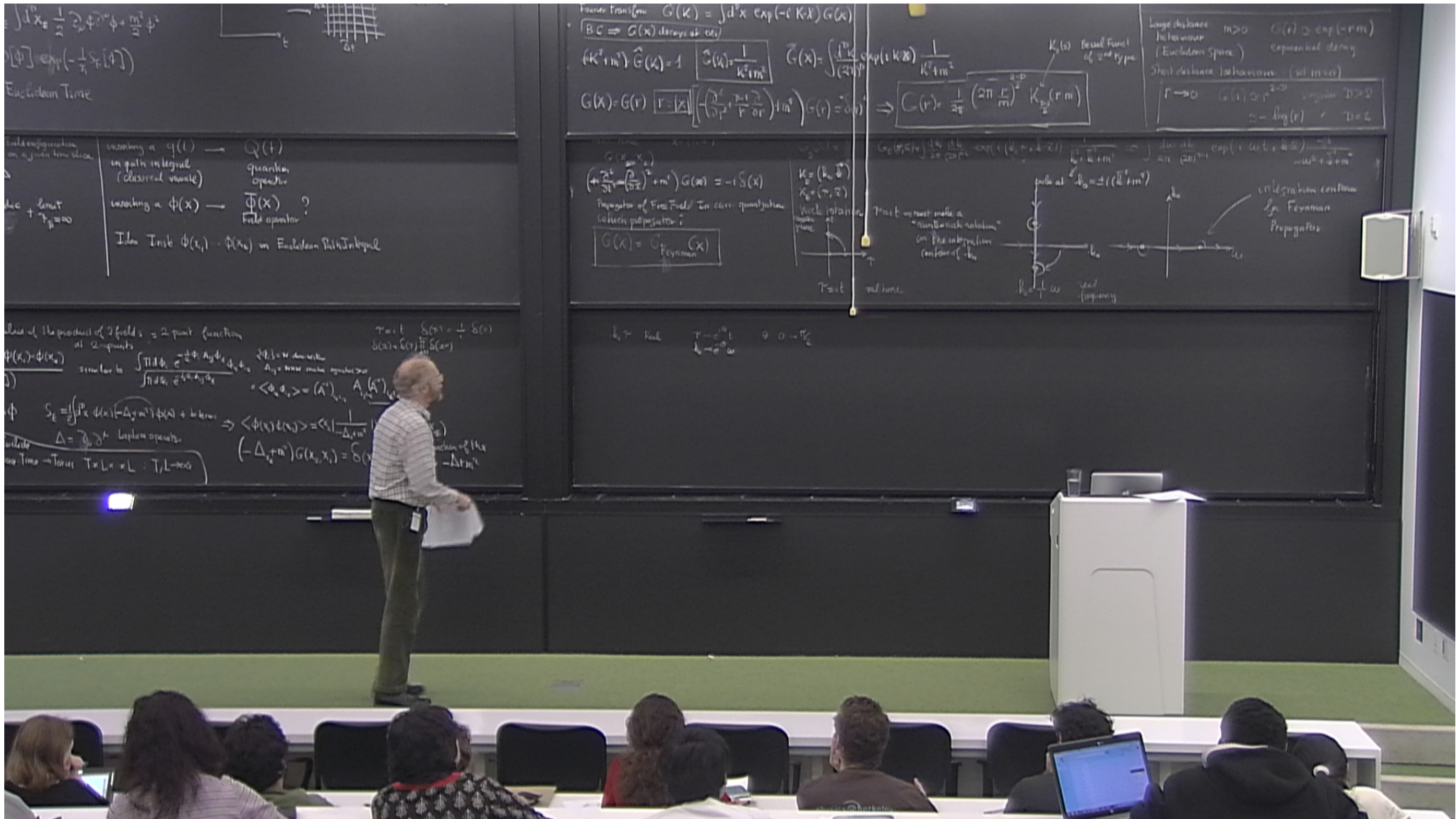
Wick rotation $\tau = it \Rightarrow$ must make a
 "counterclockwise rotation"
 in the integration
 contour of k_0



integrieren contour
 für Feynman
 Propagator







Euclidean Time

$\int d^d x \frac{1}{2} \dot{\phi}^2 + \frac{m^2}{2} \phi^2$
 $\langle \phi \rangle = \exp(-\frac{1}{2} S_E[\phi])$

inserting $q(t) \rightarrow Q(t)$
 in path integral (classical variable) \rightarrow quantum operator
 inserting $\phi(x) \rightarrow \hat{\Phi}(x)$? \rightarrow Field operator
 Idea: Insert $\phi(x_1) \dots \phi(x_n)$ in Euclidean Path Integral

Idea of the product of 2 fields = 2-point function at 2-points
 $\langle \phi(x_1) \phi(x_2) \rangle$ similar to $\int \prod d\phi_i \frac{e^{-\frac{1}{2} \phi_i A_{ij} \phi_j}}{\int \prod d\phi_i e^{-\frac{1}{2} \phi_i A_{ij} \phi_j}}$
 $\Rightarrow \langle \phi_i \phi_j \rangle = (A^{-1})_{ij}$

$S_E = \int d^d x \phi(x) (-\Delta + m^2) \phi(x) + h.c.$
 $\Delta = \nabla^2$ Laplacian operator.
 $(-\Delta + m^2) G(x_2, x_1) = \delta(x_2 - x_1)$

Fourier Transform $G(K) = \int d^d x \exp(-i Kx) G(x)$
 B.C. $\Rightarrow G(x)$ decays at ∞
 $(K^2 + m^2) \hat{G}(K) = 1 \Rightarrow \hat{G}(K) = \frac{1}{K^2 + m^2}$
 $G(x) = \int \frac{d^d k}{(2\pi)^d} \exp(i kx) \frac{1}{k^2 + m^2}$
 $G(x) = G(r) \left[\frac{1}{(2\pi)^d} \int \frac{d^d p}{p^2 + m^2} \right] G(r) = \tilde{G}(r)$
 $\Rightarrow C(r) = \frac{1}{2\pi} \left(\frac{2\pi}{m} \right)^{\frac{d-1}{2}} K_{\frac{d-1}{2}}(r m)$

$G(x) = \int \frac{d^d k}{(2\pi)^d} \exp(i kx) \frac{1}{k^2 + m^2}$
 $\Rightarrow G(x) = \int \frac{d^d k}{(2\pi)^d} \exp(i kx) \frac{1}{k^2 + m^2}$
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$G(x) = \int \frac{d^d k}{(2\pi)^d} \exp(i kx) \frac{1}{k^2 + m^2}$
 $\Rightarrow G(x) = \int \frac{d^d k}{(2\pi)^d} \exp(i kx) \frac{1}{k^2 + m^2}$

