Title: General Relativity for Cosmology - Lecture 21

Date: Nov 26, 2013 04:00 PM

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Abstract:

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GR for Cosmology, Achim Kempf, Fall 2013, Lecture 21

Note Title

Recall: A key prediction of GR is its own downfall in singularities. Or is it? Can one prove that GR has generic situations that must lead to a singularity, even in the absence of any symmetry?

The plan:

1. Define and study suitable notions of:

   - Causality
   - Horizons (next to define: "Cauchy horizons")

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The plan:

1. Define and study suitable notions of:

   - Causality
   - Horizons (next to define: "Cauchy horizons")
   - Singularities

2. Develop singularity theorems.

So travellers cannot go on cyclic paths

Recall: We assume that spacetime is stably causal.

i.e. paths without endpoint $p \in M$

Intuition: Therefore, inextendible paths either:

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So travellers cannot go on cyclic paths

Recall: We assume that spacetime is stably causal.

i.e. paths without endpoint $p \in \mathcal{M}$

Intuition: Therefore, inextendible paths either:

a.) go to ∞, or

b.) end in a singularity

$\rightsquigarrow$ Continue to study inextendible curves

$\rightsquigarrow$ Arrive at key concepts of Cauchy horizon and global hyperbolicity.

Recall:

"the causal future"

- We considered the set of points $J^+(S)$ that can somehow be reached from a set S. (i.e. the set of points that are affected by

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⇝ Continue to study inextendible curves

⇝ Arrive at key concepts of Cauchy horizon and global hyperbolicity.

Recall:

"the causal future"

- We considered the set of points $J^+(S)$ that can somehow be reached from a set S. (i.e. the set of points that are affected by S)
- Now consider set of points that can only be reached from S: (i.e. the set of events that depend on S and only S)

Definition:

i.e., a set of events among which no object could travel

Assume $S \subset \mathcal{M}$ is a closed achronal set.

Then, the "future domain of dependence of S"

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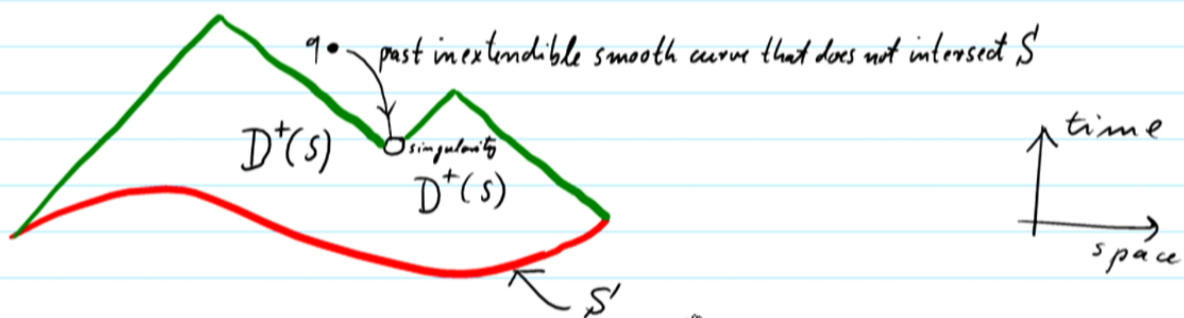
i.e., a set of events among which no object could travel

Definition:

Assume $S \subset M$ is a closed achronal set.
Then, the "future domain of dependence of S"
is defined as:

$$D^+(S') := \left\{ p \in M \;\middle|\; \begin{array}{l} \text{Every past inextendible causal} \\ \text{curve through } p \text{ intersects } S \end{array} \right\}$$

Example:



