Title: General Relativity for Cosmology - Lecture 16

Date: Nov 07, 2013 04:00 PM

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Abstract:

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Note Title

GR for Cosmology, Achim Kempf, Fall 2013, Lecture 16

Fridmann-Lemaître cosmological solutions

Experimental evidence:

Hubble, Humason 1929

- The universe is and has been expanding.
- It appears to be spatially essentially isotropic and homogeneous on scales larger than a few (3-4) billion light years.

(see e.g. Sloan Digital Sky Survey (SDSS) at www.sdss.org)

Idealizing models:

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1

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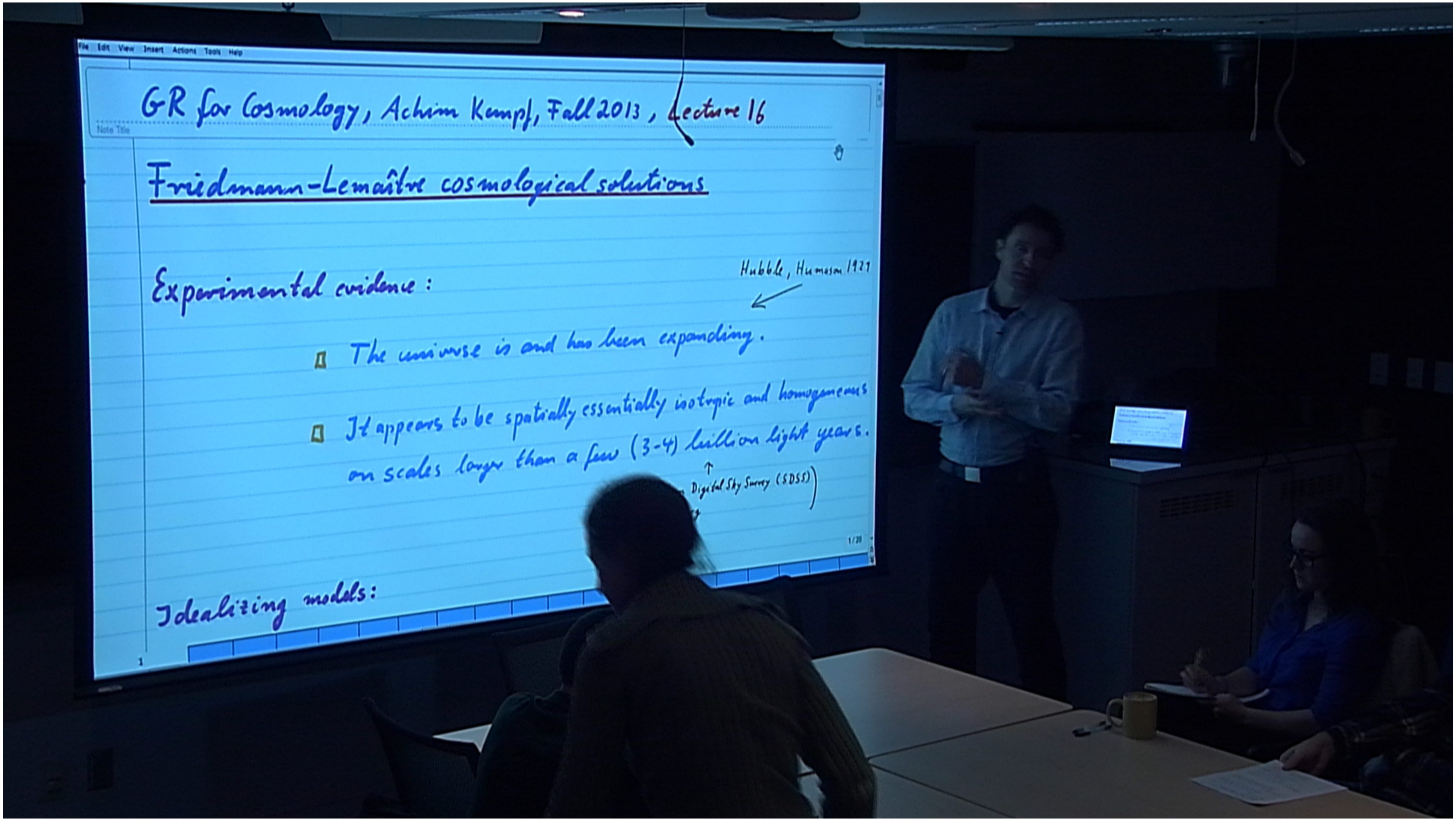
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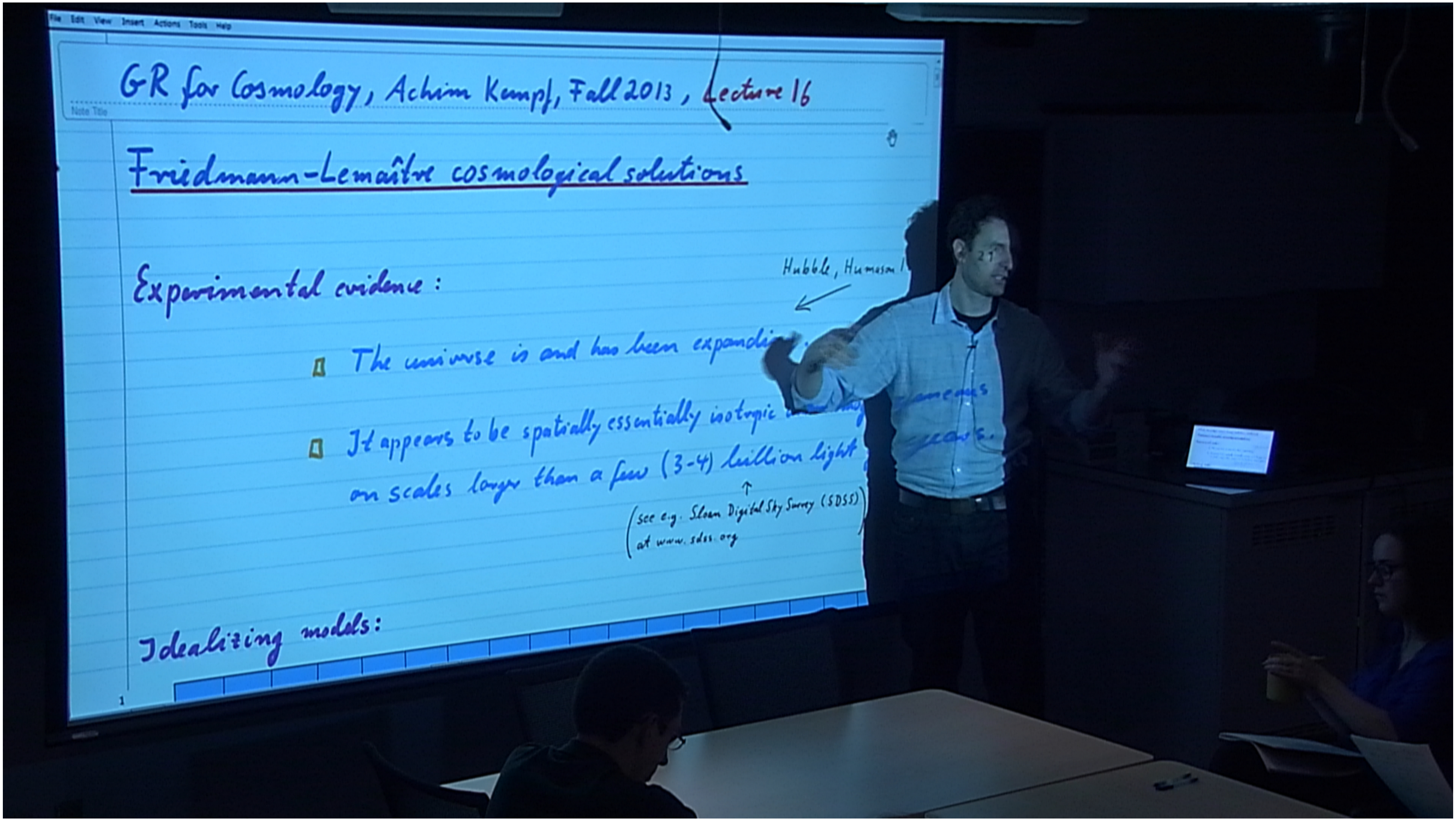
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Idealizing models:

- Assume perfect spatial isotropy and homogeneity:
- ⇝ "Friedmann & Lemaître" (later Robertson & Walker) spacetimes

Concretely:

We assume we can model spacetime as a manifold (M, g) with:

$$M = I \times \Sigma$$

$$g = -dt^2 + a^2(t)\,\bar{g}$$

(we will later use an ON frame so that $g_{\mu\nu} = \eta_{\mu\nu}$)

In the basis $\{dx^\mu\}$ which comes with the coordinate system.

Here:

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1

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Here:

- J is an interval, $J \subset \mathbb{R}$, and $t \in J$ is called "cosmic time". $a(t)$ is called the "scale factor".

- $(\Sigma, \bar{g})$ is a fully isotropic & homogeneous Riemannian manifold, i.e., a manifold of

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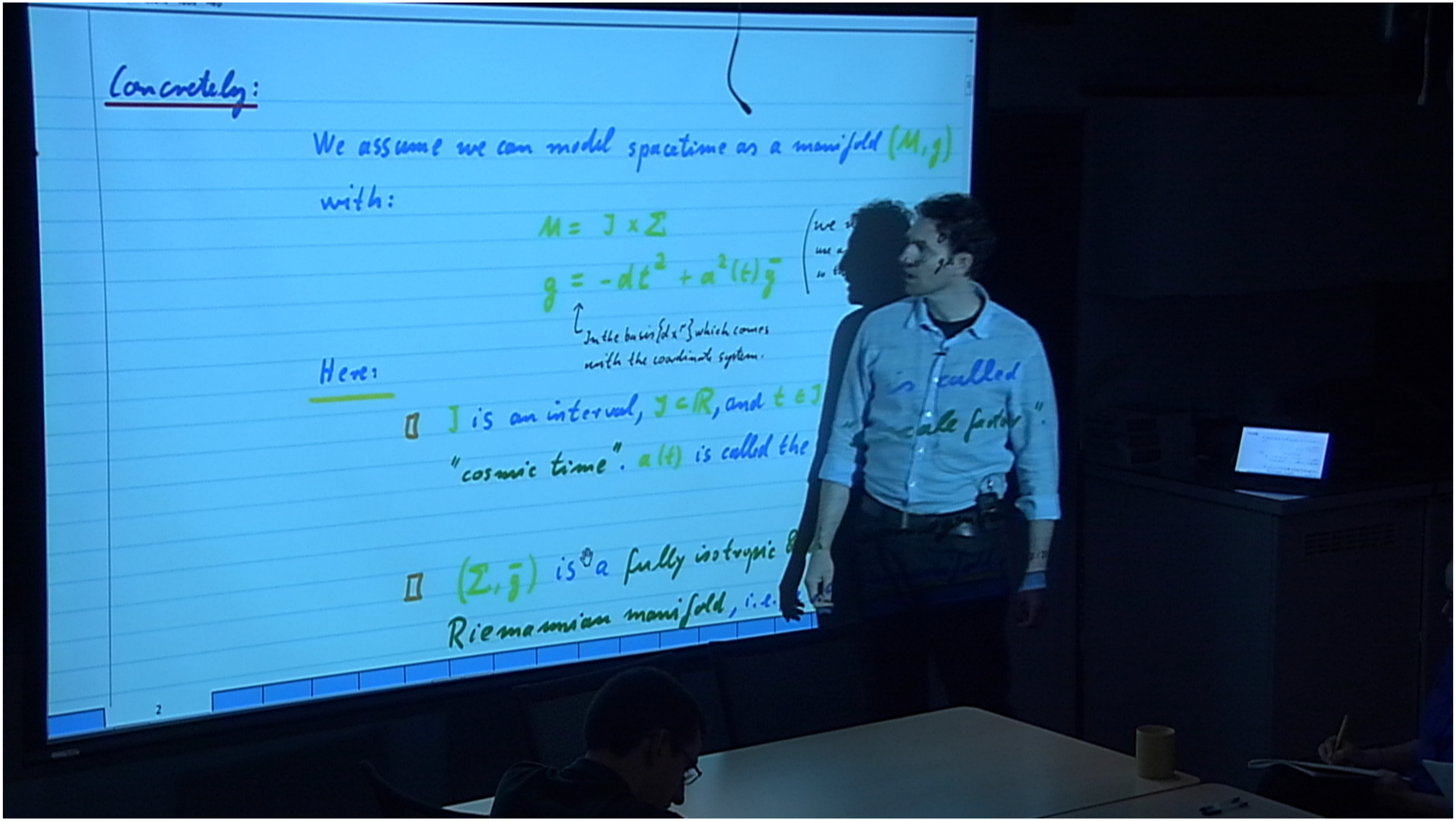
2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M = J × Σ
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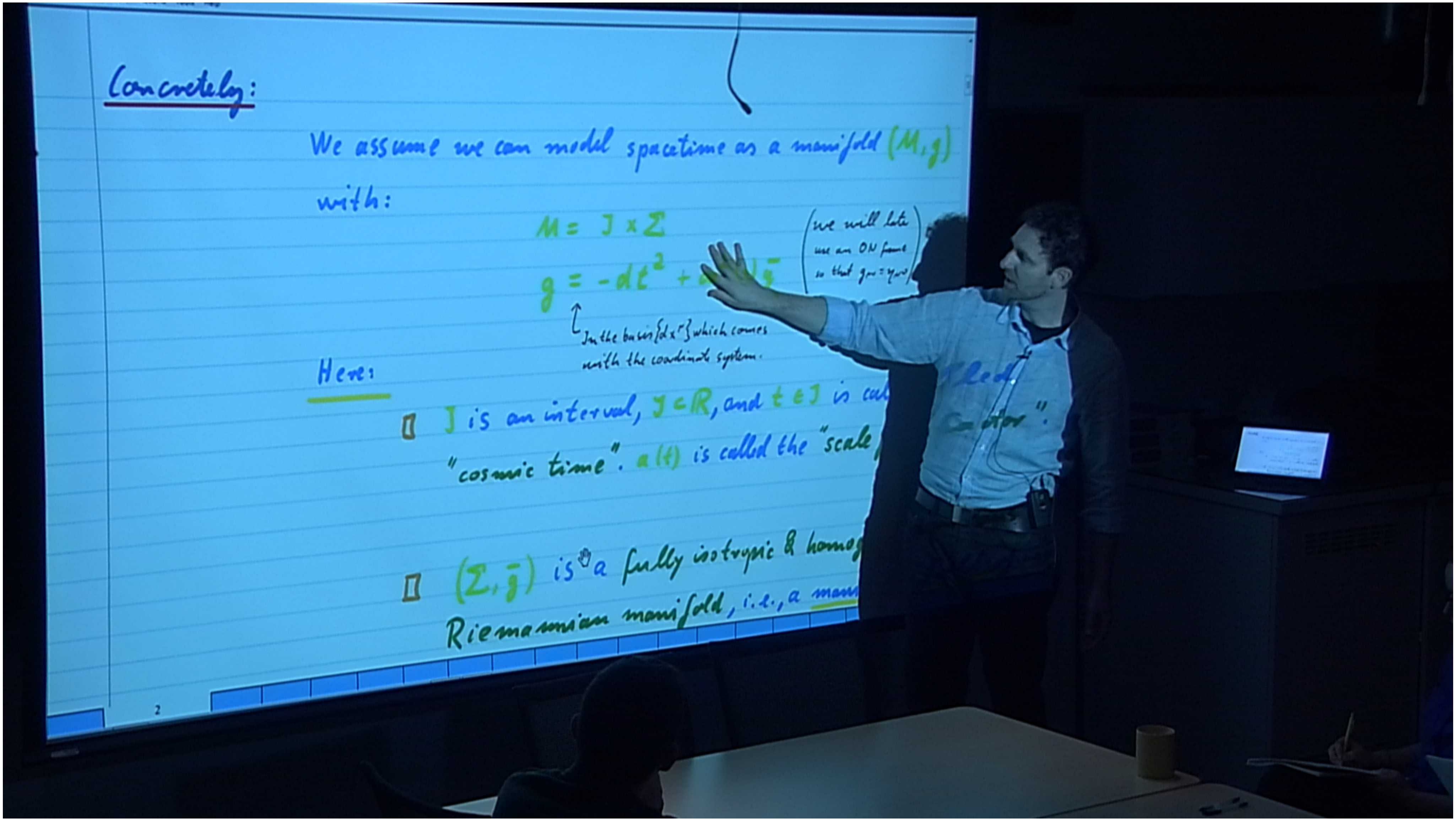
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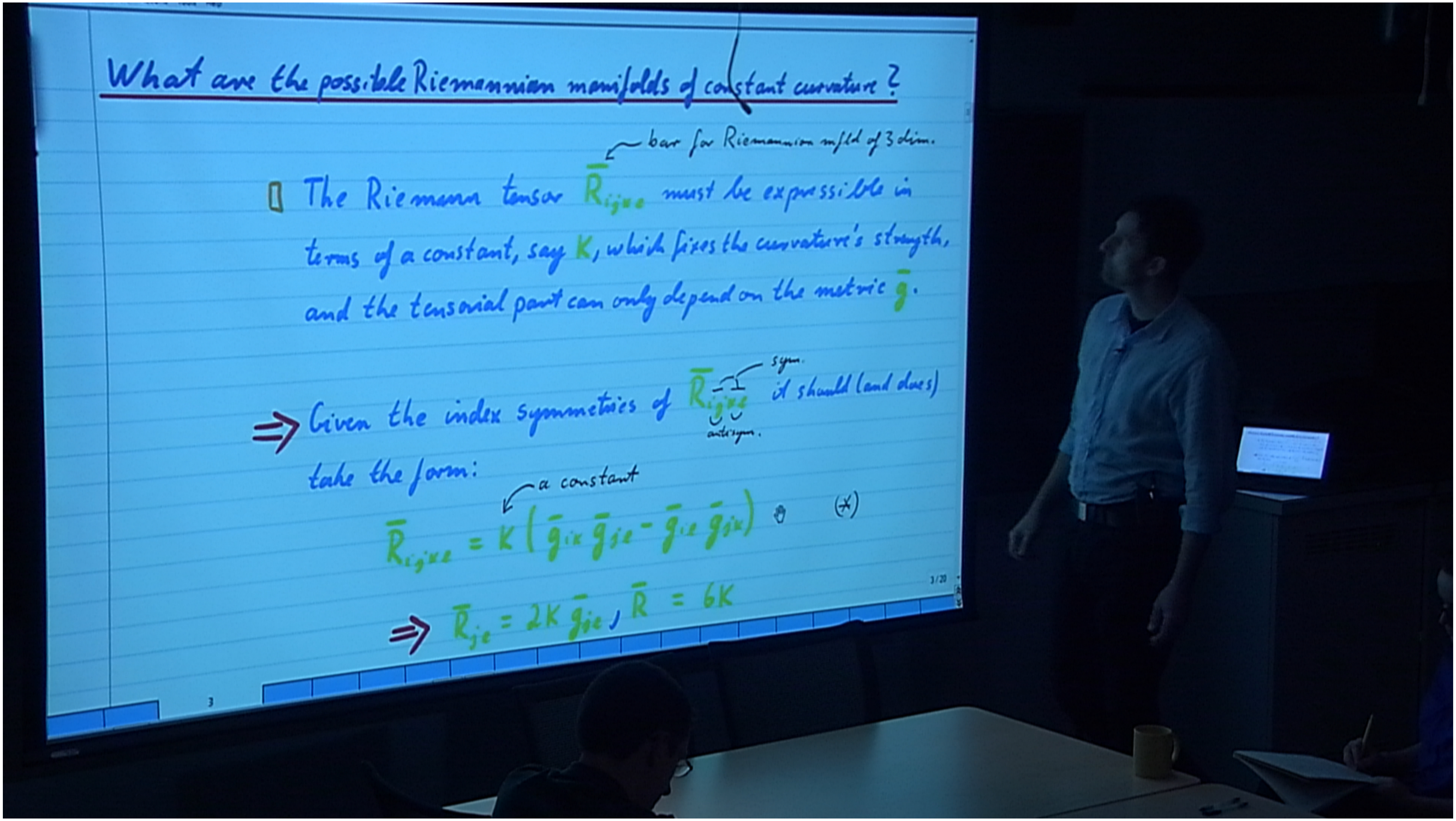
What are the possible Riemannian manifolds of constant curvature ?
bar for Riemannian mfld of 3 dim.
The Riemann tensor $\bar{R}_{ijkl}$ must be expressible in terms of a constant, say K, which fixes the curvature's strength, and the tensorial part can only depend on the metric $\bar{g}$.
sym.
$\Rightarrow$ Given the index symmetries of $\bar{R}_{ijkl}$ it should (and does)
anti-sym.
take the form:
a constant
$\bar{R}_{ijkl} = K(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk})$ (*)
$\Rightarrow \bar{R}_{jl} = 2K\bar{g}_{jl}, \bar{R} = 6K$
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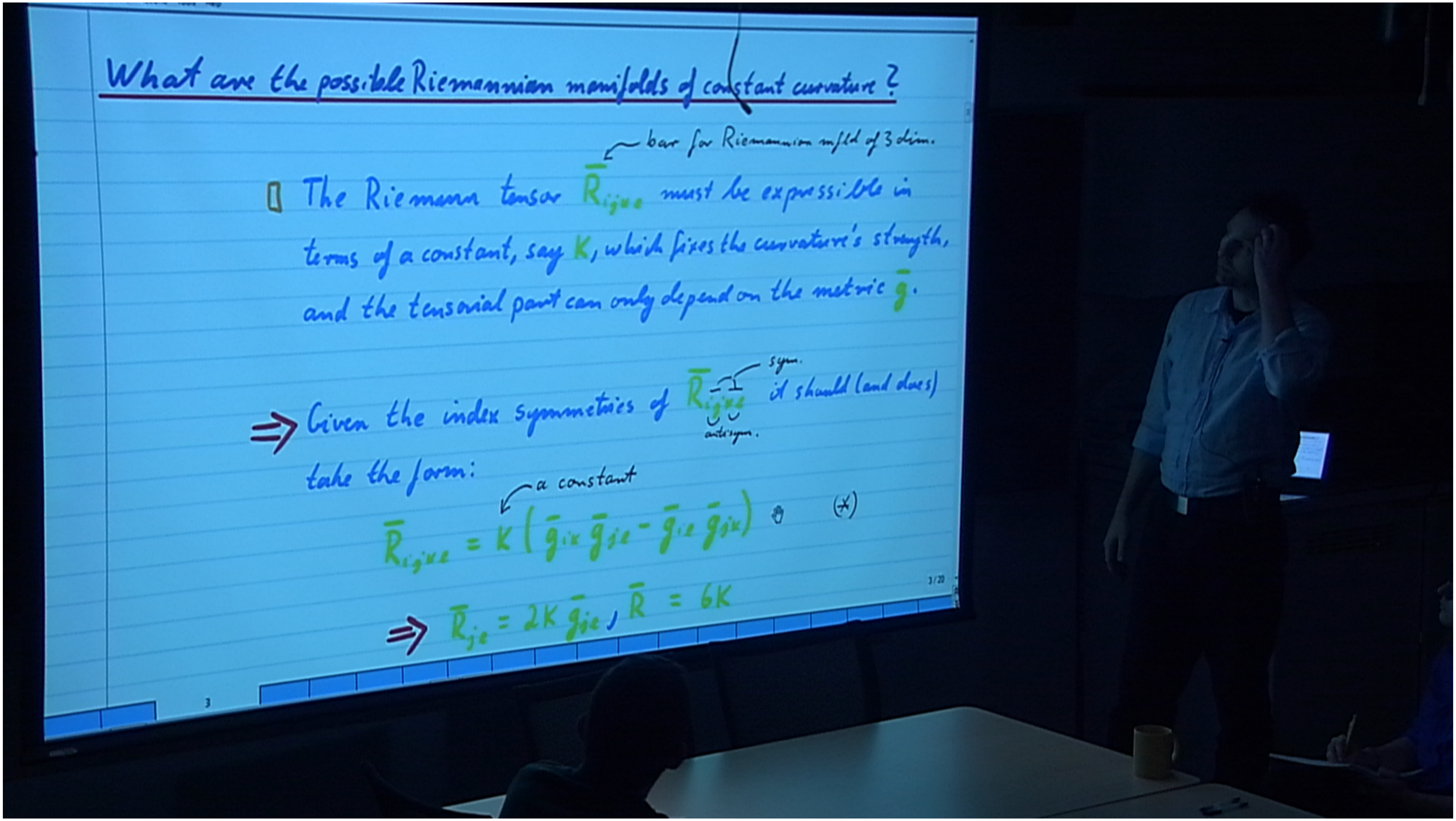
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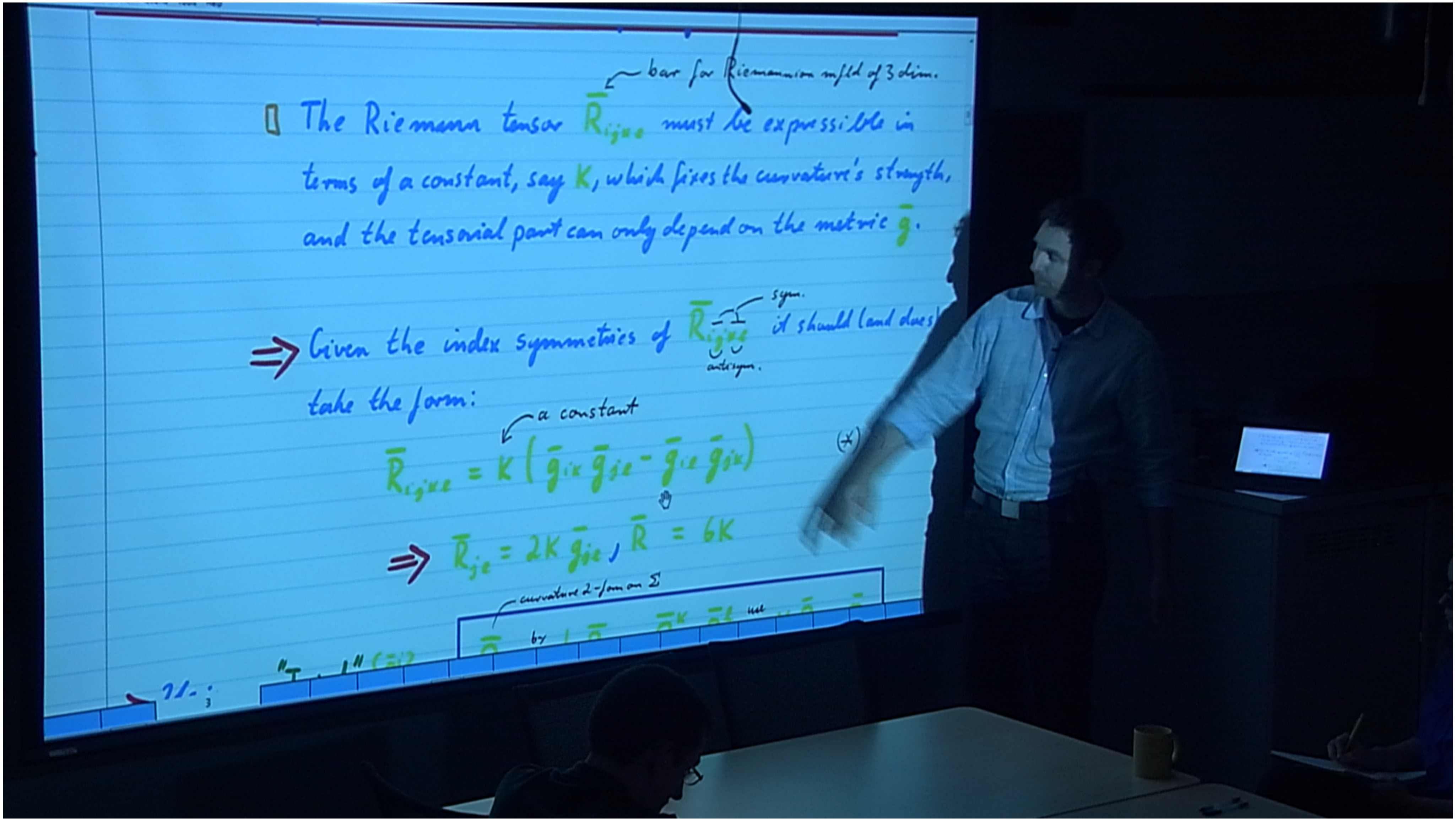
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curvature 2-form on Σ

