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Abstract:

Derived Categories and Variation of Geometeric Invariant Theory Quotients

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University of Alberta

Oct. 24, 2013

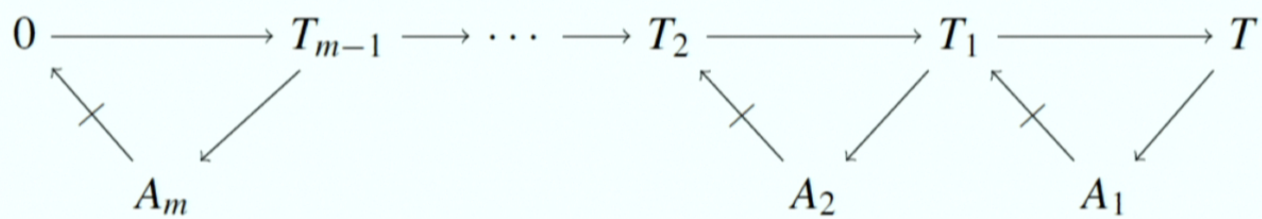
Attributions

Based on joint work with M. Ballard (U. South Carolina) and Ludmil Katzarkov (U. Miami and U. Vienna).
Available at http://arxiv.org/abs/1203.6643.

Semi-orthogonal decompositions

Definition

A **semi-orthogonal decomposition** of a triangulated category, $\mathcal{T}$, is a sequence of full triangulated subcategories, $\mathcal{A}_1, \ldots, \mathcal{A}_m$, in $\mathcal{T}$ such that $\mathcal{A}_i \subset \mathcal{A}_j^{\perp}$ for $i < j$ and, for every object $T \in \mathcal{T}$, there exists a diagram:



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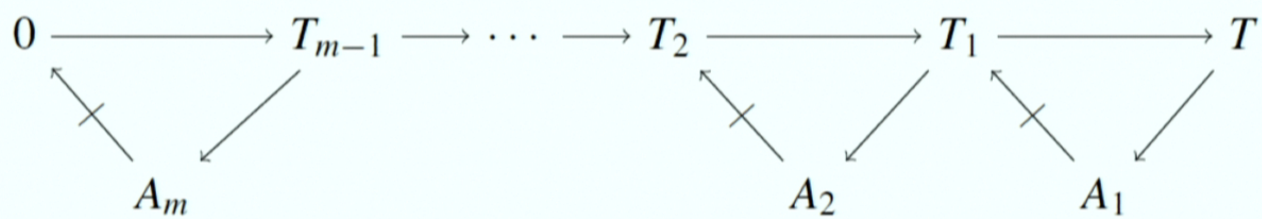

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Exceptional Collections

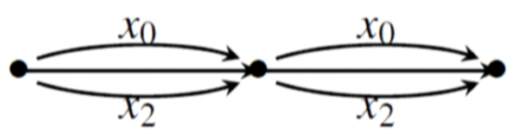
When each category $\mathcal{A}_i$ is equivalent to the derived category of a point, generated by a single object, we write:

$$\mathcal{T} = \langle E_1, \ldots, E_s \rangle.$$

For example

Theorem (Beilenson)

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathbb{P}^n) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$$



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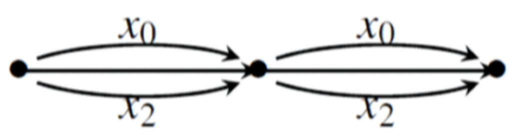

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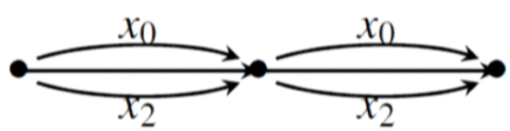

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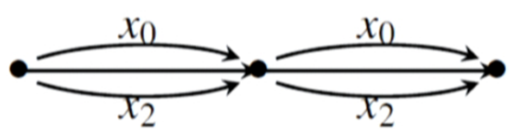

Homological Mirror Symmetry and Birational Geometry (an example)

The mirror to $\mathbb{P}^2$ is the LG model $(\mathbb{A}^2, x + y + \frac{1}{xy})$.

There are 3-singular fibers, each is an ordinary double point, and each gives a unique Lefschetz thimble up to isotopy.

There is a semi-orthogonal decomposition

$$\mathrm{Fuk}(\mathbb{A}^2, x + y + \frac{1}{xy}) = \langle E_1, E_2, E_3 \rangle,$$

where each of the E_i is equivalent to the simplest possible derived category, the category of graded vector spaces.

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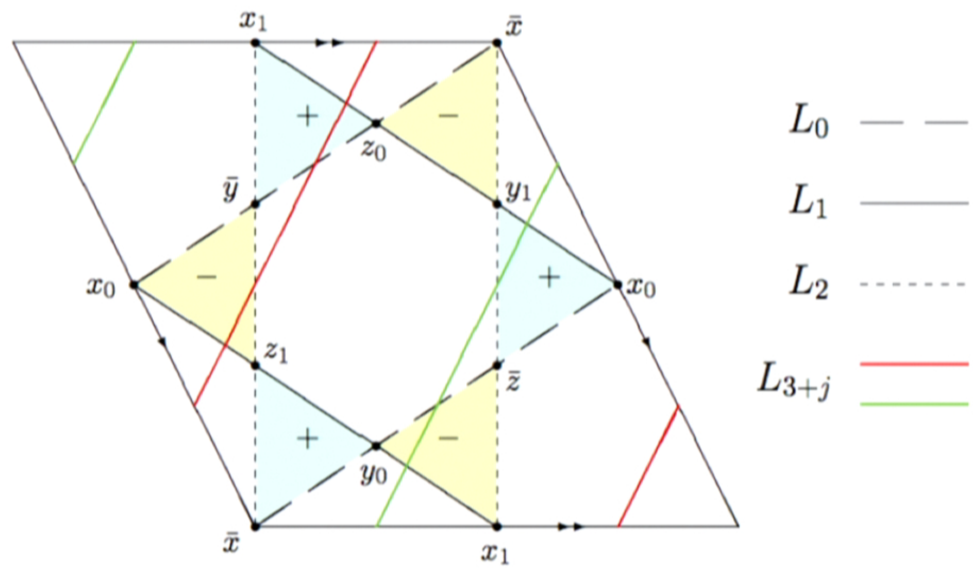
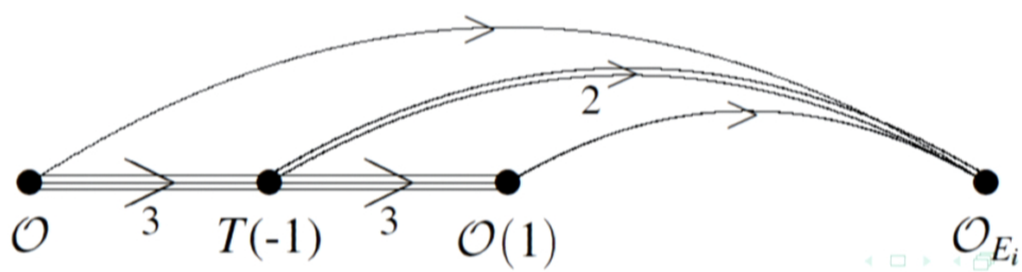
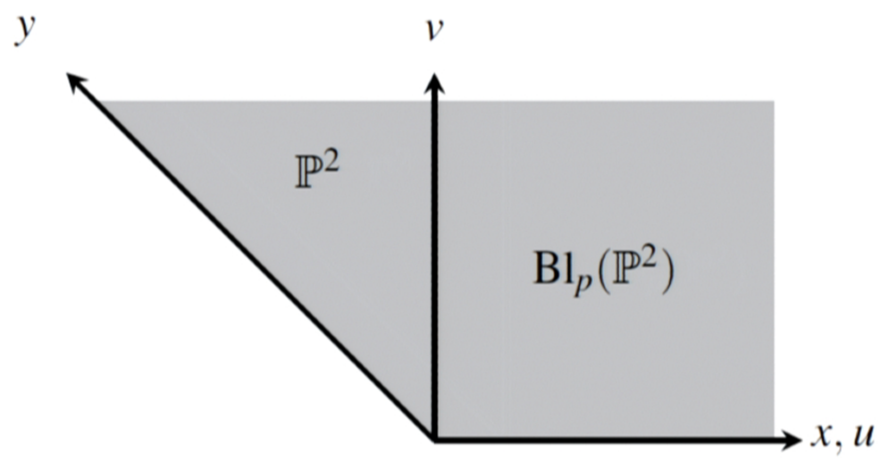
Mirror to $\mathrm{Bl}_p\,\mathbb{P}^2$





