Title: General Relativity for Cosmology - Lecture 22

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Abstract:

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GR for Cosmology, Achim Kempf, Fall 2013, Lecture 22

Note Title

Recall:

Given, in particular, the strong energy condition, our singularity theorem claimed that geodesics meet a divergence of a quantity called expansion, θ, in finite proper time in the past and this will mean a big bang singularity:

important notion also e.g. in study of grav. collaps of stars.

The "expansion", θ:

- Consider a "congruence of timelike geodesics"

... freely falling dust.

1 / 18

1

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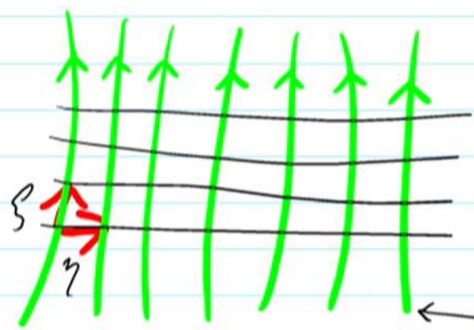
exactly one through each $p \in \Sigma$: (Σ is a Cauchy surface)

- We consider a one-parameter sub-family of these geodesics:

$$\gamma(\tau, s)$$

$\uparrow$ eigentime; s — parameter of family of neighboring geodesics.



- Then, we define the deviation vector to a neighboring geodesic:

1 / 18

1

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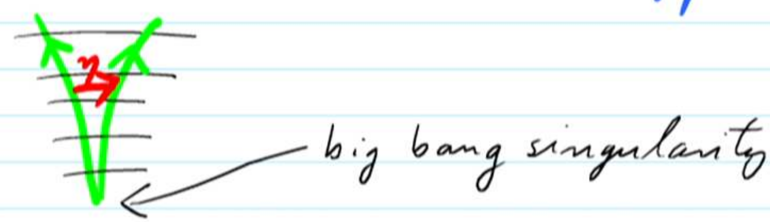
- Then, we define the deviation vector to a neighboring geodesic:

$$\eta := \frac{d}{ds}$$

- The singularity theorem claims that this happened in the past:



How does η change along a past-directed timelike geodesic with tangent ξ?

We showed:

$$\xi^\mu \eta^\nu{}_{;\mu} = \eta^\alpha B^\nu{}_\alpha \quad \text{where} \quad B^\nu{}_\mu := \xi^\nu{}_{;\mu}$$

2 / 18

2

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geodesic, i.e., a set of constant s value

- Then, we define the deviation vector to a neighboring geodesic:

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big bang singularity

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2/18

2

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We showed:

$$\boxed{\xi^\mu \eta^\nu{}_{;\mu} = \eta^\alpha B^\nu{}_\alpha \quad \text{where} \quad B^\nu{}_\mu := \xi^\nu{}_{;\mu}}$$

$\Rightarrow$ Along the geodesic, ξ, the deviation vector η^μ changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

□ The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely:

Symmetric and trace = 0

3

3 / 18

Explicitly:

Volume preserving:

$$\omega_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} - B_{\nu\mu}\right)$$ Twist:

$$\sigma_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} + B_{\nu\mu}\right) - \frac{1}{3}\theta h_{\mu\nu}$$ Shear:

Volume changing:

$$t_{\mu\nu} = \frac{1}{3}\theta\, h_{\mu\nu}$$ Expansion:

Here, we defined: $\theta := B^{\mu\nu} g_{\mu\nu}$ and $h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$

I.e., the Expansion, θ, is the trace of B, which we showed is also equal to the magnitude of the spatial part of B: $\theta = B^{\mu\nu} h_{\mu\nu}$.

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The Raychaudhuri equation

For the derivation, we will use:

A) Definition of B is: $B_{\mu\nu} := \xi_{\mu;\nu}$

B) The curvature tensor obeys the Ricci equation:

$$\boxed{\xi^a{}_{;bc} - \xi^a{}_{;cb} = R^a{}_{bcd}\,\xi^d}$$

c) ξ is tangent to a geodesic, i.e., it obeys: $\nabla_\xi \xi = 0$

5 / 18

5

Now calculate the rate of change of B along the geodesic:

$$\underbrace{\xi^c B_{ab;c}}_{\nabla_\xi B} \overset{(A)}{=} \xi^c \xi_{a;bc}$$

$$\overset{(B)}{=} \xi^c \xi_{a;cb} + \xi^c R_{abcd}\,\xi^d$$

$$\overset{\text{Leibniz rule}}{=} \underbrace{\left(\xi^c \xi_{a;c}\right)_{;b}}_{=0} - \xi^c{}_{;b}\,\xi_{a;c} + R_{abcd}\,\xi^c\xi^d$$

$$\overset{(C)}{=} -\xi^c{}_{;b}\,\xi_{a;c} + R_{abcd}\,\xi^c\xi^d$$

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In summary, we derived:

$$\xi^c B_{ab;c} = -B^c{}_b B_{ac} + R_{abcd}\xi^c\xi^d \qquad (*)$$

The trace of (*) will be the Raychaudhuri equation.

But first, we recall:

- $\xi = \frac{d}{d\tau}$

- $\mathrm{Tr}\, B = B_{\mu\nu} g^{\mu\nu} = \theta$

$\Rightarrow$ Trace(LHS) of (*) reads $\frac{d}{d\tau}\theta$!

7 / 18

7

$$B^c{}_b B_{ac} = \omega^c{}_b\left(\underline{\omega_{ac}} + \sigma_{ac} + \tfrac{1}{3}\theta h_{ac}\right)$$

$$+ \sigma^c{}_b\left(\omega_{ac} + \underline{\sigma_{ac}} + \tfrac{1}{3}\theta h_{ac}\right)$$

$$+ \tfrac{1}{3}\theta h^c{}_b\left(\omega_{ac} + \sigma_{ac} + \underline{\tfrac{1}{3}\theta h_{ac}}\right)$$

When taking the trace, $g^{ab}B^c{}_b B_{ac}$, only the diagonal terms survive:

Exercise: show it is 3

$$Tr(BB) = g^{ab}B^c{}_b B_{ac} = \omega_{ab}\omega^{ab} + \sigma_{ab}\sigma^{ab} + \tfrac{1}{9}\theta^2 \overbrace{h_{ab}h^{ab}}^{3}$$

The Raychandhuri equation is then the trace of Eq. (*) :

recall: Ricci tensor is $R_{cd} = R_{cda}{}^{a}$

$$\frac{d\theta}{\ } = -\frac{1}{\ }\theta^2 - \sigma_{ab}\sigma^{ab} - \omega_{ab}\omega^{ab} - R_{cd}\xi^c\xi^d$$

8 / 18

8

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The Raychaudhuri equation is then the trace of Eq. (*):

recall: Ricci tensor is $R_{cd} = R_{cda}{}^{a}$

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \underbrace{\sigma_{ab}\sigma^{ab}}_{\text{always positive}} - \underbrace{\omega_{ab}\omega^{ab}}_{\text{always positive (and vanishes if choose congruence }\perp\Sigma)} - \underbrace{R_{cd}\xi^c\xi^d}_{\text{pos. or neg?}}$$

?

Dynamics

- Assume that

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \quad \text{for all timelike } \xi$$

8 / 18

8

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Dynamics

- Assume that

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \quad \text{for all timelike } \xi$$

i.e., using the Einstein equation

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^a{}_a\right)$$

we are assuming that

$$T_{\mu\nu}\xi^\mu\xi^\nu - \frac{1}{2}\underbrace{\xi^\mu\xi_\mu}_{=-1} T \geq 0 \quad \text{whenever } \xi^\mu\xi_\mu < 0$$

9 / 18

9

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$R_{\mu\nu} \xi^\mu \xi^\nu \geq 0$ for all timelike ξ

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i.e. the **Strong Energy Condition**.

9 / 18

9

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Thus, assuming the strong energy condition:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0$$

i.e., $$-\frac{1}{\theta^2}\frac{d\theta}{d\tau} - \frac{1}{3} \geq 0$$

i.e., $$\boxed{\frac{d}{d\tau}\theta^{-1} \geq \frac{1}{3}}$$ (+)

Consider the cases when the geodesics are initially all either

a.) diverging, i.e., $\theta(\tau_0) > 0$ (expanding universe) or

b.) converging, i.e., $\theta(\tau_0) < 0$ (contracting universe)

10 / 18

10

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10 / 18

10

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(This is reformulating the theorem's assumption that the extrinsic curvature (i.e. the expansion or contraction at some time exceeds a certain finite value everywhere)

a.)