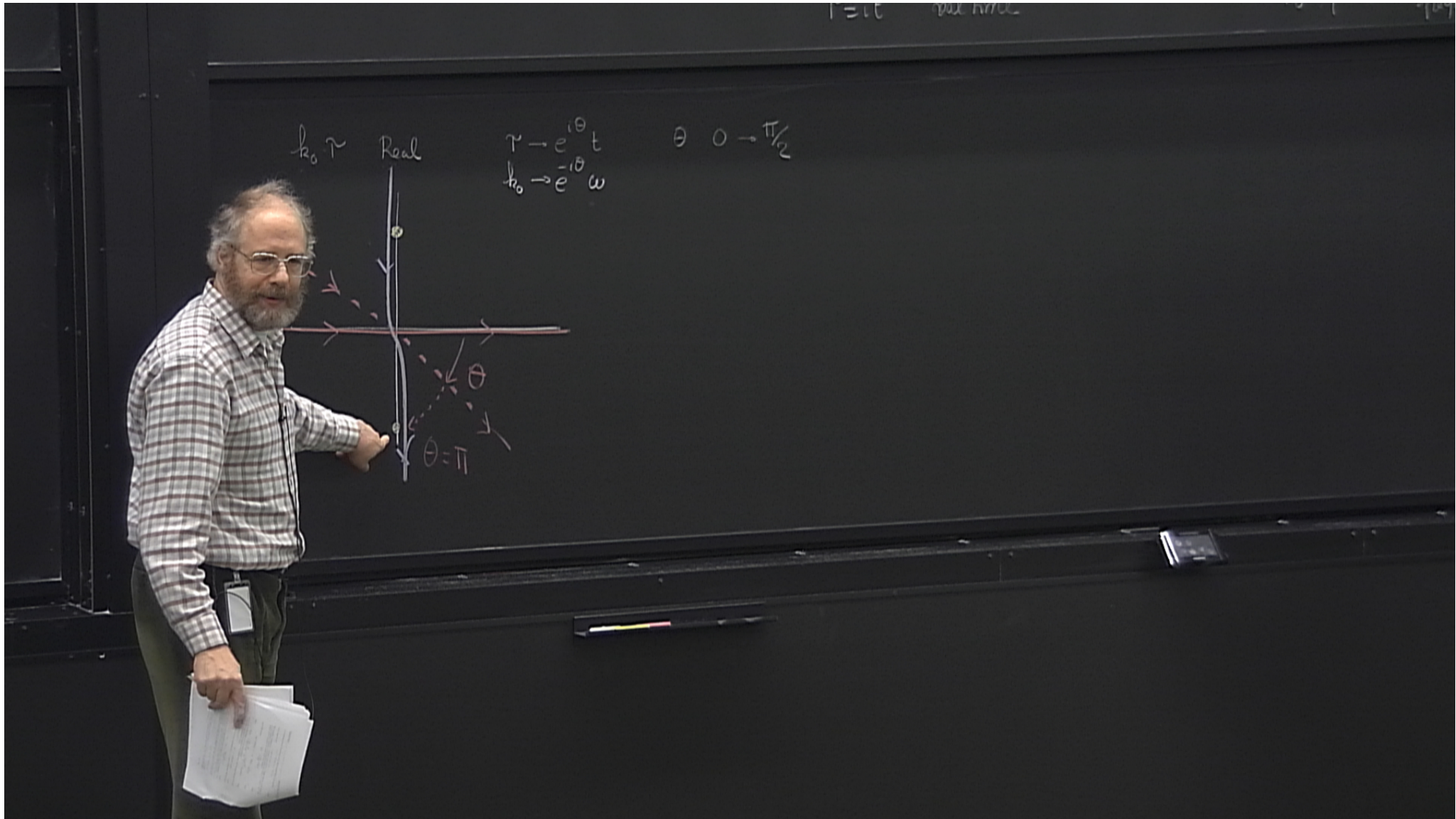
Large distance $m > 0 \Rightarrow G(x) \sim \exp(-r m)$
 Heaviside (Euclidean Space) exponential decay
 Short distance behaviour (ultra-violet)
 $r \rightarrow 0 \Rightarrow G(r) \sim r^{2-D} \sim \log(r) \quad D=2$

$G_2(p, \omega) = \int \frac{d^d k}{(2\pi)^d} \exp(i(k, p + \omega x)) \frac{1}{k^2 + m^2} \Rightarrow \int \frac{d^d k}{(2\pi)^d} \exp(i kx) \frac{1}{k^2 + m^2}$
 $\Rightarrow G_2(p, \omega) = \int \frac{d^d k}{(2\pi)^d} \exp(i(k, p + \omega x)) \frac{1}{k^2 + m^2}$
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pole at $k_0 = \pm i(\vec{k}^2 + m^2)^{1/2}$
 $\Rightarrow G_2(p, \omega) = \int \frac{d^d k}{(2\pi)^d} \exp(i(k, p + \omega x)) \frac{1}{k^2 + m^2}$
 $\Rightarrow G_2(p, \omega) = \int \frac{d^d k}{(2\pi)^d} \exp(i(k, p + \omega x)) \frac{1}{k^2 + m^2}$

that \rightarrow must make a "non-trivial relation" on the integration contour $-k_0$
 integration contour for Feynman Propagator

k_0 real $\tau = -i^0$ $\omega = 0 - i\epsilon$
 $k_0 \rightarrow \omega$

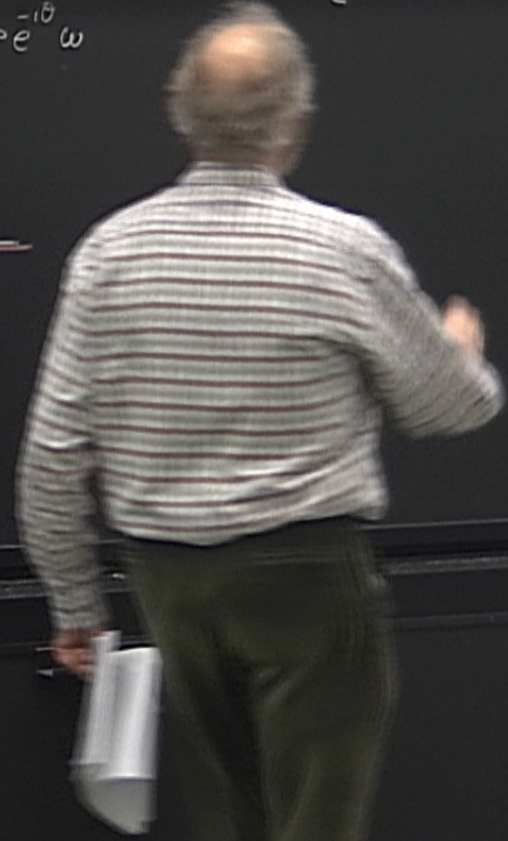
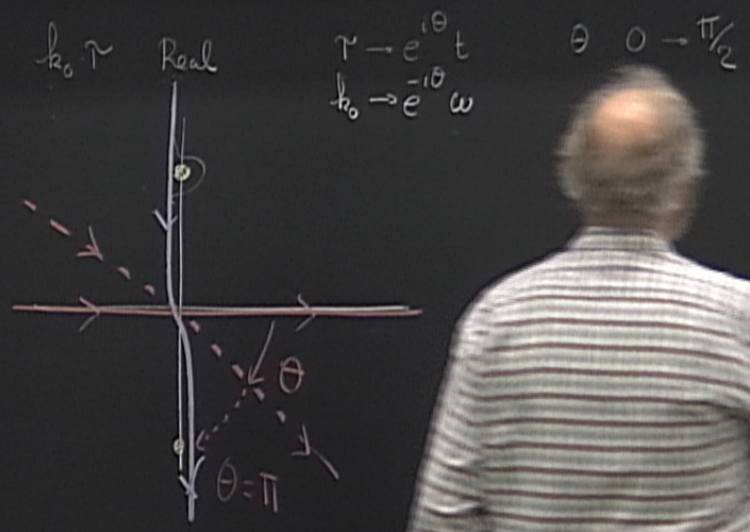


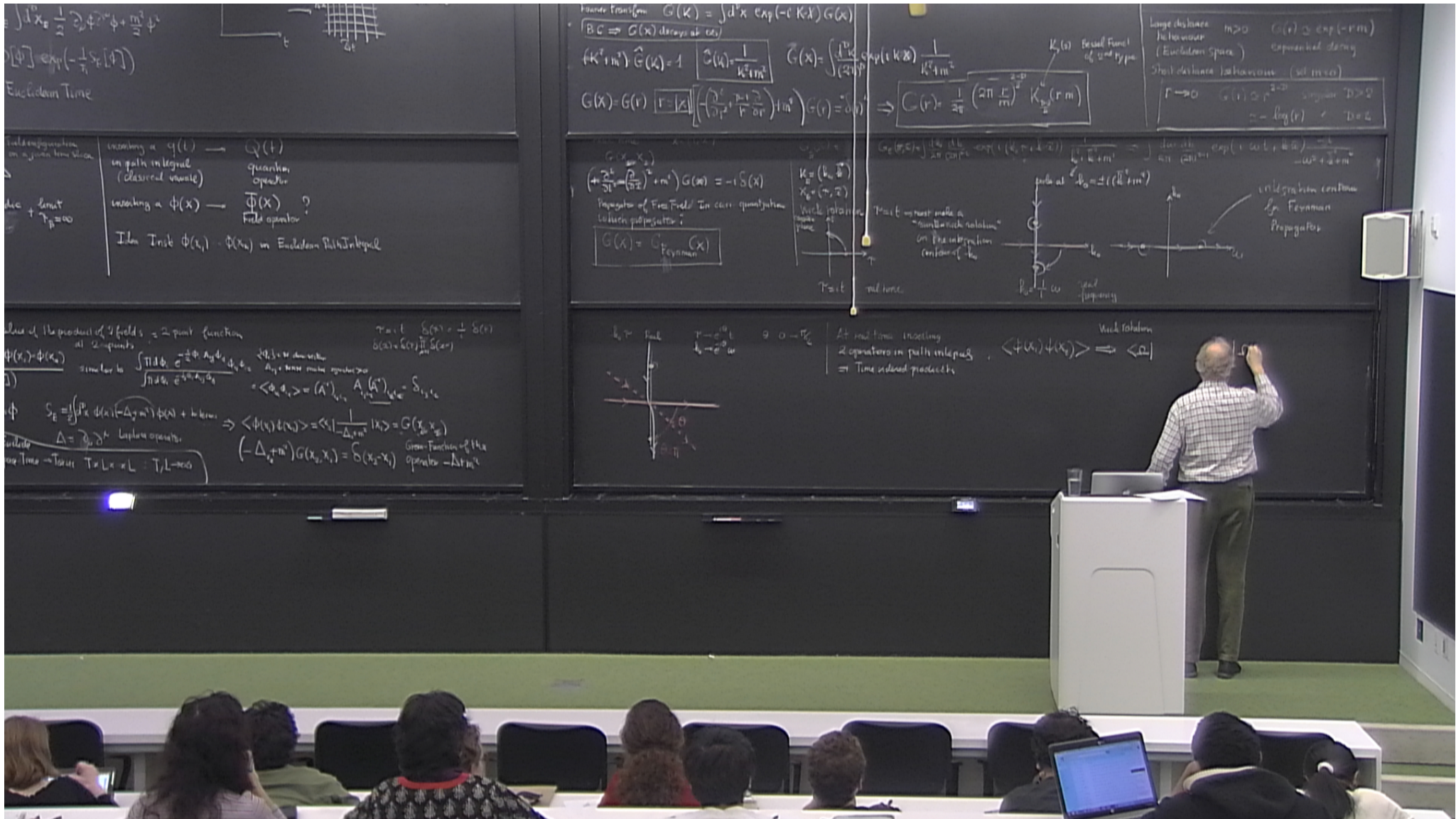
$$\delta(\tau) = \frac{1}{i} \delta(t)$$