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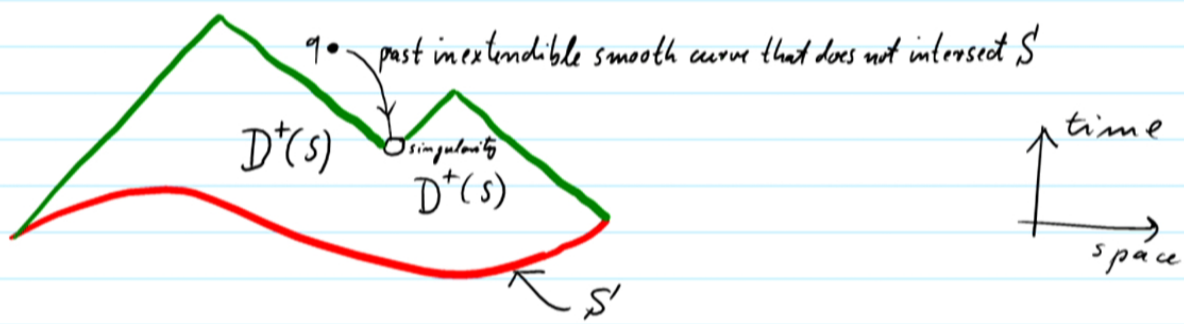

Why $q \notin D^+(S)$? Some of its past inextendible causal curves do not intersect S because they get stuck at the hole!

(q is affected by events in the "shadow" of the singularity)

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Definition:

Analogously, the "past domain of dependence of S'" is:

$$D^-(S') := \left\{ p \in M \;\middle|\; \begin{array}{l}\text{Every future inextendible causal} \\ \text{curve through } p \text{ intersects } S\end{array} \right\}$$

(the set of events p that affect only S')

Definition:

The "full domain of dependence of S'" is:

$$D(S') := D^+(S') \cup D^-(S')$$

Definition: (set of latest events that are affected only by S'? How far have initial conditions on S full predictive power?

The "future Cauchy horizon of S'"

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Definition: (set of latest events that are affected only by S'? How far have initial conditions on S' full predictive power?

The "future Cauchy horizon of S'", denoted $H^{+}(S)$ is:

$$H^{+}(S') := \overline{D^{+}(S)} - \underbrace{I^{-}}_{\text{chronological past}}(D^{+}(S))$$

(Note: $\Rightarrow H^{+}(S')$ is achronal. Why?)

Example: $H^{+}(S')$

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Definition: (set of latest events that are affected only by S? How far have initial conditions on S full predictive power?

The "future Cauchy horizon of S", denoted $H^+(S)$ is:

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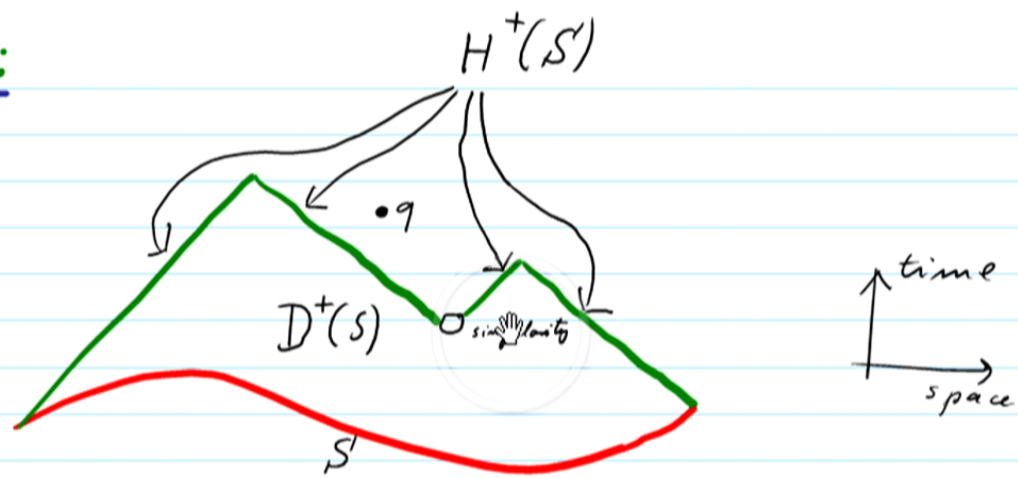

