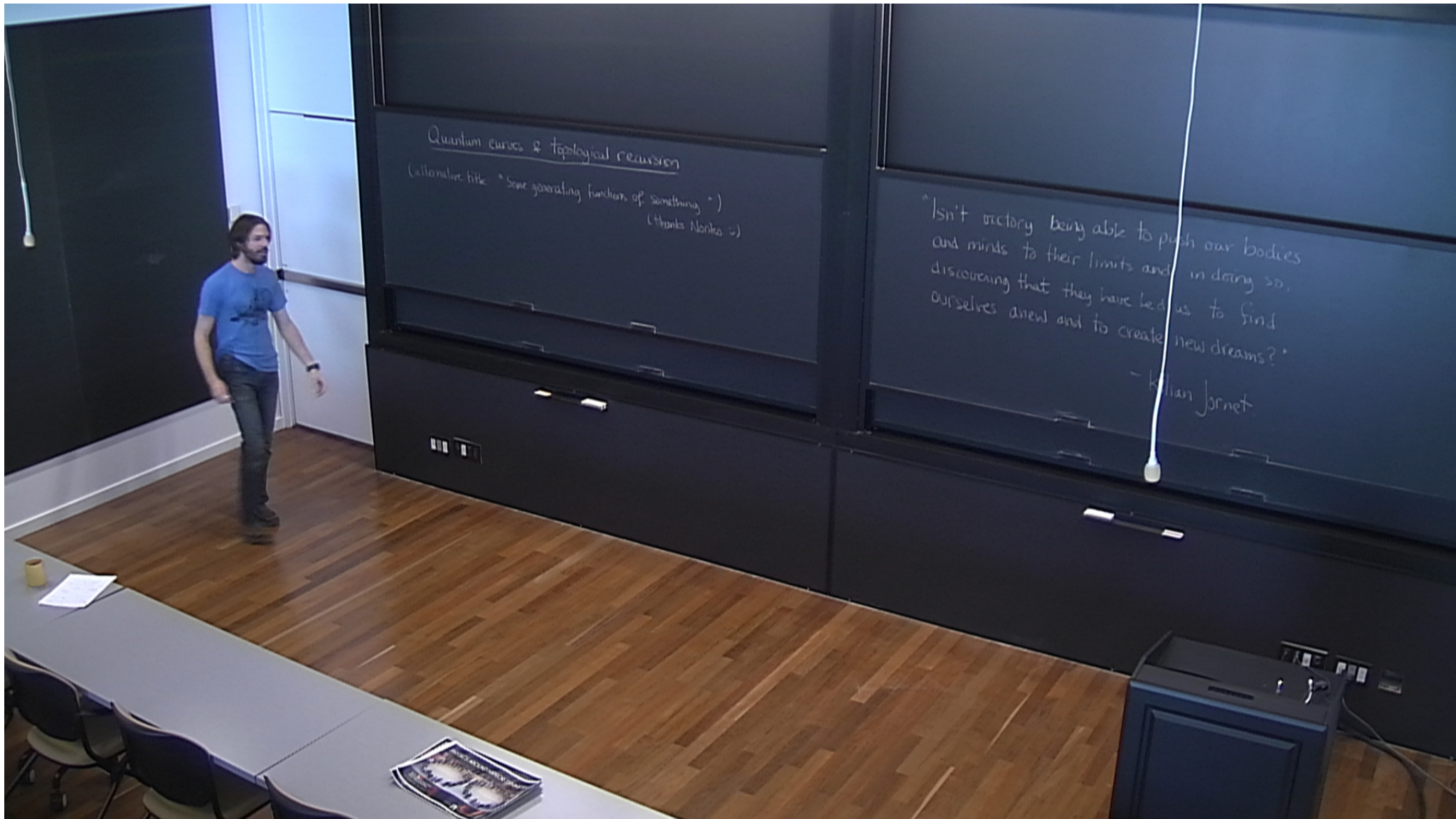


Title: TBA

Date: Oct 24, 2013 10:00 AM

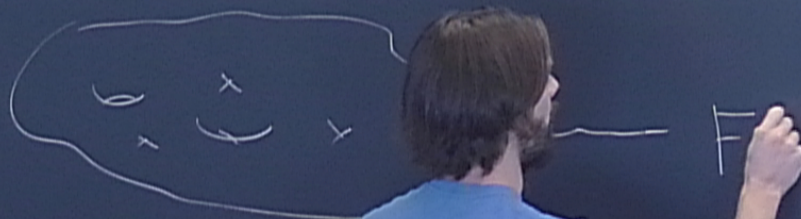
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