Blow-up - Blow-down



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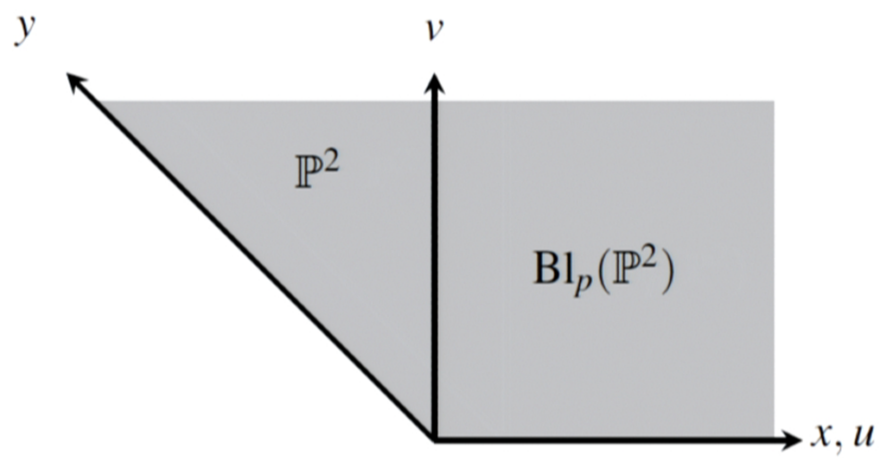
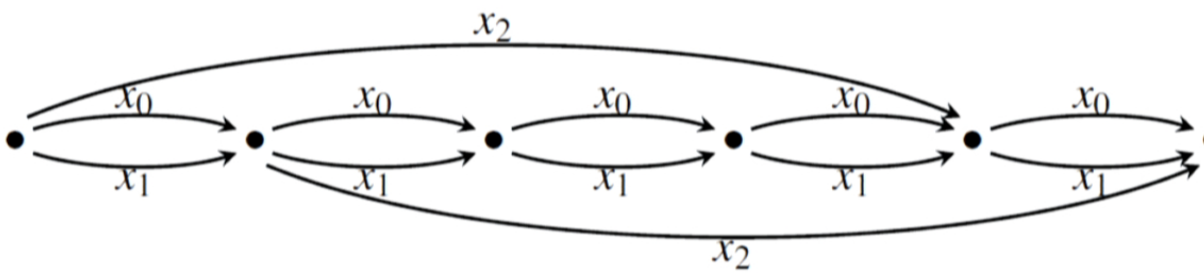


Weighted Projective Stacks

Consider $\mathbb{P}(1:1:n)$ as a smooth Deligne-Mumford stack.

$$T := \bigoplus_{i=0}^{n+2} \mathcal{O}(i)$$

This "quiver for $\mathbb{P}(1:1:4)$" is a picture of the algebra $\mathrm{End}(T)$:

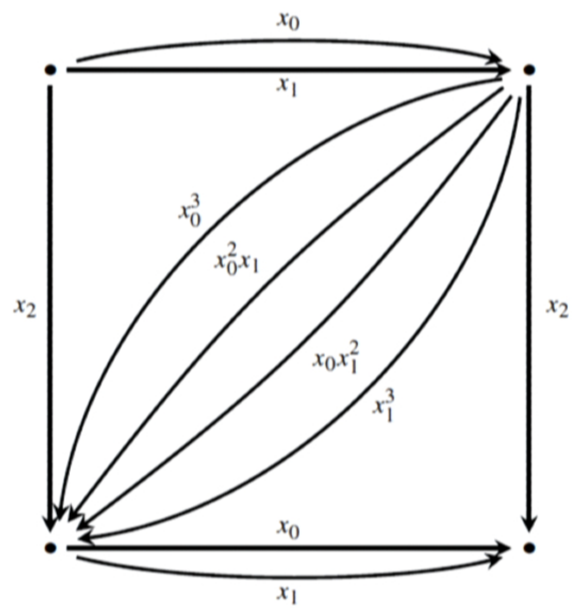


Hirzebruch Surfaces

$\pi : \mathbb{F}_n \to \mathbb{P}^1$ is a Hirzebruch surface

$$T := \mathcal{O} \oplus \pi^*\mathcal{O}(1) \oplus \mathcal{O}_\pi(1) \oplus \pi^*\mathcal{O}(1) \otimes \mathcal{O}_\pi(1)$$

Here's a picture of $\mathrm{End}(T)$ for $n = 4$:



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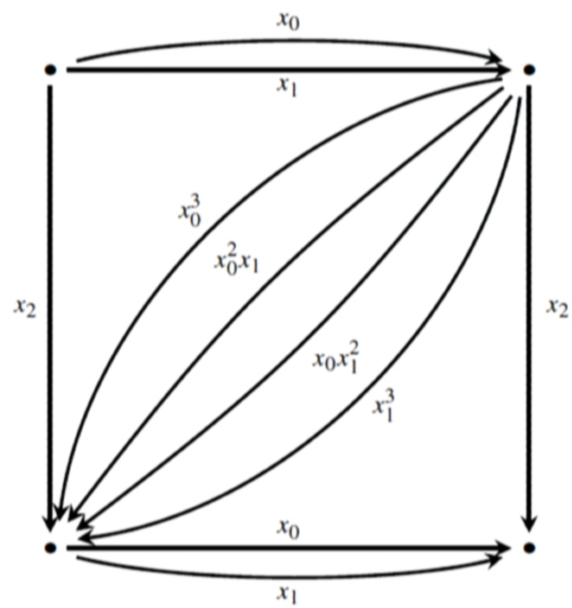