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curvature 2-form on Σ

$\Rightarrow$ Using a "Triad" $\{\bar{\theta}^i\}$: $\boxed{\bar{\Omega}_{ij} \overset{\text{by}}{\underset{\text{Def.}}{=}} \frac{1}{2}\bar{R}_{ijkl}\,\bar{\theta}^k \wedge \bar{\theta}^l \overset{\text{use}}{\underset{(*)}{=}} K\,\bar{\theta}_i \wedge \bar{\theta}_j}$

(ON bases of $T_p(\Sigma)$, $\forall p$)

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Note: 4-dim pseudo-Riemannian manifolds with constant

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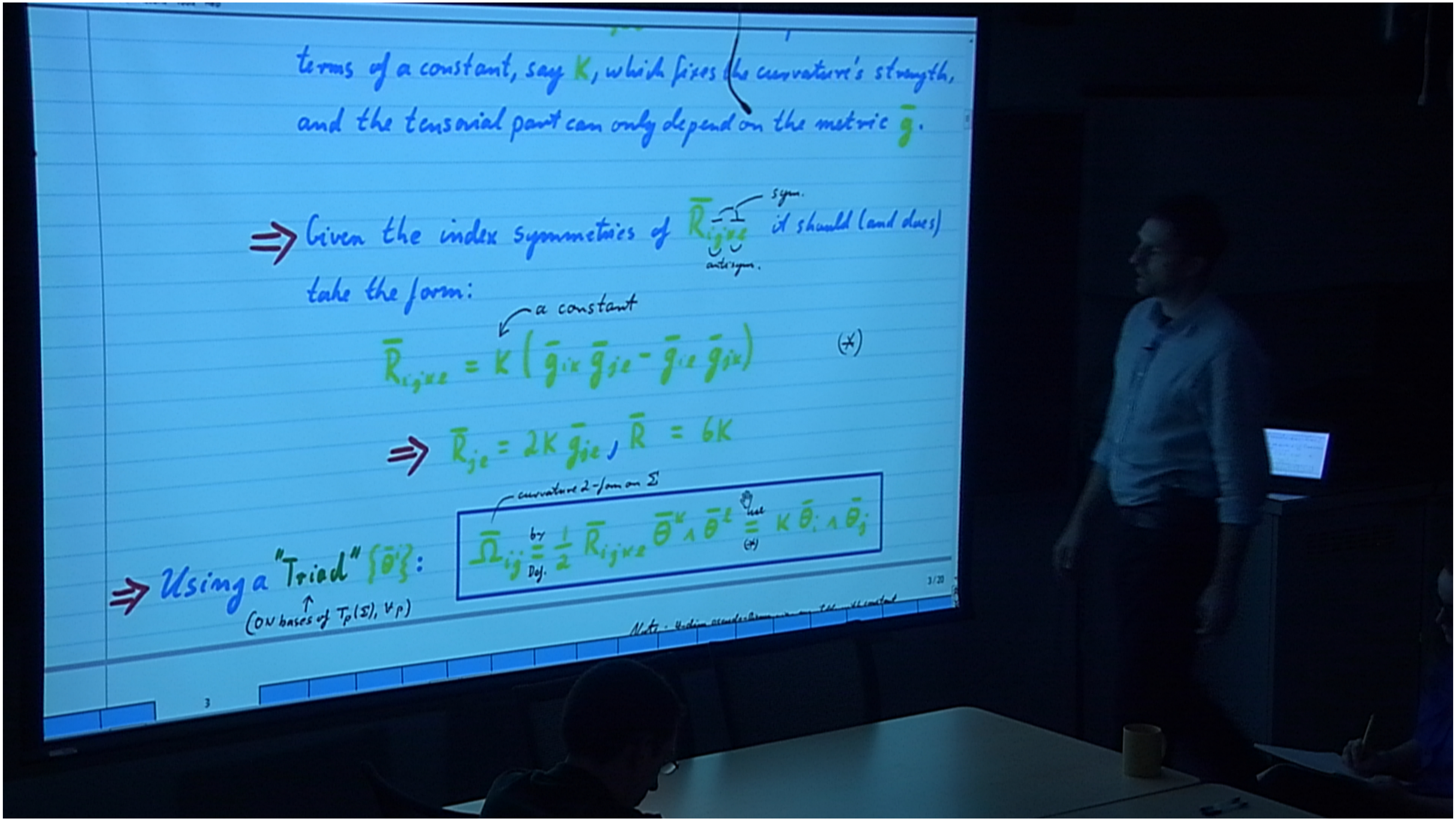
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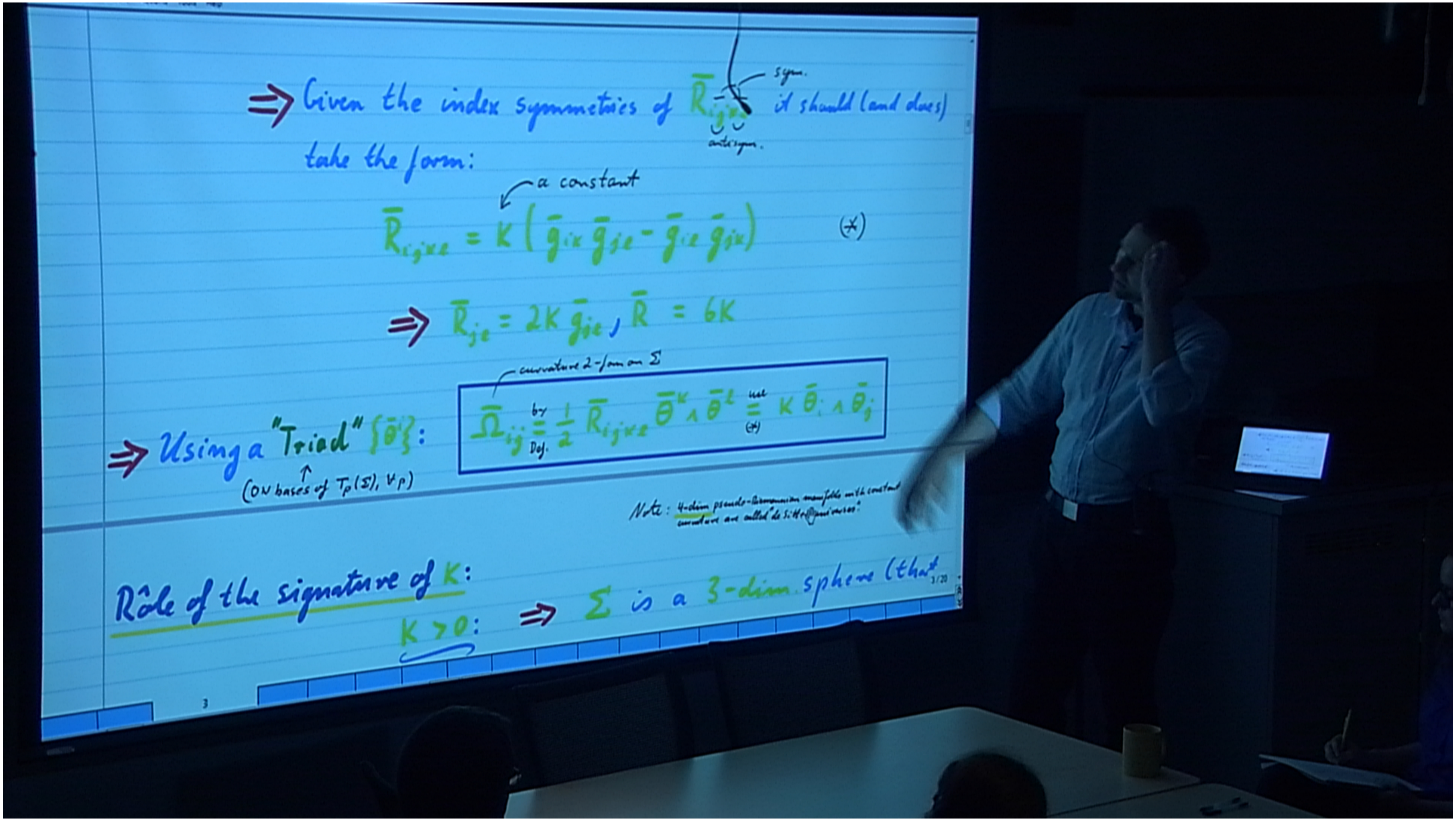
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Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called "de Sitter universes"
Rôle of the signature of K:
K > 0: ⇒ Σ is a 3-dim. sphere (that
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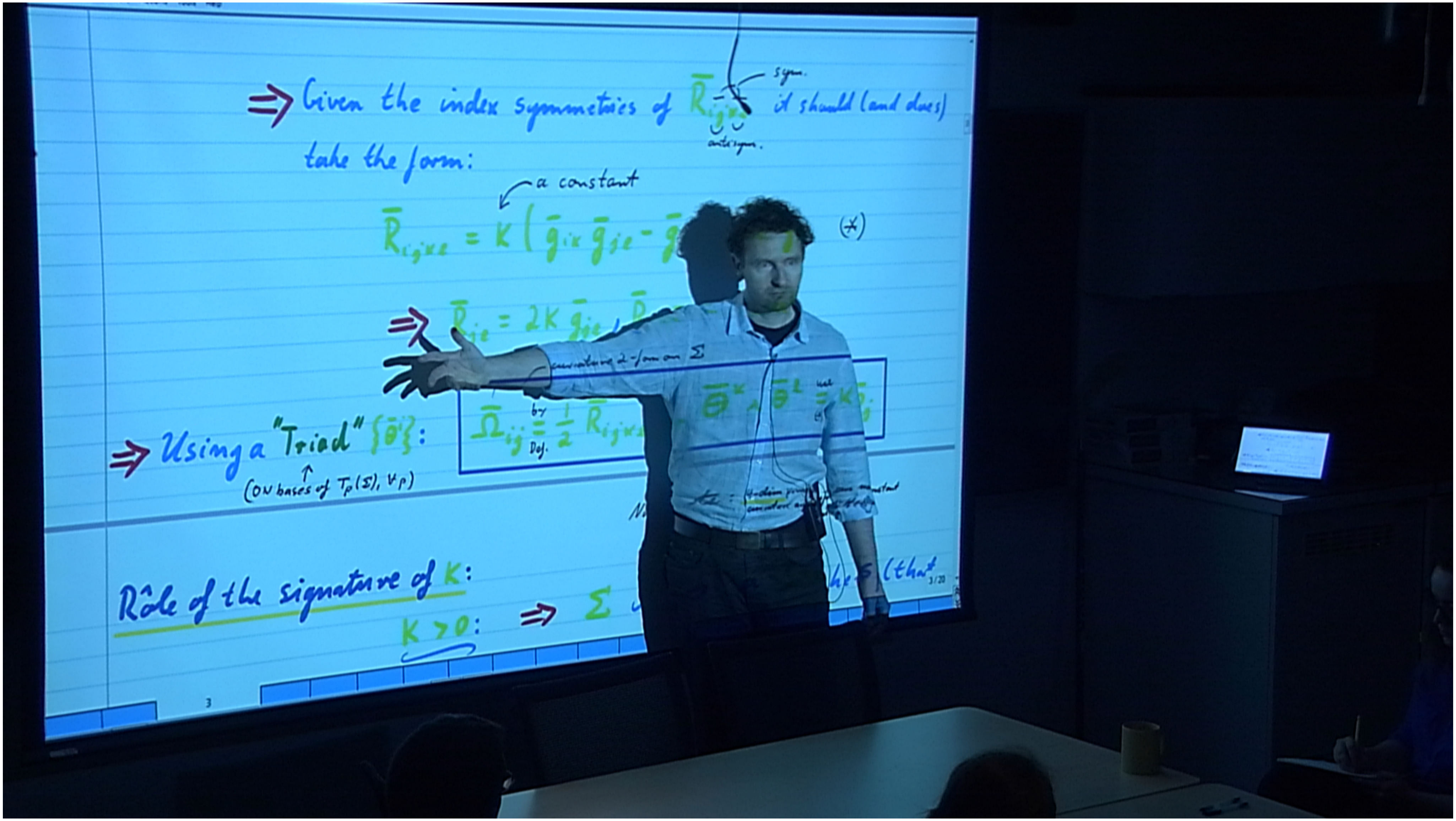
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Rôle of the signature of K:

$K > 0$: $\Rightarrow$ Σ is a 3-dim. sphere (that can be embedded e.g. in a 4 dim euclidean (i.e. flat) space: closed universe

$K = 0$: $\Rightarrow$ Σ is euclidean $\mathbb{R}^3$. flat, infinite universe

$K < 0$ $\Rightarrow$ Σ is a 3-dim. "pseudo sphere". The constant negative curvature means it is everywhere "like a saddle". These Σ's also possess ∞ volume.

Note: $\bar{R}$ and therefore K have units $\frac{1}{(\text{length})^2}$. Thus,

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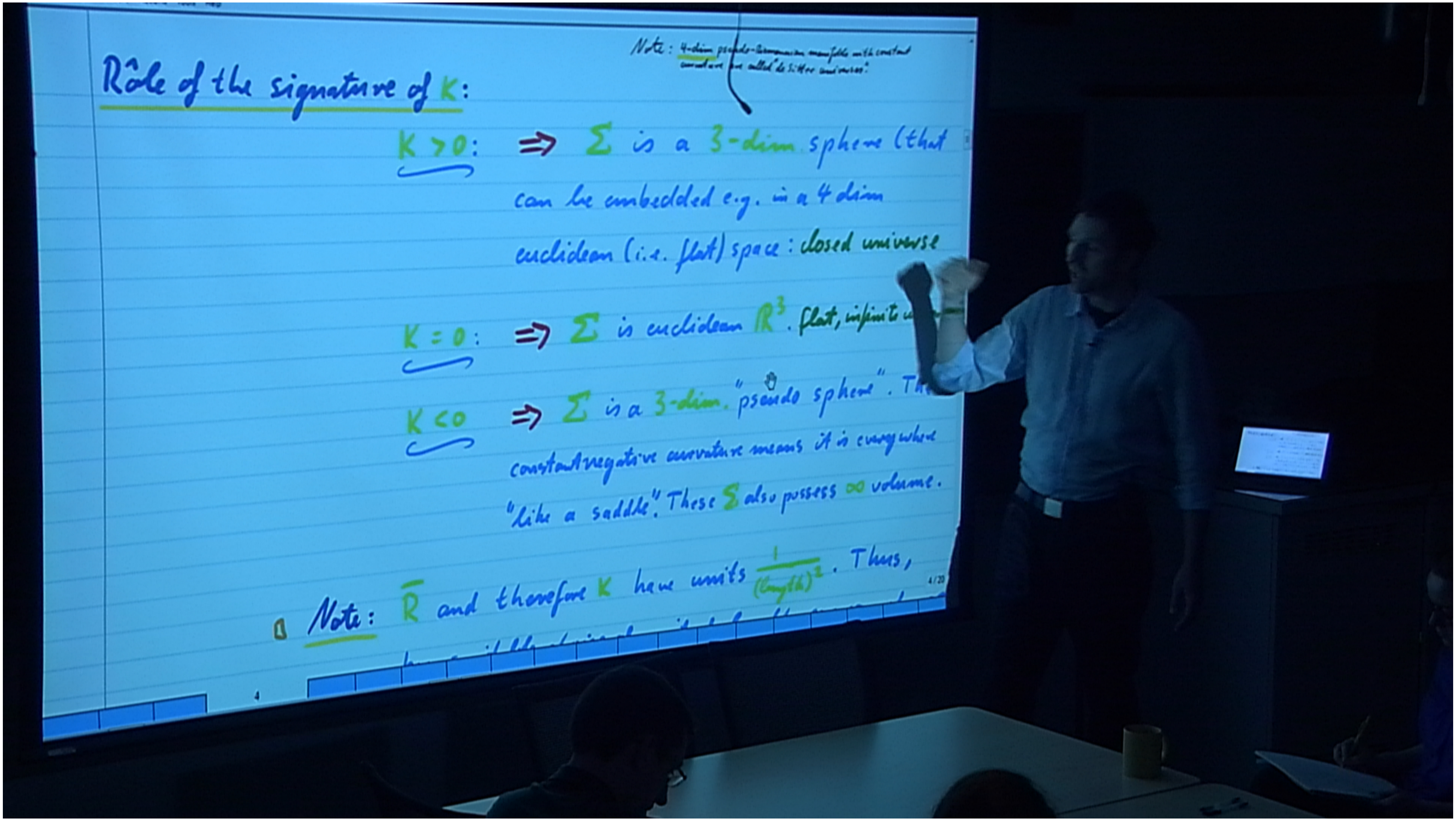
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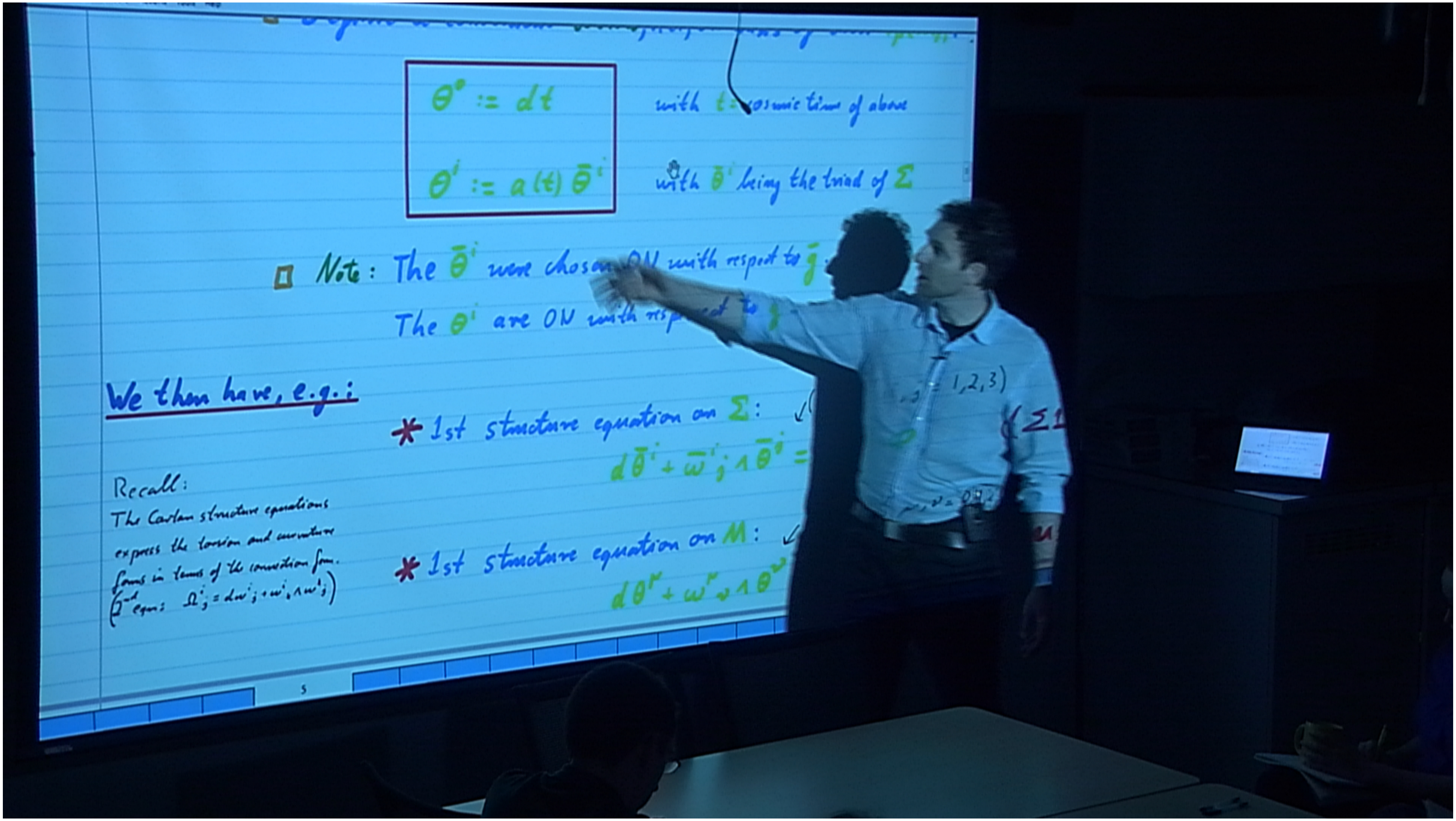

$\theta^0 := dt$
with t = cosmic time of above
$\theta^i := a(t)\,\bar{\theta}^i$
with $\bar{\theta}^i$ being the triad of Σ
Note: The $\bar{\theta}^i$ were chosen ON with respect to $\bar{g}$
The θ^i are ON with respect to g
We then have, e.g.:
1st structure equation on Σ:
$d\bar{\theta}^i + \bar{\omega}^i{}_j \wedge \bar{\theta}^j =$
1st structure equation on M:
$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu$
Recall:
The Cartan structure equations express the torsion and curvature forms in terms of the connection forms.
(2nd eqn: $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$)

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$$\boxed{\begin{aligned}\theta^0 &:= dt \\ \theta^i &:= a(t)\,\bar{\theta}^i\end{aligned}}$$

with $t =$ cosmic time of above

with $\bar{\theta}^i$ being the triad of Σ

- Note: The $\bar{\theta}^i$ were chosen ON with respect to $\bar{g}$.
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Recall:
The Cartan structure equations express the torsion and curvature forms in terms of the connection form.
(2^{nd} eqn: $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$)

* 1st structure equation on Σ: ✓ $(i, j = 1, 2, 3)$

$$d\bar{\theta}^i + \bar{\omega}^i{}_j \wedge \bar{\theta}^j = 0 \qquad (\Sigma 1)$$