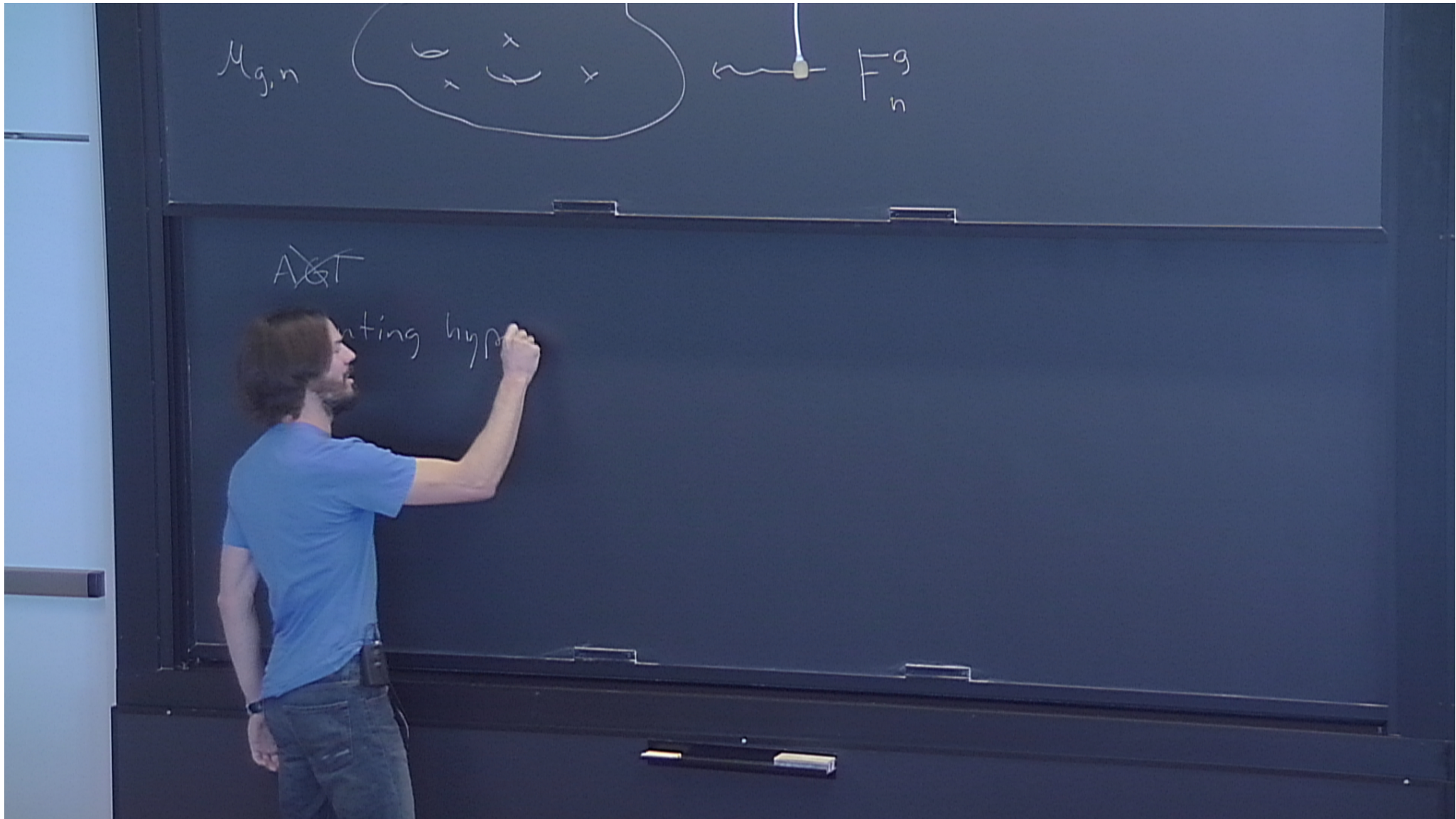
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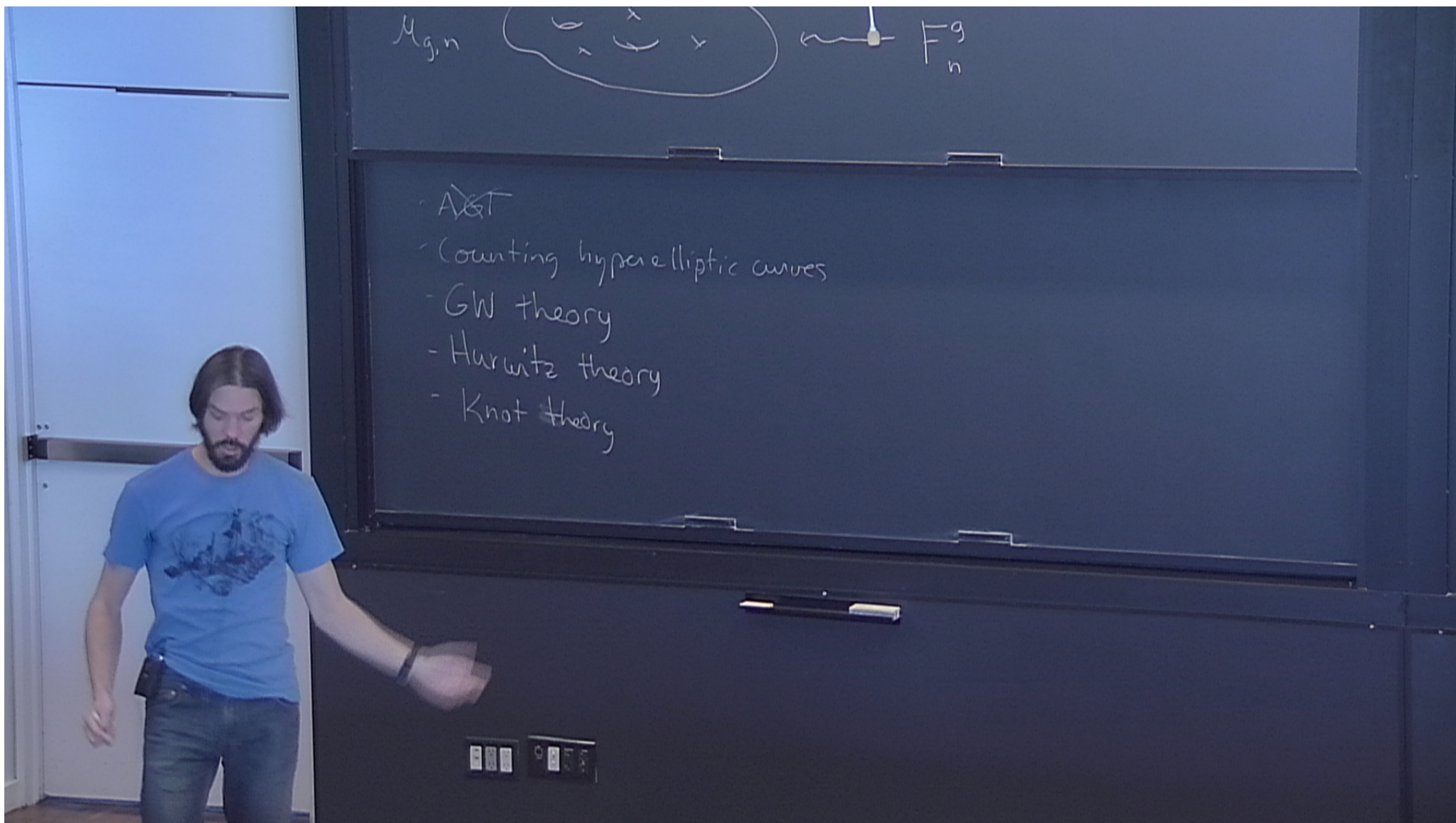