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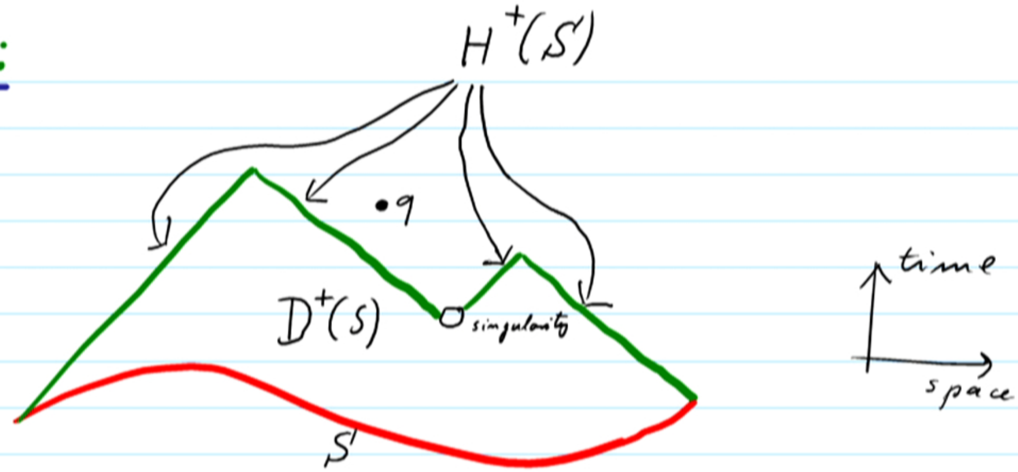

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The "past Cauchy horizon of S'", denoted H^-...

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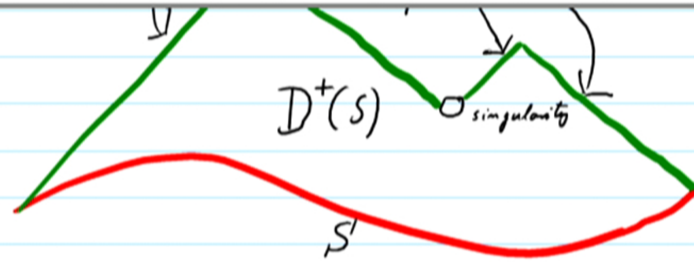

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Definition:

The "past Cauchy horizon of S'", denoted $H^-(S)$

is: $H^-(S') := \overline{D^-(S)} - I^+(D^-(S))$ (set of earliest events that affect only S)

Definition:

The "full Cauchy horizon of S'" is defined as:

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Definition:

The "full Cauchy horizon of S'" is defined as:

$$H(S') := H^+(S) \cup H^-(S)$$

Proposition:

$$H(S') = \dot{D}(S')$$

Definition:

A closed, achronal set S' is called a

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Definition:

A closed, achronal set S' is called a "**Cauchy surface**", if its full Cauchy horizon vanishes, i.e., if

a.) $H(S) = \phi$ ← empty set or equivalently if

b.) $\dot{D}(S) = \phi$ or equivalently if

c.) $D(S') = \mathcal{M}$

Hawking, Ellis, Geroch et al.

Note: This follows Wald. The definitions by others are equivalent.

but more tedious

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Remarks:

- Cauchy surfaces are important because if the

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Remarks:

- Cauchy surfaces are important because if the conditions on a Cauchy surface are known, then everything on $\mathcal{M}$ can be predicted and retrodicted.
  Note: E.g., anti de Sitter space has no Cauchy surfaces!

- Since a Cauchy surface is achronal, it can be viewed as an "instant in time".