* 1st structure equation on M: ✓ $(\mu, \nu = 0, 1, 2, 3)$

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0 \qquad (M 1)$$

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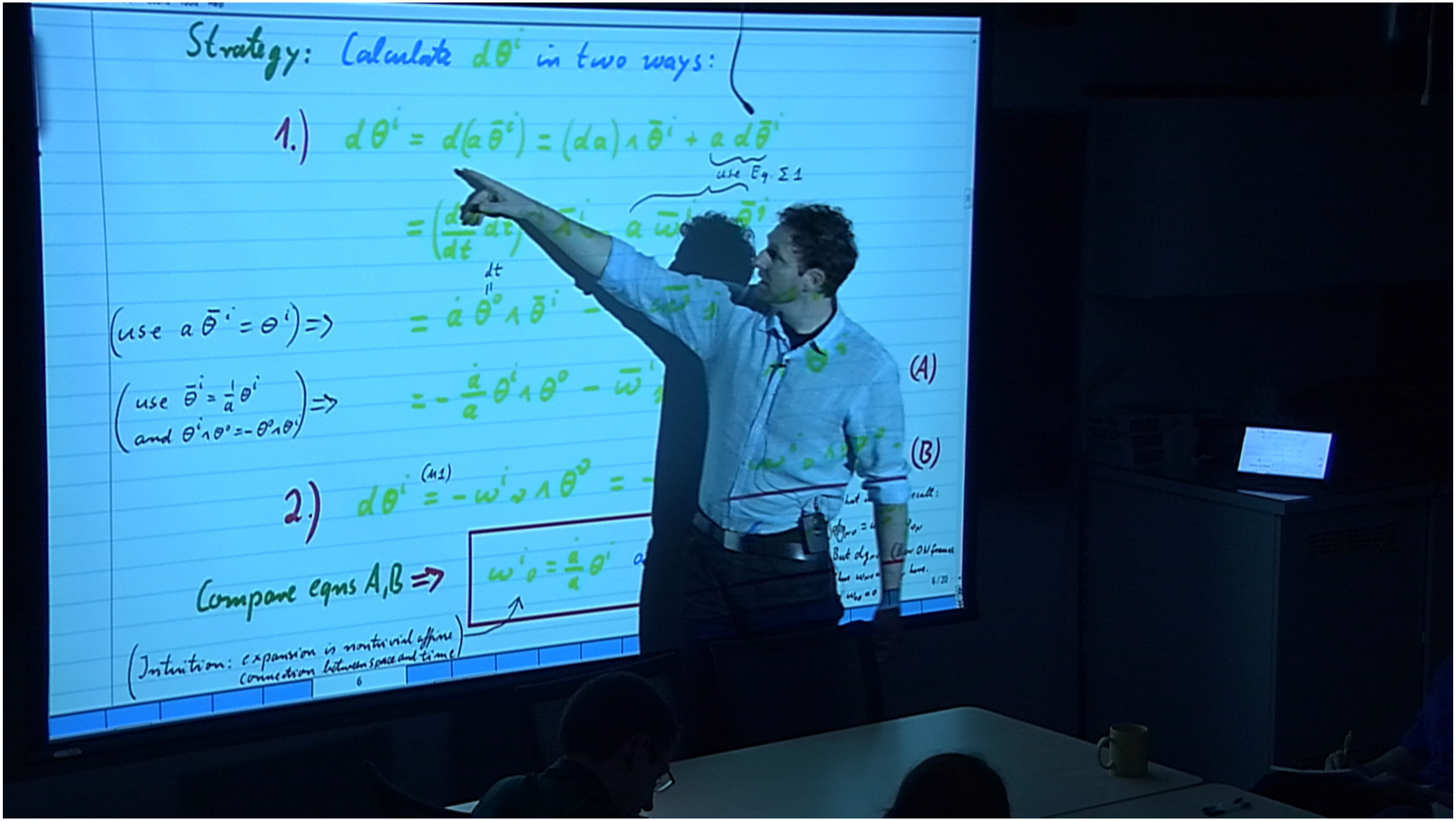
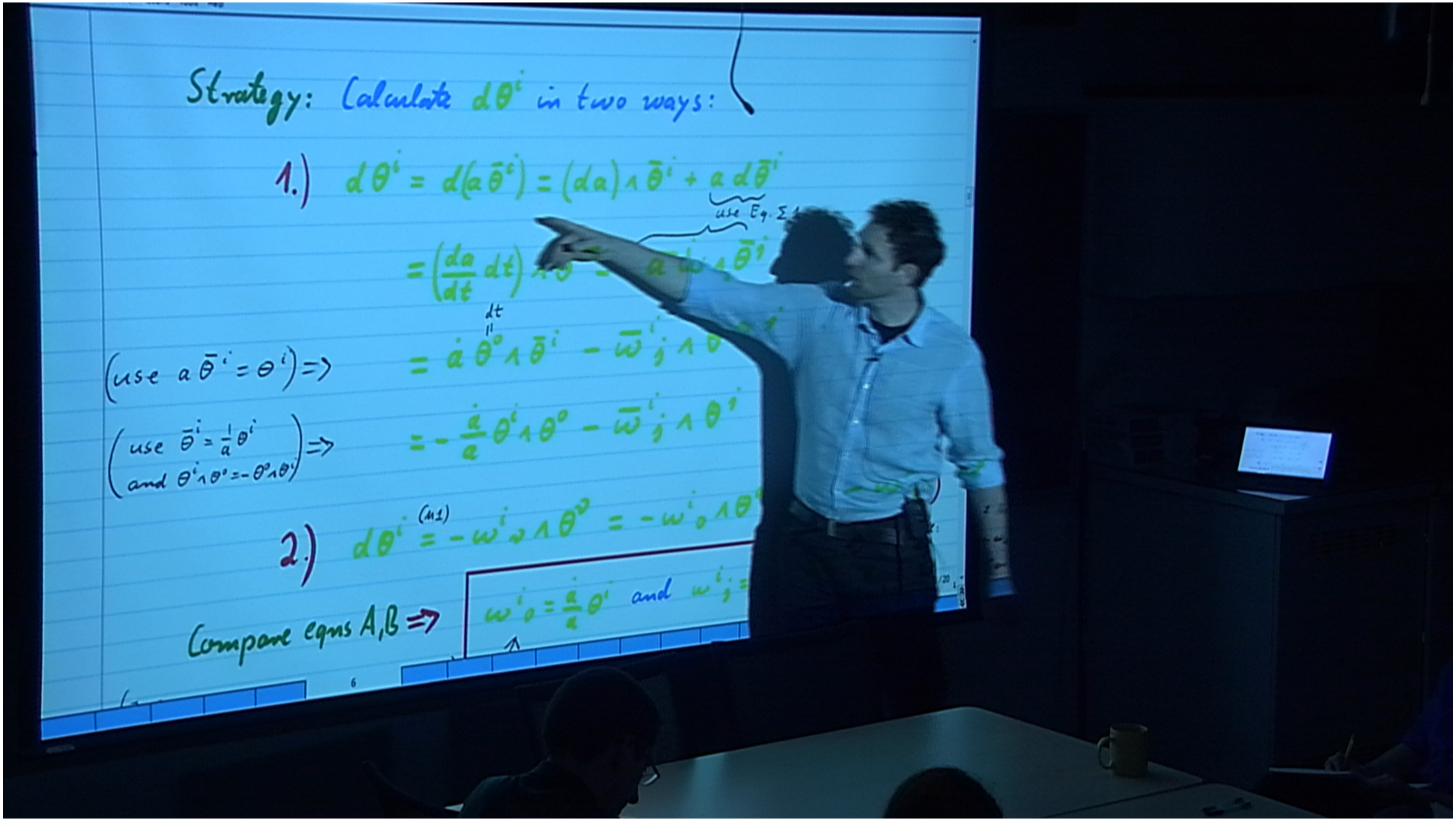

Strategy: Calculate $d\theta^i$ in two ways:
1.) $d\theta^i = d(a\bar\theta^i) = (da)\wedge\bar\theta^i + a\,d\bar\theta^i$
use Eq. Σ1
(use $a\bar\theta^i = \theta^i$) =>
(use $\bar\theta^i = \frac{1}{a}\theta^i$ and $\theta^i\wedge\theta^0 = -\theta^0\wedge\theta^i$) =>
(A)
(B)
2.) $d\theta^i = -\omega^i{}_0\wedge\theta^0$
Compare eqns A,B =>
$\omega^i{}_0 = \frac{\dot a}{a}\theta^i$
(Intuition: expansion is nontrivial affine connection between space and time)



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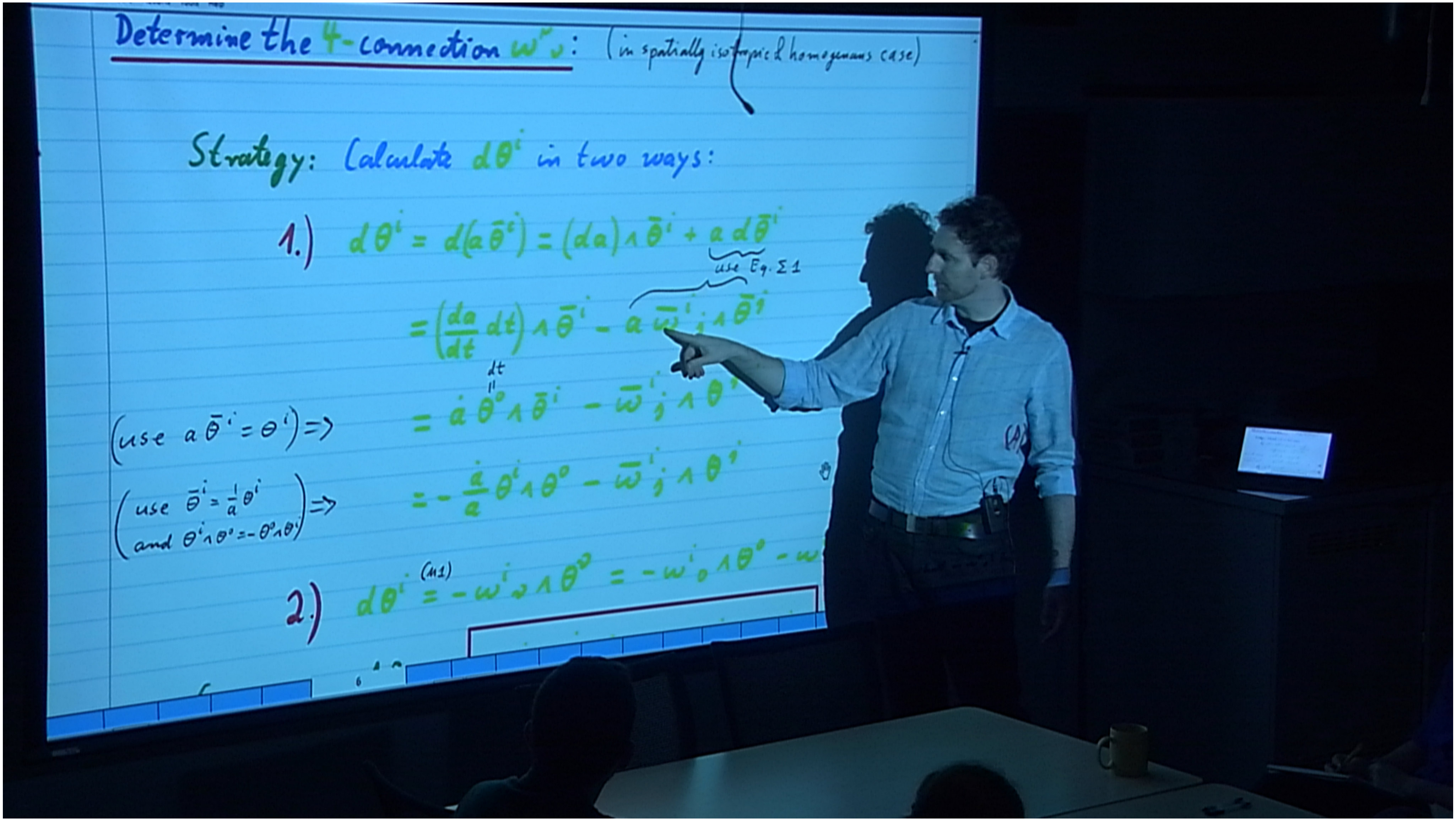
use Eq.
$= \left(\frac{da}{dt}dt\right)\wedge$
(use $a\bar\theta^i = \theta^i$) $\Rightarrow$ $= \dot a\,\theta^0\wedge\bar\theta^i - \bar\omega^i{}_j\wedge\theta$
(use $\bar\theta^i = \frac{1}{a}\theta^i$ and $\theta^i\wedge\theta^0 = -\theta^0\wedge\theta^i$) $\Rightarrow$ $= -\frac{\dot a}{a}\theta^i\wedge\theta^0 - \bar\omega^i{}_j\wedge\theta^j$
2.) $d\theta^i = -\omega^i{}_\nu\wedge\theta^\nu = -\omega^i{}_0\wedge\theta^0$
Compare eqns A,B $\Rightarrow$
$\omega^i{}_0 = \frac{\dot a}{a}\theta^i$ and $\omega^i{}_j =$



Determine the 4-connection $\omega^\mu{}_\nu$: (in spatially isotropic & homogeneous case)
Strategy: Calculate $d\theta^i$ in two ways:
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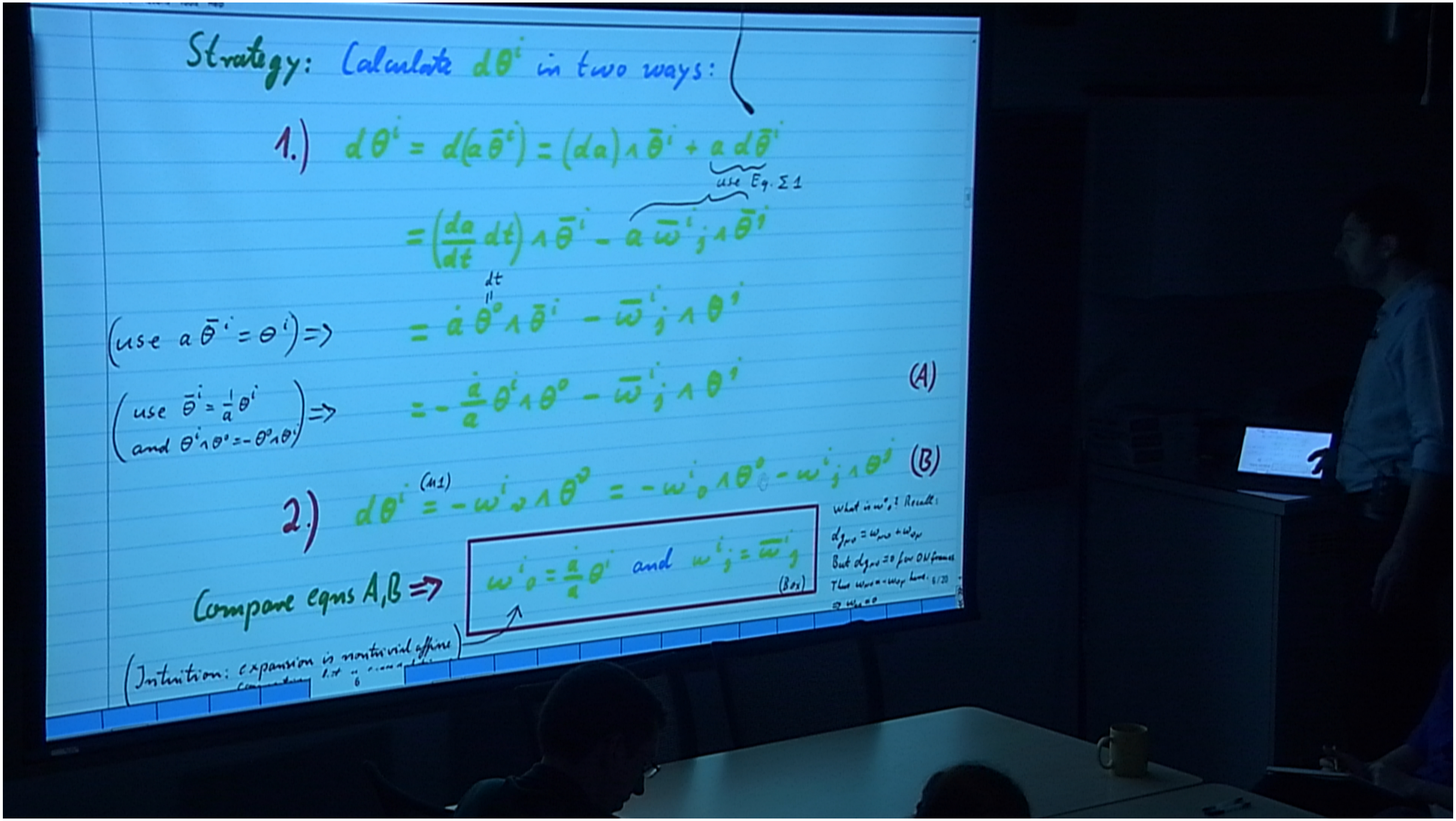
use Eq. Σ1
$= \left(\frac{da}{dt}dt\right)\wedge\bar\theta^i - a\,\bar\omega^i{}_j\wedge\bar\theta^j$
dt
(use $a\bar\theta^i = \theta^i$) => $= \dot a\,\theta^0\wedge\bar\theta^i - \bar\omega^i{}_j\wedge\theta^j$
(use $\bar\theta^i = \frac{1}{a}\theta^i$ and $\theta^i\wedge\theta^0 = -\theta^0\wedge\theta^i$) => $= -\frac{\dot a}{a}\theta^i\wedge\theta^0 - \bar\omega^i{}_j\wedge\theta^j$
2.) $d\theta^i \overset{(M1)}{=} -\omega^i{}_\nu\wedge\theta^\nu = -\omega^i{}_0\wedge\theta^0 - \omega$



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(use $a\bar{\theta}^i = \theta^i$) ⇒ $= \dot{a}\,\theta^0\wedge\bar{\theta}^i - \bar{\omega}^i{}_j\wedge\theta^j$
(use $\bar{\theta}^i = \frac{1}{a}\theta^i$ and $\theta^i\wedge\theta^0 = -\theta^0\wedge\theta^i$) ⇒ $= -\frac{\dot{a}}{a}\theta^i\wedge\theta^0 - \bar{\omega}^i{}_j\wedge\theta^j$ (A)
2.) $d\theta^i \overset{(M1)}{=} -\omega^i{}_\nu\wedge\theta^\nu = -\omega^i{}_0\wedge\theta^0 - \omega^i{}_j\wedge\theta^j$ (B)
Compare eqns A,B ⇒ $\omega^i{}_0 = \frac{\dot{a}}{a}\theta^i$ and $\omega^i{}_j = \bar{\omega}^i{}_j$ (Box)
(Intuition: expansion is nontrivial affine
What is $\omega^0{}_0$? Recall:
$dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$
But $dg_{\mu\nu} = 0$ for ON frames
Thus $\omega_{\mu\nu} = -\omega_{\nu\mu}$ here.
⇒ $\omega_{00} = 0$

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1.) $$d\theta^i = d(a\bar\theta^i) = (da)\wedge\bar\theta^i + \underbrace{a\,d\bar\theta^i}_{\text{use Eq. }\Sigma 1}$$

$$= \left(\frac{da}{dt}dt\right)\wedge\bar\theta^i - \overbrace{a\,\bar\omega^i{}_j\wedge\bar\theta^j}$$

(use $a\bar\theta^i = \theta^i$) $\Rightarrow$ $$= \dot a\,\overset{\overset{dt}{\|}}{\theta^0}\wedge\bar\theta^i - \bar\omega^i{}_j\wedge\theta^j$$

(use $\bar\theta^i = \frac{1}{a}\theta^i$ and $\theta^i\wedge\theta^0 = -\theta^0\wedge\theta^i$) $\Rightarrow$ $$= -\frac{\dot a}{a}\theta^i\wedge\theta^0 - \bar\omega^i{}_j\wedge\theta^j \qquad (A)$$

2.) $$d\theta^i \overset{(M1)}{=} -\omega^i{}_\nu\wedge\theta^\nu = -\omega^i{}_0\wedge\theta^0 - \omega^i{}_j\wedge\theta^j \qquad (B)$$

Compare eqns A,B $\Rightarrow$ $$\boxed{\omega^i{}_0 = \frac{\dot a}{a}\theta^i \quad \text{and} \quad \omega^i{}_j = \bar\omega^i{}_j} \quad \text{(Box)}$$

(Intuition: expansion is nontrivial affine connection between space and time)

What is $\omega^0{}_0$? Recall:
$dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$
But $dg_{\mu\nu} = 0$ for ON frames.
Thus $\omega_{\mu\nu} = -\omega_{\nu\mu}$ here.
$\Rightarrow \omega_{00} = 0$

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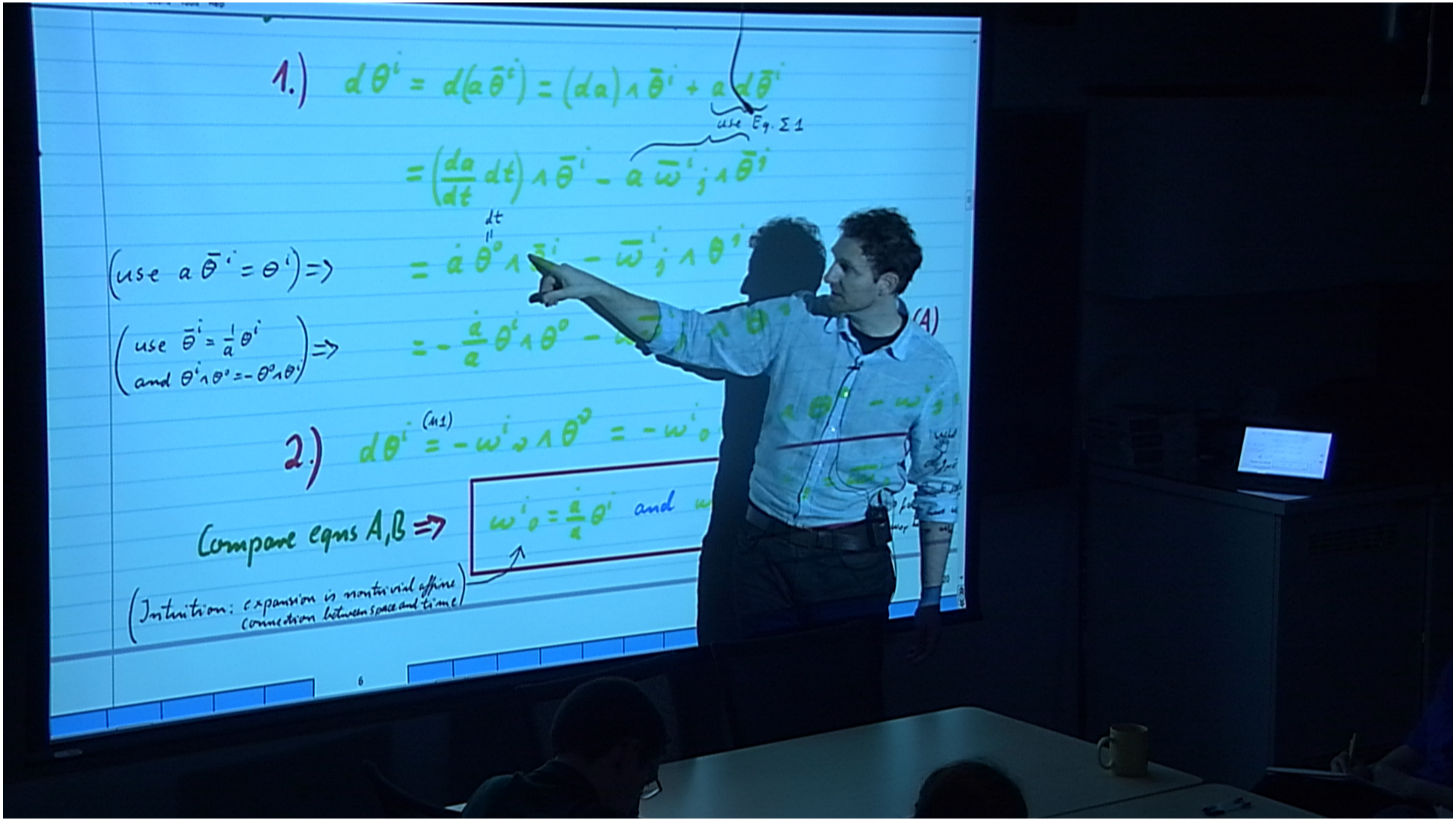
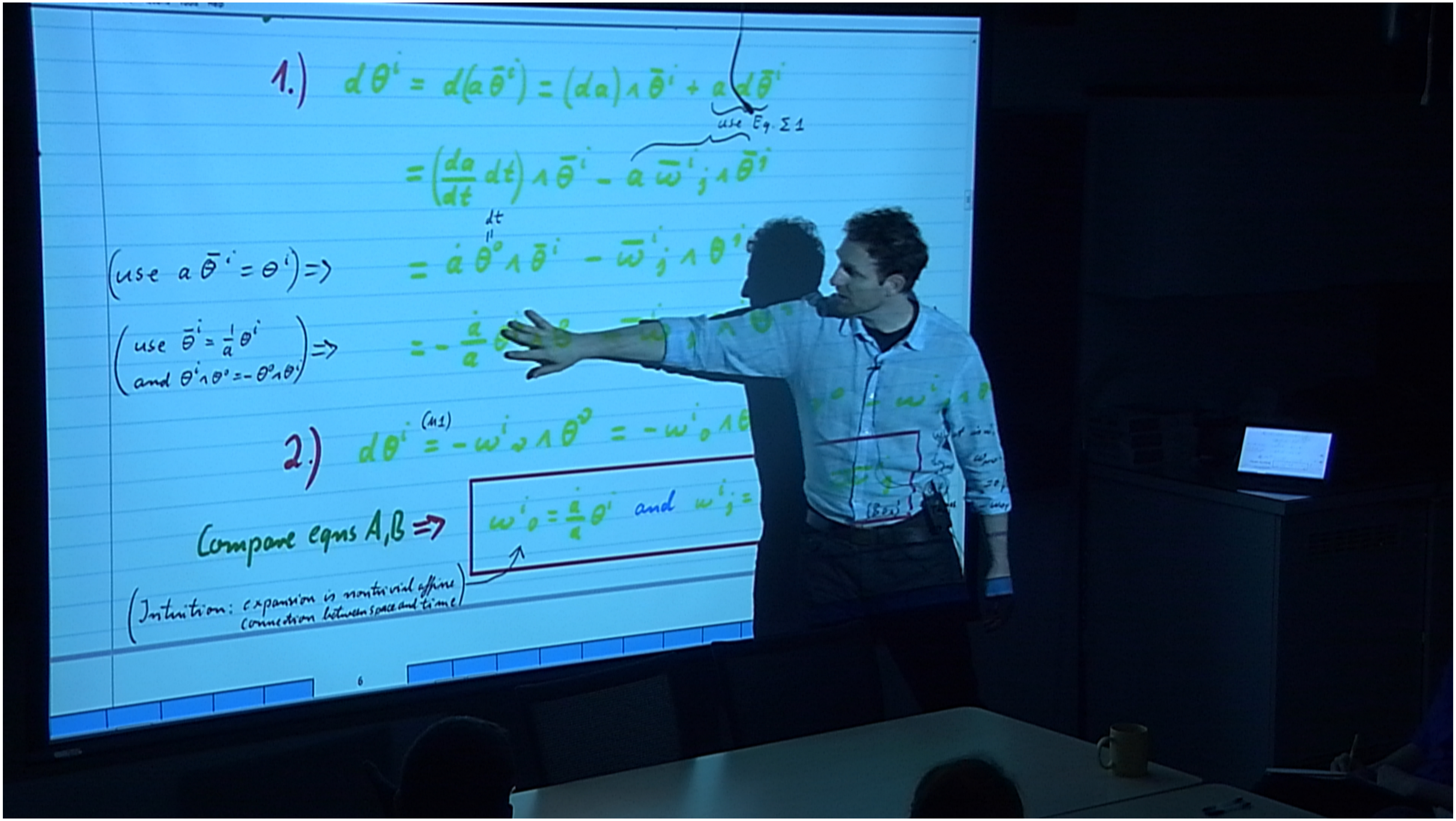