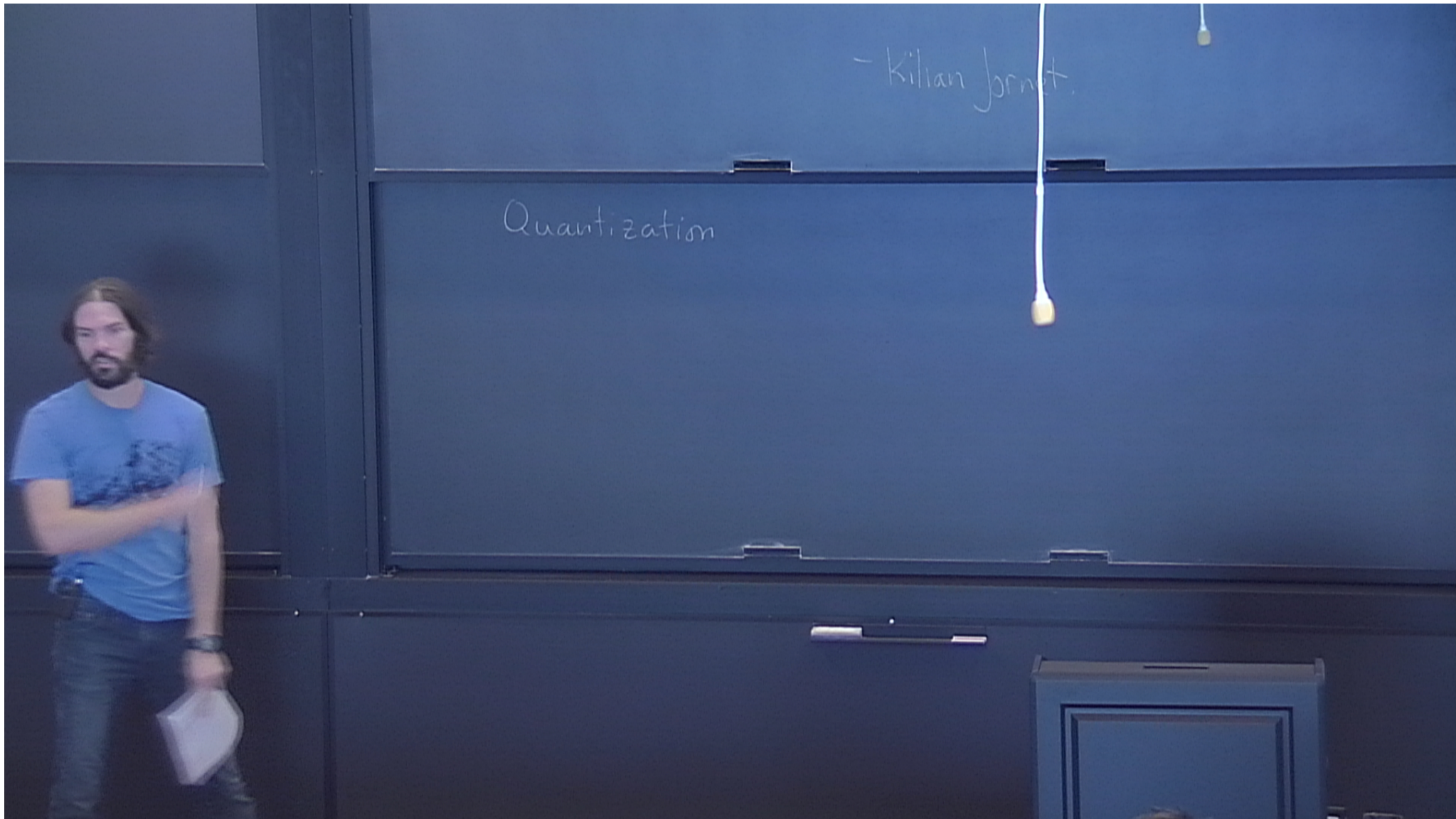
Quantum curves & topological recursion

(alternative title: "Some generating functions of something")
(thanks Noriko :))









- Killian Jornet.