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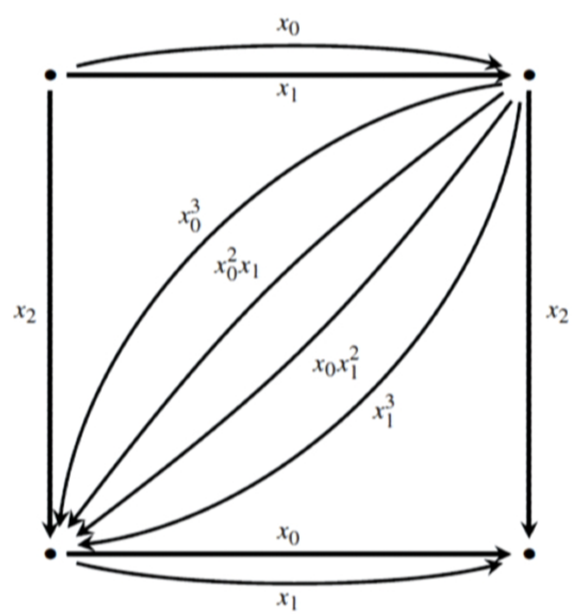
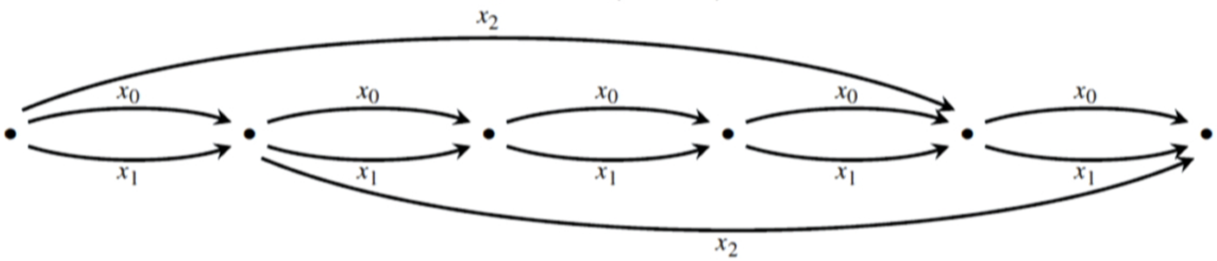
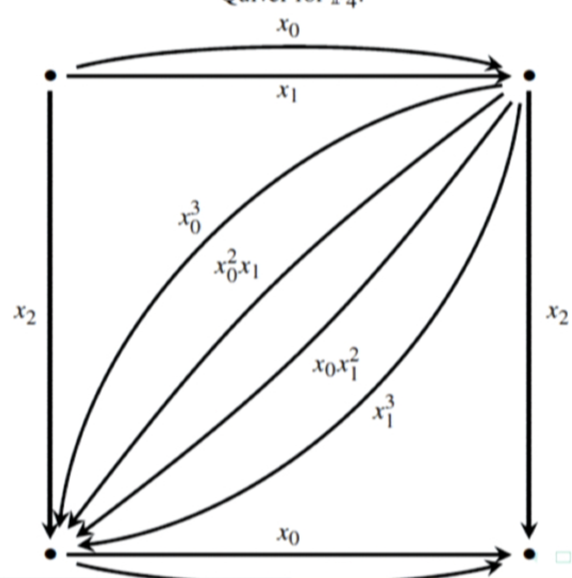


Comparing

Quiver for $\mathbb{P}(1:1:4)$:



Quiver for $\mathbb{F}_4$:



The observation that these two quivers differ by 2 vertices can be written as a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathbb{P}(1,1,4)) = \langle E_1, E_2, \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathbb{F}_4) \rangle$$

Relation to Mirror Symmetry

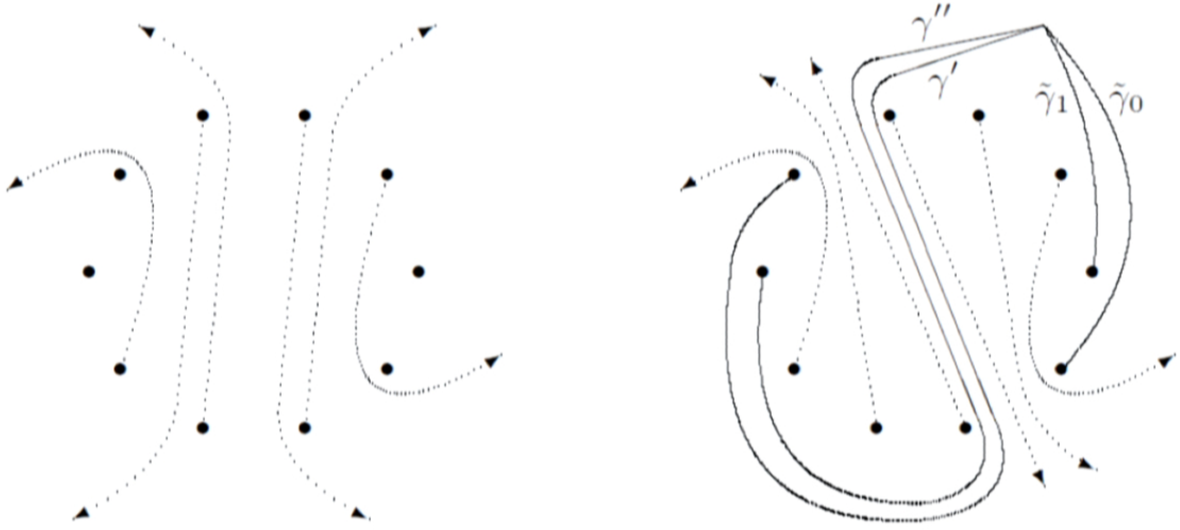


Figure: The deformation $b \to 0$ of $w = x + y + \frac{1}{x} + \frac{b}{x^8 y}$.

Setup

- X is a smooth quasi-projective variety over $\mathbb{C}$
- G is a linearly-reductive linear algebraic group acting on X
- $\lambda : \mathbb{C}^* \to G$ is a one parameter subgroup

$$Z_\lambda^0 := \text{a chosen connected component of the fixed locus of } \lambda$$
$$Z_\lambda^+ := \{x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x \text{ exists}\}$$
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Elementary wall crossings

Definition

Let X be a smooth, quasi-projective variety equipped with a G-action. An **elementary wall-crossing** is a one parameter subgroup

$$\lambda : \mathbb{C}^* \to G$$

and a connected component of the fixed locus Z_λ^0 such that

- $S_{\pm\lambda}$ are both smooth and closed
- There are natural isomorphisms

$$[G \times Z_{\pm\lambda}/P(\pm\lambda)] \xrightarrow{\sim} S_{\pm\lambda},$$

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Setup

Where

$$P(\lambda) := \{g \in G \mid \lim_{\alpha \to 0} \lambda(\alpha) g \lambda(\alpha)^{-1} \text{ exists}\}.$$

acting on $G \times Z_\lambda$ by

$$p \cdot (g, z) = (gp^{-1}, pz).$$

Mumford's stability function

- X is a variety with a G action
- $\lambda : \mathbb{C}^* \to G$ is a one-parameter subgroup
- $\mathcal{L}$ is a G-equivariant line bundle
- Choose a connected component of the fixed locus Z_λ^0
- $x \in Z_\lambda^0$.

Definition

$\mu(\mathcal{L}, \lambda, x)$ is the weight of the $\mathbb{C}^*$-action on the fiber of the geometric vector bundle associated to $\mathcal{L}$ induced by λ. It is called **Mumford's numerical function**.

Given an elementary wall-crossing we get a number:

$$\mu := \mu(\omega_{S_{-\lambda}|X}, -\lambda, x) - \mu(\omega_{S_\lambda|X}, \lambda, x)$$

Main comparison theorem (no potential)

Theorem

For an elementary wall-crossing, fix $d \in \mathbb{Z}$.