1.) dθ^i = d(a θ̄^i) = (da) ∧ θ̄^i + a dθ̄^i
use Eq. Σ1
= (da/dt dt) ∧ θ̄^i − a ω̄^i_j ∧ θ̄^j
(use a θ̄^i = θ^i) =>
(use θ̄^i = 1/a θ^i and θ^i∧θ^0 = −θ^0∧θ^i) =>
2.) dθ^i = −ω^i_a ∧ θ^a = −ω^i_0
Compare eqns A,B =>
ω^i_0 = ȧ/a θ^i and
(Intuition: expansion is nontrivial affine connection between space and time)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(use θ̄^i = 1/a θ^i and θ^i ∧ θ^0 = −θ^0 ∧ θ^i) =>
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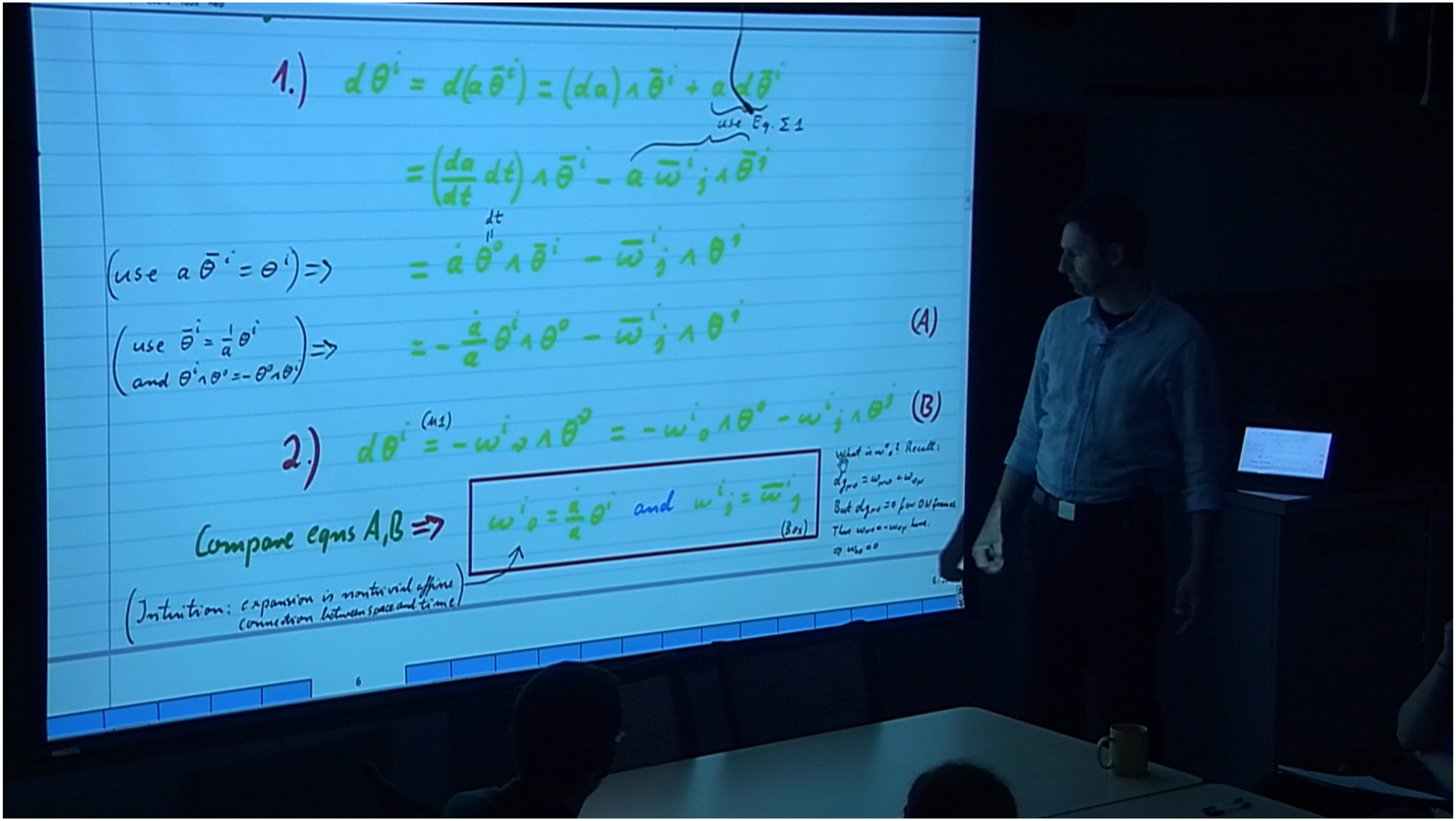
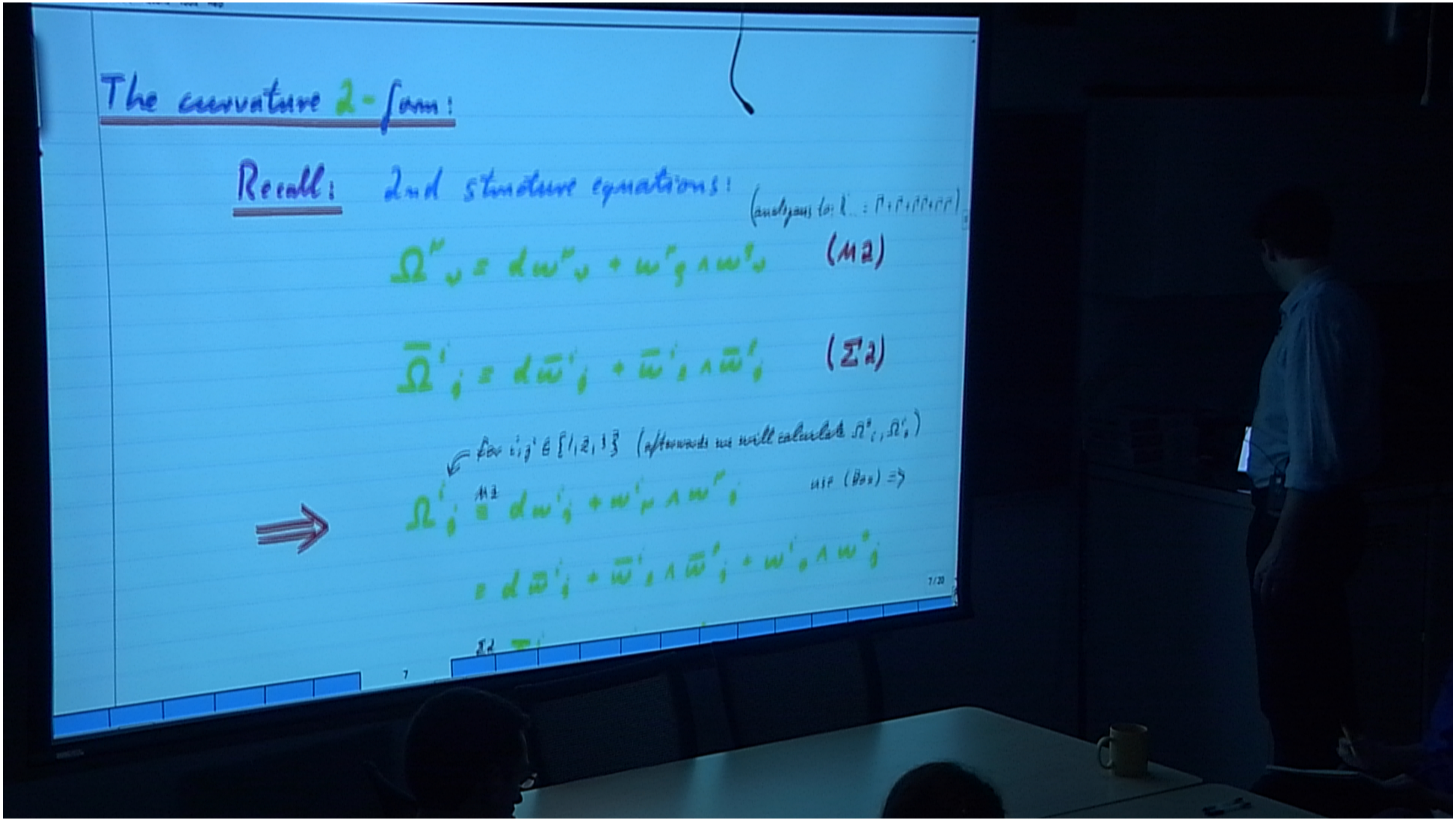
ω^i_0 = ȧ/a θ^i and ω^i_j =
(Intuition: expansion is nontrivial affine connection between space and time)



1.) $d\theta^i = d(a\bar\theta^i) = (da)\wedge\bar\theta^i + a\,d\bar\theta^i$
use Eq. Σ1
$= \left(\frac{da}{dt}dt\right)\wedge\bar\theta^i - a\,\bar\omega^i{}_j\wedge\bar\theta^j$
(use $a\bar\theta^i = \theta^i$) ⇒ $= \dot a\,\theta^0\wedge\bar\theta^i - \bar\omega^i{}_j\wedge\theta^j$
(use $\bar\theta^i = \frac{1}{a}\theta^i$ and $\theta^i\wedge\theta^0 = -\theta^0\wedge\theta^i$) ⇒ $= -\frac{\dot a}{a}\theta^i\wedge\theta^0 - \bar\omega^i{}_j\wedge\theta^j$ (A)
2.) $d\theta^i \overset{(a1)}{=} -\omega^i{}_\alpha\wedge\theta^\alpha = -\omega^i{}_0\wedge\theta^0 - \omega^i{}_j\wedge\theta^j$ (B)
Compare eqns A,B ⇒ $\omega^i{}_0 = \frac{\dot a}{a}\theta^i$ and $\omega^i{}_j = \bar\omega^i{}_j$ (Box)
(Intuition: expansion is nontrivial affine connection between space and time)
What is $\omega^0{}_0$? Recall:
$dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$
But $dg_{\mu\nu} = 0$ for ON frames.
Thus $\omega_{\mu\nu} = -\omega_{\nu\mu}$ here.
⇒ $\omega_{00} = 0$


The curvature 2-form:
Recall: 2nd structure equations:
(M2)
(Σ2)
use (Box) ⇒
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The curvature 2-form:

Recall: 2nd structure equations:

(analogous to: $R^{\cdot}_{\cdot\cdot\cdot} = \Gamma + \Gamma + \Gamma\Gamma + \Gamma\Gamma$)

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu \qquad (M2)$$

$$\bar{\Omega}^i{}_j = d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j \qquad (\Sigma 2)$$

for $i,j \in \{1,2,3\}$ (afterwards we will calculate $\Omega^0{}_i, \Omega^i{}_0$)

$$\Rightarrow \quad \Omega^i{}_j \overset{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j \qquad \text{use (Box)} \Rightarrow$$

$$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

$$\overset{\Sigma 2}{=} \bar{\Omega}^i{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

Recall also:

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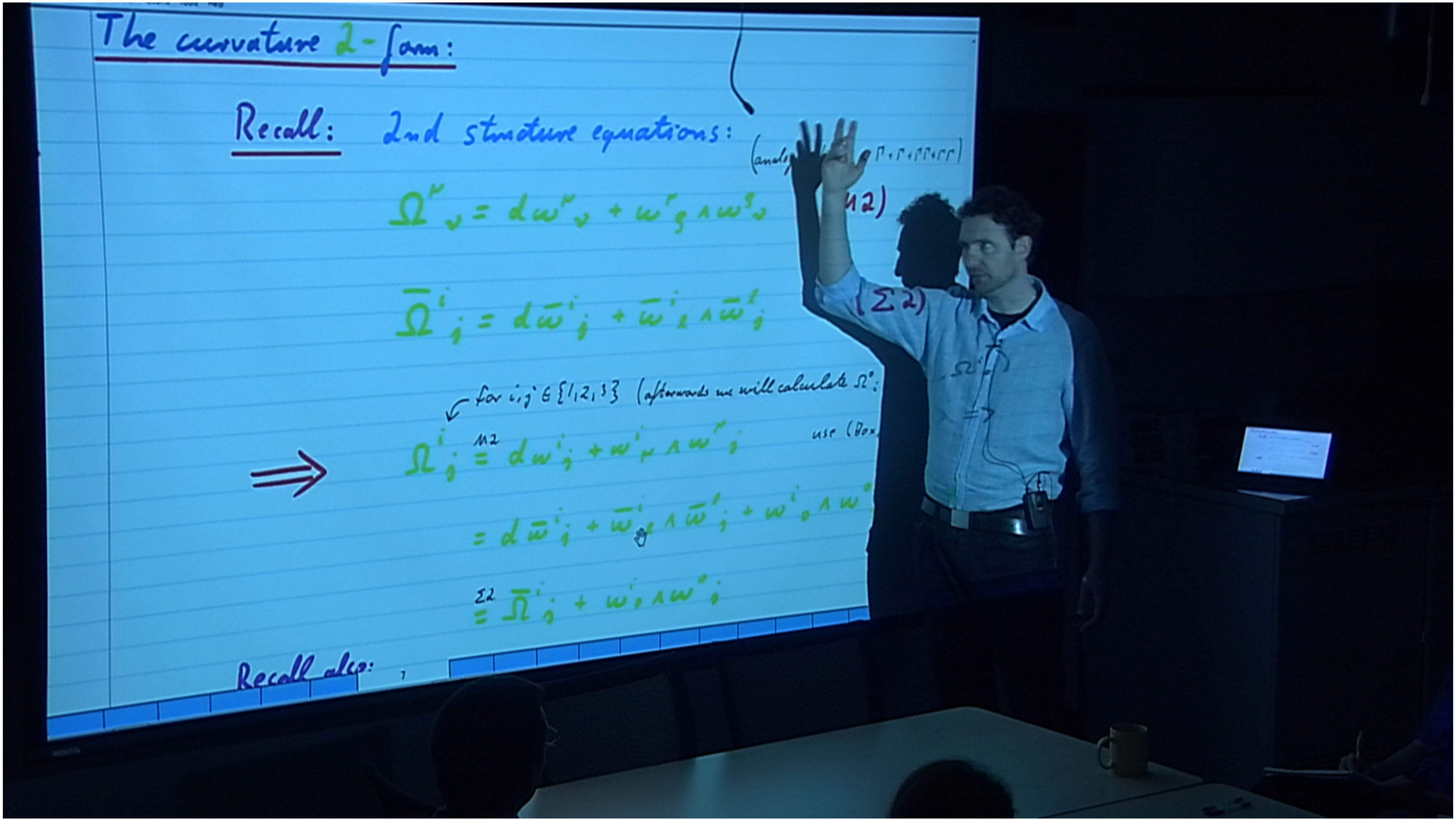
The curvature 2-form:
Recall: 2nd structure equations:
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$\bar{\Omega}^i{}_j = d\bar{\omega}^i{}_j + \bar{\omega}^i{}_\ell \wedge \bar{\omega}^\ell{}_j$
for $i,j \in \{1,2,3\}$ (afterwards we will calculate Ω^0:
$\Omega^i{}_j \overset{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j$
$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_\ell \wedge \bar{\omega}^\ell{}_j + \omega^i{}_0 \wedge \omega^0{}_j$
$\overset{\Sigma 2}{=} \bar{\Omega}^i{}_j + \omega^i{}_0 \wedge \omega^0{}_j$
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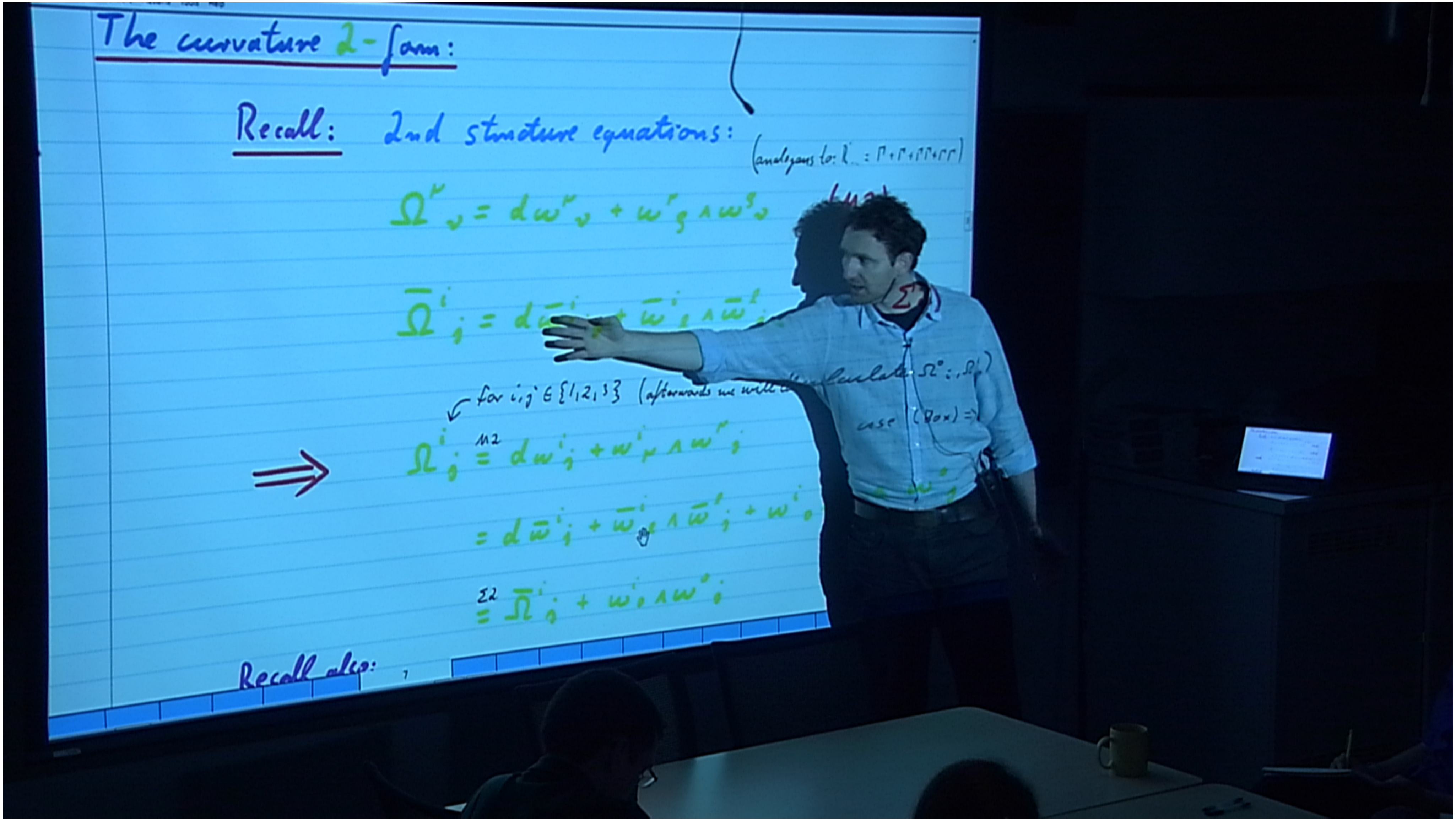
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Recall: 2nd structure equations:
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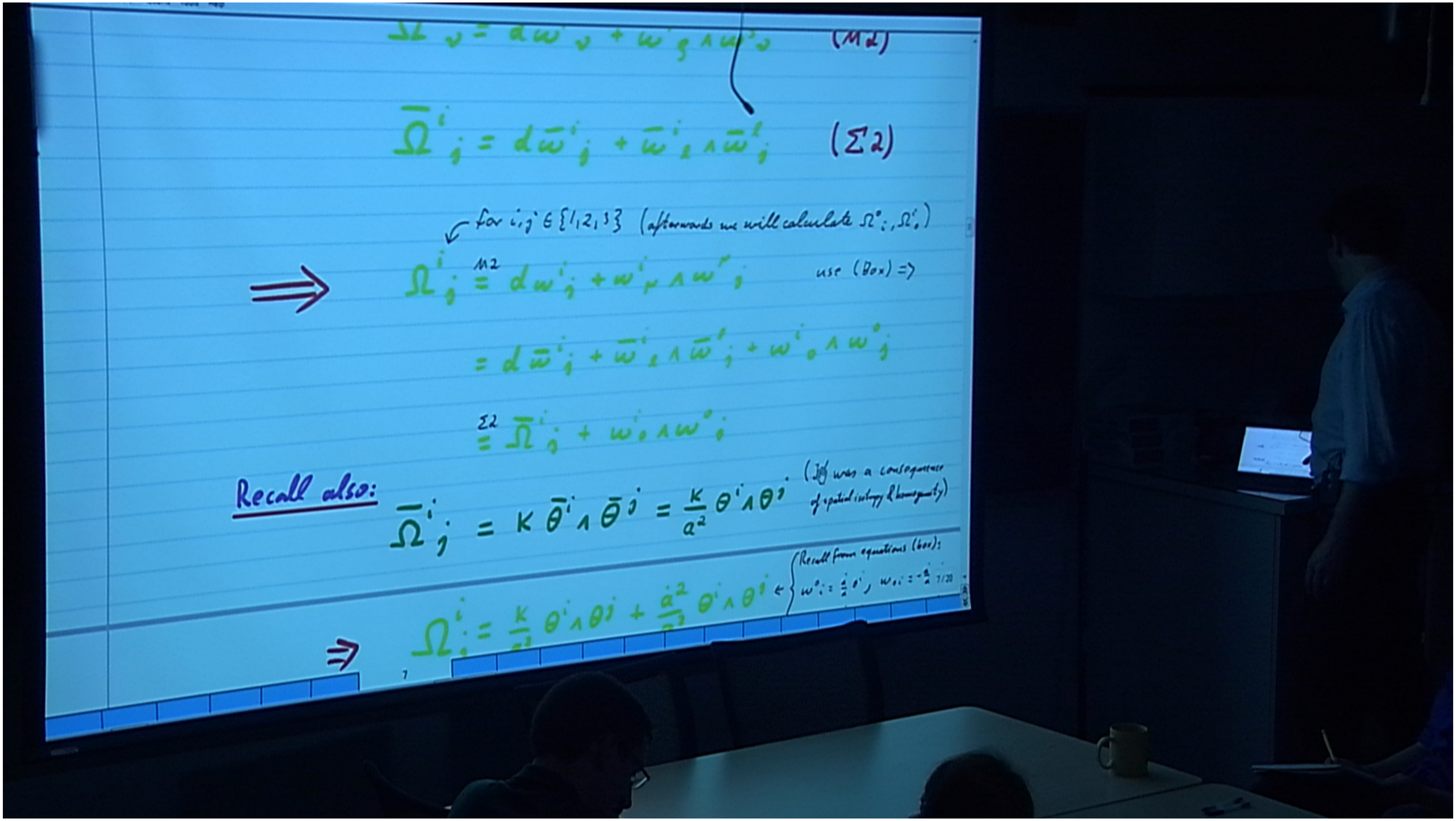