$$\prod_{\mu=1}^n \delta(x^\mu)$$

$$\delta_{1312}$$

(X_1, X_2)
 Green-Function of the
 operator $-\Delta + m^2$





Euclidean Time

$\int d^4x \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2$
 $\langle \phi \rangle = \exp(-\frac{1}{\hbar} S_E[\phi])$

inserting a $Q(t)$ on path integral (classical variable) → quantum operator
 inserting a $\Phi(x)$ → $\overline{\Phi(x)}$? Field operator

Idea: Insert $\phi(x_1) \dots \phi(x_n)$ in Euclidean Path Integral

Line of the product of 2 fields = 2-point function at 2-points
 $\langle \phi(x_1) \phi(x_2) \rangle \sim \frac{\int \prod d\phi \phi(x_1) \phi(x_2) e^{-\frac{1}{\hbar} S_E[\phi]}}{\int \prod d\phi e^{-\frac{1}{\hbar} S_E[\phi]}}$

$S_E = \int d^4x \phi(x) (-\Delta + m^2) \phi(x) + \dots$
 $\Delta = \partial_\mu \partial_\mu$ Laplace operator.

$\langle \phi(x_1) \phi(x_2) \rangle = \langle \phi(x_1) \phi(x_2) \rangle = \langle \phi(x_1) \phi(x_2) \rangle = \langle \phi(x_1) \phi(x_2) \rangle$

Fourier Transform $G(K) = \int d^4x \exp(-iKx) G(x)$
 $B.C. \Rightarrow G(x) \text{ decays at } \infty$
 $(K^2 + m^2) \hat{G}(K) = 1 \Rightarrow \hat{G}(K) = \frac{1}{K^2 + m^2}$
 $G(x) = \int \frac{d^4K}{(2\pi)^4} \exp(iKx) \frac{1}{K^2 + m^2}$

$G(x) = G(r) \frac{1}{|x|^D} \left[\left(\frac{\partial}{\partial r} + \frac{D-1}{2r} \right) + m^2 \right] G(r) = 0$
 $\Rightarrow C(r) = \frac{1}{2\pi} \left(\frac{2\pi}{m} \right)^{\frac{D-1}{2}} K_{\frac{D-1}{2}}(r/m)$

Propagator of Free Field in canon. quantization. Which propagator? $G(x) = G_{\text{Feynman}}(x)$

$k_0 = \omega$ pole
 $\omega = \sqrt{k^2 + m^2}$
 $\omega = 0 \rightarrow \omega_c$

At real time insertion 2 operators in path integral \Rightarrow Time ordered product

Large distance behaviour (Euclidean space) $G(r) \sim \exp(-r/m)$ exponential decay
 Short distance behaviour (at $m=0$) $G(r) \sim r^{2-D}$ singular $D > 2$
 $\sim -\log(r)$ $D=2$

$G_E(x) = \int \frac{d^4k}{(2\pi)^4} \frac{\exp(-i(kx + k_0 t))}{k^2 + m^2} \Rightarrow \int \frac{d^3k}{(2\pi)^3} \frac{\exp(-i(kx + k_0 t))}{k^2 + m^2}$

Wick rotation $t \rightarrow -i\tau$
 $k_0 = i\omega$ pole at $k_0 = \pm i(\sqrt{k^2 + m^2})$
 contour of k_0

Wick rotation $\langle \phi(x_1) \phi(x_2) \rangle \Rightarrow \langle \Omega | \dots | \Omega \rangle$

$\tau = it$ real time

$k_0 = \frac{1}{i} \omega$ real frequency

$\theta \rightarrow \pi/2$

At real time inserting
2 operators in path integral
 \Rightarrow Time ordered products

Wick rotation

$$\langle \phi(x_1) \phi(x_2) \rangle \Rightarrow \langle \mathcal{Q} | \bar{\Phi}(x) \mathcal{Q} \rangle$$

$\tau = it$ real time

$k_0 = \frac{1}{i} \omega$ real frequency

$$\tau \rightarrow e^{i\theta} t$$
$$k_0 \rightarrow e^{-i\theta} \omega$$

$$\theta \quad 0 \rightarrow \pi/2$$

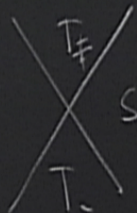
At real time inserting
2 operators in path integral
 \Rightarrow Time ordered products

Wick rotation

$$\langle \phi(x_1) \phi(x_2) \rangle \Rightarrow \langle \Omega | T [\phi(x_1) \phi(x_2)] | \Omega \rangle$$

↑
Time ordered product

$$\Phi(x) = \Phi(t, \vec{x}) \quad [\Phi(x_1), \phi(x_2)] \neq 0 \text{ if } t^2 > x^2$$



$\tau = it$ real time

$k_0 = \frac{1}{i} \omega$ real frequency

$\tau \rightarrow e^{i\theta} t$
 $k_0 \rightarrow e^{-i\theta} \omega$

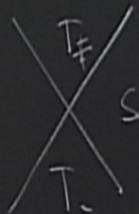
$\theta \ 0 \rightarrow \pi/2$

At real time inserting
2 operators in path integral
 \Rightarrow Time ordered products

Wick rotation
 $\langle \phi(x_1) \phi(x_2) \rangle \Rightarrow \langle \Omega | T [\Phi(x_1) \Phi(x_2)] | \Omega \rangle$
 \uparrow Time ordered product

$\Phi(x) = \Phi(t, \vec{x})$ $[\Phi(x_1), \Phi(x_2)] \neq 0$ if $t^2 > x^2$

Singularity at $r=0$
 \Downarrow
relations have a δ



$\Phi(x_1) \phi(x_2)$ if $t_1 > t_2$ in T_+
 $\Phi(x_2) \Phi(x_1)$ if $t_1 < t_2$ in T_-
Whatever space region

$\tau = it$ real time

$k_0 = \frac{1}{i} \omega$ real frequency

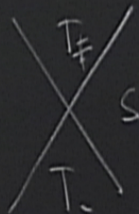
$\tau \rightarrow e^{i\theta} t$
 $k_0 \rightarrow e^{-i\theta} \omega$

$\theta \ 0 \rightarrow \pi/2$

At real time inserting
2 operators in path integral
 \rightarrow Time ordered products

$\Phi(x) = \underline{\Phi}(t, \vec{x})$ $[\underline{\Phi}(x_1), \underline{\Phi}(x_2)] \neq 0$ if $t^2 > x^2$

Singularity at $r=0$
 \Downarrow
commutation relations have a $\delta^{(4)}$



Wick rotation

$\langle \phi(x_1) \phi(x_2) \rangle \Rightarrow \langle \Omega | T [\underline{\Phi}(x_1) \underline{\Phi}(x_2)] | \Omega \rangle$