- If $\mu > 0$, then there exist fully-faithful exact functors,

$$\Phi_d : \mathrm{D^b}(\mathrm{coh}\, X/\!\!/-) \to \mathrm{D^b}(\mathrm{coh}\, X/\!\!/+)$$
$$\Upsilon_i^- : \mathrm{D^b}(\mathrm{coh}[Z_\lambda^0/C(\lambda)])_i \to \mathrm{D^b}(\mathrm{coh}\, X/\!\!/+),$$

for $-d \leq i \leq \mu - d - 1$ and a semi-orthogonal decomposition,

$$\mathrm{D^b}(\mathrm{coh}\, X/\!\!/+) = \langle \Upsilon_{-d}^-, \ldots, \Upsilon_{\mu-d-1}^-, \Phi_d \rangle.$$

When the quotient,

$$C(\lambda) \to C(\lambda)/\lambda,$$

splits by χ then

$$\mathrm{D^b}(\mathrm{coh}[Z_\lambda^0/C(\lambda)])_i \cong \mathrm{D^b}(\mathrm{coh}[Z_\lambda^0/(C(\lambda)/\lambda)])$$

for all i and

$$\Upsilon_i^+ \cong (\mathcal{O}(\chi) \otimes \bullet) \circ \Upsilon_{i-1}^+.$$

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Projective space/ bundles

- B be a quasi-projective algebraic variety
- $\mathcal{E}$ be a vector bundle over B
- $X = \operatorname{tot} \mathcal{E}$
- $G = \mathbb{C}^*$ acts on X by dilating the fibers.
- $\lambda = \mathrm{Id}$
- Z_λ^0 is the zero section
- $S_\lambda = Z_\lambda^0$
- $S_{-\lambda} = X$
- $C(\lambda)/\lambda = 1$

1. $X /\!\!/ + = \mathbb{P}(\mathcal{E})$
2. $X /\!\!/ - = \emptyset$

This yields

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathbb{P}(\mathcal{E})) = \langle \pi^* \mathrm{D}^{\mathrm{b}}(\operatorname{coh} B), \ldots, \pi^* \mathrm{D}^{\mathrm{b}}(\operatorname{coh} B)(\mathcal{O}_\pi(n-1)) \rangle.$$

Hirzebruch surfaces

We can realize $\mathbf{F}_n$ as a GIT quotient of the spectrum of the Cox ring $X = \mathbb{A}^4$ by the subgroup

$$G = (\mathbb{C}^*)^2 = \{(r, r^{-n}s, r, s) : r, s \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^4.$$

Write $\mathbf{k}[x, y, u, v]$ for the ring of regular functions on $\mathbb{A}^4$.

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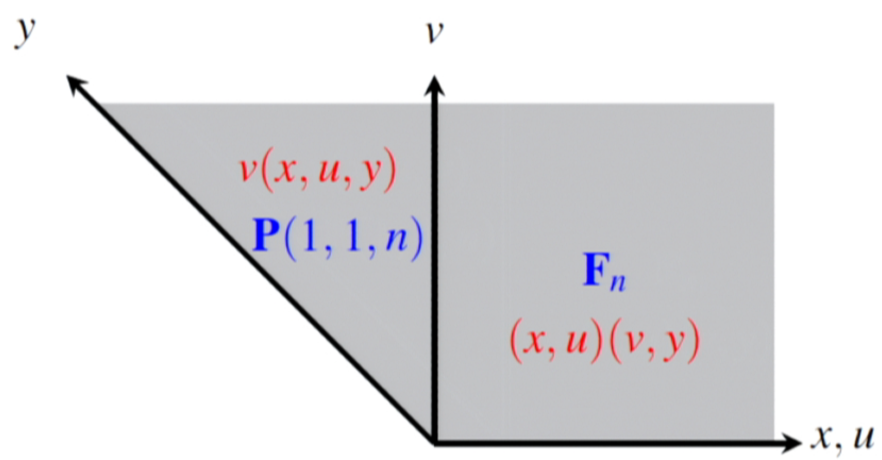
The GIT fan for this quotient is

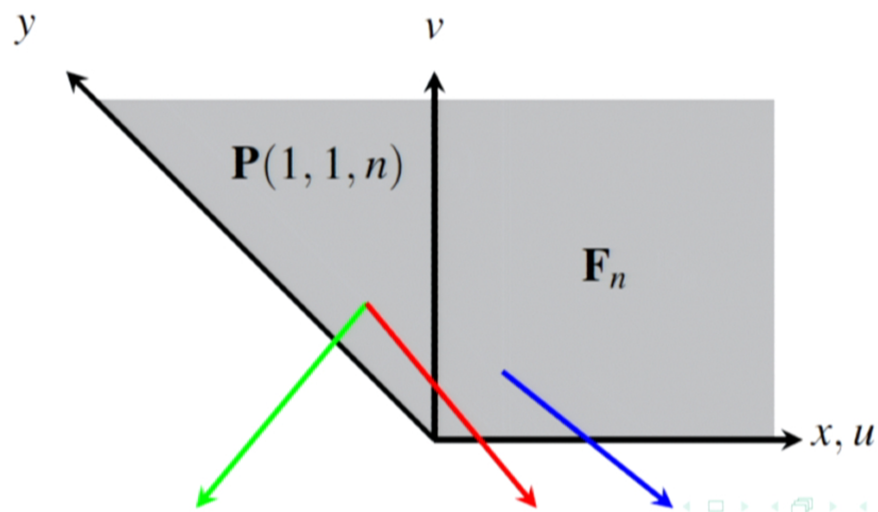


A theorem of Kawamata

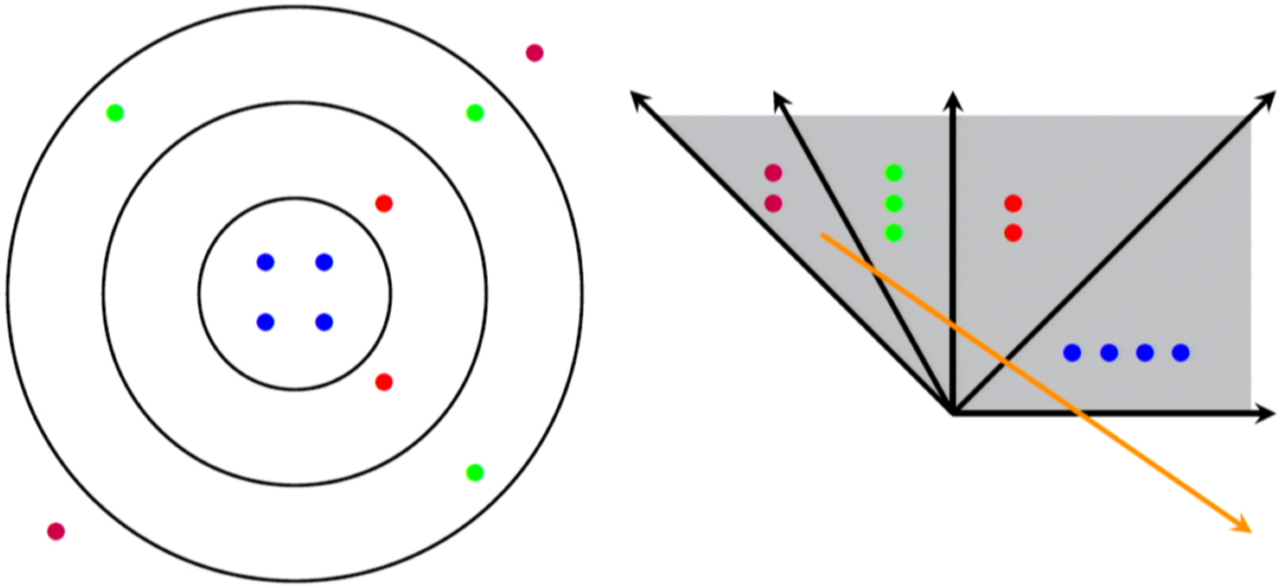
Theorem

Let X be a smooth projective toric DM stack. Then $\mathrm{D^b}(\mathrm{coh}\, X)$ admits a full exceptional collection.