$\bar{\Omega}^i{}_j = d\bar{\omega}^i{}_j + \bar{\omega}^i{}_\ell \wedge \bar{\omega}^\ell{}_j$ $(\Sigma 2)$
for $i,j \in \{1,2,3\}$ (afterwards we will calculate $\Omega^0{}_i, \Omega^i{}_0$)
$\Rightarrow \Omega^i{}_j \overset{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j$
use (Box) =>
$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_\ell \wedge \bar{\omega}^\ell{}_j + \omega^i{}_0 \wedge \omega^0{}_j$
$\overset{\Sigma 2}{=} \bar{\Omega}^i{}_j + \omega^i{}_0 \wedge \omega^0{}_j$
Recall also:
$\bar{\Omega}^i{}_j = K \bar{\theta}^i \wedge \bar{\theta}^j = \frac{K}{a^2} \theta^i \wedge \theta^j$.
(This was a consequence of spatial isotropy & homogeneity)
Recall from equations (box):
$\omega^0{}_i = \frac{\dot{a}}{a}\theta^i$, $\omega_{0i} = -\frac{\dot{a}}{a}$
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$\Rightarrow \Omega^i{}_j = \frac{K}{a^2} \theta^i \wedge \theta^j + \frac{\dot{a}^2}{a^2} \theta^i \wedge \theta^j$

(analogous to: $R_{\dots} = \Gamma + \Gamma + \Gamma\Gamma + \Gamma\Gamma$)

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu \qquad (M2)$$

$$\bar{\Omega}^i{}_j = d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j \qquad (\Sigma 2)$$

for $i,j \in \{1,2,3\}$ (afterwards we will calculate $\Omega^0{}_i$, $\Omega^i{}_0$)

$$\Rightarrow \quad \Omega^i{}_j \overset{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j$$

use (Box) $\Rightarrow$

$$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

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Recall also:

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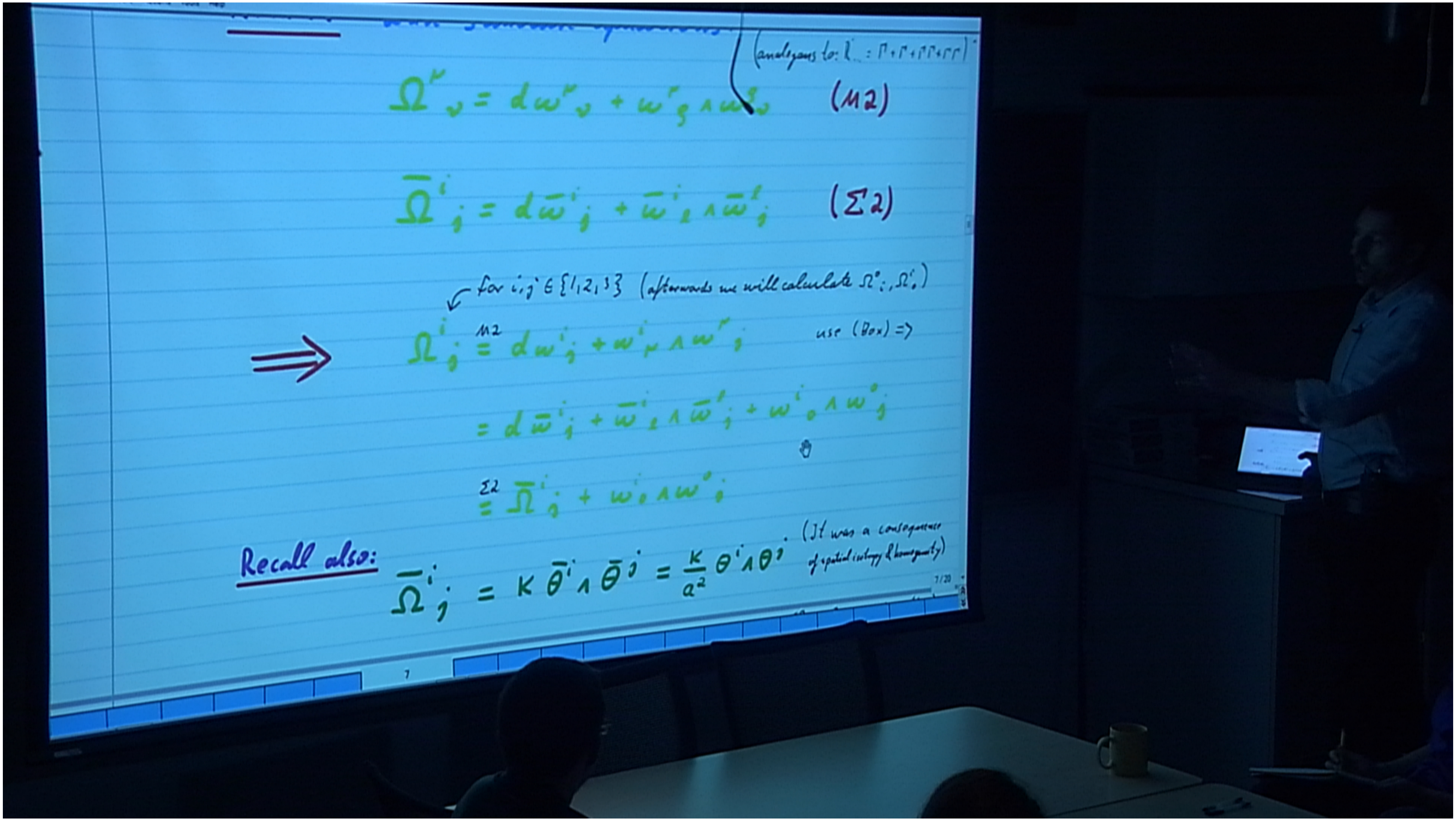
(analogous to: $R_{..} = \Gamma + \Gamma + \Gamma\Gamma + \Gamma\Gamma$)
$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$ (M2)
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for $i,j \in \{1,2,3\}$ (afterwards we will calculate $\Omega^0{}_i, \Omega^i{}_0$)
$\Rightarrow \Omega^i{}_j \overset{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j$
use (Box) $\Rightarrow$
$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j + \omega^i{}_0 \wedge \omega^0{}_j$
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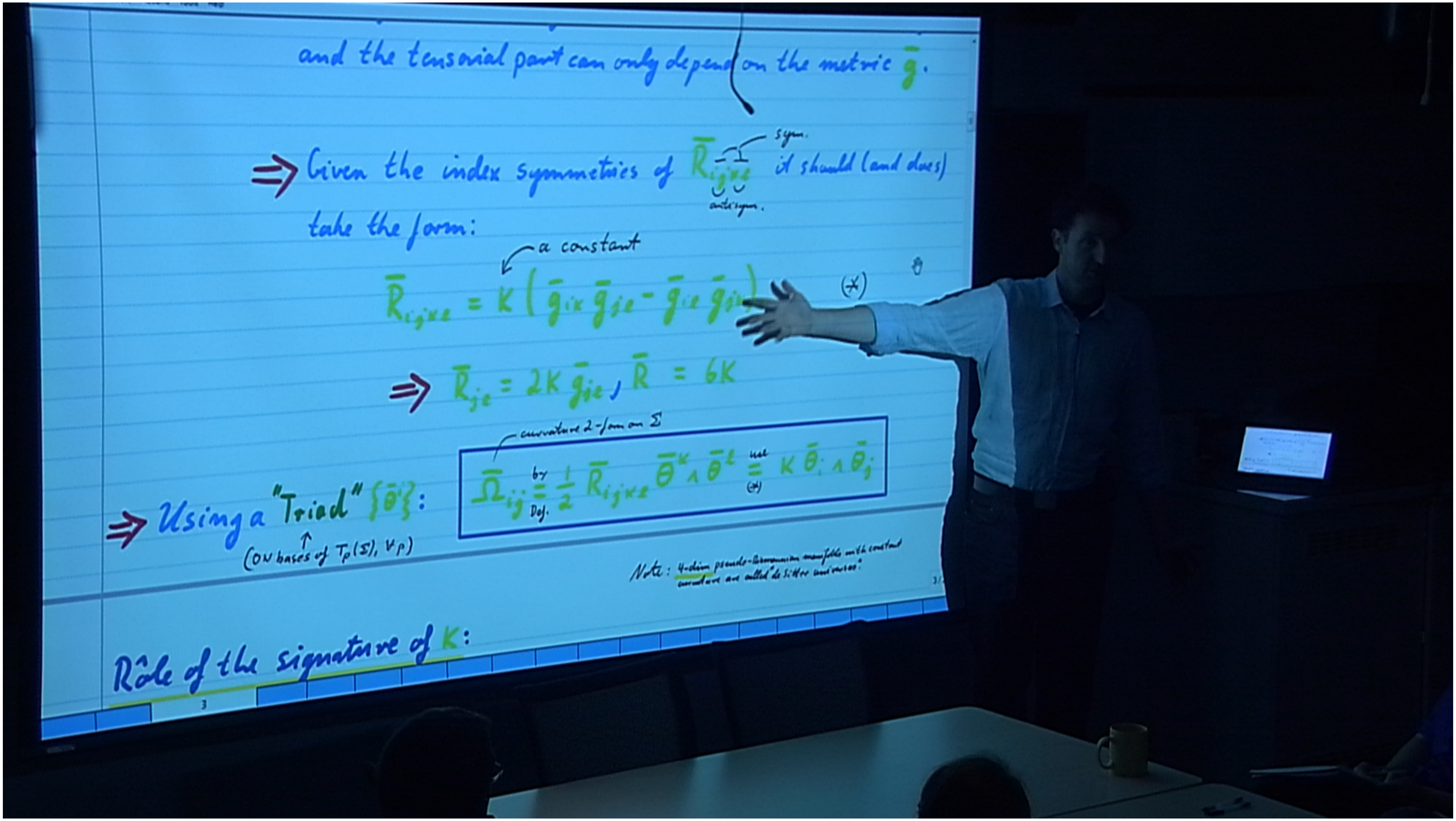
and the tensorial part can only depend on the metric $\bar{g}$.
$\Rightarrow$ Given the index symmetries of $\bar{R}_{ijkl}$ (symm., anti-symm.) it should (and does) take the form:
a constant
$\bar{R}_{ijkl} = K\left(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}\right)$ (*)
$\Rightarrow \bar{R}_{jl} = 2K\bar{g}_{jl}, \ \bar{R} = 6K$
curvature 2-form on Σ
$\Rightarrow$ Using a "Triad" $\{\bar{\theta}^i\}$:
(ON bases of $T_p(\Sigma)$, $\forall p$)
$\bar{\Omega}_{ij} \overset{\text{by Def.}}{=} \frac{1}{2}\bar{R}_{ijkl}\bar{\theta}^k \wedge \bar{\theta}^l \overset{\text{use (*)}}{=} K\bar{\theta}_i \wedge \bar{\theta}_j$
Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called "de Sitter universes"
Rôle of the signature of K:

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$$\boxed{\bar{\Omega}_{ij} \overset{\text{by}}{\underset{\text{Def.}}{=}} \frac{1}{2}\bar{R}_{ijkl}\,\bar{\theta}^k \wedge \bar{\theta}^l \overset{\text{use}}{\underset{(*)}{=}} K\,\bar{\theta}_i \wedge \bar{\theta}_j}$$

Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called "de Sitter universes".

Rôle of the signature of K:

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(analogous to: ...

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu \qquad (M2)$$

$$\bar{\Omega}^i{}_j = d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j \qquad (\Sigma 2)$$

for $i,j \in \{1,2,3\}$ (afterwards we will calculate $\Omega^0{}_i$, $\Omega^i{}_0$)

$$\Rightarrow \quad \Omega^i{}_j \overset{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j \qquad \text{use (Box)} \Rightarrow$$

$$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

$$\overset{\Sigma 2}{=} \bar{\Omega}^i{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

Recall also:

$$\bar{\Omega}^i{}_j = K\, \bar{\theta}^i \wedge \bar{\theta}^j = \frac{K}{a^2}\, \theta^i \wedge \theta^j$$

(It was a consequence of spatial isotropy & homogeneity)

(Recall from equations (box, 7/20

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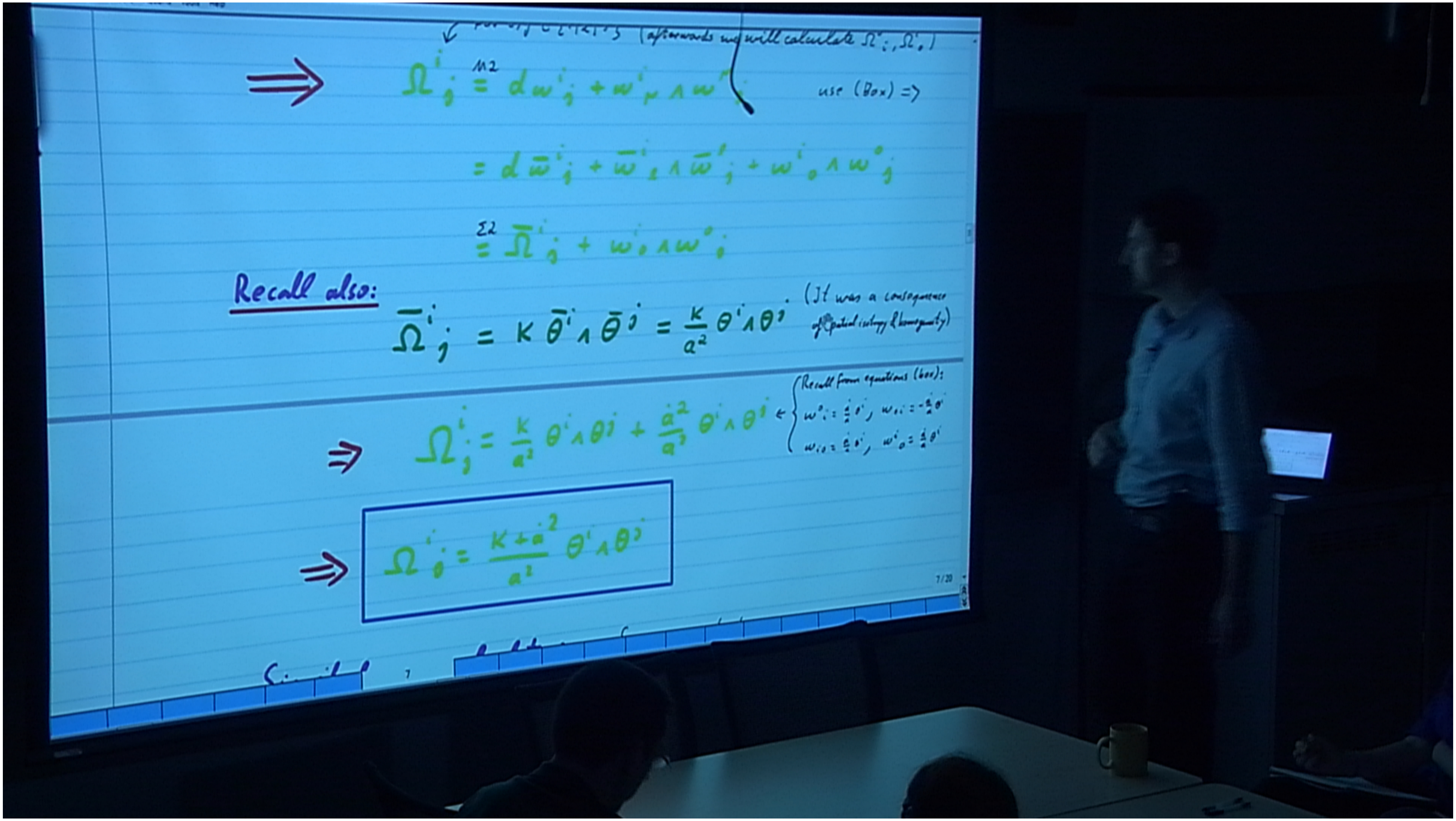
(afterwards we will calculate $\Omega^i{}_0, \Omega^0{}_i$)
$\Rightarrow \Omega^i{}_j = d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j$
use (Box) $\Rightarrow$
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(It was a consequence of spatial isotropy & homogeneity)
$\Rightarrow \Omega^i{}_j = \frac{K}{a^2} \theta^i \wedge \theta^j + \frac{\dot{a}^2}{a^2} \theta^i \wedge \theta^j$
Recall from equations (box):
$\omega^0{}_i = \frac{\dot{a}}{a}\theta^i$, $\omega_{0i} = -\frac{\dot{a}}{a}\theta^i$
$\omega_{i0} = \frac{\dot{a}}{a}\theta^i$, $\omega^i{}_0 = \frac{\dot{a}}{a}\theta^i$
$\Rightarrow \boxed{\Omega^i{}_j = \frac{K + \dot{a}^2}{a^2} \theta^i \wedge \theta^j}$
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2.) $d\theta^i = -\omega^i{}_\nu \wedge \theta^\nu = -\omega^i{}_0 \wedge \theta^0 - \omega^i{}_j \wedge \theta^j$ (B)

Compare eqns A,B $\Rightarrow$ $\boxed{\omega^i{}_0 = \frac{\dot a}{a}\theta^i \text{ and } \omega^i{}_j = \bar\omega^i{}_j}$ (Box)

(Intuition: expansion is nontrivial affine connection between space and time)

What is $\omega^0{}_0$? Recall: $dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$

But $dg_{\mu\nu} = 0$ for ON frames. Thus $\omega_{\mu\nu} = -\omega_{\nu\mu}$ here.

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for $i,j \in \{1,2,3\}$ (afterwards we will calculate $\Omega^0{}_{\cdot}$, $\Omega^i{}_{\cdot}$)

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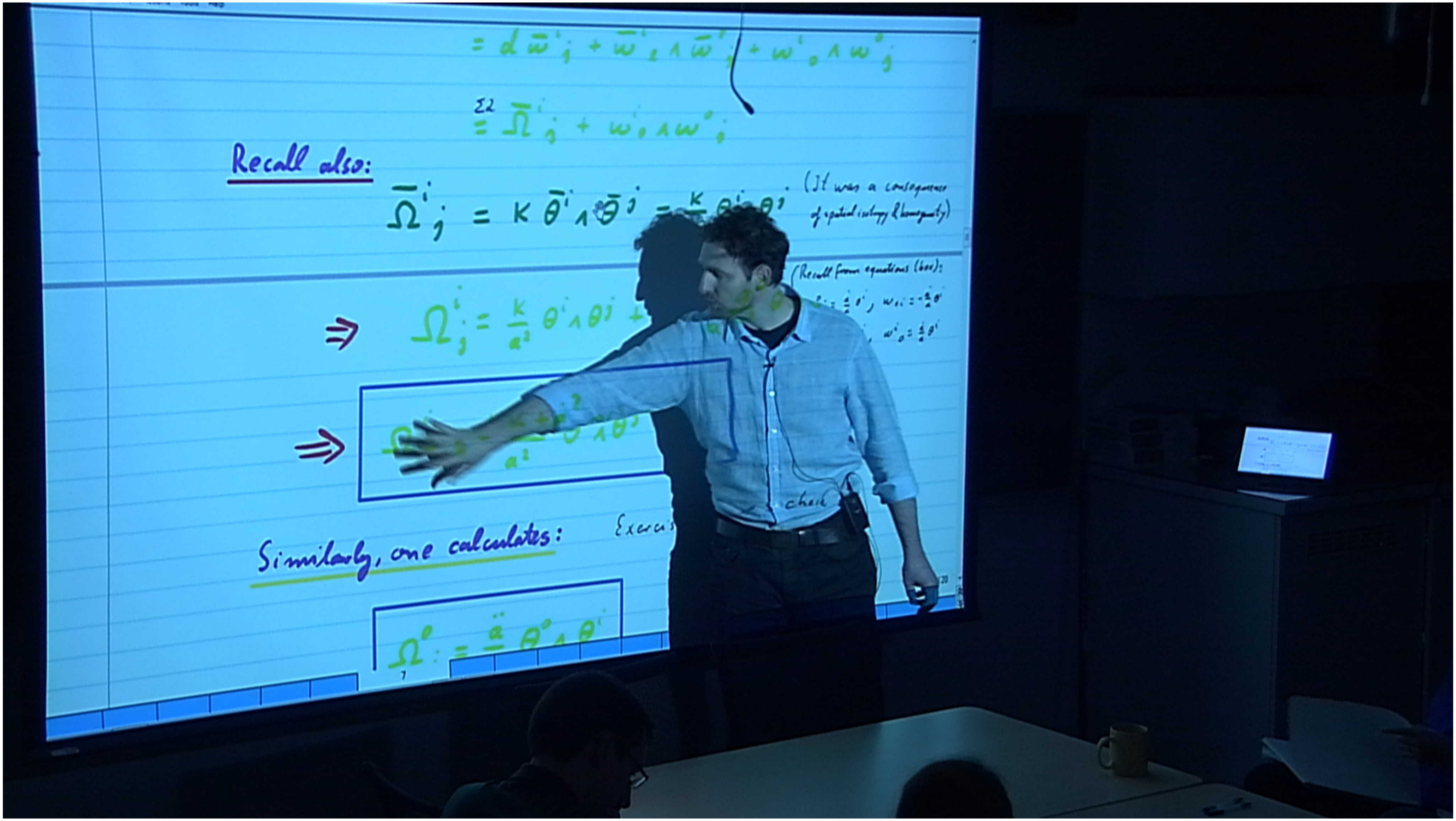
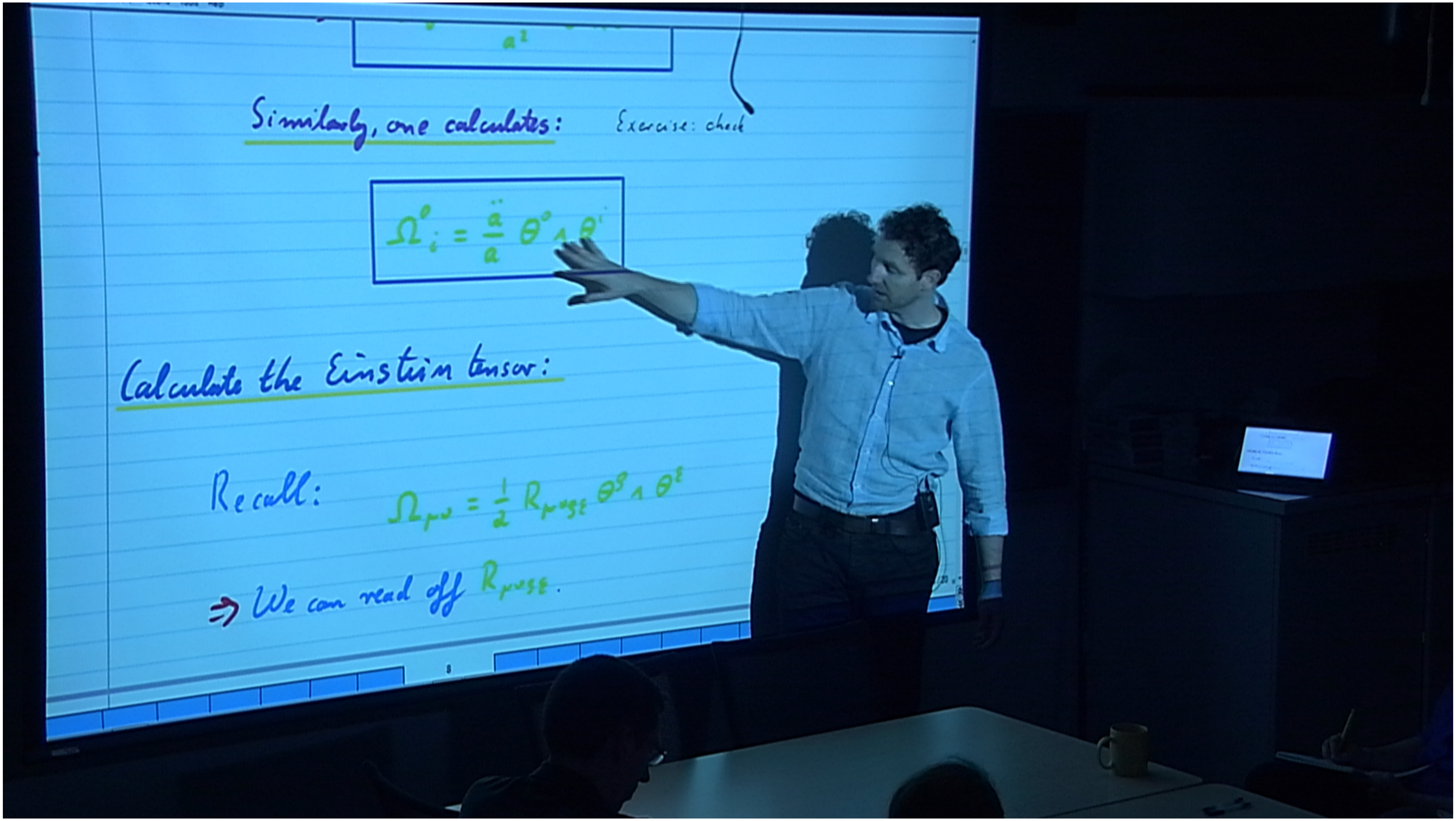
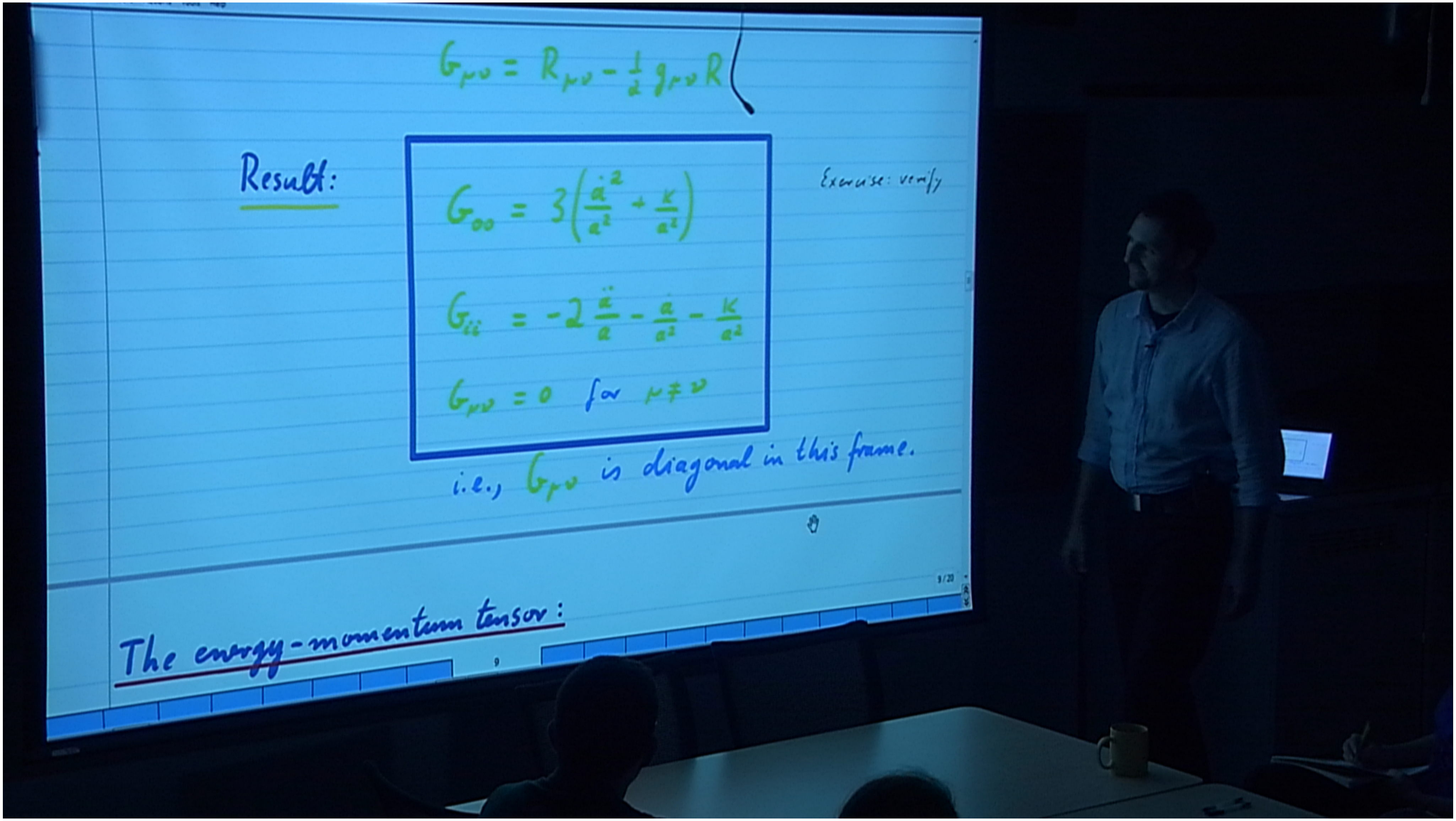


Recall also:
(It was a consequence of spatial isotropy & homogeneity)
(Recall from equations (box):
Similarly, one calculates:



Similarly, one calculates:
Exercise: check
$\Omega^0{}_i = \frac{\ddot{a}}{a}\, \theta^0 \wedge \theta^i$
Calculate the Einstein tensor:
Recall:
$\Omega_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma$
$\Rightarrow$ We can read off $R_{\mu\nu\rho\sigma}$.