\uparrow Time ordered product

$\underline{\Phi}(x_1) \underline{\Phi}(x_2)$ if $t_1 > t_2$ in T_+

$\underline{\Phi}(x_2) \underline{\Phi}(x_1)$ if $t_1 < t_2$ in T_+

Whatever

space region

$\int d^4x \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2$
 $[\phi] \sim \exp(-\frac{1}{\Lambda} S_E[\phi])$
 Euclidean Time

Path integral on a given time slice
 inserting a $q(t) \rightarrow Q(t)$ on path integral (classical variable) \rightarrow Quantum operator
 inserting a $\phi(x) \rightarrow \hat{\Phi}(x)$? Field operator
 Idea: Insert $\phi(x_1) \cdot \phi(x_2)$ on Euclidean Path Integral

Fourier Transform $G(K) = \int d^4x \exp(-iKx) G(x)$
 $B, C \Rightarrow G(x)$ decays at ∞
 $(K^2 + m^2) \hat{G}(K) = 1 \Rightarrow \hat{G}(K) = \frac{1}{K^2 + m^2}$
 $G(x) = \int \frac{d^4K}{(2\pi)^4} \exp(iKx) \frac{1}{K^2 + m^2}$
 $G(x) = G(r) \frac{1}{|x|} \left[\left(\frac{\partial}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial}{\partial r} \right] G(r) = 0$
 $\Rightarrow C(r) = \frac{1}{2\pi} \int_0^\infty \frac{d\omega}{\omega} K_0(r, m)$

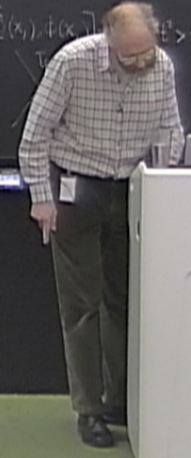
Large distance behavior (Euclidean Space) $m > 0 \Rightarrow G(r) \sim \exp(-r m)$ Exponential decay
 Short distance behavior (at $m=0$) $\Gamma \rightarrow 0 \Rightarrow G(r) \sim r^{2-D}$ singular $D > 2$
 $\sim \log(r) \quad D=2$

$G_2(x, y) = \int \frac{d^4k}{(2\pi)^4} \exp(i(k_0 t + \vec{k} \cdot \vec{x})) \frac{1}{k_0^2 + \vec{k}^2 + m^2} = \int \frac{d^3k}{(2\pi)^3} \exp(i\vec{k} \cdot \vec{x}) \frac{1}{2\omega} \frac{1}{\omega^2 + \vec{k}^2 + m^2}$
 $\omega = \sqrt{\vec{k}^2 + m^2}$
 Wick rotation $t \rightarrow i\tau$
 $k_0 = i\omega$
 Wick rotation in k_0 plane
 Pole at $k_0 = \pm i(\sqrt{\vec{k}^2 + m^2})$
 Contour in k_0 plane
 Residue at $k_0 = i\omega$
 Integration contour for Feynman Propagator

$(\square + m^2) G(x) = -\delta(x)$
 Propagator of Free Field in canon. quantization
 Wick rotation
 $G(x) = G_{\text{Feynman}}(x)$
 $T=1$

Level the product of 2 fields = 2 point function at 2-points
 $\langle \phi(x_1) \phi(x_2) \rangle$ similar to $\int \int \frac{d^4x_1 d^4x_2}{\Omega^2} \frac{-\frac{1}{2} \phi_1 \Delta \phi_2}{\int \int \frac{d^4x}{\Omega} \frac{1}{2} \phi \Delta \phi} = \langle \phi_1 \phi_2 \rangle = (A^{-1})_{12} = A_{21}^{-1} \delta_{12}$
 $S_E = \int \int \frac{d^4x}{\Omega} \phi(x) (-\Delta + m^2) \phi(x) + b \phi(x)$
 $\Delta = \partial_\mu \partial_\mu$ Laplace operator
 $\langle \phi(x) \phi(x') \rangle = \langle \phi | \frac{1}{-\Delta + m^2} | \phi \rangle = G(x, x')$
 $(-\Delta + m^2) G(x, x') = \delta(x - x')$ Green-Function of H.K. operator $-\Delta + m^2$

$\langle \phi(x_1) \phi(x_2) \rangle \Rightarrow \langle \phi | T[\hat{\Phi}(x_1) \hat{\Phi}(x_2)] | \Omega \rangle$
 $\hat{\Phi}(x) = \hat{\Phi}(t, \vec{x})$
 $\langle \hat{\Phi}(x_1) \hat{\Phi}(x_2) \rangle$
 Singularity at $r=0$
 commutator relation $\hat{\Phi}(x) \hat{\Phi}(y) = \hat{\Phi}(y) \hat{\Phi}(x) + i \Delta(x-y)$
 Time ordered product
 $\hat{\Phi}(x_1) \hat{\Phi}(x_2) \in t_1 > t_2$ in T_+
 $\hat{\Phi}(x_2) \hat{\Phi}(x_1) \in t_1 < t_2$ in T_-
 Wick rotation
 Liekear
 equal signs