Idea of Proof:
Choose a run of the toric minimal model program beginning in your chamber:



Relationship to Mirror Symmetry



Runs of the toric minimal model program deform the mirror LG model to the boundary of the moduli space of toric LG models constructed by Diemer, Katzarkov, Kerr.

Gauged LG-models

- X is a smooth quasi-projective variety
- G is a group acting on X
- $\mathcal{L}$ is a G-equivariant line bundle
- $w \in \mathrm{H}^0(\mathcal{L})^G$

Definition

A **gauged Landau-Ginzburg model** (gauged LG-model) is a quadruple $(X, G, \mathcal{L}, w)$.

B-model for Gauged LG-model

"Coherent sheaves" on a gauged LG-model, $(X, G, \mathcal{L}, w)$ are called factorizations.

Definition

A **factorization** of a gauged LG-model, $(X, G, \mathcal{L}, w)$, consists of a pair of coherent G-equivariant sheaves, $\mathcal{E}^{-1}$ and $\mathcal{E}^0$, and a pair of G-equivariant $\mathcal{O}_X$-module homomorphisms,

$$\phi_{\mathcal{E}}^{-1} : \mathcal{E}^0 \otimes \mathcal{L} \to \mathcal{E}^{-1}$$
$$\phi_{\mathcal{E}}^{0} : \mathcal{E}^{-1} \to \mathcal{E}^0$$

such that the compositions, $\phi_{\mathcal{E}}^0 \circ \phi_{\mathcal{E}}^{-1} : \mathcal{E}_0 \otimes \mathcal{L} \to \mathcal{E}^0$ and $\phi_{\mathcal{E}}^{-1} \otimes \mathcal{L} \circ \phi_{\mathcal{E}}^0 : \mathcal{E}^{-1} \to \mathcal{E}^{-1} \otimes \mathcal{L}$, are isomorphic to multiplication by w.

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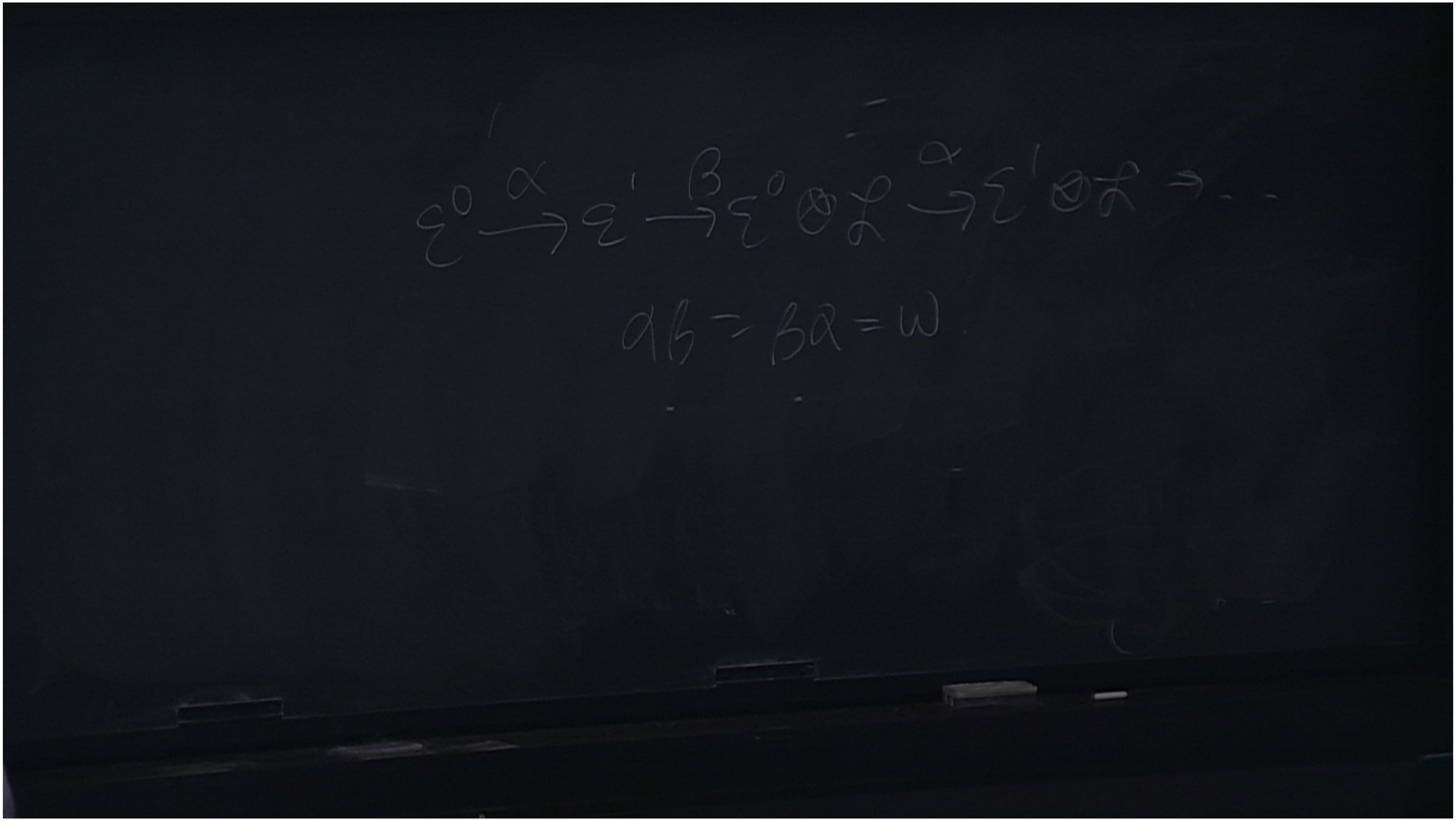
$\mathcal{E}^0 \xrightarrow{\alpha} \mathcal{E}^1 \xrightarrow{\beta} \mathcal{E}^0 \otimes \mathcal{L} \xrightarrow{\alpha} \mathcal{E}^1 \otimes \mathcal{L} \to \cdots$
$\alpha\beta = \beta\alpha = \omega$

Main comparison theorem (with potential)

Theorem

For an elementary wall-crossing, fix $d \in \mathbb{Z}$.

- If $\mu > 0$, then there exist fully-faithful exact functors,

$$\Phi_d : \text{Fact}(X \backslash S_{-\lambda}, G, w_-) \to \text{Fact}(X \backslash S_\lambda, G, w_+)$$
$$\Upsilon_i^- : \text{Fact}(Z_\lambda^0, C(\lambda), w_\lambda)_i \to \text{Fact}(X \backslash S_{-\lambda}, G, w_+),$$

for $-d \leq i \leq \mu - d - 1$ and a semi-orthogonal decomposition,

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Main comparison theorem (with potential)

Theorem

Fix $d \in \mathbb{Z}$.

- If $\mu = 0$, then there exists an exact equivalence,

$$\Phi_d : \text{Fact}(X \backslash S_{-\lambda}, G, w_-) \to \text{Fact}(X \backslash S_{\lambda}, G, w_+).$$

Isik-Shipman

Theorem (Isik, Shipman)

Let X be a variety and let $\sigma : \mathcal{O}_X \to E$ be a regular section of a vector bundle, E. Let Z denote the zero locus of σ. There is an equivalence

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z) \cong \operatorname{Fact}(\operatorname{tot} E^{\vee}, \mathbb{C}^*, w)$$

where w is the regular function induced by σ and the $\mathbb{C}^*$ is the dilation action on the fibers of $\operatorname{tot} E^{\vee}$.