- The term "surface" is motivated by a theorem:

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conditions on a Cauchy surface are known, then everything on M can be predicted and retrodicted.

Note: E.g., anti de Sitter space has no Cauchy surfaces!

- Since a Cauchy surface is achronal, it can be viewed as an "instant in time".

- The term "surface" is motivated by a <u>**theorem:**</u>

Every Cauchy surface, Σ, is a 3-dimensional C^0 submanifold of M.

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Definition:

If (M, g) possesses a Cauchy surface then it is called "globally hyperbolic".

Remark: We'll need this notion later for a cosmological singularity theorem.

Proposition:

If (M, g) is globally hyperbolic, then:

- There exists a "global time function f" so that every surface of constant f is a Cauchy surface.

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Proposition:

If (M,g) is globally hyperbolic, then:

- There exists a "global time function f" so that every surface of constant f is a Cauchy surface.
- (M,g) is stably (and therefore also strongly) causal.

Recall: Plan is to study inextendible geodesics in order to detect singularities.

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Recall: Plan is to study inextendible geodesics in order to detect singularities.

Now: How to identify these geodesics which are inextendible because they end at a singularity in the manifold?

First: Avoid trivial cases where manifold is ending but could be extended.

Definition:

We say that (M, g) is inextendible, if it is not

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Definition:

We say that (M, g) is inextendible, if it is not isometric to a proper subset of another spacetime (M', g').

→ We will always assume that (M, g) is inextendible.

Definition:

A geodesic which is inextendible but possesses a finite range of its affine parameter is called "incomplete".

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Definition:

- We say that (M, g) possesses a "singularity" if it possesses an incomplete geodesic.

  $\rightsquigarrow$ We distinguish singularities of null, spacelike and timelike type.

- When going along an incomplete geodesic towards a "singularity", 3 things can happen:

  I) A scalar constructed from $R^{\mu\nu}{}_{\sigma\delta}$,

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▫ When going along an incomplete geodesic towards a "singularity", 3 things can happen:

I) A scalar constructed from $R^{\mu\nu}{}_{\sigma s}$,

e.g. R, $R^{\mu\nu}R_{\mu\nu}$, etc diverges.

⇝ We say it is a "scalar curvature singularity".

II) In a parallel transported tetrad frame, a scalar component of $R^{\mu\nu}_{\sigma s}$ or its covariant derivatives diverge.

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III) None of the above. Example: "Conical singularity".

(This way mfld can be)

(cut out a suitable piece and identify the boundaries of the cut)

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→ We say it is a non-curvature singularity

<u>Fundamental problem:</u>

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III) None of the above. Example: "Conical singularity".

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Fundamental problem:

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are obviously present.
- But these spacetimes are highly symmetric.

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Fundamental problem:

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are obviously present.

- But these spacetimes are highly symmetric.

  Do more realistic, i.e. perturbed spacetimes also show these singularities?

- Example:

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Example:

Spherically symmetric dust shell infall.

In Newton gravity: Use catastrophe theory

⇒ e.g., predict ∞ mass density to occur,
but not if symmetry perturbed!

In Einstein gravity: Use singularity theorems

⇒ e.g., predict black hole singularity to occur,
even if symmetry is perturbed,

Remark:
Black holes provide finite energy endpoint of grav. collaps, thus stabilizing GR energetically.
Note: In Q.M. charges driven

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Remark:
Black holes provide finite energy endpoint of grav. collaps, thus stabilizing GR energetically.
Note: In Q.M., charge driven collaps is bounded at finite energy by uncertainty principle.

⇒ e.g., predict black hole singularity to occur, even if symmetry is perturbed, (if assuming e.g. dominant energy cond. etc.)

or also: post-dict a cosmological singularity

Remark:

Singularity theorems ⇒ prediction of singularities is robust.

Thus: If quantum gravity is to resolve singularities, it will have to overcome this robustness!

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Remark:

Singularity theorems $\Rightarrow$ prediction of singularities is robust.