Typical free field configurations

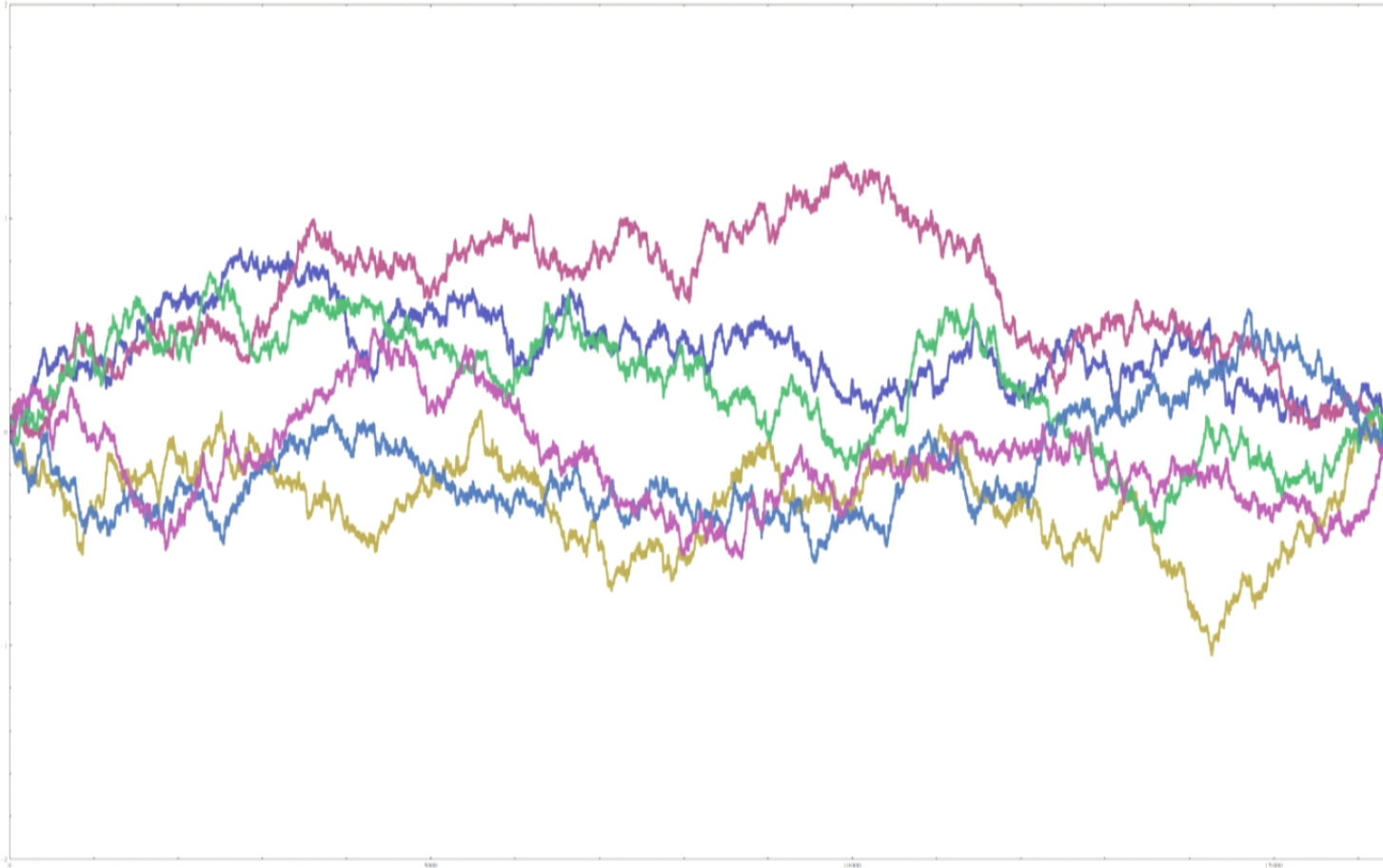
$$D = 1$$

$$\phi(t) \simeq q(t)$$

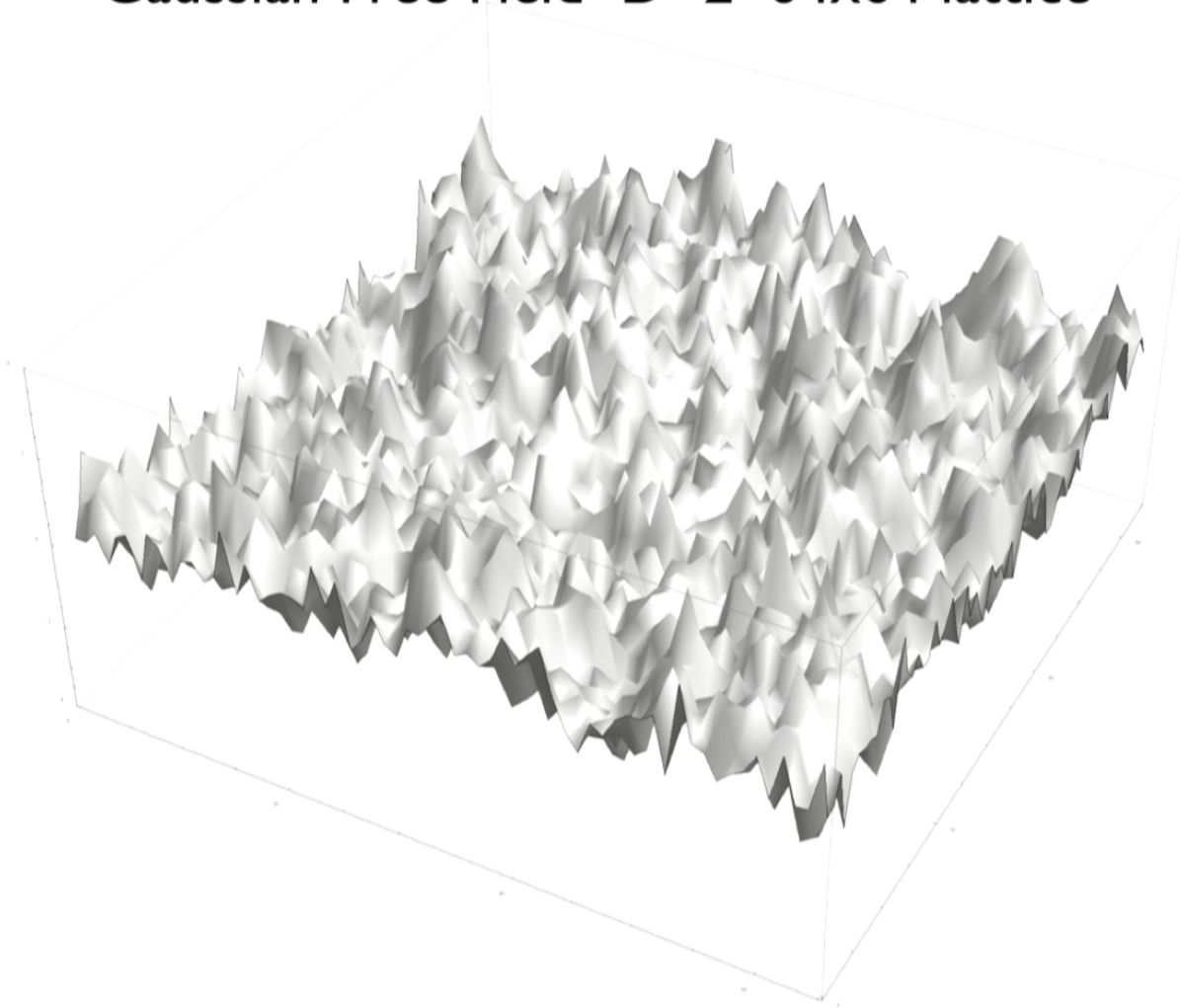
positiva β an

harmonic co

$D=1$: Gaussian Free Fields = Random Walk
(i.e. Brownian or Wiener Process)



