

Title: General Relativity for Cosmology - Lecture 22

Date: Nov 28, 2013 04:00 PM

URL: <http://pirsa.org/13110007>

Abstract:

GR for Cosmology, Achim Kempf, Fall 2013, Lecture 22

Note Title

Recall:

Given, in particular, the strong energy condition, our singularity theorem claimed that geodesics meet a divergence of a quantity called **expansion**, θ , in finite proper time in the past and this will mean a big bang singularity:

The "expansion", θ : important notion also e.g. in study of grav. collapse of stars.

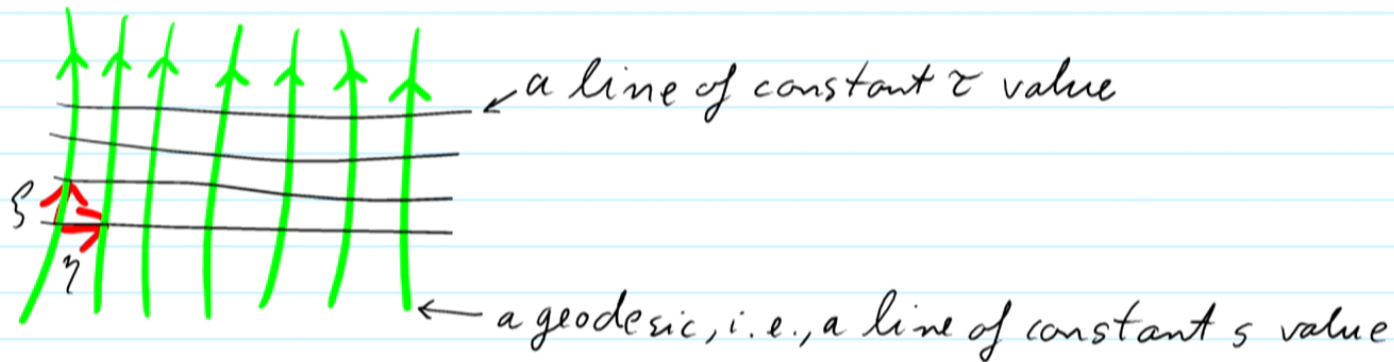
□ Consider a "congruence of timelike geodesics" ← freely falling dust.

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exactly one through each $p \in \Sigma$: (Σ is a (andly surface))

□ We consider a one-parameter sub-family of these geodesics:

$\gamma(\tau, s)$
↑ parameter of family of neighboring geodesics.
↑ eigentime

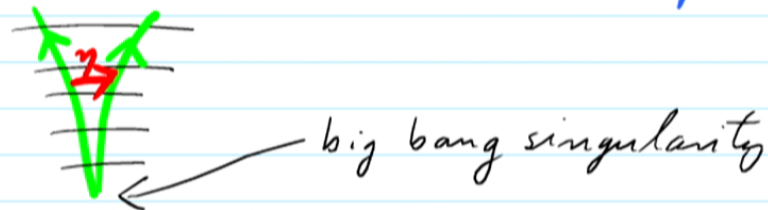


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$$\eta := \frac{d}{ds}$$

□ The singularity theorem claims that this happened in the past:



How does η change along a past-directed timelike geodesic with tangent ξ ?

We showed:

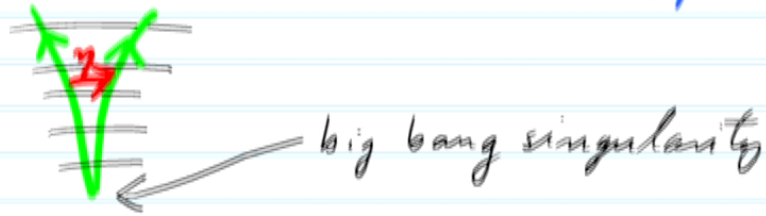
$$\xi^\mu \eta^\nu_{;\mu} = \eta^\mu B^\nu_{;\mu} \quad \text{where} \quad B^\nu_{;\mu} := \xi^\nu_{;\mu}$$

→ geodesic, i.e., a curve of constant ξ value

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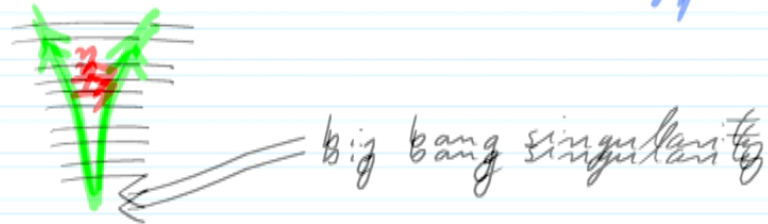
We showed:

$$\xi^\mu \nabla_{\mu} \eta^{\nu} = \eta^{\mu} R^{\nu}_{\mu} \quad \text{where} \quad R^{\nu}_{\mu} = \xi^{\nu} \xi_{\mu}$$