Quantization

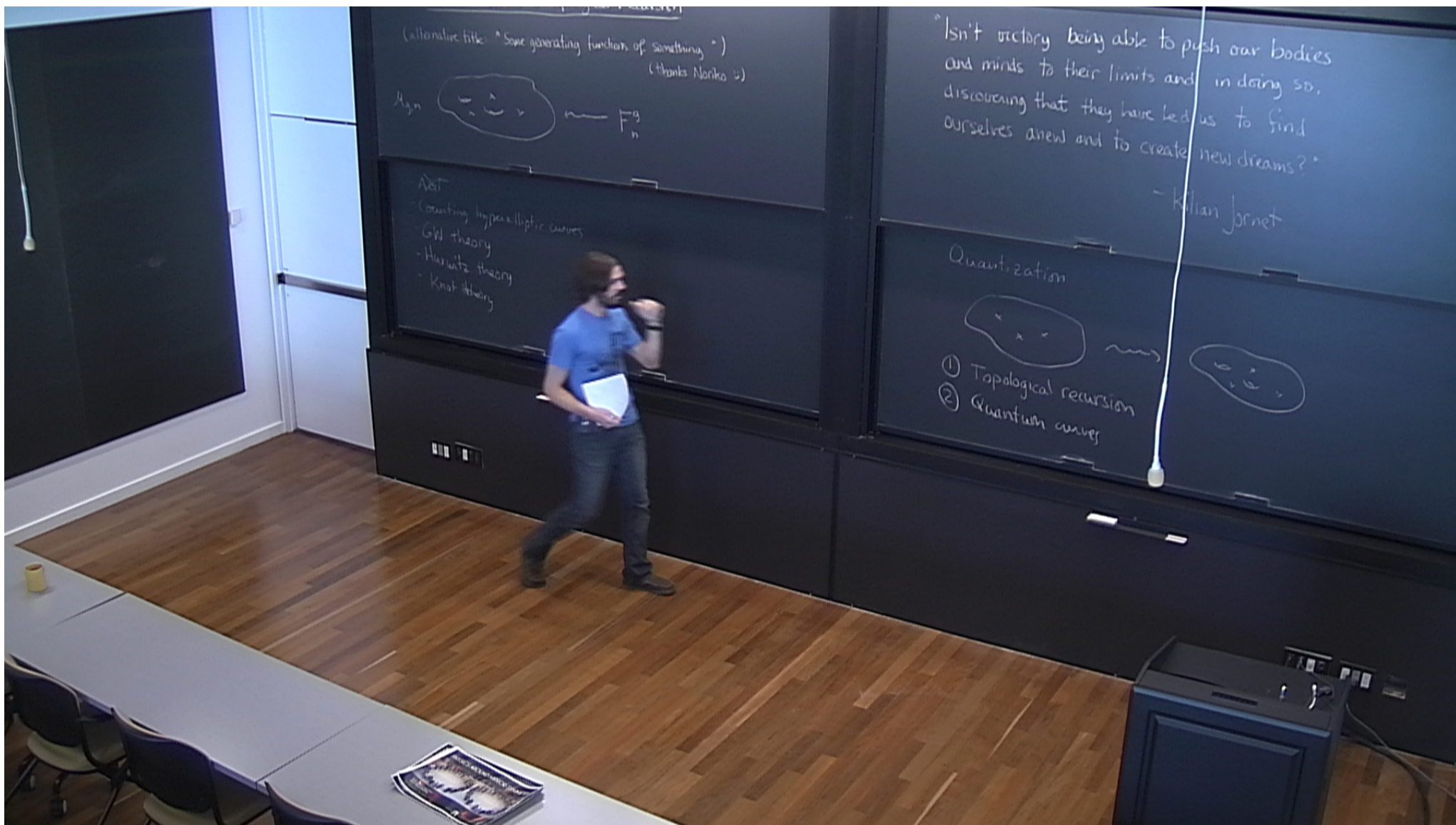


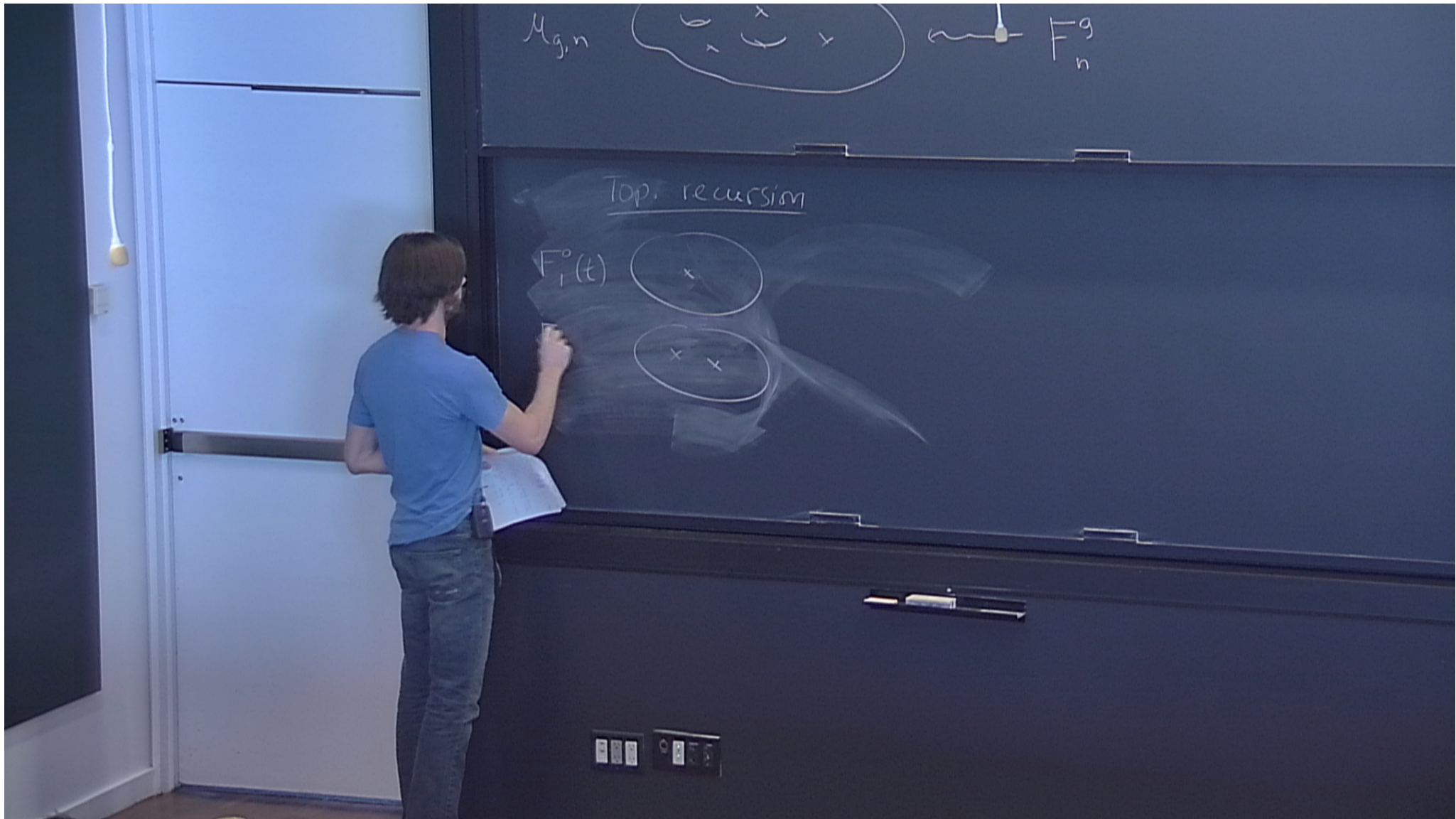
- Kilian Jornt.