Gaussian Free Field $D=2$ 64x64 lattice



$$\langle \phi(x)\phi(x) \rangle = \infty$$

$$D \geq 2$$

typical $\phi(x)$

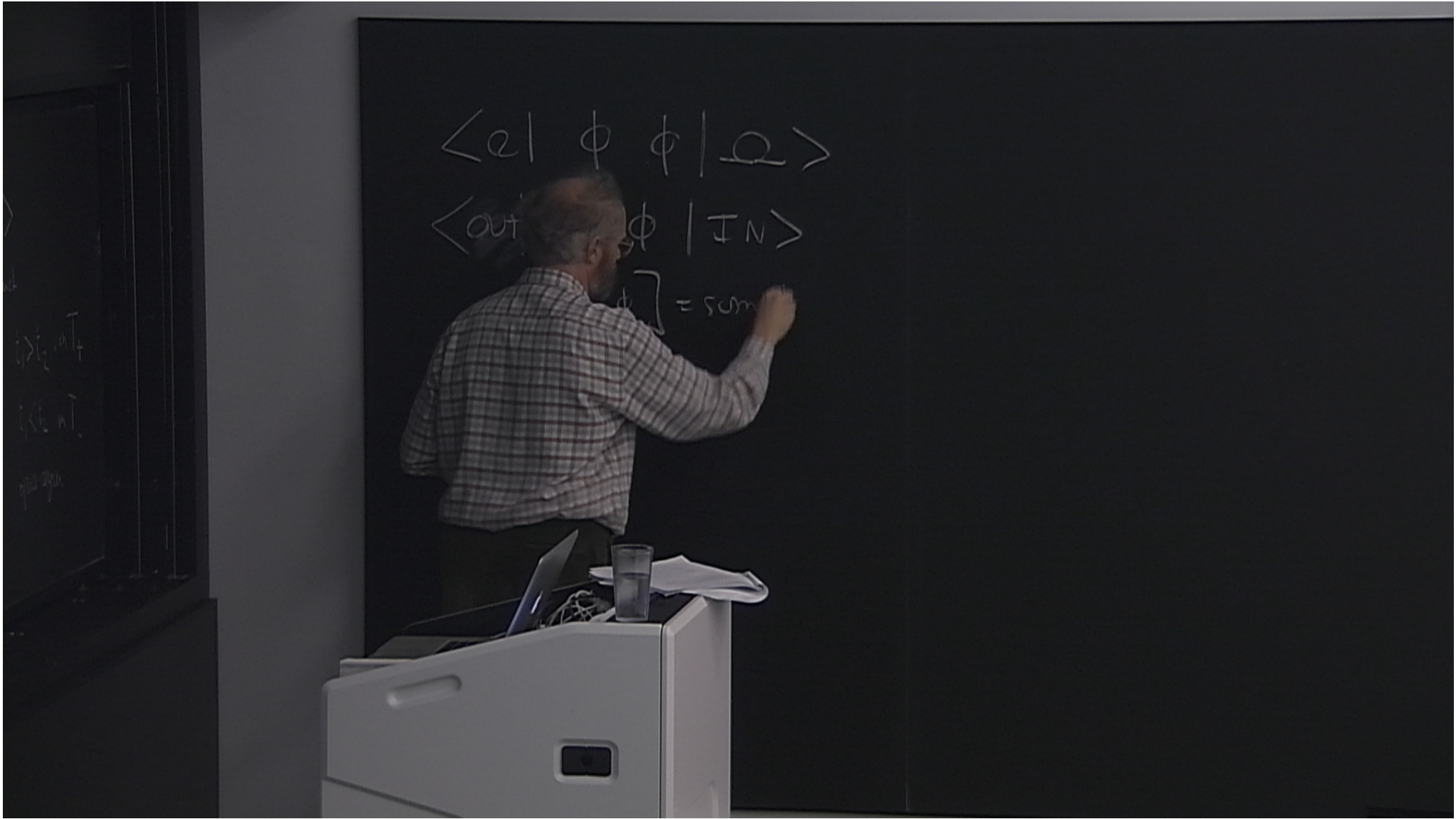
continuous, not

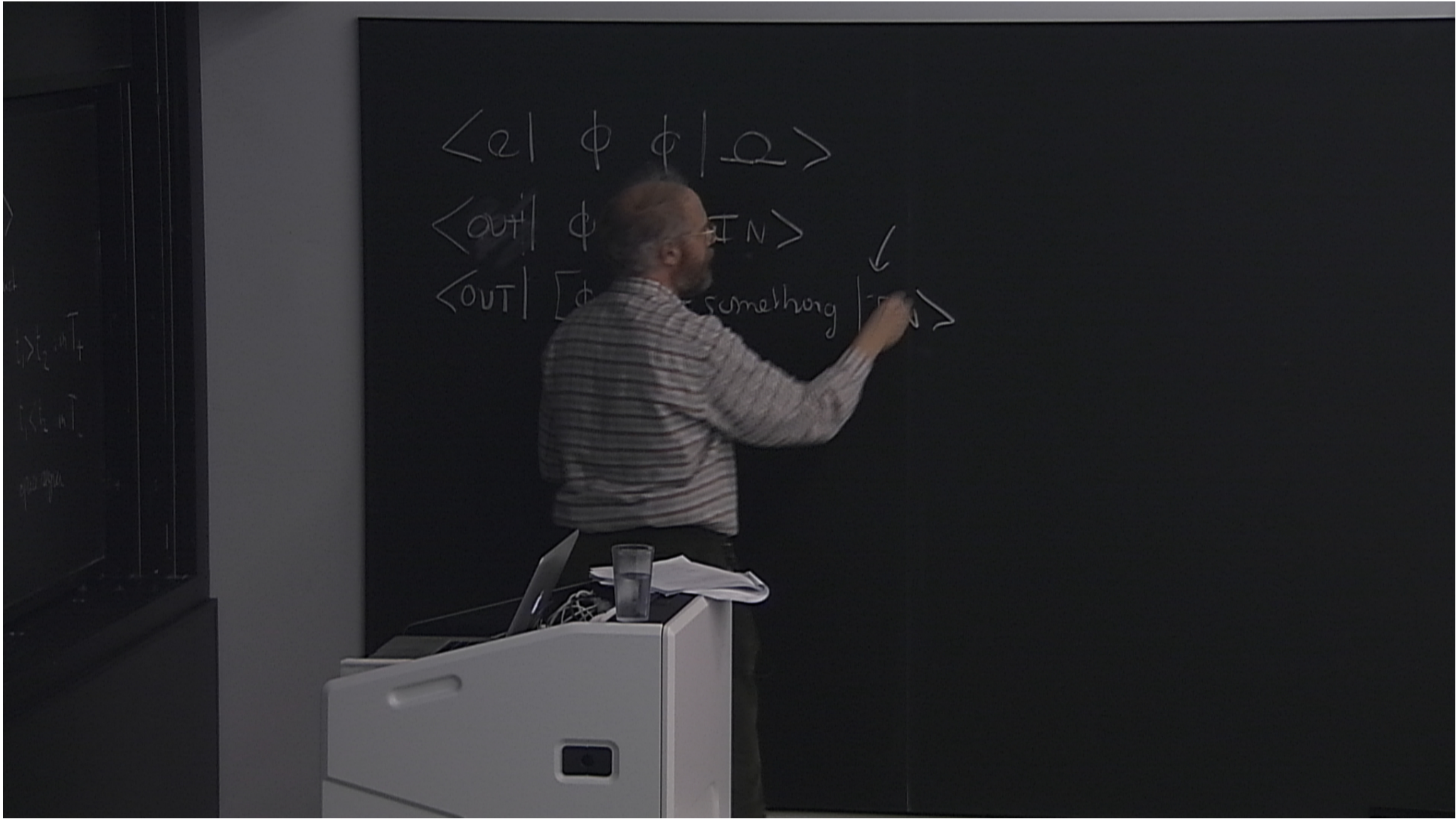
it is a distribution

$$D=1$$

$$\phi(t) \simeq q(t)$$

position q of an
harmonic oscillator





$$\phi(x_1) \phi(x_2)$$

$$x \in \mathbb{R} - (x_1, x_2) \in \mathbb{R}^2$$

$$\phi \rightarrow \phi \otimes \phi$$

~~$$\phi \rightarrow \phi^2$$~~

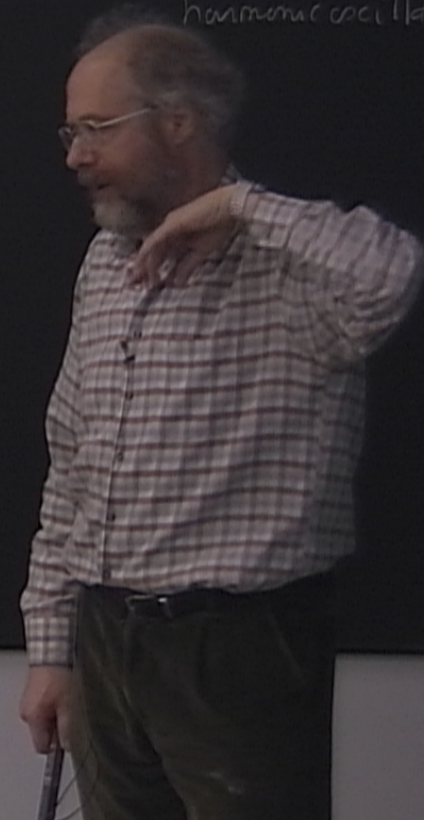
$$\langle \phi(x) \phi(x) \rangle = \infty$$

$$D \geq 2$$

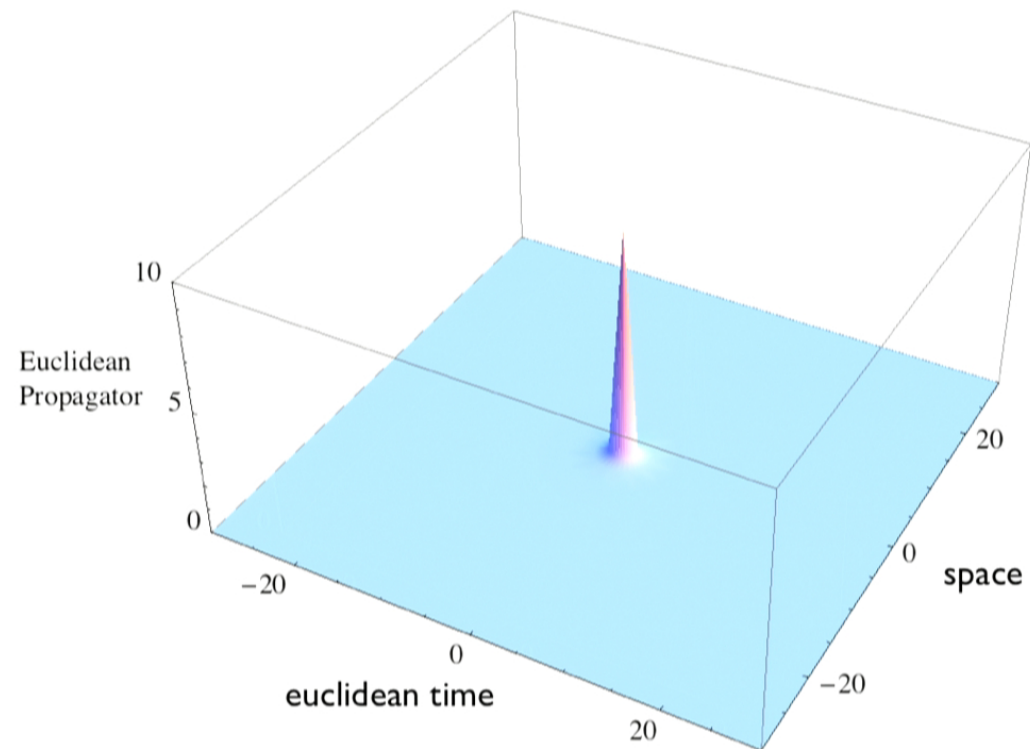
typical $\phi(x)$ is not
continuous, not a function
it is a distribution (Schwartz)

$$D = 1$$

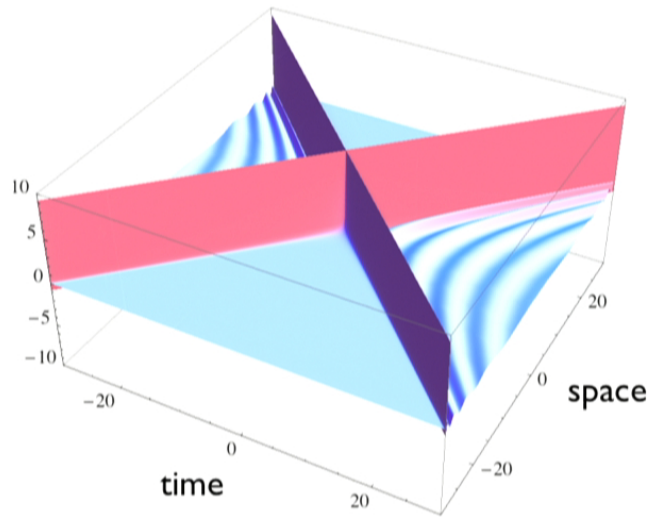
$\phi(t) \simeq q(t)$
position for an
harmonic oscillator