Thus: If quantum gravity is to resolve singularities, it will have to overcome this robustness!

Strategy for singularity theorems:

a) Focus attention on singularities that can be identified by the existence of incomplete inextendible timelike (or null) geodesics.

Why? It is clear that these are important singularities

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Strategy for singularity theorems:

a) Focus attention on singularities that can be identified by the existence of incomplete inextendible timelike (or null) geodesics.

Why? It is clear that these are important singularities because observers travelling such a geodesic have their eigentime bounded above and/or below.

Other singularities? (e.g. singularities identified through incomplete spacelike geodesics or singularities identified by some other criterion.)

May well exist in addition but the

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b.) Basic idea:

Singularities can be in the way of geodesics.

$\Rightarrow$

The presence of singularities interferes with the property of geodesics of being extremal length curves.

c.) Recall:

(Euler)

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c.) Recall:

(Euler Lagrange equation)

Extremizing curve length $\Longrightarrow$ geodesic equation

The geodesic equation is a differential equation.

Thus:

At least locally, geodesics are paths of extremal length:

- Space-like geodesics are curves of shortest proper distance.

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At least locally, geodesics are paths of extremal length:

- Space-like geodesics are curves of shortest proper distance.

- Time-like geodesics are curves of maximal proper time (i.e. of maximal eigentime).

Why maximal?

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- Space-like geodesics are curves of shortest proper distance.

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Why maximal?

If there is a timelike curve between two events p, q, then there are timelike curves with shorter eigentime: just take a spatially longer path and travel it faster.

d.) Prove that, even in generic spacetimes:

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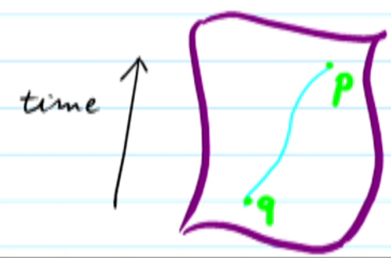
□ Time-like geodesics are curves of maximal proper time (i.e. of maximal eigentime).

Why maximal?

If there is a timelike curve between two events p, q, then there are timelike curves with shorter eigentime: just take a spatially longer path and travel it faster.

d.) Prove that, even in generic spacetimes:

There always exist curves of maximal length between two events.



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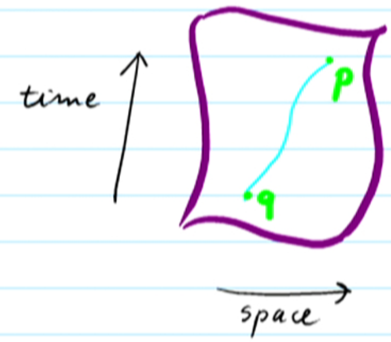

What assumptions are needed?

E.g., the assumption that spacetime is globally hyperbolic suffices.

e) Prove that in the circumstances that one choose

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approximate suffices.

e) Prove that in the circumstances that one choose to consider, these extremal length curves cannot be geodesics with eigentime larger than a certain finite amount either into the past or future.

e.g., if the universe has a finite age

What assumptions needed?

E.g., that matter obeys a suitable energy condition.

c) Conclude that there are incomplete geodesics ...

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E.g., that matter obeys a suitable energy condition.

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A singularity theorem:

Assume that: ▫ (M, g) is a globally hyperbolic spacetime

▫ The energy-momentum tensor of matter obeys the "S... ..."

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A singularity theorem:

Assume that: ▯ $(\mathcal{M}, g)$ is a globally hyperbolic spacetime

▯ The energy-momentum tensor of matter obeys the "Strong energy condition":

$$\left(T_{\mu\nu} - \frac{1}{2} T^{\rho}{}_{\rho}\, g_{\mu\nu}\right) \xi^{\mu} \xi^{\nu} \geq 0 \quad \forall \text{ timelike } \xi.$$

Notice: Since the Einstein equation can be brought in the form $kR_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}$, the strong energy condition is a condition on the Ricci tensor too. This will be the use of the strong energy condition.