$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$
Result:
$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$
Exercise: verify
$G_{ii} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2} - \frac{k}{a^2}$
$G_{\mu\nu} = 0$ for $\mu \neq \nu$
i.e., $G_{\mu\nu}$ is diagonal in this frame.
The energy-momentum tensor:
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$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Result:

$$\boxed{\begin{aligned} G_{00} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \\ G_{ii} &= -2\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2} - \frac{k}{a^2} \\ G_{\mu\nu} &= 0 \quad \text{for } \mu \neq \nu \end{aligned}}$$

Exercise: verify

i.e., $G_{\mu\nu}$ is diagonal in this frame.

The energy-momentum tensor:

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- From $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

we obtain that also $T_{\mu\nu}$ must also be diagonal.

- We identify the diagonal entries as usual as matter energy density ρ, matter pressure p and cosmological constant Λ:

(Why this factor here? Because Λ was traditionally put on the LHS, with the curvature)

Any diagonal $T_{\mu\nu}$ can be expanded uniquely this way.

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix} - \frac{1}{8\pi G}\,\Lambda \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$\Rightarrow$ The only nontrivial dynamics of matter is here its equation of state: $\boxed{\rho = \rho(p) \quad \text{or} \quad p = p(\rho)}$!

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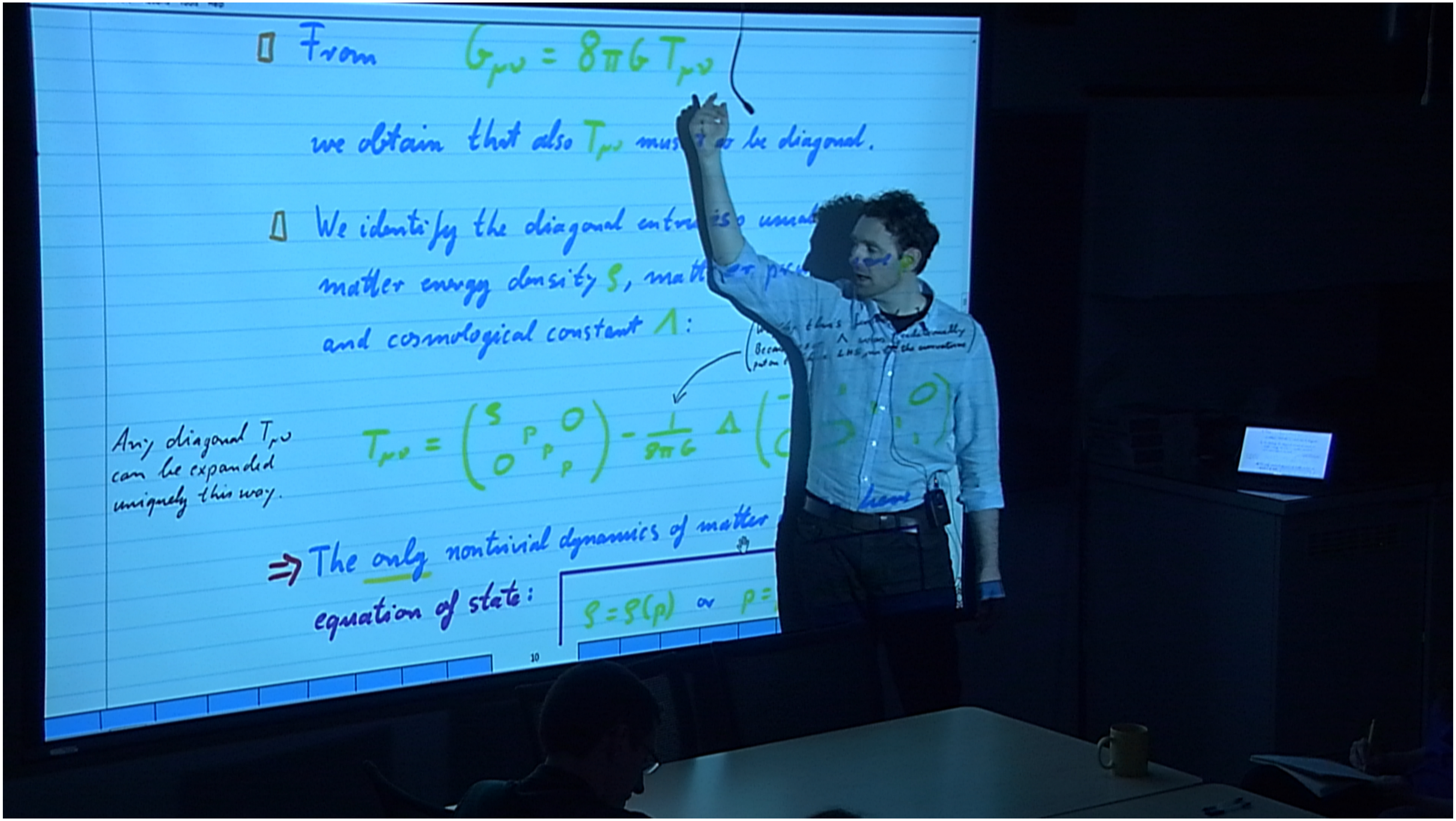
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The only nontrivial dynamics of matter
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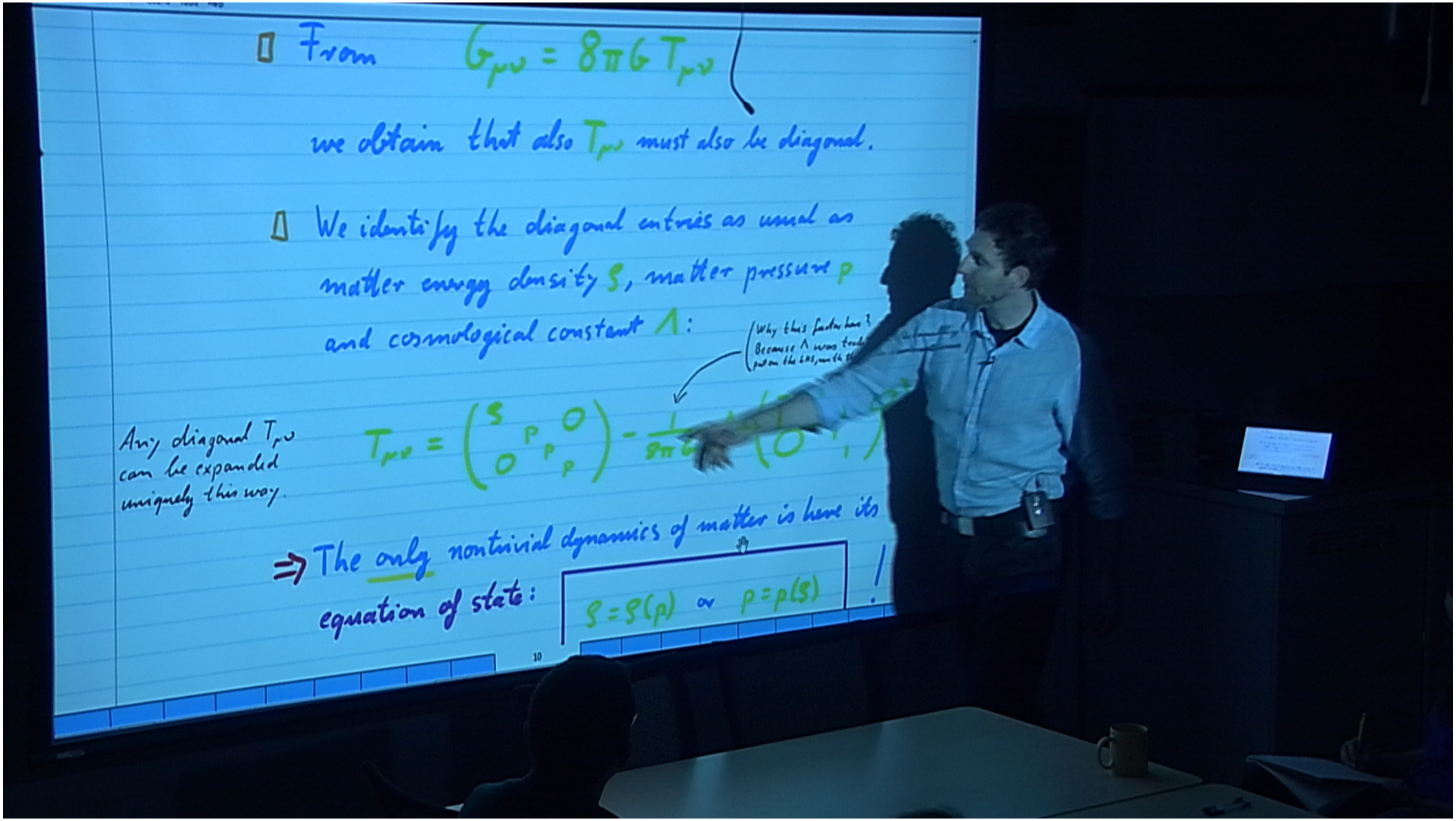
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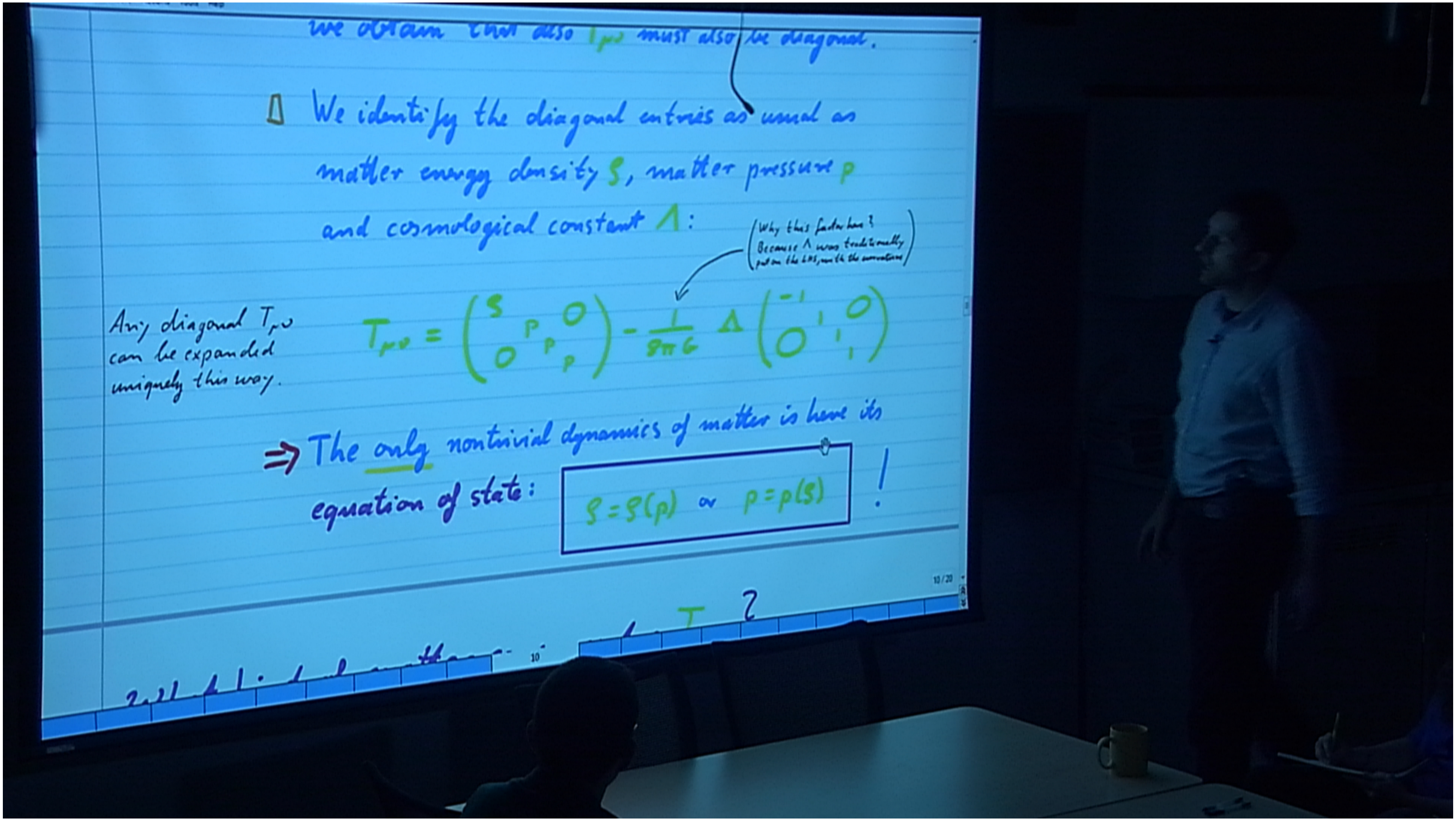
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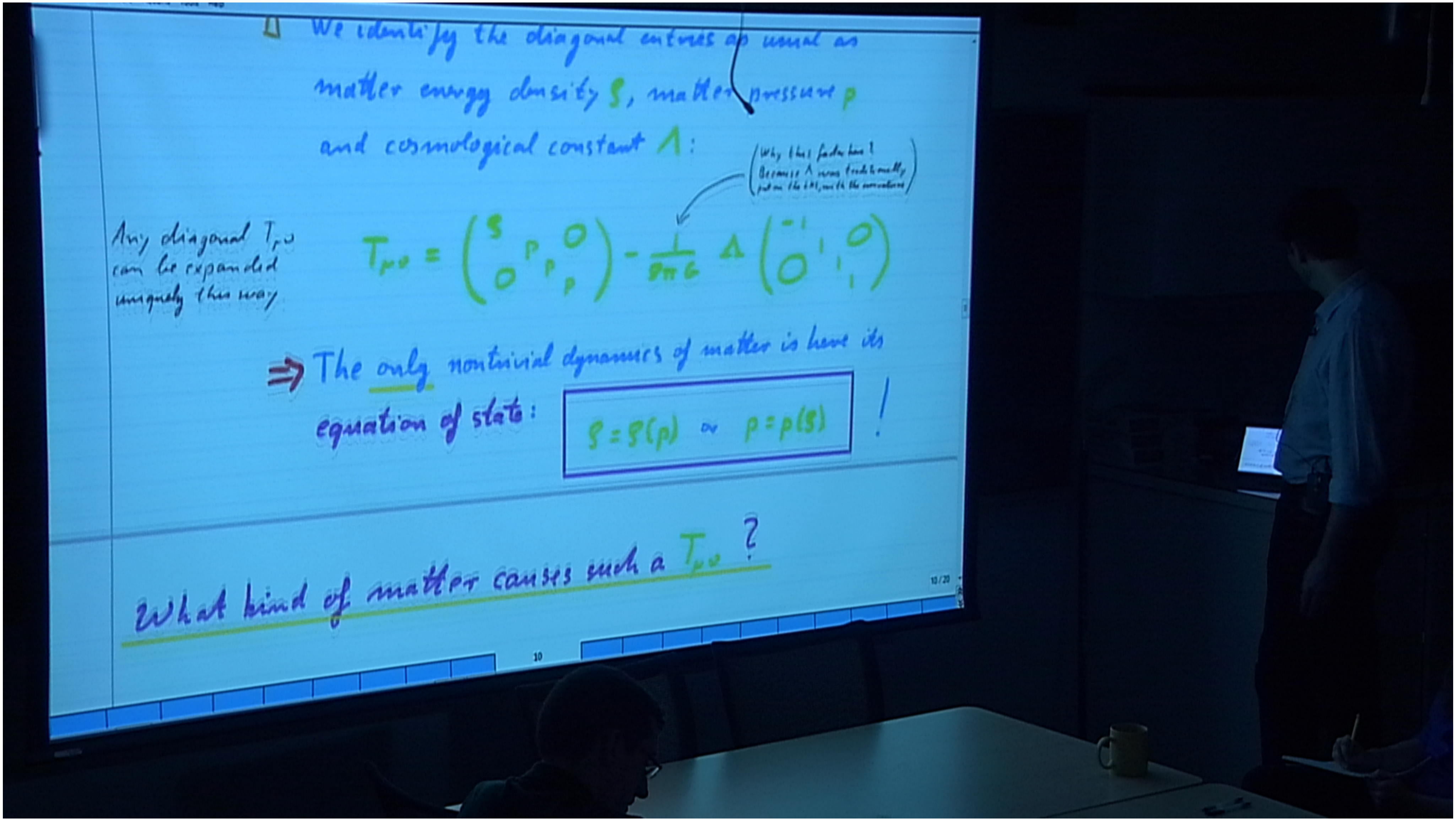
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What kind of matter causes such a $T_{\mu\nu}$?
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The energy-momentum tensor:

- From $$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$ we obtain that also $T_{\mu\nu}$ must also be diagonal.

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$\Rightarrow$ The only nontrivial dynamics of matter is here its

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What kind of matter causes such a $T_{\mu\nu}$?

Proposition:

The $T_{\mu\nu}$ of any F.L. spacetime is always of the form of that of a perfect fluid.

* So while the matter doesn't have to be fluid - it could also be e.g. a suitable quantum field, the

* But the high symmetry of a F.L. spacetime requires that the matter's $T_{\mu\nu}$ matches that of a perfect fluid.

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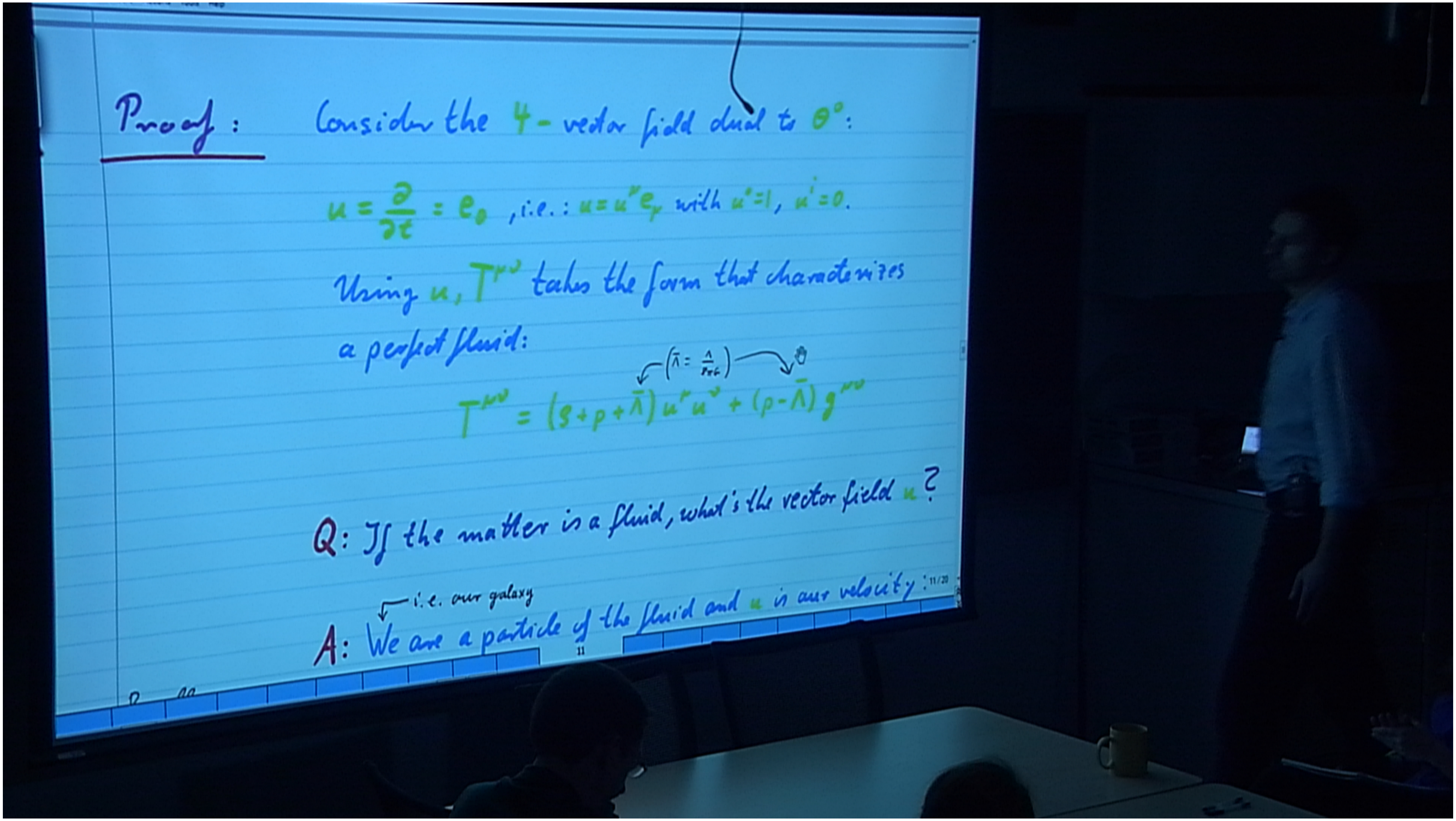
Proof:
Consider the 4-vector field dual to θ^0:
$u = \frac{\partial}{\partial t} = e_0$, i.e.: $u = u^\nu e_\nu$ with $u^0 = 1$, $u^i = 0$.
Using u, $T^{\mu\nu}$ takes the form that characterizes a perfect fluid:
$(\bar{\Lambda} = \frac{\Lambda}{8\pi G})$
$T^{\mu\nu} = (\rho + p + \bar{\Lambda}) u^\mu u^\nu + (p - \bar{\Lambda}) g^{\mu\nu}$
Q: If the matter is a fluid, what's the vector field u?
i.e. our galaxy
A: We are a particle of the fluid and u is our velocity:
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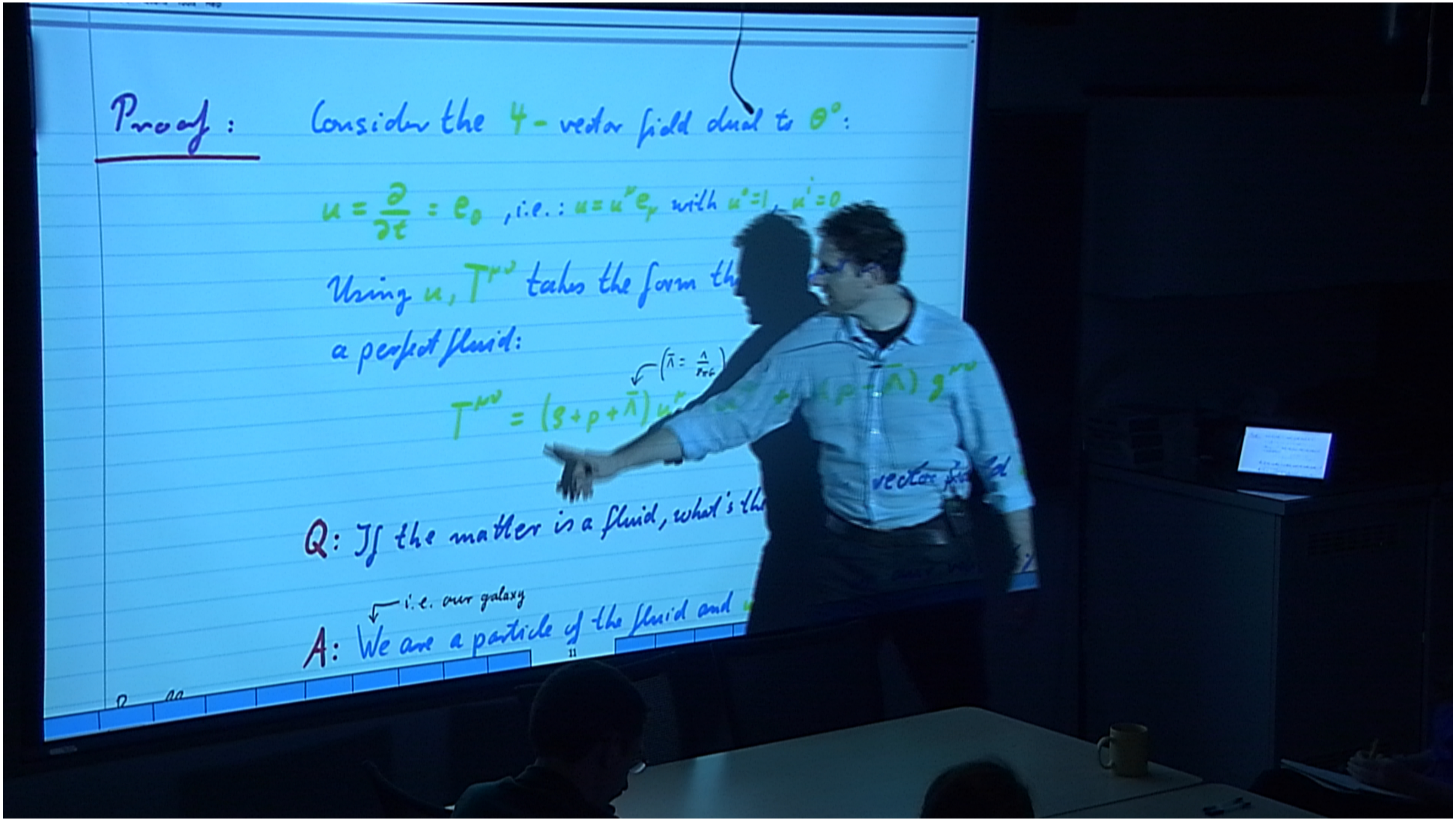
Proof:
Consider the 4-vector field dual to θ°:
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Using u, T^{μν} takes the form th
a perfect fluid:
T^{μν} = (ρ + p + Λ̄) u
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Proof: Consider the 4-vector field dual to θ^0:

$u = \frac{\partial}{\partial t} = e_0$, i.e.: $u = u^\mu e_\mu$ with $u^0 = 1$, $u^i = 0$.