Orlov's Theorem

The following concept was explained physically in the work of Herbst, Hori, and Page, and was described mathematically by Segal followed by Shipman.

$$\begin{array}{ccc} \mathrm{Fact}(\mathrm{tot}\, E^\vee, \mathbb{C}^*, w_+) & \xleftrightarrow{\text{VGIT}} & \mathrm{Fact}(\text{Another GIT Quotient}, \mathbb{C}^*, w_-) \\ \text{Isik/Shipman} \updownarrow & \overset{\text{Orlov}}{\swarrow \!\!\! \nearrow} & \\ \mathrm{D^b}(\mathrm{coh}\, Z) & & \end{array}$$

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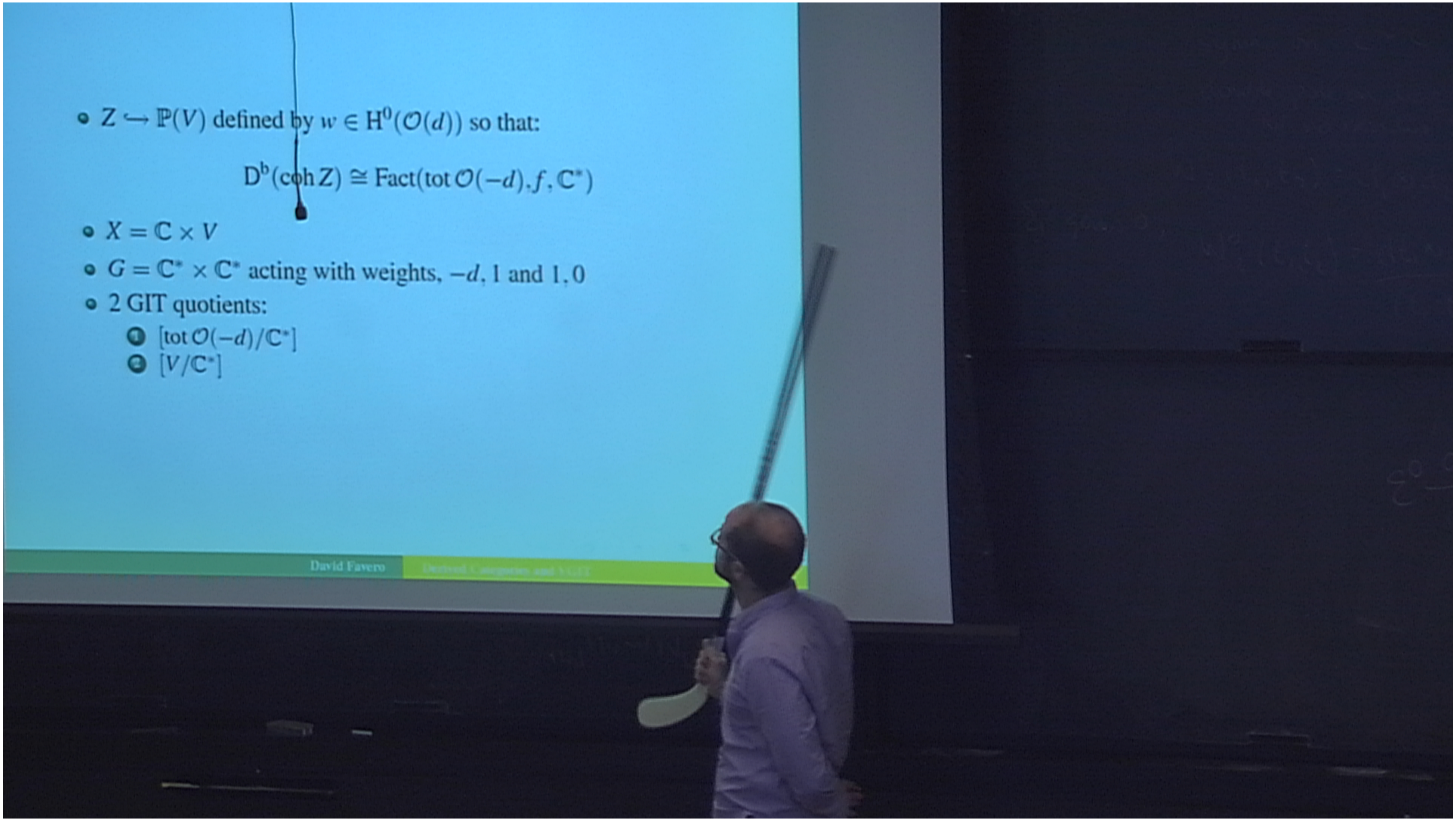
- $Z \hookrightarrow \mathbb{P}(V)$ defined by $w \in \mathrm{H}^0(\mathcal{O}(d))$ so that:

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Z) \cong \mathrm{Fact}(\mathrm{tot}\, \mathcal{O}(-d), f, \mathbb{C}^*)$$

- $X = \mathbb{C} \times V$
- $G = \mathbb{C}^* \times \mathbb{C}^*$ acting with weights, $-d, 1$ and $1, 0$
- 2 GIT quotients:
  1. $[\mathrm{tot}\, \mathcal{O}(-d)/\mathbb{C}^*]$
  2. $[V/\mathbb{C}^*]$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David Favero

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Orlov's Theorem (hypersurface/commutative case)

Theorem (Orlov, hypersurface/commutative case)

1. If $n+1-d>0$, there is a semi-orthogonal decomposition,
$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z) = \langle \mathcal{O}_Z(d-n), \ldots, \mathcal{O}_Z, \operatorname{Fact}(V, \mathbb{C}^*, w)\rangle.$$
2. If $n+1-d=0$, there is an equivalence of triangulated categories,
$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z) = \langle \operatorname{Fact}(V, \mathbb{C}^*, w)\rangle.$$
3. If $n+1-d<0$, there is a semi-orthogonal decomposition,
$$\operatorname{Fact}(V, \mathbb{C}^*, w) \cong \left\langle k, \ldots, k(n+2-d), \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z)\right\rangle.$$

Homological Projective Duality

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\begin{array}{ccc}
\mathrm{Fact}(\mathrm{tot}\,\mathcal{O}(-1,-1), \mathbb{C}^*, \langle , \rangle) & \overset{\text{VGIT}}{\longleftrightarrow} & \mathrm{Fact}(\mathcal{Y}, \mathbb{C}^*, \langle , \rangle) \\
\text{Isik/Shipman} \updownarrow & \overset{\text{Orlov}}{\swarrow\!\!\nearrow} & \updownarrow \text{Isik/Shipman} \\
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- $X \to P(V)$ a projective variety
- Start with
$$\mathrm{D^b}(\mathrm{coh}\, X) = \langle \mathcal{A}_0, \ldots, \mathcal{A}_t(t) \rangle$$
with $\mathcal{A}_t \subseteq \ldots \subseteq A_0$
- $L \subseteq V^*$ with $\dim L = r$
- $$\mathrm{D^b}(\mathrm{coh}\, X_L) = \langle \mathcal{C}_L, \mathcal{A}_r, \ldots, \mathcal{A}_t(t) \rangle$$
- A HPD looks like a space $Y \to \mathbb{P}(V^*)$ with
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Grassmannian/Pffafian