▯ There exists a C^2 spacelike Cauchy surface Σ, on which the trace of the extrinsic

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$(T_{\mu\nu}$

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- The energy-momentum tensor of matter obeys the "Strong energy condition":

$$\left(T_{\mu\nu} - \frac{1}{2} T^{\rho}{}_{\rho}\, g_{\mu\nu}\right) \xi^{\mu} \xi^{\nu} \geq 0 \qquad \forall \text{ timelike } \xi.$$

- There exists a C^2 spacelike Cauchy surface Σ, on which the trace of the extrinsic curvature, K, is bounded from above by a negative constant C:

$$K(p) \leq C < 0 \quad \text{for all } p \in \Sigma$$

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Then:

No past-directed timelike curve from Σ can have eigentime, i.e., proper length, larger than $\frac{3}{c}$.

All past-directed timelike geodesics are incomplete.

because all past-directed paths end on it.

$\Longrightarrow$ There is a cosmological singularity in

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No past-directed timelike curve from Σ can have eigentime, i.e., proper length, larger than $\frac{3}{C}$.

All past-directed timelike geodesics are incomplete.

because all past-directed paths end on it.

$\Longrightarrow$ There is a cosmological singularity in the finite past!

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Extrinsic curvature?

later more on this

- The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between the spacelike hypersurface and the time dimension.

Intuitively: it is the rate of the expansion of spacetime, more precisely its negative.

$\forall p \in \Sigma$

Thus: Assuming $K(p) \leq C < 0$ meant

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The strong energy condition?

Recall:

- The "weak energy condition":

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v: \; g(v,v) < 0$$

Meaning? For an observer with unit tangent v the local energy density is: $T_{\mu\nu} v^\mu v^\nu \geq 0$

- The "dominant energy condition":

$$\underbrace{T_{\mu\nu} v^\mu v^\nu \geq 0}_{\text{weak energy condition}} \quad \text{and} \quad K_\mu K^\mu \leq 0$$

[i.e. $T_{\mu\nu} v^\nu$ is non-space-like.

where v is any timelike vector and $K_\mu := T_{\mu\nu} v^\nu$

Meaning? The local energy-momentum flow

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$T_{\mu\nu} v^\mu v^\nu \geq 0$ for all timelike v: $g(v,v) < 0$

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i.e. $T_{\mu\nu} v^\nu$ is non-space-like.

where v is any timelike vector and $K_\mu := T_{\mu\nu} v^\nu$

Meaning? The local energy-momentum flow vector K may not be conserved but has to be non-space-like: Flow should be into the future ← need for causality.

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The "strong energy condition"

Matter is said to obey the strong energy condition iff:

$$\left(T_{\mu\nu} - \frac{1}{2} T^{\rho}{}_{\rho}\, g_{\mu\nu}\right) \xi^{\mu} \xi^{\nu} \geq 0 \ \forall \text{ timelike } \xi.$$

- Intuition? Excludes matter that causes accelerated expansion. (as we will discuss below)

- Plausible? Yes, obeyed by known matter. (but not by dark energy)

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- Intuition? Excludes matter that causes accelerated expansion. (as we will discuss below)
- Plausible? Yes, obeyed by known matter. (but not by dark energy)
- Relationship? Independent of weak and dominant

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$$\left(T_{\mu\nu} - \frac{1}{2} T^{\rho}{}_{\rho}\, g_{\mu\nu}\right) \xi^{\mu} \xi^{\nu} \geq 0 \ \forall \text{ timelike } \xi.$$

- **Intuition?** Excludes matter that causes accelerated expansion. (as we will discuss below)

- **Plausible?** Yes, obeyed by known matter. (but not by dark energy)

- **Relationship?** Independent of weak and dominant energy condition.