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We showed:

$$\xi^\mu \eta_{;\mu}^{\nu} = \eta^{\nu\alpha} B_{\alpha}^{\beta} \xi^{\mu} \quad \text{where} \quad B_{\mu}^{\nu} := \xi^{\alpha} \eta_{;\alpha}^{\nu}$$

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We showed:

$$\xi^\nu \eta^\alpha{}_{;\nu} = \eta^\alpha B^\nu{}_\alpha \quad \text{where} \quad B^\nu{}_\mu := \xi^\nu{}_{;\mu}$$

\Rightarrow Along the geodesic, ξ , the deviation vector η^α changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

\square The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely:

Symmetric and trace = 0

Explicitly:

Volume preserving →

$$\omega_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} - B_{\nu\mu})$$

Twist: $\circ \rightarrow \odot$

Volume changing:

$$\sigma_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \theta h_{\mu\nu}$$

Shear: $\circ \rightarrow \ell$

$$t_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu}$$

Expansion: $\circ \rightarrow \bigcirc$

Here, we defined: $\theta := B^{\mu\nu} g_{\mu\nu}$ and $h_{\mu\nu} := g_{\mu\nu} + \xi_{\mu} \xi_{\nu}$

I.e., the Expansion, θ , is the trace of B , which we showed is also equal to the magnitude of the spatial part of B : $\theta = B^{\mu\nu} h_{\mu\nu}$.

The Raychaudhuri equation

For the derivation, we will use:

A) Definition of B is: $B_{\mu\nu} := \xi_{\mu;\nu}$

B) The curvature tensor obeys the Ricci equation:

$$\xi^a{}_{;jbc} - \xi^a{}_{;jcb} = R^a{}_{bcd} \xi^d$$

C) ξ is tangent to a geodesic, i.e., it obeys: $\nabla_{\xi} \xi = 0$

Now calculate the rate of change of B along the geodesic:

$$\xi^c B_{ab;c} \stackrel{(A)}{=} \xi^c \xi_{a;bc}$$

$\nabla_{\xi} B$ \nearrow

$$\stackrel{(B)}{=} \xi^c \xi_{a;cb} + \xi^c R_{abcd} \xi^d$$

$$\stackrel{\text{Leibniz rule}}{=} \underbrace{(\xi^c \xi_{a;bc})_{;b}}_0 - \xi^c \xi_{ib} \xi_{a;c} + R_{abcd} \xi^c \xi^d$$

$$\stackrel{(C)}{=} -\xi^c \xi_{ib} \xi_{a;c} + R_{abcd} \xi^c \xi^d$$

$$\stackrel{(A)}{=} \dots$$

In summary, we derived:

$$\xi^c B_{ab;c} = -B^c{}_b B_{ac} + R_{abcd} \xi^c \xi^d \quad (*)$$

The trace of (*) will be the Raychaudhuri equation.

But first, we recall:

$$\square \xi = \frac{d}{d\tau}$$

$$\square \text{Tr } B = B_{\mu\nu} g^{\mu\nu} = \Theta$$

$$\Rightarrow \text{Trace(LHS) of } (*) \text{ reads } \frac{d}{d\tau} \Theta!$$

$$\begin{aligned}
 B^c_b B_{ac} &= \omega^c_b (\omega_{ac} + \sigma_{ac} + \frac{1}{3} \theta h_{ac}) \\
 &+ \sigma^c_b (\omega_{ac} + \sigma_{ac} + \frac{1}{3} \theta h_{ac}) \\
 &+ \frac{1}{3} \theta h^c_b (\omega_{ac} + \sigma_{ac} + \frac{1}{3} \theta h_{ac})
 \end{aligned}$$

When taking the trace, $g^{ab} B^c_b B_{ac}$, only the diagonal terms survive:

$$\text{Tr}(BB) = g^{ab} B^c_b B_{ac} = \omega_{ab} \omega^{ab} + \sigma_{ab} \sigma^{ab} + \frac{1}{9} \theta^2 h_{ab} h^{ab}$$

Exercise:
show it is 3

The Raychaudhuri equation is then the trace of Eq. (*):