- W is a vector space of dimension 7
- $X = \mathrm{Gr}(2, W) \to \mathbb{P}(\wedge^2 W)$
- $\mathcal{A}_0 = \ldots = \mathcal{A}_6 = \langle \mathcal{O}, \mathcal{U}, S^2\mathcal{U} \rangle$
- $\mathrm{D^b}(\mathrm{coh}\, X) = \langle \mathcal{A}_0, \ldots, \mathcal{A}_6(6) \rangle$
- $Y \sim \mathrm{Pff}(4, W^*) \subseteq \mathbb{P}(\wedge^2 W^*)$ is the set of singular hyperplane sections of $\mathrm{Gr}(2, W)$ realized as degenerate skew-symmetric forms (rank ≤ 4)
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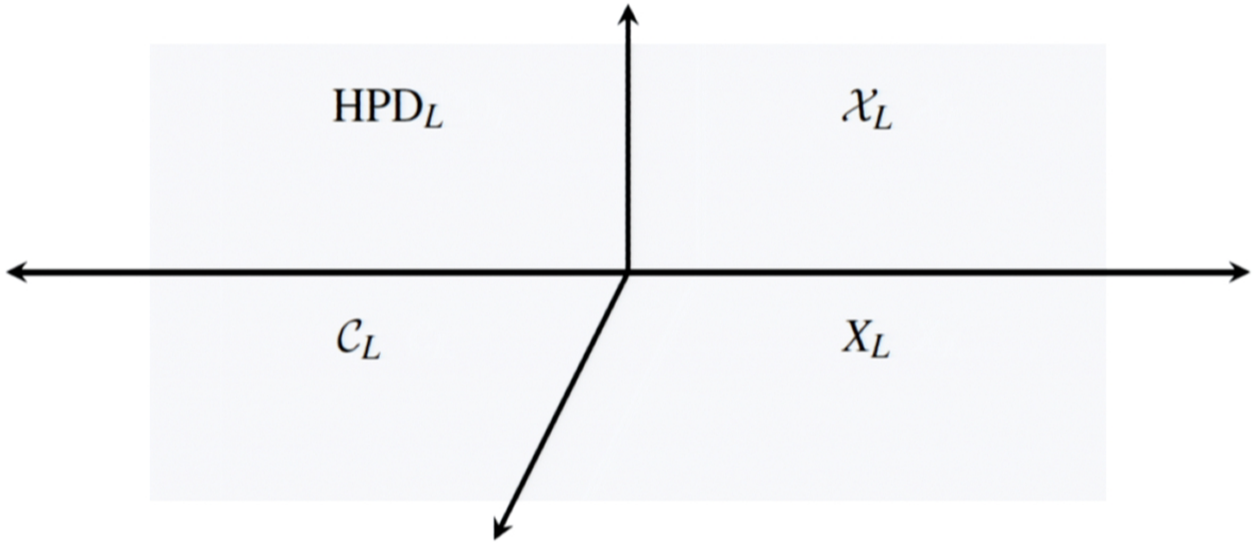
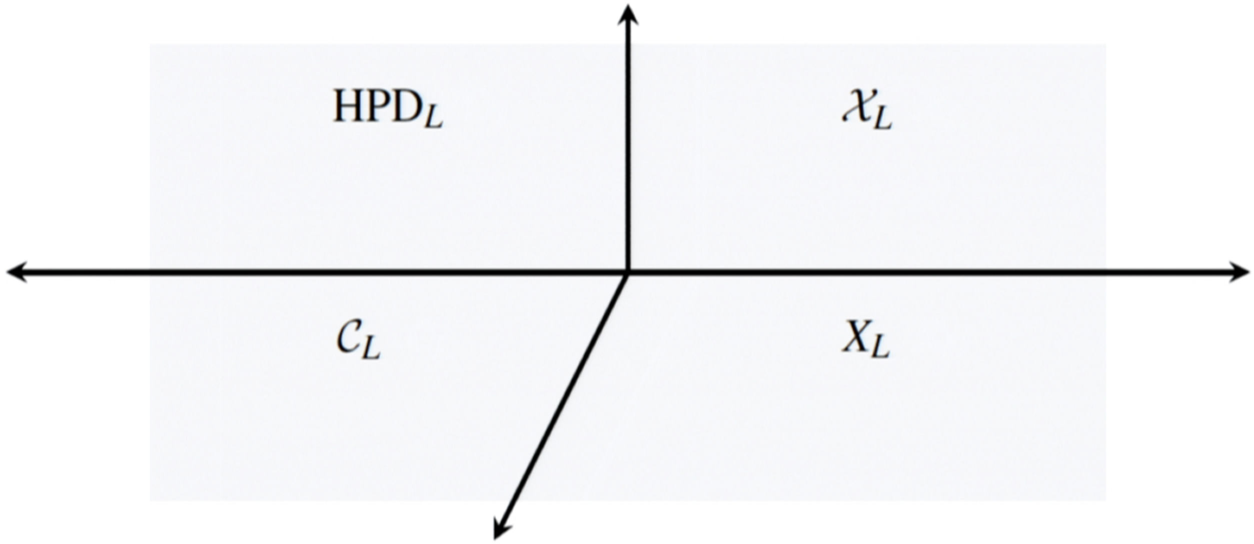