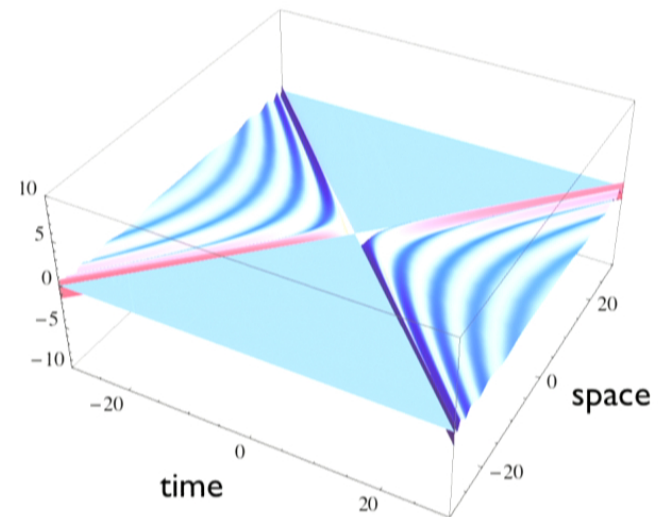
Free field mass m , dimension $D=2$ propagator in Euclidean space



Free field mass m , dimension $D=2$
propagator in Minkowski space
= Feynman propagator



Real part



Imaginary part

$\langle \phi(x_1) \dots \phi(x_4) \rangle$ 4 pts function

$\langle \phi(x_1) \dots \phi(x_4) \rangle$
Gaussian Variables

4 pts function

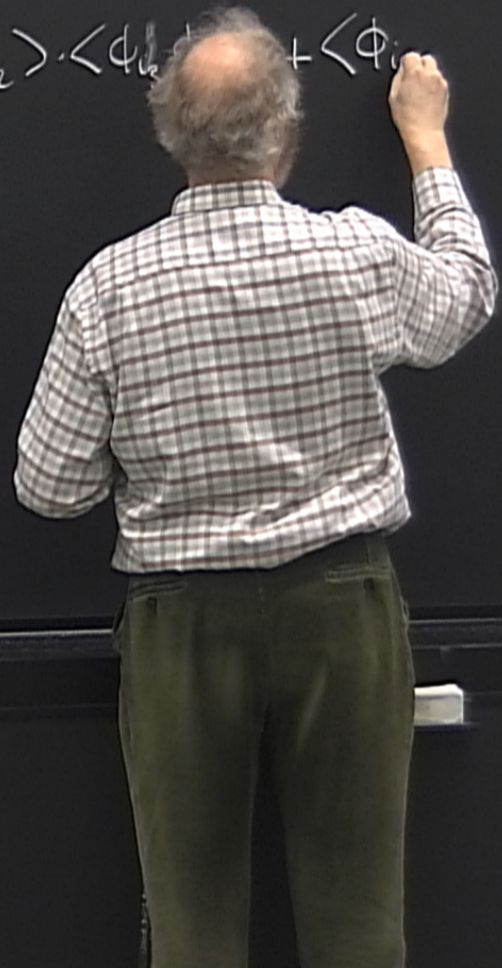
General Theorem
for Gaussian
Variables

General Theorem
for Gaussian
Variables

$$\langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \rangle = \langle \phi_{i_1} \phi_{i_2} \rangle \langle \phi_{i_3} \phi_{i_4} \rangle + \langle \phi_{i_1} \phi_{i_3} \rangle \langle \phi_{i_2} \phi_{i_4} \rangle + \langle \phi_{i_1} \phi_{i_4} \rangle \langle \phi_{i_2} \phi_{i_3} \rangle$$

Gaussian

$$\langle \phi_i \rangle = 0$$



4 pts function

General Theorem
Gaussian
variables

Theorem in a
0

$$\begin{aligned} \langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \rangle &= \langle \phi_{i_1} \phi_{i_2} \rangle \langle \phi_{i_3} \phi_{i_4} \rangle + \langle \phi_{i_2} \phi_{i_3} \rangle \langle \phi_{i_1} \phi_{i_4} \rangle \\ &\quad + \langle \phi_{i_1} \phi_{i_3} \rangle \langle \phi_{i_2} \phi_{i_4} \rangle \\ &= \sum_{\text{all pairings of the 4}} \langle \phi\phi \rangle \langle \phi\phi \rangle \end{aligned}$$

$$\langle \phi(x_1) \dots \phi(x_4) \rangle \quad \leftarrow \text{pts function}$$

Gaussian Variable

$$= \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle$$

+ x x₄

+ x x₄

General Theorem
for Gaussian
Variables

$$\langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \rangle = \langle \dots \rangle_{\text{Gaussian}}$$

$$\langle \phi_i \rangle = 0$$

Wick's Theorem in a
functional integral setting

4 pt. $\langle \phi(x_1) \dots \phi(x_4) \rangle$ 4 pts function
 Gaussian Variables

General Theorem
 for Gaussian
 Variables

$$\langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \rangle = \langle \phi_{i_1} \phi_{i_2} \rangle \langle \phi_{i_3} \phi_{i_4} \rangle + \dots$$

Gaussian

$$\langle \phi_i \rangle = 0$$

$$= \sum_{\text{all pairings of the 4}} \langle \phi \phi \rangle$$

$$= \langle \phi(x_1) \phi(x_2) \rangle \langle \phi(x_3) \phi(x_4) \rangle$$

4 pt

+	x_1, x_3	x_2, x_4
+	x_2, x_4	x_1, x_3

\Leftrightarrow same relative than

Wick Theorem
 function

$$4\text{pt. } \langle \phi(x_1) \dots \phi(x_4) \rangle$$

Gaussian Variables

4 pts function

General Theorem
for Gaussian
Variables

$$\langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \rangle = \langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \rangle_{\text{Gaussian}}$$

$$\langle \phi_i \rangle = 0$$

$$= \sum_{\text{all pairings of the 4}}$$

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle$$

$$= \langle X_2 X_4 \rangle$$

$$= \langle X_2 X_3 \rangle$$

Wick Theorem in a
functional integral setting

⇒ same rules than Green Functions
in canonical quantization

$\dots \times L \times L \dots, L \rightarrow \infty$

4 pts function

General Theorem
for Gaussian
variables

$$\langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \rangle = \langle \phi_{i_1} \phi_{i_2} \rangle \langle \phi_{i_3} \phi_{i_4} \rangle + \langle \phi_{i_1} \phi_{i_3} \rangle \langle \phi_{i_2} \phi_{i_4} \rangle + \langle \phi_{i_1} \phi_{i_4} \rangle \langle \phi_{i_2} \phi_{i_3} \rangle$$

Gaussian

$$\langle \phi_i \rangle = 0$$

$$= \sum_{\text{all pairings of the 4}} \langle \phi\phi \rangle \langle \phi\phi \rangle$$

$\langle \phi(x_3) \phi(x_4) \rangle$
 $x_2 = x_1$
 $x_2 = x_1$

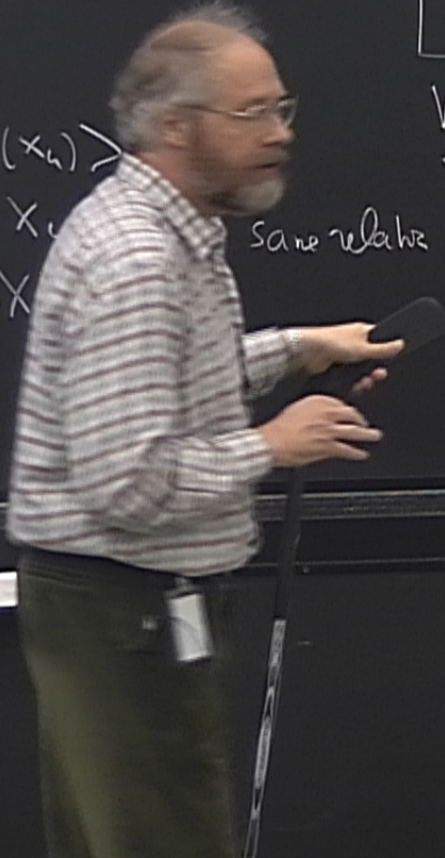
Wick's Theorem in a
functional integral setting