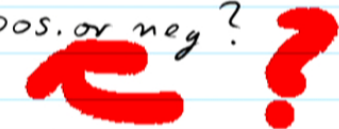
$$\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma_{ab} \sigma^{ab} - \omega_{ab} \omega^{ab} - R_{cd} \xi^c \xi^d$$

recall: Ricci tensor is
 $R_{cd} = R_{da}^a$

The Raychaudhuri equation is then the trace of Eq. (*):

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \underbrace{G_{ab}G^{ab}}_{\text{always positive}} - \underbrace{\omega_{ab}\omega^{ab}}_{\text{always positive (and vanishes if choose congruence } \perp \Sigma)} - \underbrace{R_{cd}\xi^c\xi^d}_{\text{pos. or neg?}}$$

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Dynamics

□ Assume that

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \quad \text{for all timelike } \xi$$

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i.e., using the Einstein equation

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we are assuming that

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i.e. the Strong Energy Condition.

TL ...

... condition ... 9 / 18

Thus, assuming the strong energy condition:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0$$

i.e., $-\frac{1}{\theta^2} \frac{d\theta}{d\tau} - \frac{1}{3} \geq 0$

i.e., $\frac{d}{d\tau} \theta^{-1} \geq \frac{1}{3}$ ⊕

Consider the cases when the geodesics are initially all either

a.) diverging, i.e., $\theta(\tau_0) > 0$ (expanding universe) or

b.) converging, i.e., $\theta(\tau_0) < 0$ (contracting universe)

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(This is reformulating the theorem's assumption that the extrinsic curvature (i.e. the expansion or contraction at some time exceeds a certain finite value everywhere)

a.)

$\theta^{-1}(\tau)$

$\theta^{-1}(\tau_0)$

curve with slope $1/3$

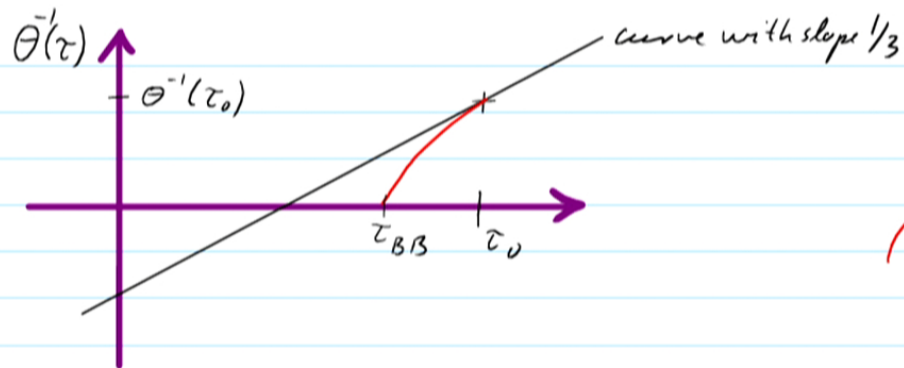
$\tau_0 = \text{e.g. today}$

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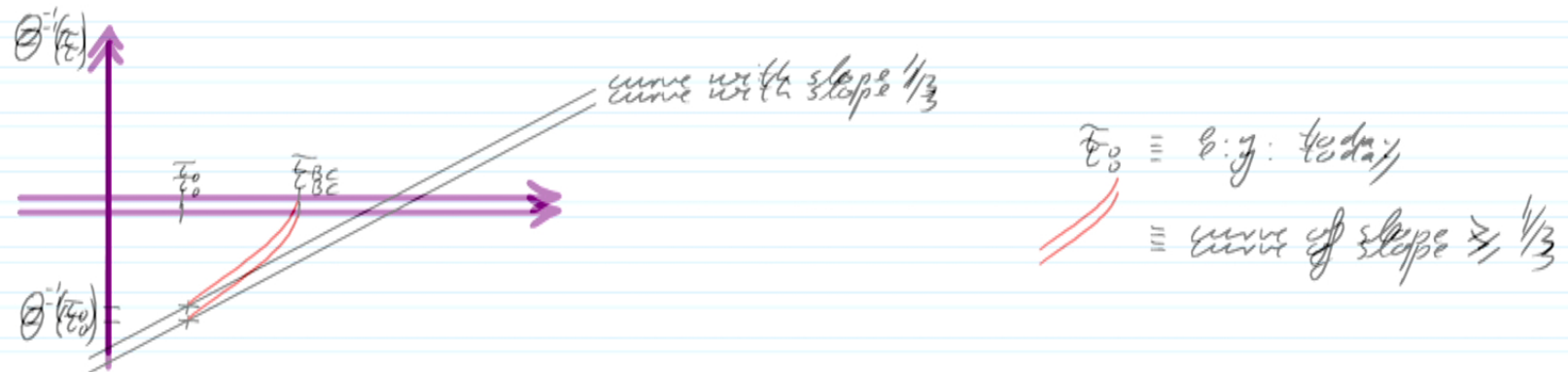
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(= curve $\Theta'(\tau)$ of slope $> 1/3$

We see that $\Theta'(\tau)$ must have hit $\Theta'(\tau) = 0$ at a finite time τ_{BB} (Big Bang).