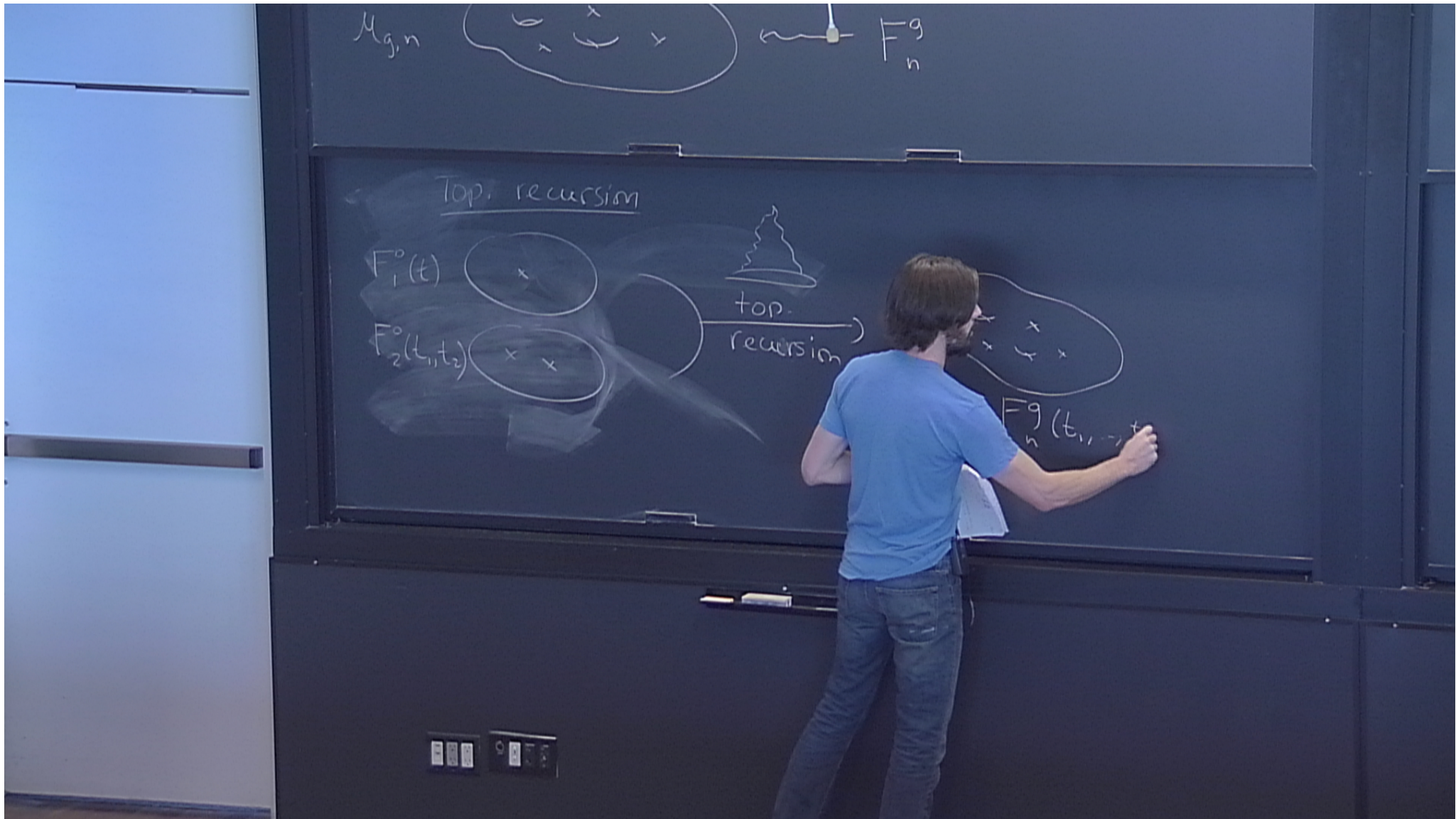
Quantization

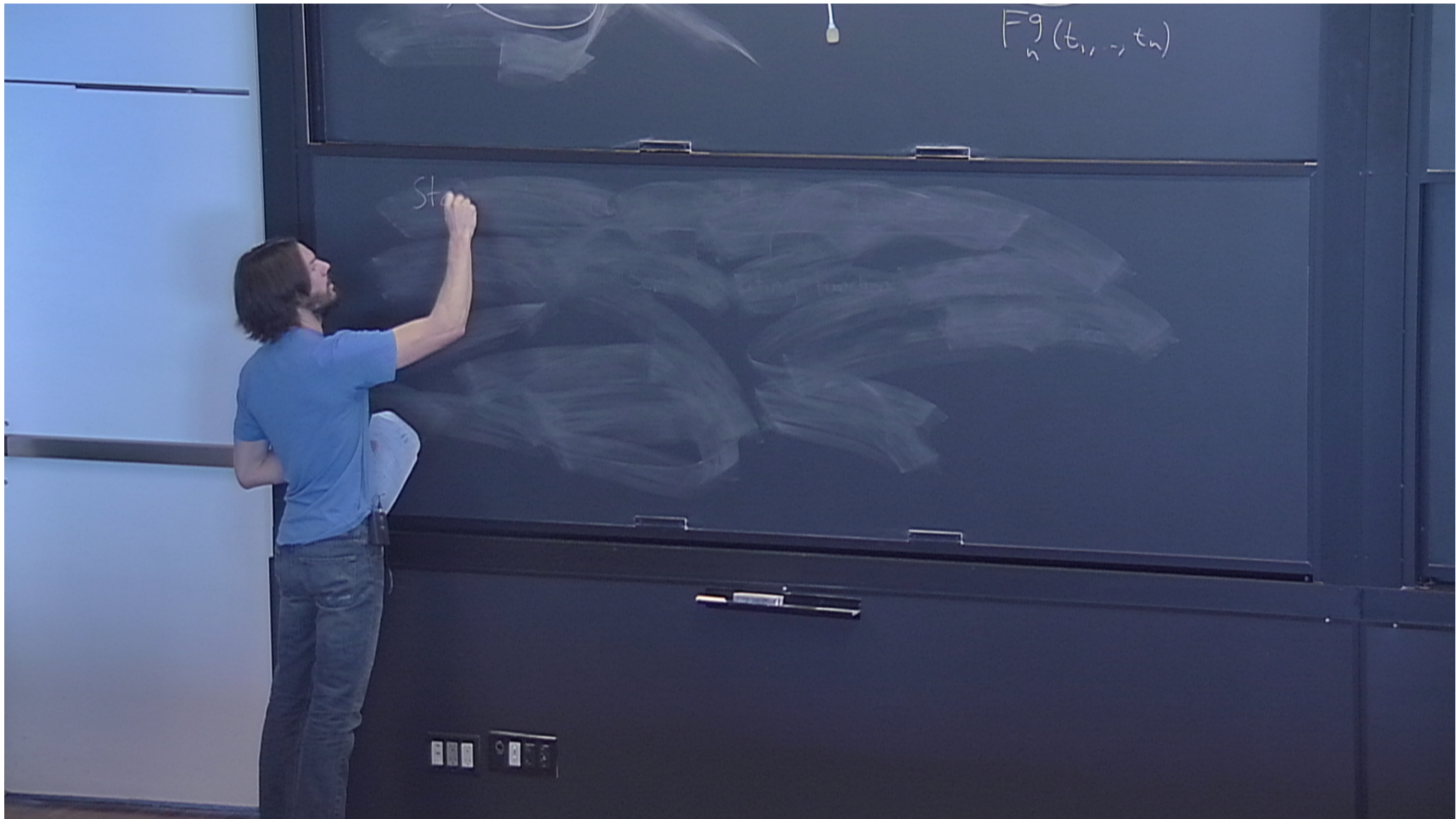


① Topological recursion









$$F_n^g(t_1, \dots, t_n)$$

Starting point

Defⁿ: A sp curve is a hol. Lagrangian immersion

$$\begin{array}{ccc} \Sigma^1 & \rightarrow & T^*\mathbb{C} \\ & & \downarrow \\ & & \mathbb{C} \end{array}$$