HPD_L
$\mathcal{X}_L$
$\mathcal{C}_L$
X_L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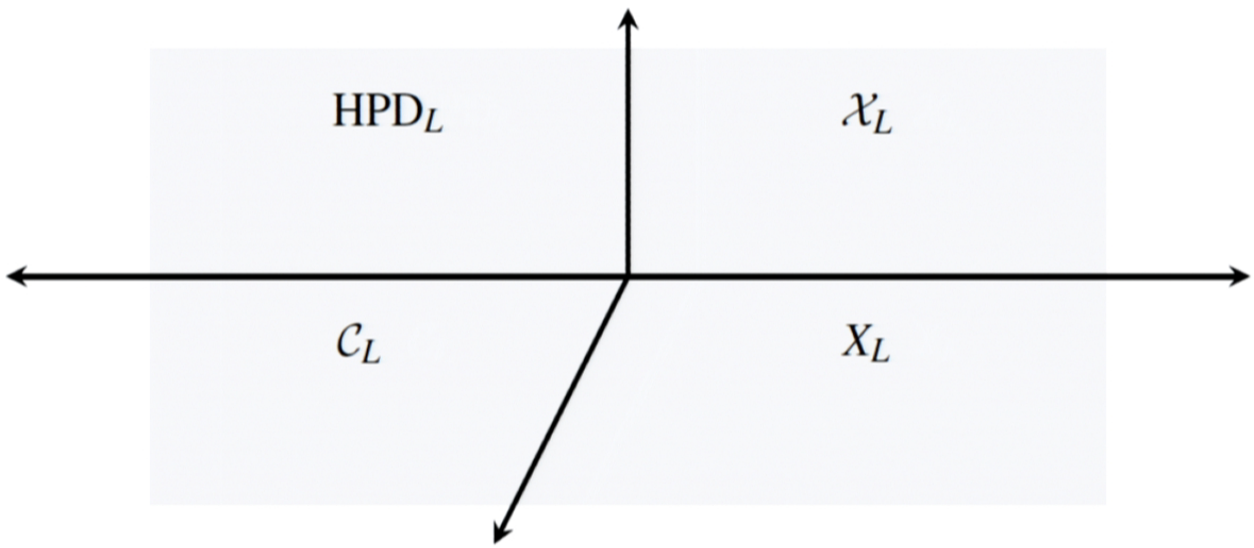
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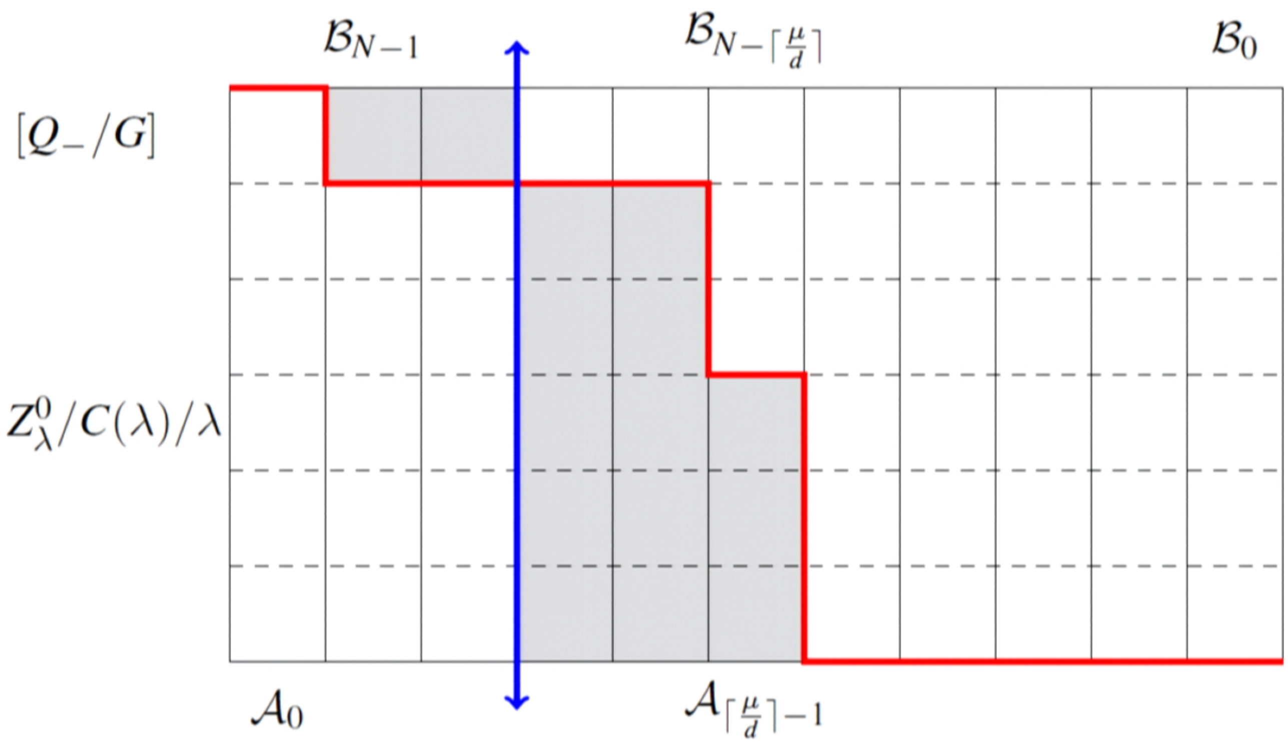
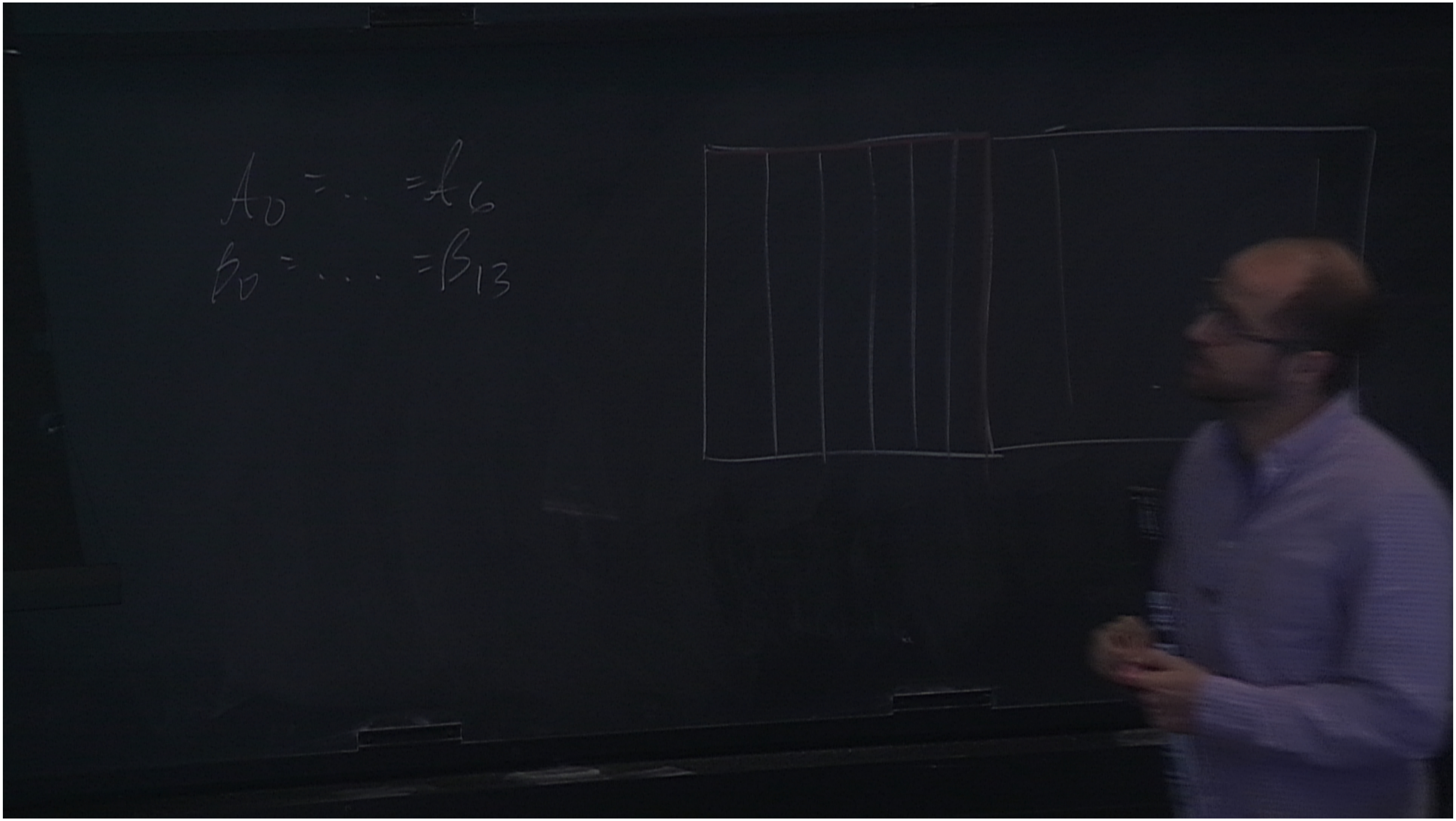

$\mathcal{B}_{N-1}$
$\mathcal{B}_{N-\lceil \frac{\mu}{d} \rceil}$
$\mathcal{B}_0$
$[Q_-/G]$
$Z^0_\lambda/C(\lambda)/\lambda$
$\mathcal{A}_0$
$\mathcal{A}_{\lceil \frac{\mu}{d} \rceil -1}$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