We see that $\theta^{-1}(\tau)$ must have hit $\theta^{-1}(\tau) = 0$ at a finite time τ_{BB} (Big Bang):

b)



We see that $\theta^{-1}(\tau)$ will hit $\theta^{-1}(\tau) = 0$ at a finite time τ_{BC} (Big Crunch)

Conclusion: Eq. (†) implies that $\theta^{-1}(\tau)$ must go through $\theta, i.e.:$

a.) for sufficiently early τ , have $\theta > +\infty, i.e.:$ Big Bang

Conclusion: Eq. (+) implies that $\Theta(\tau)$ must go through 0, i.e.:

- a.) for sufficiently early τ , have $\Theta \rightarrow +\infty$, i.e.: Big Bang
- b.) for sufficiently late τ , have $\Theta \rightarrow -\infty$, i.e.: Big Crunch

Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.

Remark: Singularity theorems suitable for black hole case

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E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.

Remark: Singularity theorems suitable for black hole case also assume a trapped surface, i.e., a surface on which both in- and outgoing null geodesics converge.

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also assume a **trapped surface**, i.e., a surface on which both in- and outgoing null geodesics converge.

Results, e.g., regarding cosmic singularity?

- Assume a set of symmetries of matter and spacetime has been chosen.
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- Assume a set of symmetries of matter and spacetime has been chosen.
 - Assume an exact solution or at least its asymptotic properties at early times have been found.
 - Assume, we choose a timelike congruence e.g. of geodesics (or of the fundamental velocity field - they may be different)
- ⇒ We can now explicitly calculate the **twist**, **shear** and **expansion** along the congruence:

The Hubble functions:

In particular, we can see how the expansion or contraction of the universe behaves dynamically, e.g. when the condition of perfect isotropy is relaxed:

□ Now we have different expansions in different directions, nonlinearly influencing another.

□ Recall:

The expansion in one direction can be say enhanced by shear, as long as shear shrinks other directions.

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Definition:

We define a rate of expansion tensor that includes shear:

$$\Theta_{\mu\nu} := \underbrace{\sigma_{\mu\nu}}_{\text{symmetric part of } B_{\mu\nu}} + \frac{1}{3} \underbrace{\theta}_{\text{expansion scalar}} \underbrace{h_{\mu\nu}}_{\text{projector } \perp \text{ to the timelike } u\text{-field}}$$

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$\Theta_{\mu\nu}$ is fully spacelike and symmetric $\Rightarrow \Theta_{\mu\nu}$ can be diagonalized in suitable ON frame $\{e_0, e_1, e_2, e_3\}$:

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \theta_1 & 0 & 0 \\ 0 & 0 & \theta_2 & 0 \\ 0 & 0 & 0 & \theta_3 \end{pmatrix} \quad \text{3 space-like directions.}$$

with the traditional expansion being the trace (because

Definition:

We use H_i, H to define local directional and general scale factors l_i, l :

The l_i, l are defined as the solutions to:

$$\frac{\dot{l}_i}{l_i} = H_i$$

$$\frac{\dot{l}}{l} = H$$

Here, the time derivative is defined as:

$$\dot{l} = u(l) = u^\mu \frac{\partial}{\partial x^\mu} l$$

recall: u is timelike.

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□ What behavior can occur in the far past?

Full set of cases not yet known.

But:

Explicit examples are known where e.g:

□ All $l_i \rightarrow 0$ as in FL cosmologies

□ $l_1, l_2 \rightarrow 0, l_3 \rightarrow \infty$ "cigar singularity"

□ $l_1, l_2 \rightarrow 0, l_3 \rightarrow \text{const}$ "barrel singularity"

□ $l_1, l_2 \rightarrow \text{const}, l_3 \rightarrow 0$ "pancake singularity"

□ Note: For homogeneous, isotropic FL models, H is

Traveling to other stars
would be great!



But do we really want
to leave all this behind?



I'd suggest: Shoot a strong neutrino beam to the sun, to modulate its fusion,
nonsymmetrically. This could accelerate the sun, and it takes us along!

Who wants to be the pilot?