$$F_g^n(t_1, \dots, t_n)$$

Starting point

Defⁿ: A spectral curve is a hol. Lagrangian immersion

$$i: \Sigma_1 \rightarrow T^*\mathbb{C} \ni (x, y)$$

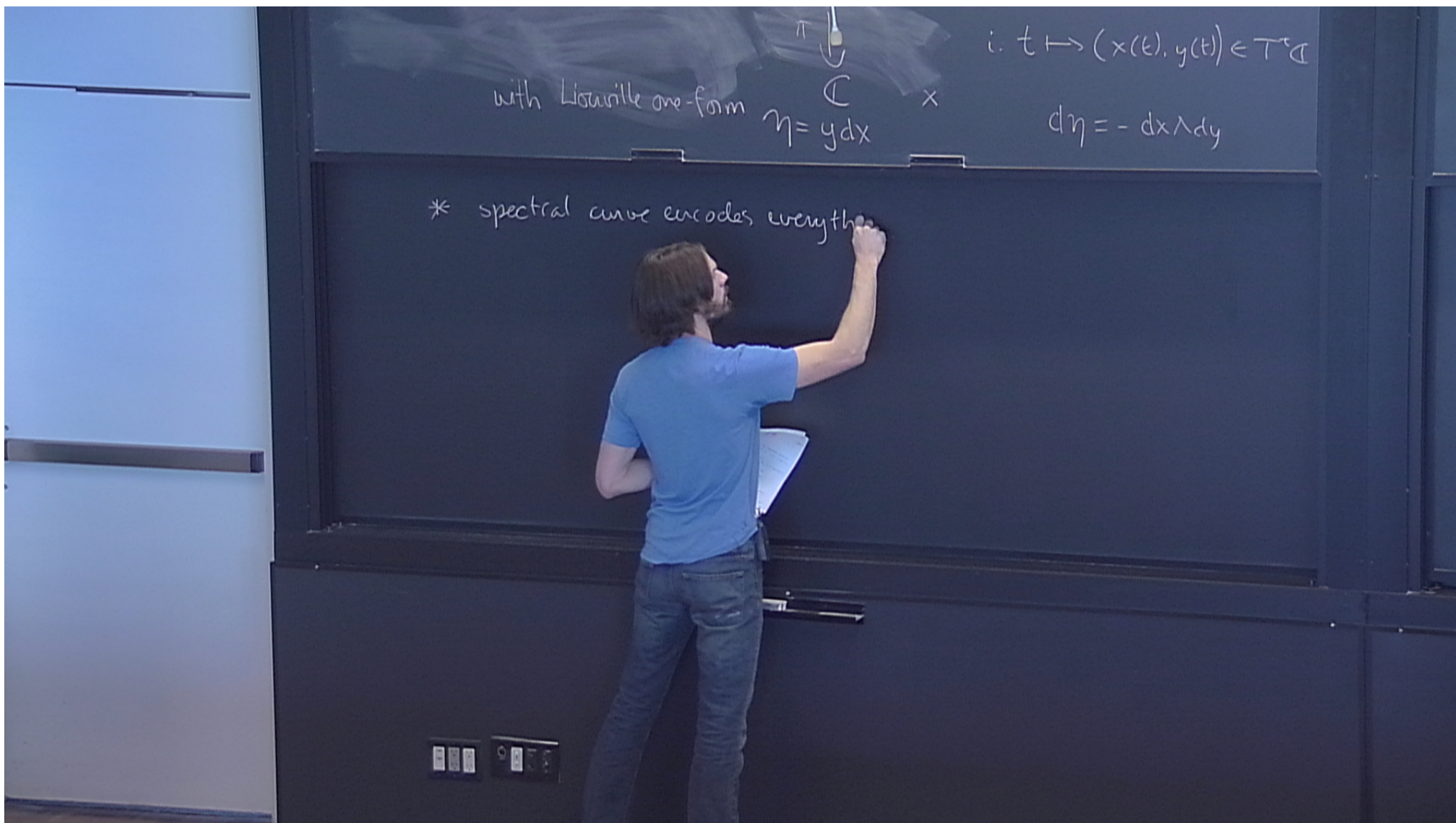
$$\pi \downarrow$$