Concretely: For known matter, $T_{\mu\nu}$ is diagonalizable to obtain:

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energy density observed by comoving observer

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & p_1 & & \\ & & p_2 & \\ 0 & & & p_3 \end{pmatrix}$$

principal pressures

The energy conditions then read:

- **Weak:** $\rho \geq 0$ and $\rho + p_i \geq 0$ for $i \in \{1,2,3\}$
- **Dominant:** $\rho \geq |p_i|$ for $i \in \{1,2,3\}$
- **Strong:** $\rho + \sum_{i=1}^{3} p_i \geq 0$ and $\rho + p_i \geq 0$ for $i \in \{1,2,3\}$

Exercise: Show this →

Note: could possibly be also negative.

Recall: A cosmological constant Λ can be viewed as a contribution to $T_{\mu\nu}$.

Indeed, there is no big bang singularity, e.g., if $w = -1$ $\forall t$,

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Exercise: Show that the strong energy condition is violated in cosmology iff $w < -\frac{1}{3}$, i.e., iff the expansion is accelerating: $\ddot{a}(t) > 0$.

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Essence of point e):

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion θ in finite proper time:

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Essence of point e):

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, θ, in finite proper time:

important notion also e.g. in study of grav. collaps of stars.

The "expansion", θ:

- Consider a "congruence of timelike geodesics" through Σ, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper

e.g., freely falling dust.

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The "expansion", θ:

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$$\xi := \frac{d}{d\tau}$$ ← proper time

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will obey : $g(\xi,\xi) = -1 \;\; \forall p$.

Consider now a one-parameter sub family of these geodesics:

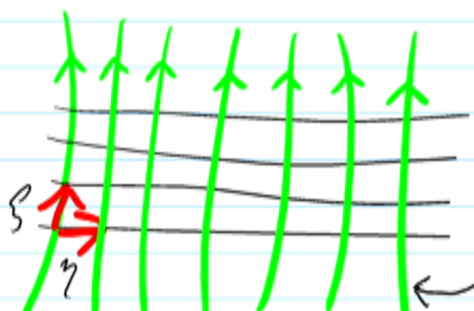
$$\gamma(\tau, s)$$

parameter of family of neighboring geodesics.

a "connecting vector field"

Then, we define the deviation vector:

$$\eta := \frac{d}{ds}$$



a line of constant τ value

a geodesic, i.e., a line of constant s value

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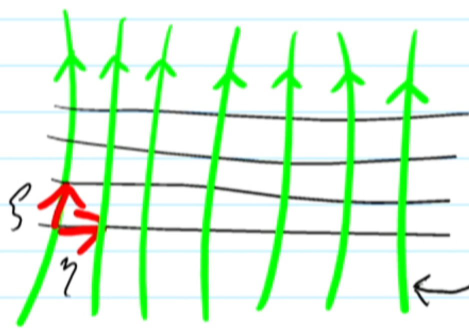

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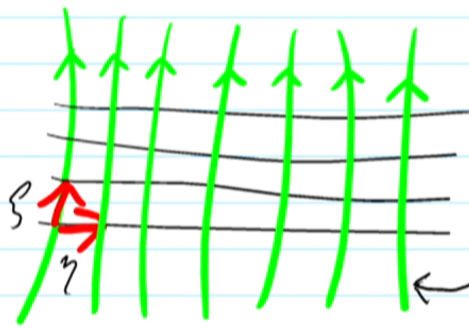

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□ How does η change along a geodesic?