Using u, $T^{\mu\nu}$ takes the form that characterizes a perfect fluid:

$\left(\bar{\Lambda} = \frac{\Lambda}{8\pi G}\right)$

$$T^{\mu\nu} = (\rho + p + \bar{\Lambda})\, u^\mu u^\nu + (p - \bar{\Lambda})\, g^{\mu\nu}$$

Q: If the matter is a fluid, what's the vector field u?

i.e. our galaxy

A: We are a particle of the fluid and u is our velocity:

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What kind of matter causes such a $T_{\mu\nu}$?

Proposition:

The $T_{\mu\nu}$ of any F.L. spacetime is always of the form of that of a perfect fluid.

* So while the matter doesn't have to be fluid - it could also be e.g. a suitable quantum field, the

* But the high symmetry of a F.L. spacetime requires that

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$$T^{\mu\nu} = (\rho + p + \bar{\Lambda}) u^\mu u^\nu + (p - \bar{\Lambda}) g^{\mu\nu}$$

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A: We are a particle of the fluid and u is our velocity:

Why? u is tangent to timelike geodesics (that stand still in space because $u \perp e_i \ \forall i = 1,2,3$)

Recall:

$\omega^0{}_i = \frac{\dot{a}}{a} \theta^i$

$\omega_{0i} = -\frac{\dot{a}}{a} \theta^i$

$\omega_{i0} = \frac{\dot{a}}{a} \theta^i$

$\omega^i{}_0 = \frac{\dot{a}}{a} \theta^i$

$\omega^0{}_0 = 0$

dual basis

$$\nabla_u u = \nabla_{e_0} e_0 = \omega^\mu{}_0 (e_0) e_\mu = \frac{\dot{a}}{a} \underbrace{\theta^i(e_0)}_{=0} e_i = 0$$

The Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

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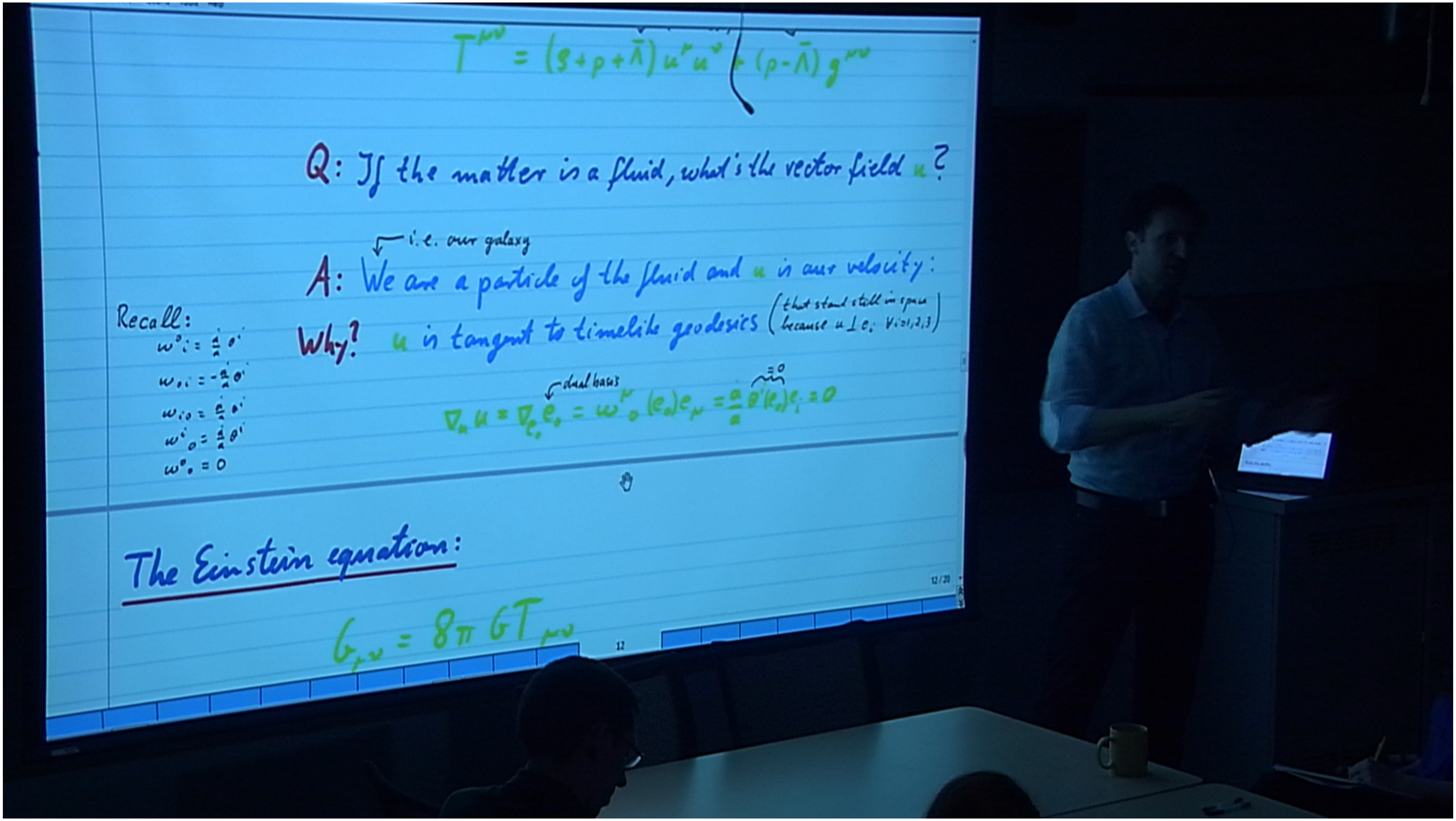
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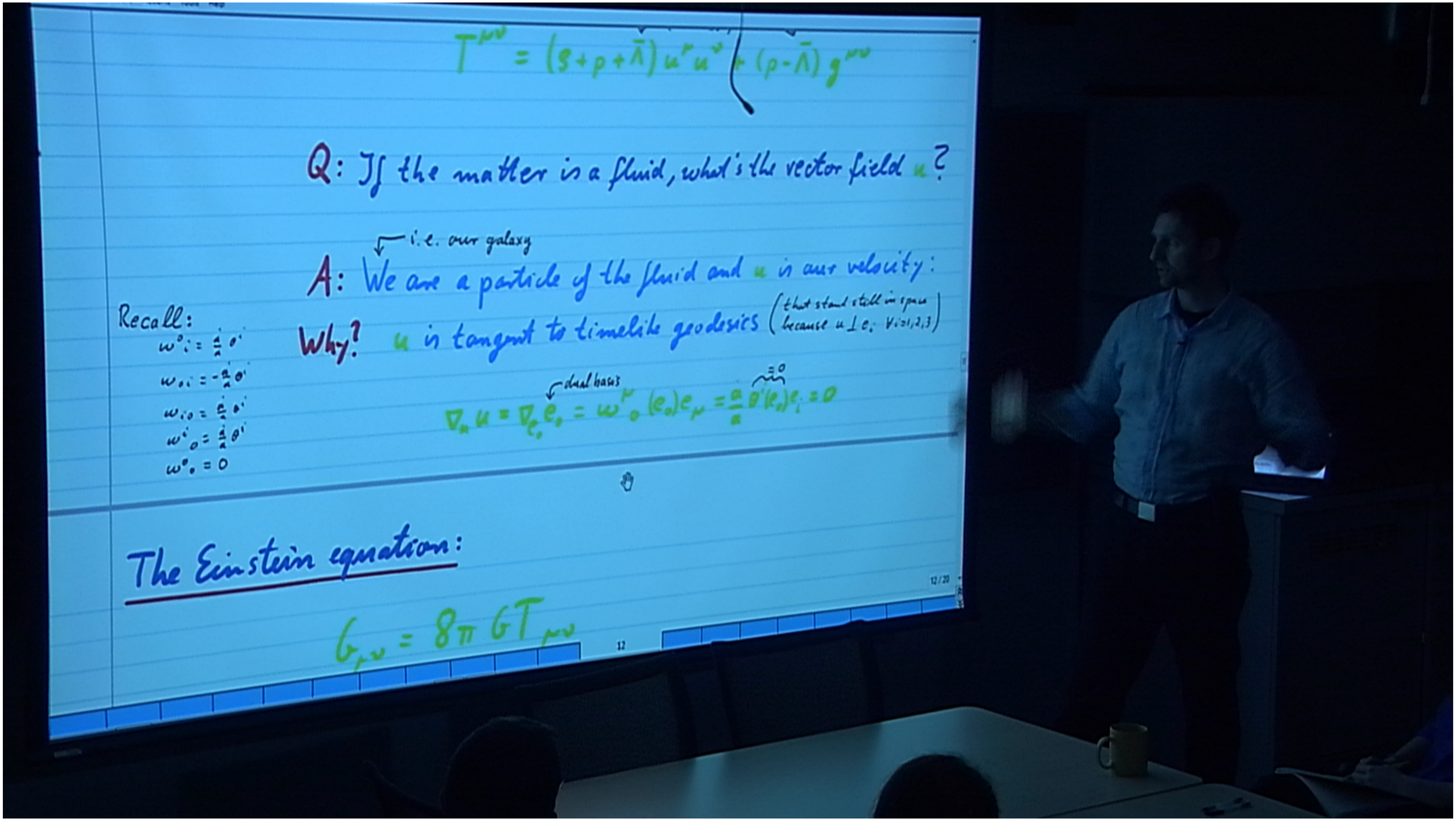
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The Einstein equation:

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now consists of merely 2 equations: Exercise: verify

$G_{00} = 8\pi G\, T_{00} \Rightarrow$
$$\boxed{3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G \rho + \Lambda} \quad \text{"Friedmann equation"} \quad (A)$$

$G_{ii} = 8\pi G\, T_{ii} \Rightarrow$
$$\boxed{-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi G p - \Lambda} \quad (B)$$

- Notice that Λ contributes
  - positively to the energy but

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The Einstein equation:

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"Friedmann equation" (A)

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(B)

Notice that Λ contributes

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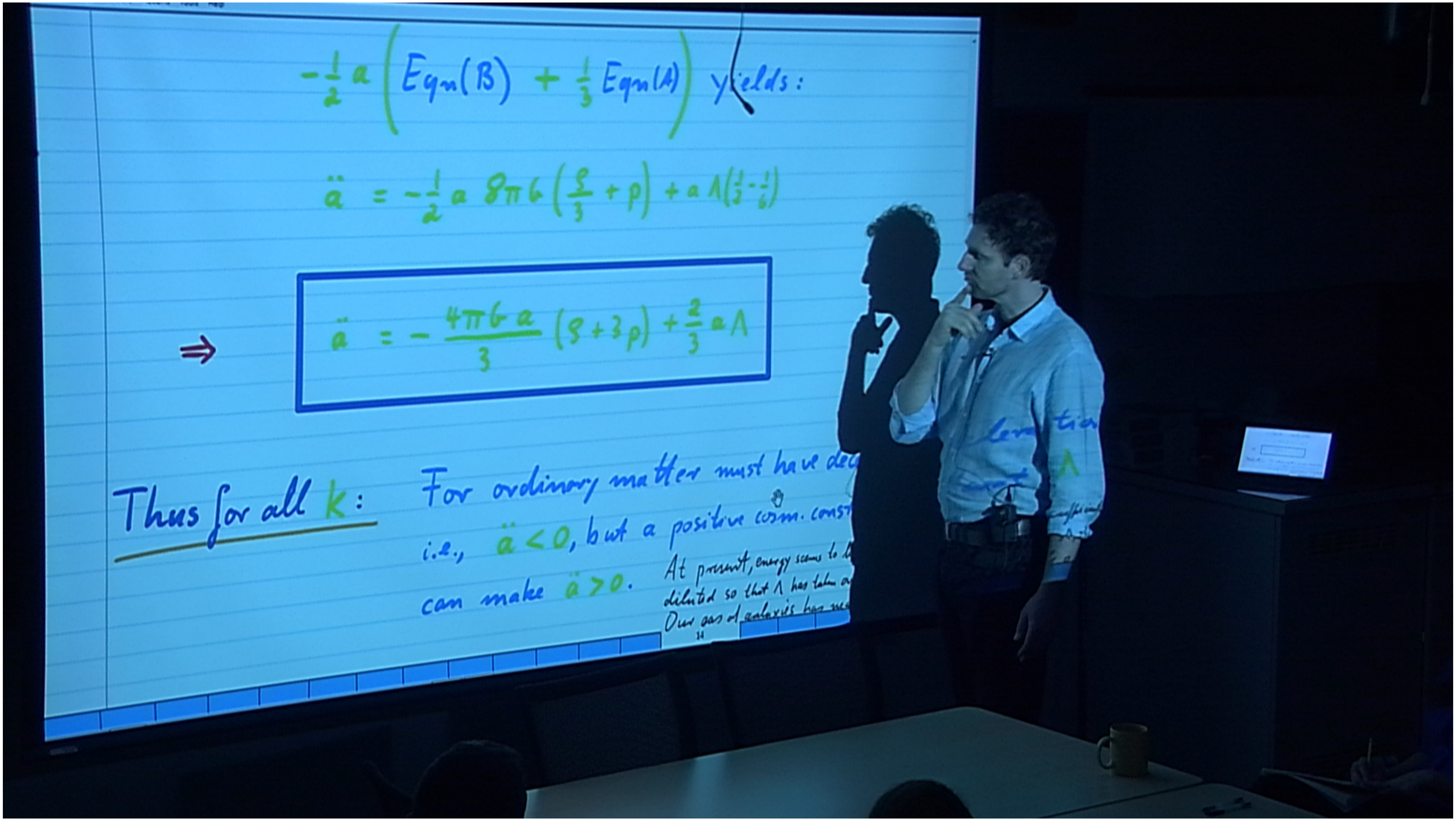
Observation: K/a^2 occurs in (A) and (B), i.e., we can eliminate it:

$$-\frac{1}{2}a\left(\text{Eqn}(B) + \frac{1}{3}\text{Eqn}(A)\right) \text{ yields:}$$

$$\ddot{a} = -\frac{1}{2}a\, 8\pi G\left(\frac{\rho}{3}+p\right) + a\Lambda\left(\frac{1}{2}-\frac{1}{6}\right)$$

$$\Rightarrow \quad \boxed{\ddot{a} = -\frac{4\pi G a}{3}(\rho+3p) + \frac{2}{3}a\Lambda}$$

Thus for all k: For ordinary matter must have deceleration,


-1/2 a (Eqn(B) + 1/3 Eqn(A)) yields:
ä = -1/2 a 8πG (ρ/3 + p) + a Λ (1/3 - 1/6)
⇒ ä = - (4πG a)/3 (ρ + 3p) + 2/3 a Λ
Thus for all k:
For ordinary matter must have
i.e., ä < 0, but a positive cosm.
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At present, energy seems to
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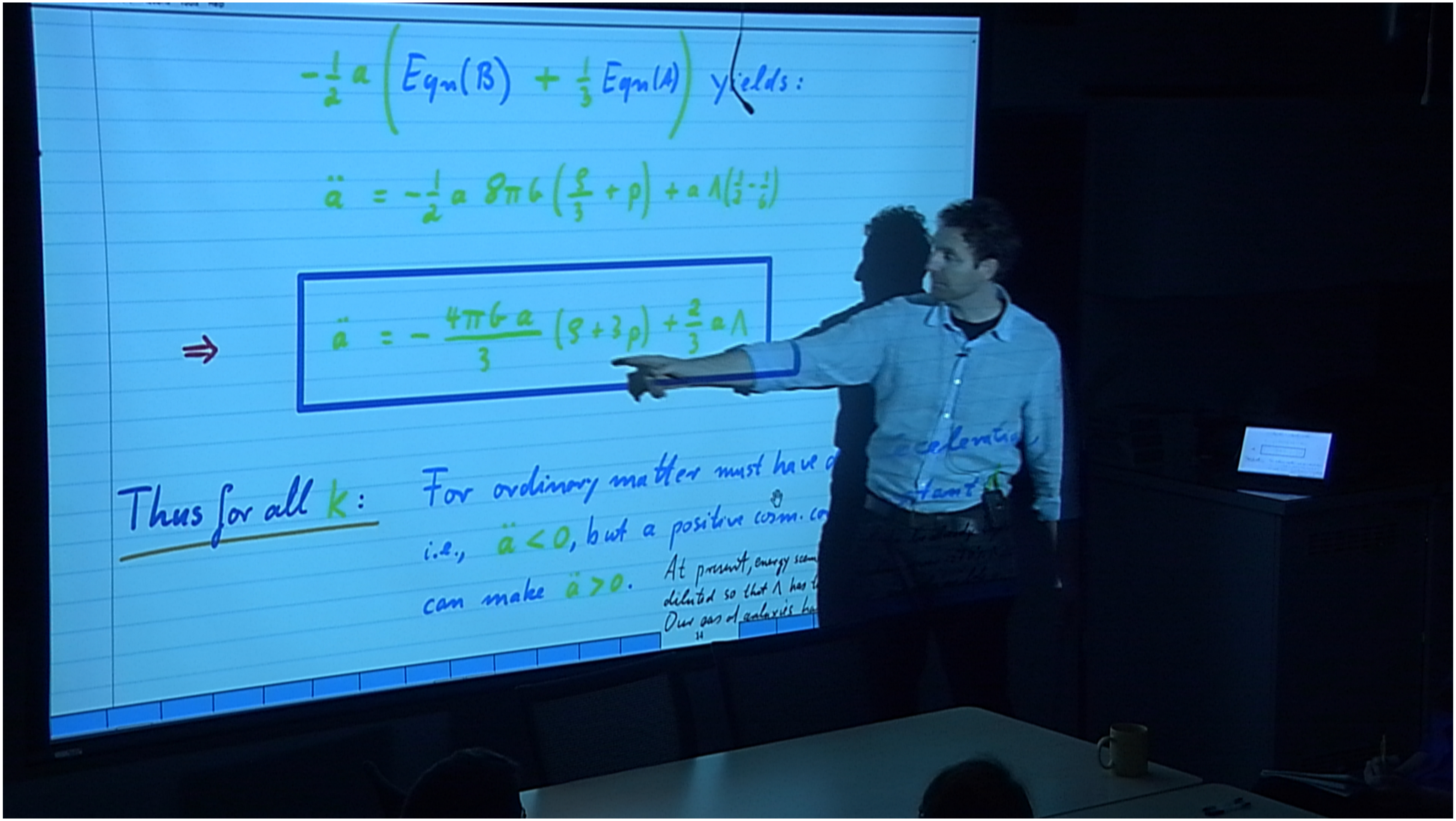
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Thus for all k: For ordinary matter must have deceleration, i.e., $\ddot{a} < 0$, but a positive cosm. constant Λ can make $\ddot{a} > 0$.

At present, energy seems to be already sufficiently diluted so that Λ has taken over: $\approx 70\%$ Λ, $\approx 30\%$ ρ

Our gas of galaxies has negligible p.