τ_0 = e.g. today

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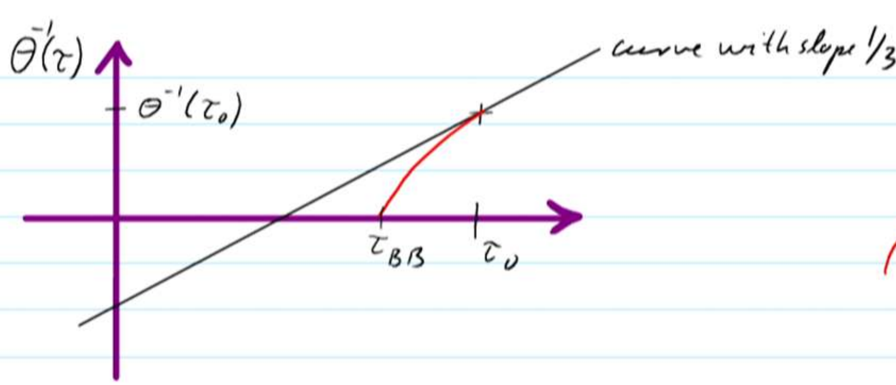


τ_0 = e.g. today

= curve $\theta^{-1}(\tau)$ of slope $\geq \frac{1}{3}$

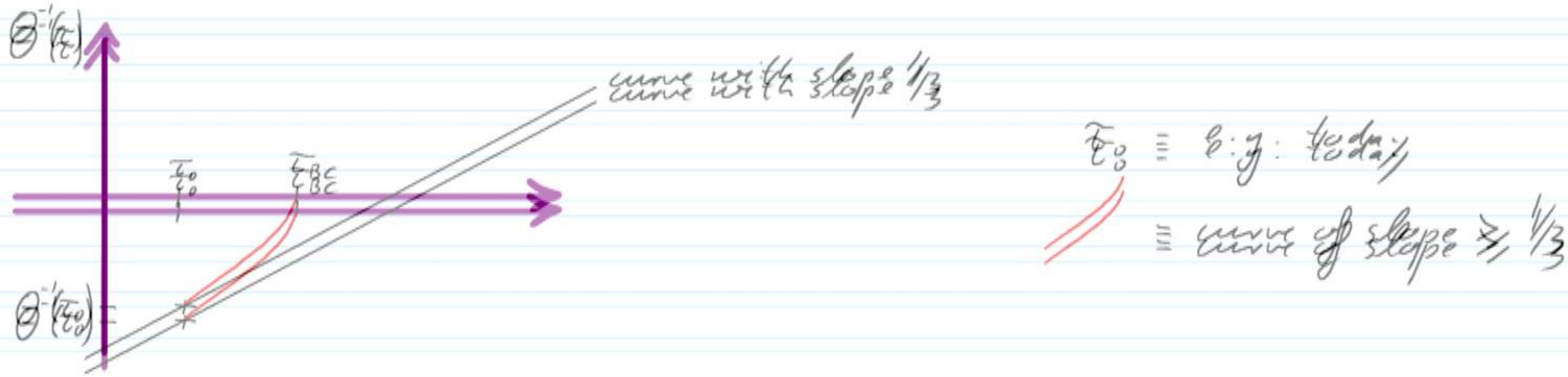
We see that $\theta^{-1}(\tau)$ must have hit $\theta^{-1}(\tau) = 0$ at a finite time τ_{BB} (Big Bang).

10 / 18

10

We see that $\theta^{-1}(\tau)$ must have hit $\theta^{-1}(\tau)=0$ at a finite time τ_{BB} (Big Bang):

b)



We see that $\theta^{-1}(\tau)$ will hit $\theta^{-1}(\tau)=0$ at a finite time τ_{BC} (Big Crunch)

Conclusion: Eq. (+) implies that $\theta^{-1}(\tau)$ must go through 0, i.e.:

a.) for sufficiently early τ, have $\theta \to +\infty$, i.e. Big Bang

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Conclusion: Eq. (+) implies that $\dot{\Theta}^{-1}(\tau)$ must go through 0, i.e.:

a.) for sufficiently early τ, have $\Theta \to +\infty$, i.e.: Big Bang

b.) for sufficiently late τ, have $\Theta \to -\infty$, i.e.: Big Crunch

Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.