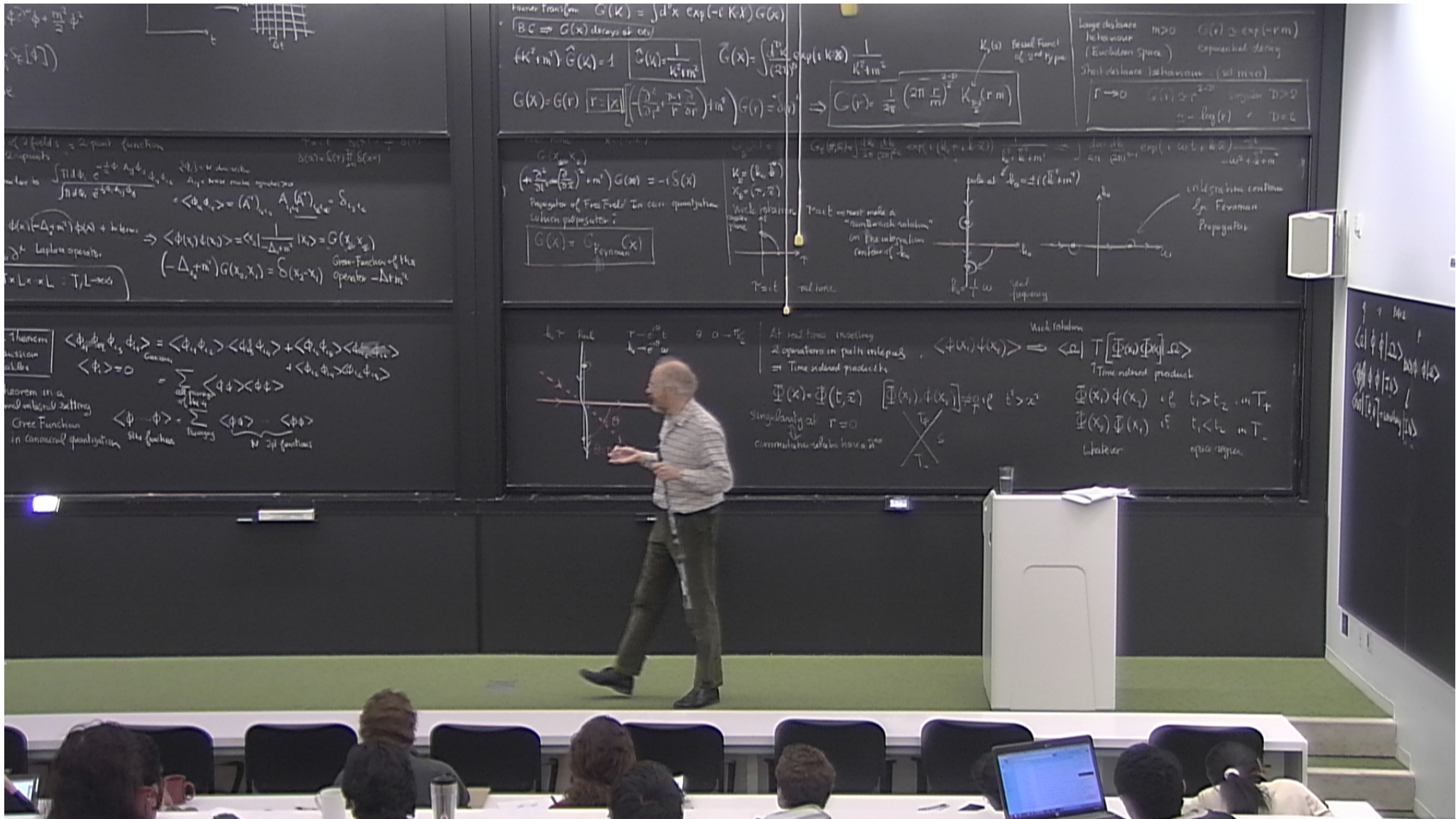
same relation than Green Functions
in canonical quantization

$$\langle \phi \dots \phi \rangle = \sum_{\text{Pairing}} \langle \phi\phi \rangle \dots \langle \phi\phi \rangle$$

2pt functions

N 2pt functions





Fourier transform $G(k) = \int d^d x \exp(-i k x) G(x)$
 B.C. $\Rightarrow G(x)$ decays at ∞
 $(k^2 + m^2) \hat{G}(k) = 1 \Rightarrow \hat{G}(k) = \frac{1}{k^2 + m^2}$
 $G(x) = \int \frac{d^d k}{(2\pi)^d} \exp(i k x) \frac{1}{k^2 + m^2}$
 $G(x) = G(r) \frac{1}{|x|^d} \left[\left(\frac{d}{2} - 1 \right) \frac{1}{r} \frac{\partial}{\partial r} + m^2 \right] G(r) = \delta(r)$
 $\Rightarrow C(r) = \frac{1}{2\pi} \left(\frac{2\pi}{m} \right)^{\frac{d-1}{2}} K_{\frac{d-1}{2}}(r m)$
 Large distance behaviour (Euclidean space) $G(r) \sim \exp(-r m)$ Exponential decay
 Short distance behaviour (sd $m \rightarrow 0$)
 $\Gamma \rightarrow 0 \Rightarrow G(r) \sim r^{2-D}$ singular $D > 2$
 $\sim -\log(r) \quad D = 2$

$G(x_1, x_2) = \int \frac{d^d k}{(2\pi)^d} \exp(i k(x_1 - x_2)) \frac{1}{k^2 + m^2}$
 $\frac{d}{dt} \frac{d^d k}{(2\pi)^d} \exp(i k(x_1 - x_2)) \frac{1}{k^2 + m^2} \Rightarrow \frac{d}{dt} \frac{d^d k}{(2\pi)^d} \exp(i k(x_1 - x_2)) \frac{1}{k^2 + m^2} = \frac{1}{\omega^2 + m^2}$
 $G(x_1, x_2) = \int \frac{d^d k}{(2\pi)^d} \exp(i k(x_1 - x_2)) \frac{1}{k^2 + m^2}$
 $(-\Delta_x + m^2) G(x) = -\delta(x)$
 Propagator of Free Field / In cov. quantization
 Wick rotation
 $G(x) = G_{\text{Feynman}}(x)$
 Wick rotation
 Tract - must make a "branch cut" on the integration contour of k_0
 integration contour for Feynman Propagator
 Wick rotation
 At real time (inserting 2 operators in path integral) \Rightarrow Time ordered product
 $\Phi(x) = \Phi(t, x) \quad [\Phi(x_1) \Phi(x_2)] = \theta(t_1 - t_2) \Phi(x_1) \Phi(x_2) + \theta(t_2 - t_1) \Phi(x_2) \Phi(x_1)$
 Singularity at $r=0$
 commutator relation has a $\delta^d(x)$
 Wick rotation
 $\langle \phi(x_1) \phi(x_2) \rangle \Rightarrow \langle \Omega | T[\Phi(x_1) \Phi(x_2)] | \Omega \rangle$
 Time ordered product
 $\Phi(x_1) \phi(x_2) \quad \text{if } t_1 > t_2 \quad \text{in } T_+$
 $\Phi(x_2) \phi(x_1) \quad \text{if } t_2 > t_1 \quad \text{in } T_+$
 whatever space region

2 fields = 2-point function
 $\langle \phi(x_1) \phi(x_2) \rangle = \langle \phi(x_1) \phi(x_2) \rangle + \langle \phi(x_1) \phi(x_2) \rangle + \dots$
 $\langle \phi(x_1) \phi(x_2) \rangle = \langle \phi(x_1) \phi(x_2) \rangle + \dots$
 Laplace operator
 $(-\Delta_x + m^2) G(x_1, x_2) = \delta(x_1 - x_2)$
 Green-Function of the operator $-\Delta + m^2$