We have $\overset{=\eta}{\frac{d}{ds}}\,\overset{=\xi}{\frac{d}{d\tau}} = \frac{d}{d\tau}\frac{d}{ds}$, i.e., in arb. coordinate system:

→ to see this, have them act on a scalar function

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$$\Rightarrow \quad \xi^\mu \nabla_{e_\mu} \eta^\nu e_\nu = \eta^\alpha \nabla_{e_\alpha} \xi^\beta e_\beta$$

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- The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely into:

Symmetric and trace = 0

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The tensor $B^{\nu}{}_{\mu}$ can be decomposed covariantly and uniquely into:

$$B_{\mu\nu} = \underbrace{\omega_{\mu\nu}}_{\text{antisymmetric}} + \overbrace{\sigma_{\mu\nu}}^{\text{symmetric and trace}=0} + \underbrace{t_{\mu\nu}}_{\text{rest}}$$

(all 3 terms are tensors because the split is covariant)

We have: $\omega_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$, clearly.

But $\sigma_{\mu\nu}, t_{\mu\nu} = ?$

In preparation: define the projector $h_{\mu\nu}$ onto $(\mathbb{R}\,\xi)^{\perp}$ i.e. onto the spatial components:

ξ is timelike

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$$h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$$

Check: is $h_{\mu\nu} w^\nu$ really always $\perp$ to ξ ?

Indeed: $\xi^\mu h_{\mu\nu} w^\nu = (\xi, w) + \underbrace{(\xi,\xi)}_{=-1}(\xi, w) = 0$

Define: The "expansion", θ, is defined as the

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Define: The "expansion", θ, is defined as the magnitude of the spatial part of B:

$$\theta := B^{\mu\nu} h_{\mu\nu}$$

Claim: $Tr(B) = \theta$

Indeed:

$$\theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \xi^\mu \xi_\nu B_\mu{}^\nu$$

$$= Tr(B) + \underbrace{\xi^\mu \xi_\nu \nabla_\mu \xi^\nu}$$

($= 0$ because $\nabla_\xi \xi = 0$ for geodesics.)

Therefore:

$$\sigma_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3}\theta h_{\mu\nu}$$

(because: $Tr(h_{\mu\nu}) = g^{\mu\nu} h_{\mu\nu} = g^{\mu\nu}(g_{\mu\nu} + \xi_\mu \xi_\nu) = 4 - 1$)

the part of $B_{\mu\nu}$ which is symmetric and traceless.

and:

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Interpretation:

a) $\omega_{\mu\nu}$ is antisymmetric: $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$\Rightarrow$ it generates Lorentz transformations for η

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a) $\omega_{\mu\nu}$ is antisymmetric: $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$\Rightarrow$ it generates Lorentz transformation for η.

but all η are $\perp$ to the time direction

$\Rightarrow$ $\omega_{\mu\nu}$ generates spatial rotations of neighboring geodesics around another. So, $\omega_{\mu\nu}$ is called

$\omega =$ "Twists tensor"

One can prove: (nontrivial)

If one chooses the congruence as

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<u>One can prove</u>: (nontrivial)

If one chooses the congruence of geodesics $\perp$ to Σ then $\omega_{\mu\nu} = 0$.

b.) $\sigma_{\mu\nu}$ is symmetric, $\sigma_{\mu\nu} = \sigma_{\nu\mu}$. (i.e. hermitean)

Consider "diagonalized", by suitable choice of cd basis.

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b) $\sigma_{\mu\nu}$ is symmetric, $\sigma_{\mu\nu} = \sigma_{\nu\mu}$. (i.e. hermitean)

Consider "diagonalized", by suitable choice of cd basis.

$\Rightarrow$ $\sigma_{\mu\nu}$ changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow → become points on an ellipsoid.

Note: Since $\underbrace{Tr(\sigma)}_{\text{infinitesimal transport along geodesics}} = 0$ we have $\det(\underbrace{e^{\alpha\sigma}}_{\text{finite transport}}) = 1$

$\Rightarrow$ The volume spanned by basis vectors

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infinitesimal transport along geodesics

finite transport

⇒ The volume spanned by basis vectors stays the same under the action of σ.

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Evolution of θ along a geodesic?

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