Remark: Singularity theorems suitable for black hole case

12 / 18

12

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Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.

Remark: Singularity theorems suitable for black hole case also assume a trapped surface, i.e., a surface on which both in- and outgoing null geodesics converge.

12 / 18

12

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also assume a trapped surface, i.e., a surface on which both in- and outgoing null geodesics converge.

Results, e.g., regarding cosmic singularity?

- Assume a set of symmetries of matter and spacetime has been chosen.
- Assume an exact solution or at least its asymptotic properties at early times have been found.
- Assume, we choose a timelike congruence

12 / 18

12

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- Assume a set of symmetries of matter and spacetime has been chosen.

- Assume an exact solution or at least its asymptotic properties at early times have been found.

- Assume, we choose a timelike congruence e.g. of geodesics (or of the fundamental velocity field - they may be different)

⇒ We can now explicitly calculate the twist, shear and expansion along the congruence:

13 / 18

13

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The Hubble functions:

In particular, we can see how the expansion or contraction of the universe behaves dynamically e.g. when the condition of perfect isotropy is relaxed:

- Now we have different expansions in different directions, nonlinearly influencing another.

- Recall:

The expansion in one direction can be say enhanced by shear, as long as shear shrinks other directions.

- Definition:

14 / 18

14

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- Now we have different expansions in different directions, nonlinearly influencing another.

- **Recall:**

The expansion in one direction can be say enhanced by shear, as long as shear shrinks other directions.

- **Definition:**

We define a rate of expansion tensor that includes shear:

shear

projector $\perp$ to the timelike u-field

$$\Theta_{\mu\nu} := \sigma_{\mu\nu} + \frac{1}{3}\,\theta\, h_{\mu\nu}$$

symmetric part of $B_{\mu\nu}$

expansion scalar.

14 / 18

14

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Definition:

We define a rate of expansion tensor that includes shear:

$$\Theta_{\mu\nu} := \underbrace{\sigma_{\mu\nu}}_{\text{shear}} + \frac{1}{3}\,\underbrace{\theta}_{\text{expansion scalar.}}\,\underbrace{h_{\mu\nu}}_{\text{projector } \perp \text{ to the timelike } u\text{-field}}$$

symmetric part of $B_{\mu\nu}$

- $\Theta_{\mu\nu}$ is fully spacelike and symmetric $\Rightarrow$ $\Theta_{\mu\nu}$ can be diagonalized in suitable ON frame $\{e_0, e_1, e_2, e_3\}$:

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & & & 0 \\ & \theta_1 & & \\ & & \theta_2 & \\ 0 & & & \theta_3 \end{pmatrix}$$

3 space-like directions.

with the traditional expansion being the trace (because

14 / 18

14

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Definition:

We use H_i, H to define local directional and general scale factors ℓ_i, ℓ:

The ℓ_i, ℓ are defined as the solutions to:

$$\frac{\dot{\ell}_i}{\ell_i} = H_i$$

$$\frac{\dot{\ell}}{\ell} = H$$

Here, the time derivative is defined as:

recall: u is timelike.

$$\dot{\ell} = u(\ell) = u^\mu \frac{\partial}{\partial x^\mu} \ell$$

16 / 18

16

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and general scale factors l_i, l:

The l_i, l are defined as the solutions to:

$$\frac{\dot{l}_i}{l_i} = H_i$$

$$\frac{\dot{l}}{l} = H$$

Here, the time derivative is defined as:

recall: u is timelike.

$$\dot{l} = u(l) = u^\mu \frac{\partial}{\partial x^\mu} l$$

- What behavior can occur in the far past?

16 / 18

16

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Full set of cases not yet known.

But:

Explicit examples are known where e.g:

- All $l_i \to 0$ as in FL cosmologies
- $l_1, l_2 \to 0$, $l_3 \to \infty$ "cigar singularity"
- $l_1, l_2 \to 0$, $l_3 \to const$ "barrel singularity"
- $l_1, l_2 \to const$, $l_3 \to 0$ "pancake singularity"

- **Note:** For homogeneous, isotropic FL models, H is

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Traveling to other stars would be great!

But do we really want to leave all this behind?

I'd suggest: Shoot a strong neutrino beam to the sun, to modulate its fusion, nonsymmetrically. This could accelerate the sun, and it takes us along!

Who wants to be the pilot?

18 / 18

18