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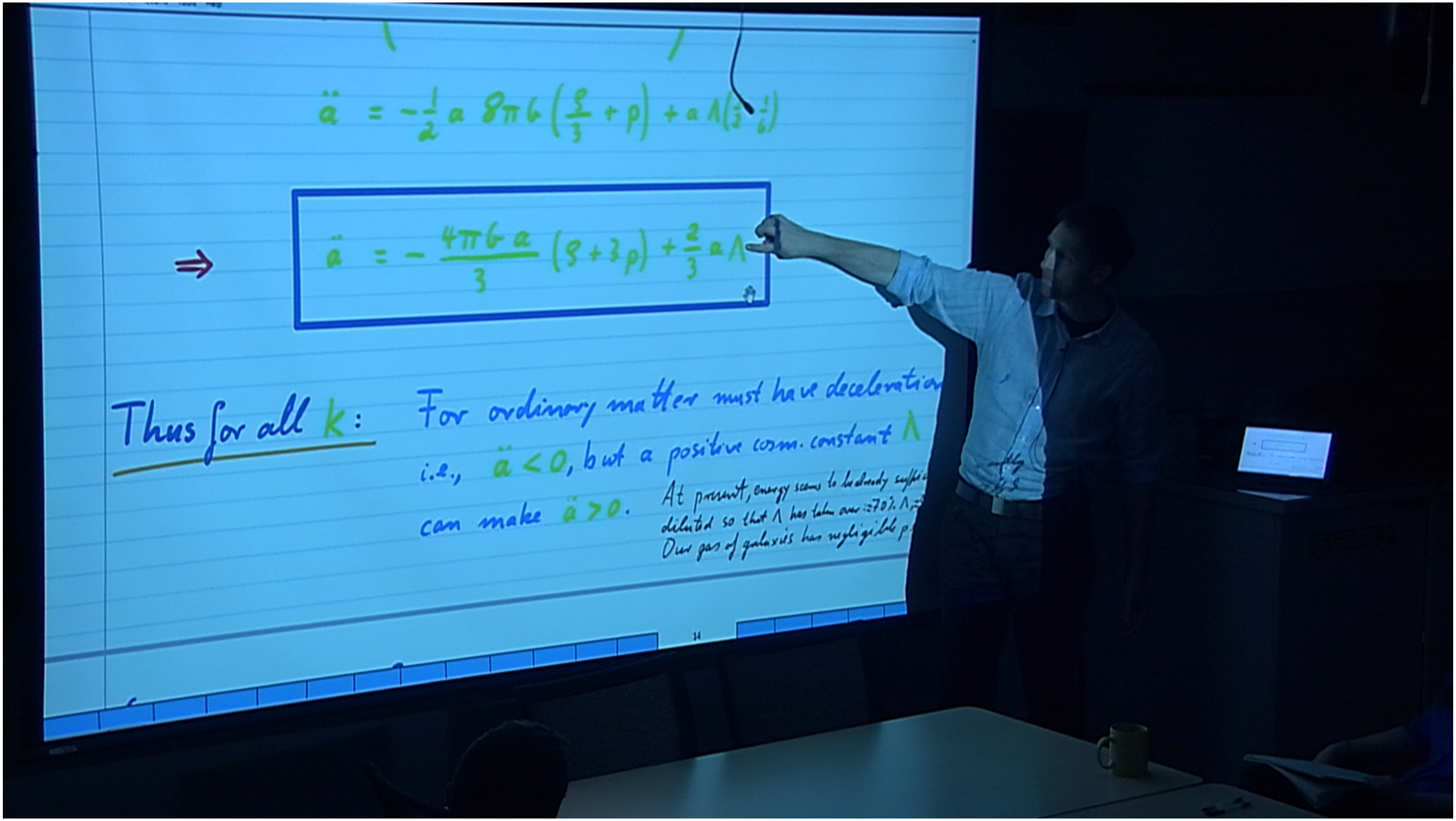
14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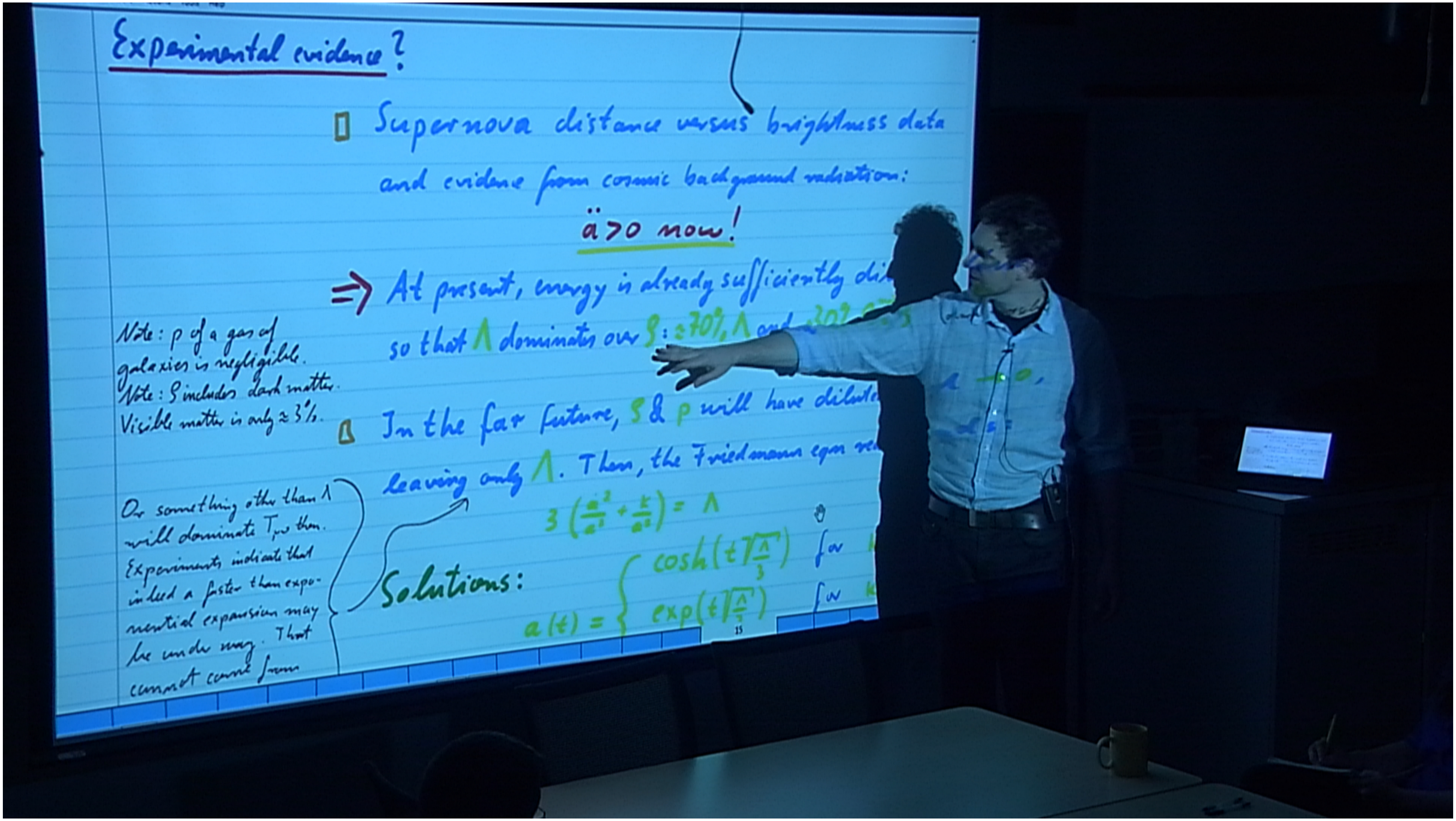
Experimental evidence?
Supernova distance versus brightness data
and evidence from cosmic background radiation:
ä>0 now!
At present, energy is already sufficiently
so that Λ dominates over
Note: p of a gas of galaxies is negligible.
Note: ρ includes dark matter.
Visible matter is only ≈ 3%.
In the far future, ρ & p will have
leaving only Λ. Then, the Friedmann eqn
Or something other than Λ will dominate Tμν then. Experiments indicate that indeed a faster than exponential expansion may be under way. That cannot come from
Solutions:
cosh
exp

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Experimental evidence?

- Supernova distance versus brightness data and evidence from cosmic background radiation:

$\ddot{a} > 0$ now!

$\Rightarrow$ At present, energy is already sufficiently diluted so that Λ dominates over ρ: $\approx 70\%$ Λ and $\approx 30\%$ ρ^{matter} (dark + visible)

Note: p of a gas of galaxies is negligible.
Note: ρ includes dark matter.
Visible matter is only $\approx 3\%$.

- In the far future, ρ & p will have diluted $\rightarrow 0$, leaving only Λ. Then, the Friedmann eqn reads:

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = \Lambda$$

Or something other than Λ will dominate then. Experiments indicate that indeed a faster than exponential expansion may be under way. That cannot come from

Solutions:

$$a(t) = \begin{cases} \cosh\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k=1 \\ \exp\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k=0 \end{cases}$$

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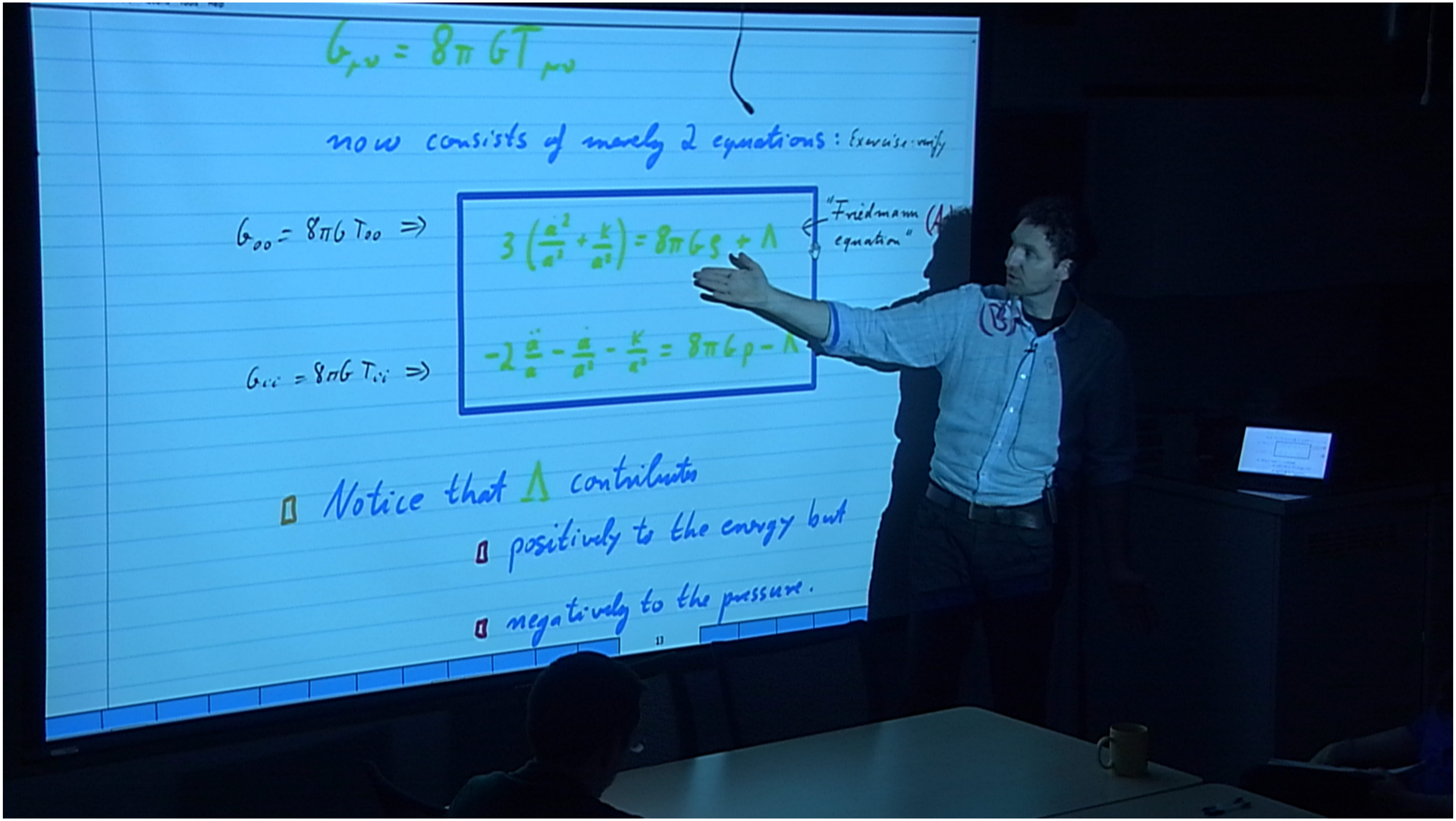
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Notice that Λ contributes
positively to the energy but
negatively to the pressure.
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$\ddot{a} > 0$ now!

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In the far future, ρ & p will have diluted $\to 0$, leaving only Λ. Then, the Friedmann eqn reads:

Or something other than Λ will dominate $T_{\mu\nu}$ then. Experiments indicate that indeed a faster than exponential expansion may be under way. That cannot come from just Λ dominance alone. See essay topic!

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = \Lambda$$

Solutions:

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$\Rightarrow$ Exponential expansion is predicted!

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General solution strategy with cosm. constant and matter:

- We have 3 unknown functions of time

$$a(t),\ \rho(t),\ p(t)$$

and we have 3 equations that they obey:

Eqs. A, B and an equation of state $p = p(\rho)$ that depends on the "matter":

$p_\Lambda(\rho_\Lambda) = -\rho_\Lambda$ for pure vacuum energy (e.g., in very early universe)

$p(\rho) = \frac{1}{3}\rho$ for pure radiation (e.g., in the early universe)

$p(\rho) = 0$ for pure dust (e.g., middle aged universe before Λ took over)

- Observation:

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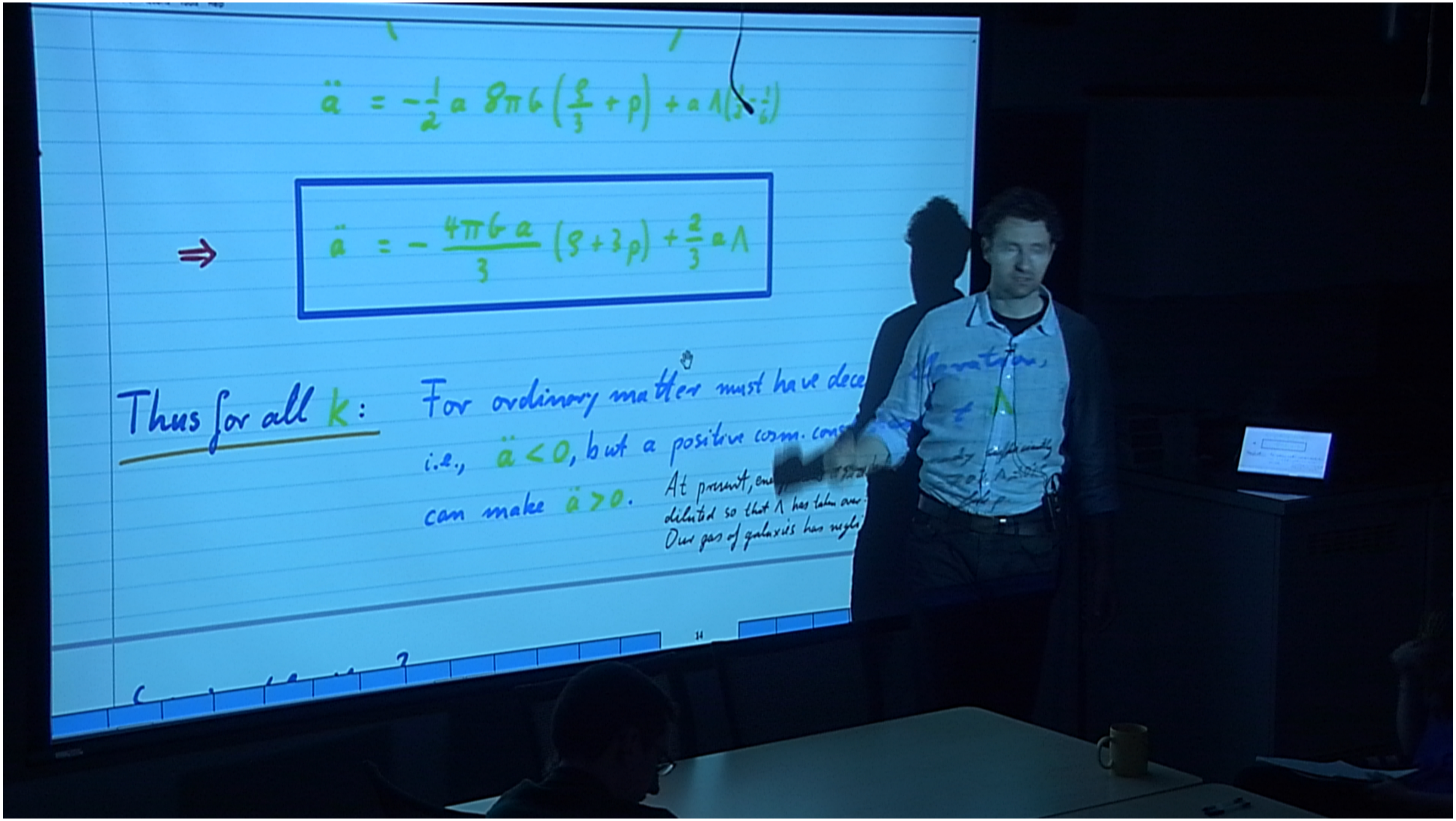

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- Observation:

(Eqn. A)

The Friedmann eqn. only contains a, ρ but not p!

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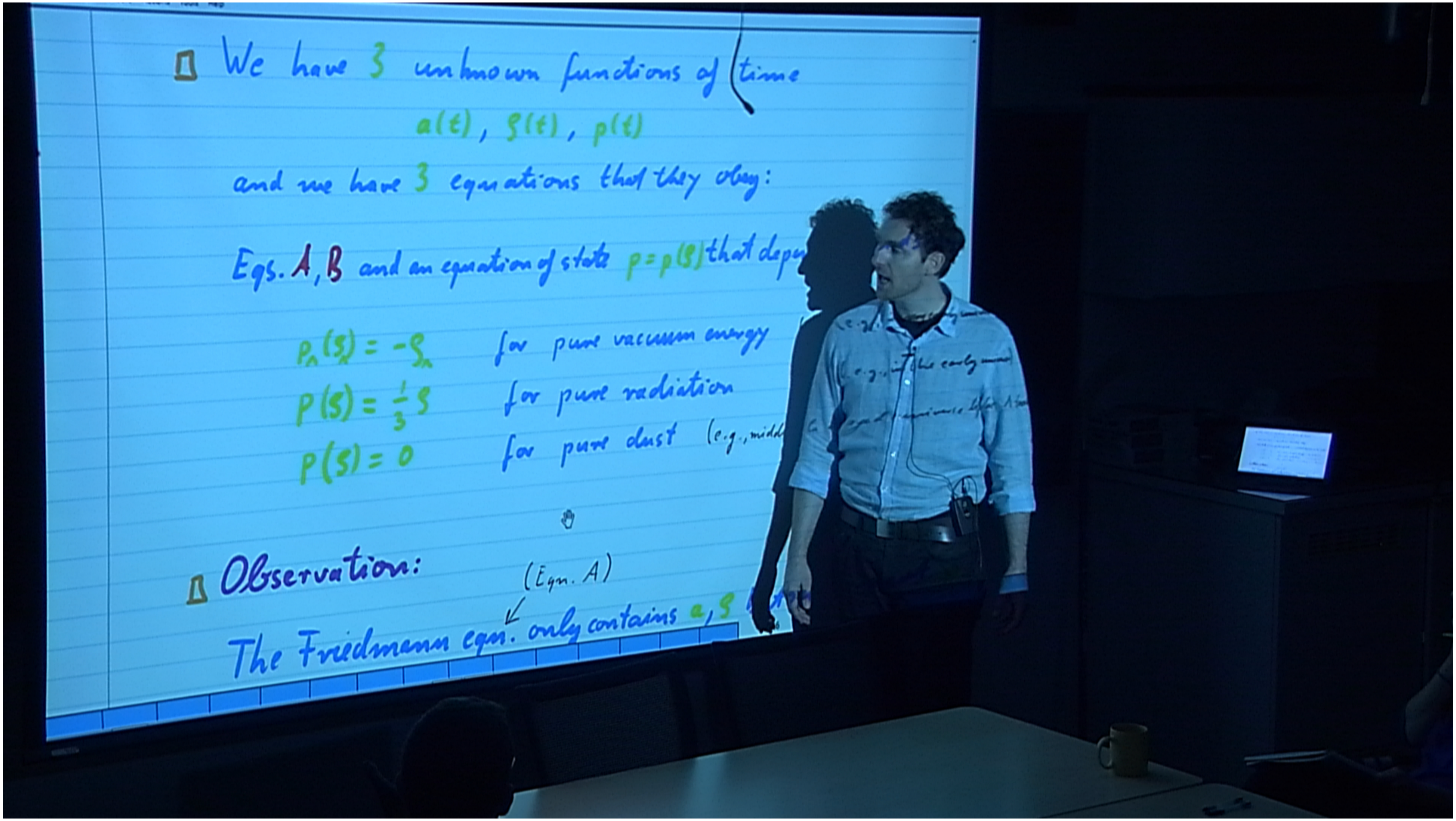

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Observation:
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Proposition: The Einstein eqns A, B imply:

$$\boxed{\frac{d}{da}\left(\rho a^3\right) = -3 p a^2} \qquad (P)$$

Indeed, when the parameter w in $p = w\rho$ is known, (P) yields $\rho(a)$:

- For dust, $p = 0$ $\Rightarrow \rho \sim a^{-3}$
- For radiation, $p = \rho/3$ $\Rightarrow \rho \sim a^{-4}$

$\longrightarrow$ ρ of radiation decays quicker than ρ of dust because radiation is not only diluted, its wavelengths are also stretched, which reduces the energy too.

- For pure Λ: $p = -\rho$ $\Rightarrow \rho = \text{const}$ $\longrightarrow$ ρ of vacuum ener...

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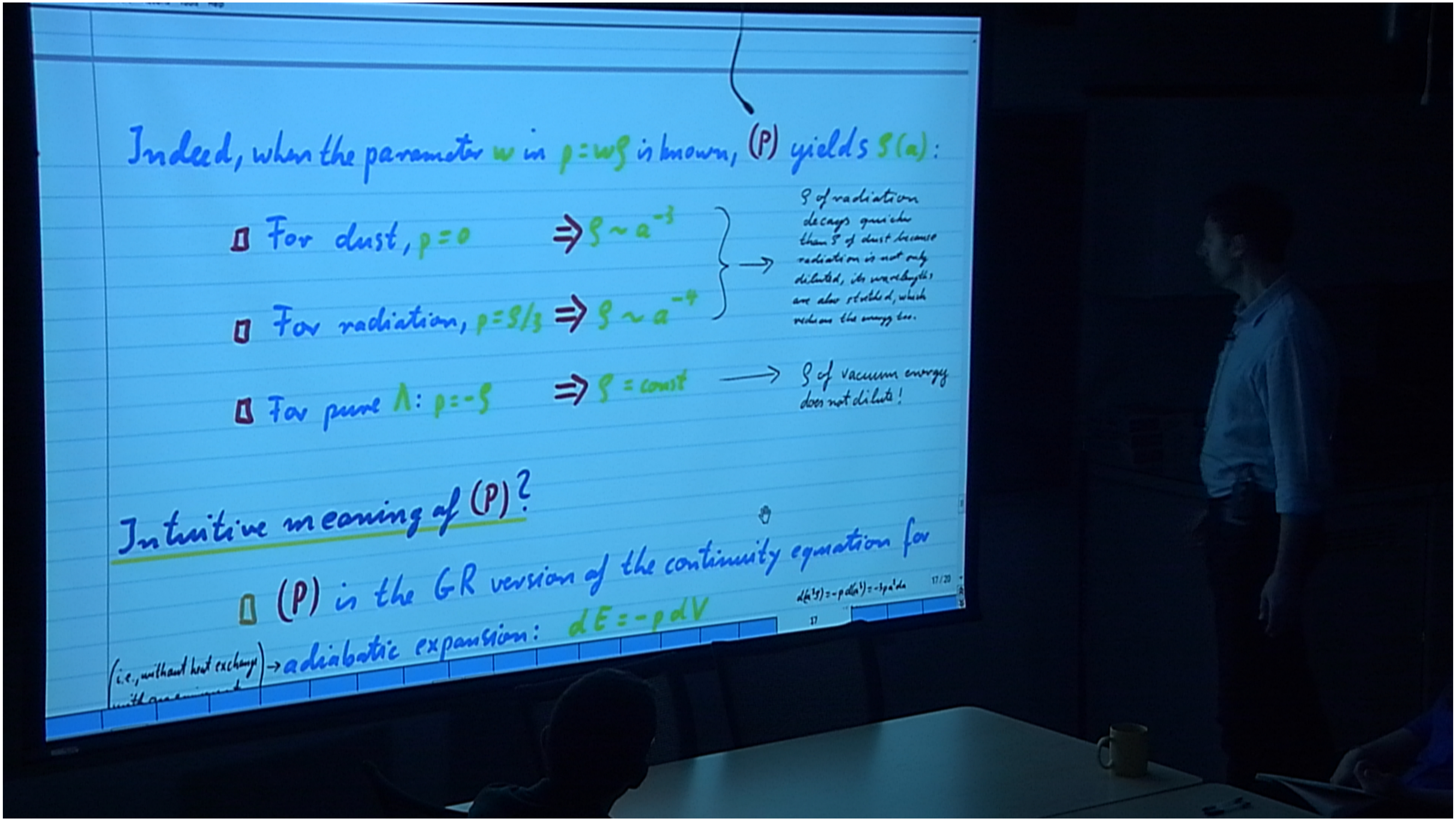

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Intuitive meaning of (P)?
(P) is the GR version of the continuity equation for
adiabatic expansion: $dE = -p\,dV$
(i.e., without heat exchange)
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Intuitive meaning of (P)?

- (P) is the GR version of the continuity equation for (i.e., without heat exchange with an environment) → adiabatic expansion: $dE = -p\,dV$

$d(a^3\rho) = -p\,d(a^3) = -3pa^2da$

so that: $\frac{d(a^3\rho)}{da} = -3pa^2$ ✓

- With $V := a^3$, $E := \rho V$ it yields:

$$d(a^3\rho) = -p\,d(a^3) = -3pa^2da$$

- Thus: $\frac{d}{da}(a^3\rho) = -3pa^2$ which is indeed (P).

Exact proof of proposition (P):

- We know that Eqs. A, B imply $T^{\mu\nu}{}_{;\nu} = 0$.

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Exact proof of proposition (P):

- We know that Eqs. A, B imply $T^{\mu\nu}{}_{;\nu} = 0$.
- Here: $T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$
- Thus:

$$0 = T^{\mu\nu}{}_{;\nu} = \overbrace{(\rho_{;\nu} + p_{;\nu}) u^\mu u^\nu + (\rho + p) u^\mu \underbrace{u^\nu{}_{;\nu}}_{\nabla\cdot u :=}}^{\text{Leibniz rule}} + p_{;\nu} g^{\mu\nu} \qquad (g^{\mu\nu}{}_{;\nu} = 0)$$

(using $\nabla_{fv} w = f \nabla_v w \Rightarrow u^\nu \rho_{;\nu} = \nabla_u \rho$) $\Rightarrow$

$$= (\nabla_u \rho + \nabla_u p) u^\mu + (\rho + p) u^\mu \nabla u + p^{;\mu} \qquad \Big| \cdot u_\mu$$

(using $u^\mu u_\mu = -1$) $\Rightarrow$

$$= -\nabla_u \rho - \cancel{\nabla_u p} - (\rho + p) \nabla\cdot u + \cancel{\underbrace{u_\mu p^{;\mu}}_{= \nabla_u p}}$$

$$\Rightarrow \quad 0 = \nabla_u \rho + (\rho + p)(\nabla\cdot u)$$

(X

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