$$i: t \mapsto (x(t), y(t)) \in T^*\mathbb{C}$$

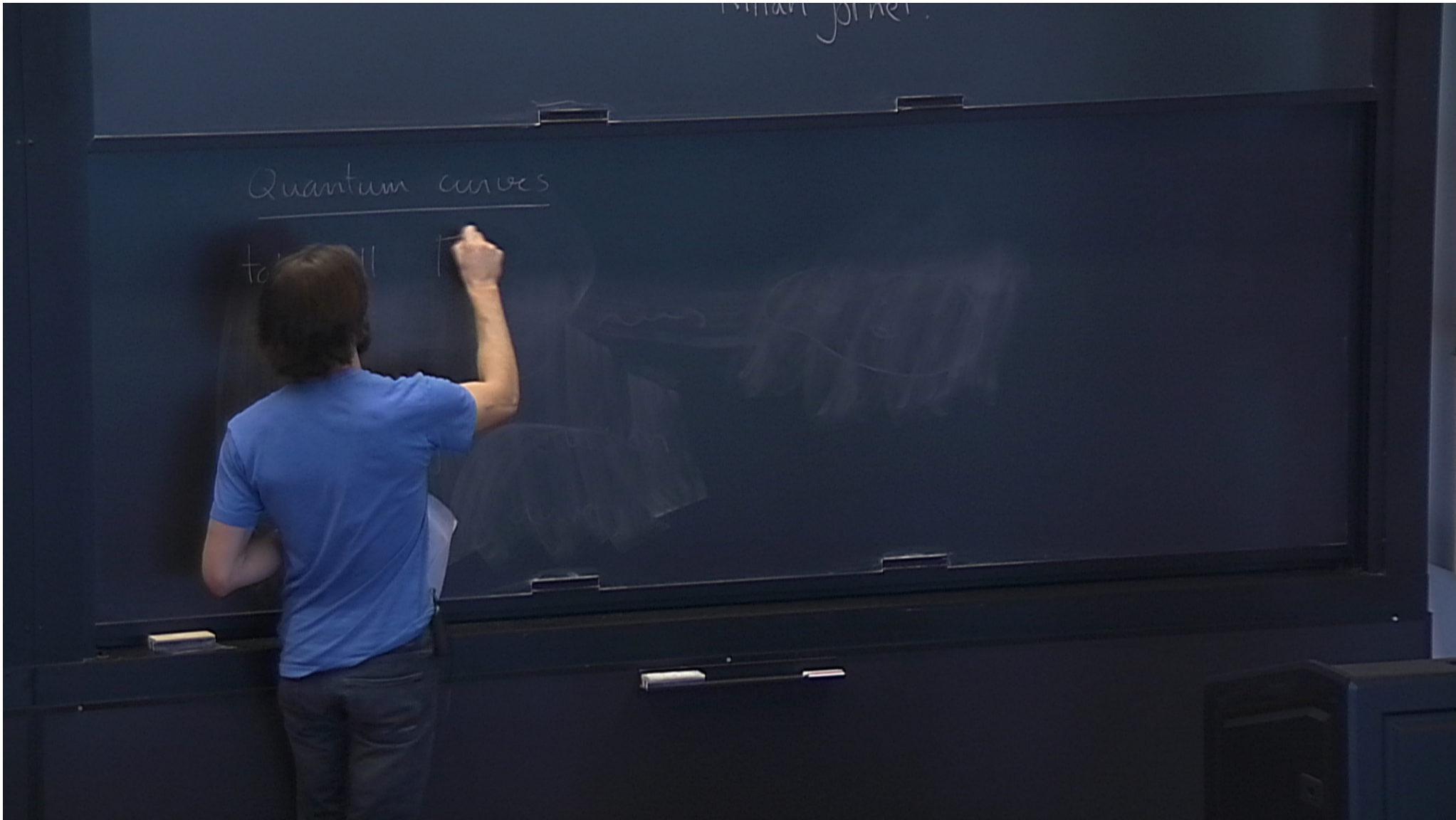
Liouville one-form

$$\mathbb{C} \ni x$$

$$\eta = y dx$$



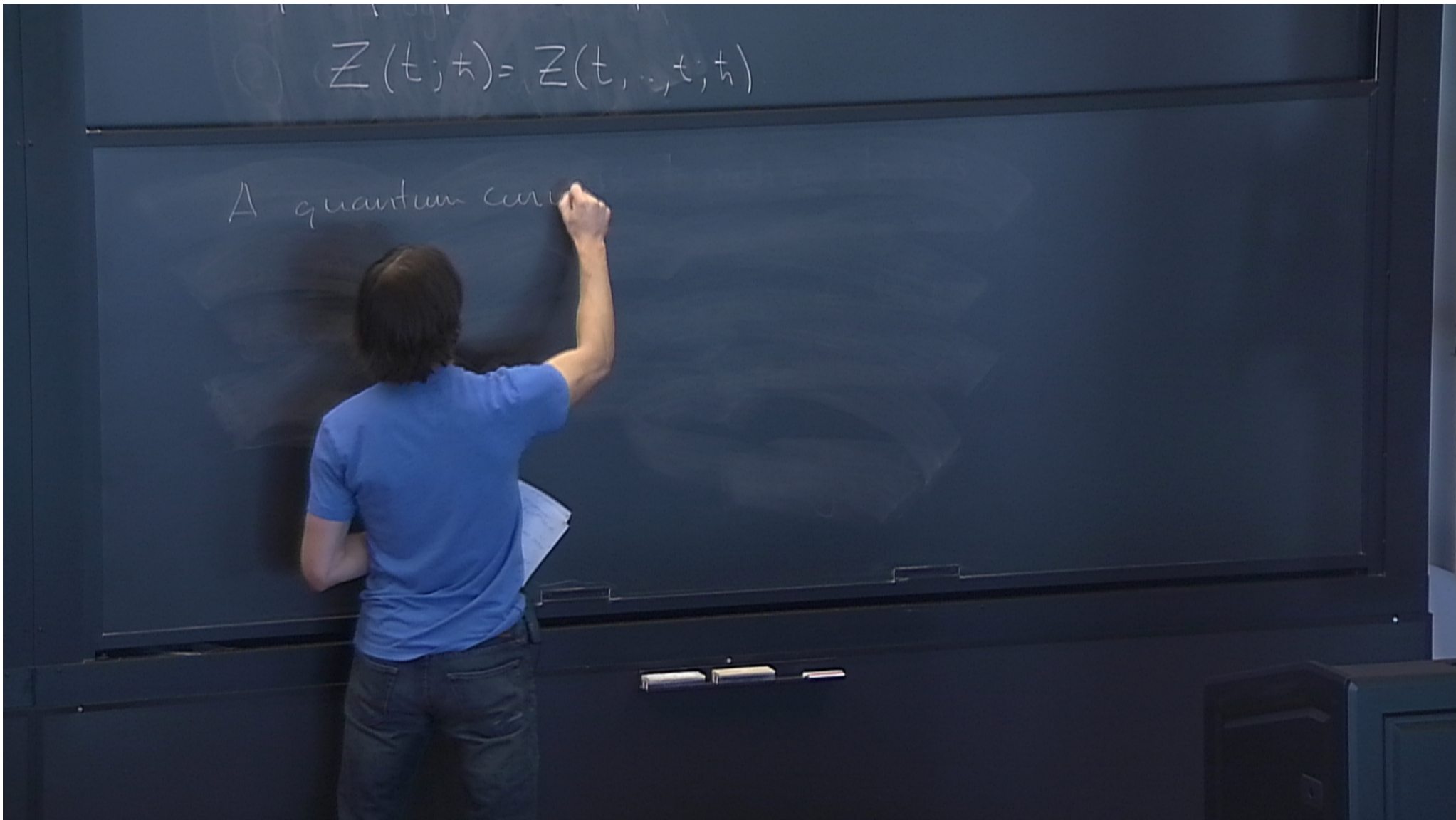


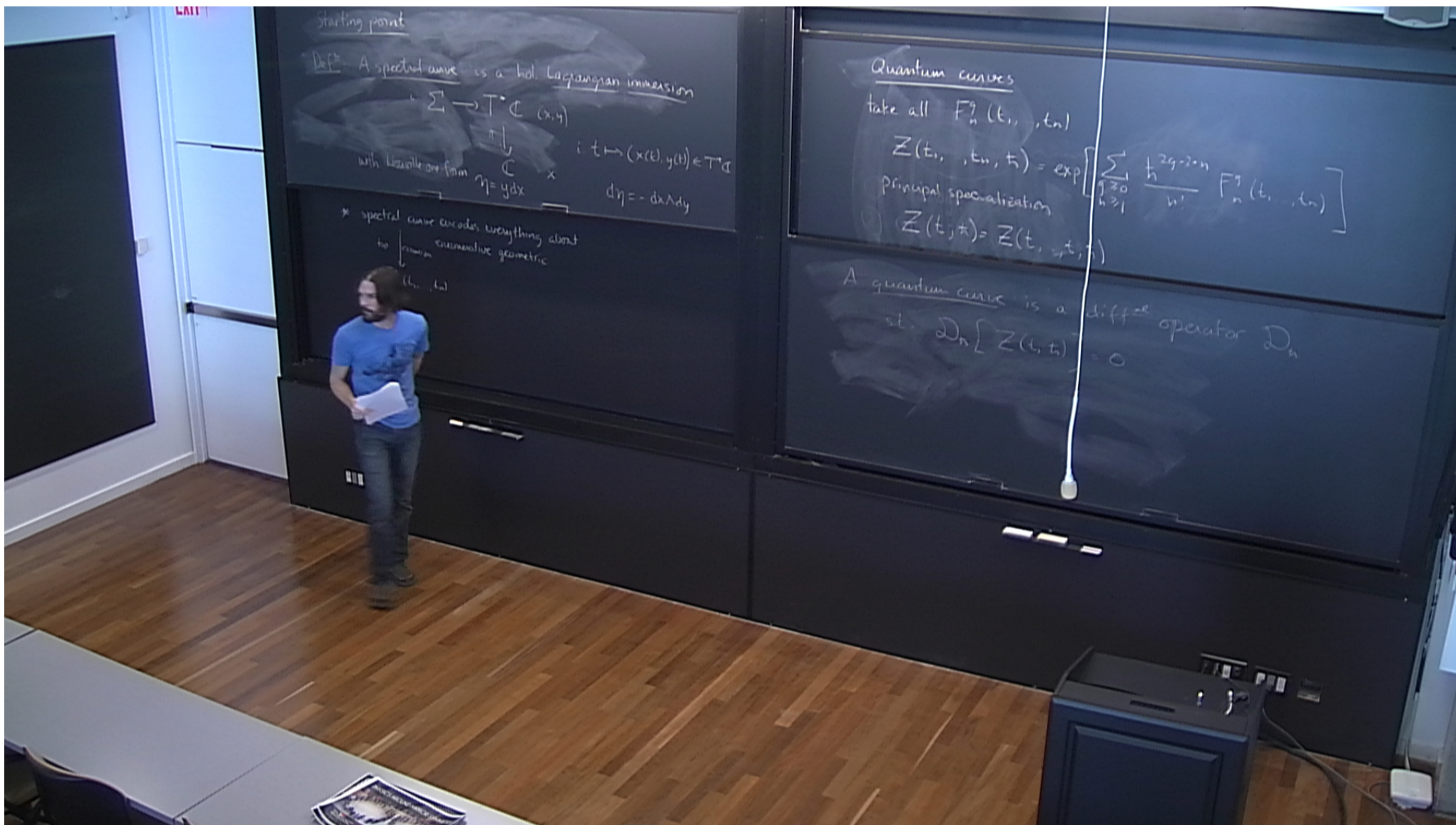


Quantum curves

take all $F_n^g(t_1, \dots, t_n)$

$$Z(t_1, \dots, t_n, \hbar) = \exp \left[\sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\hbar^{2g-2+n}}{n!} F_n^g(t_1, \dots, t_n) \right]$$

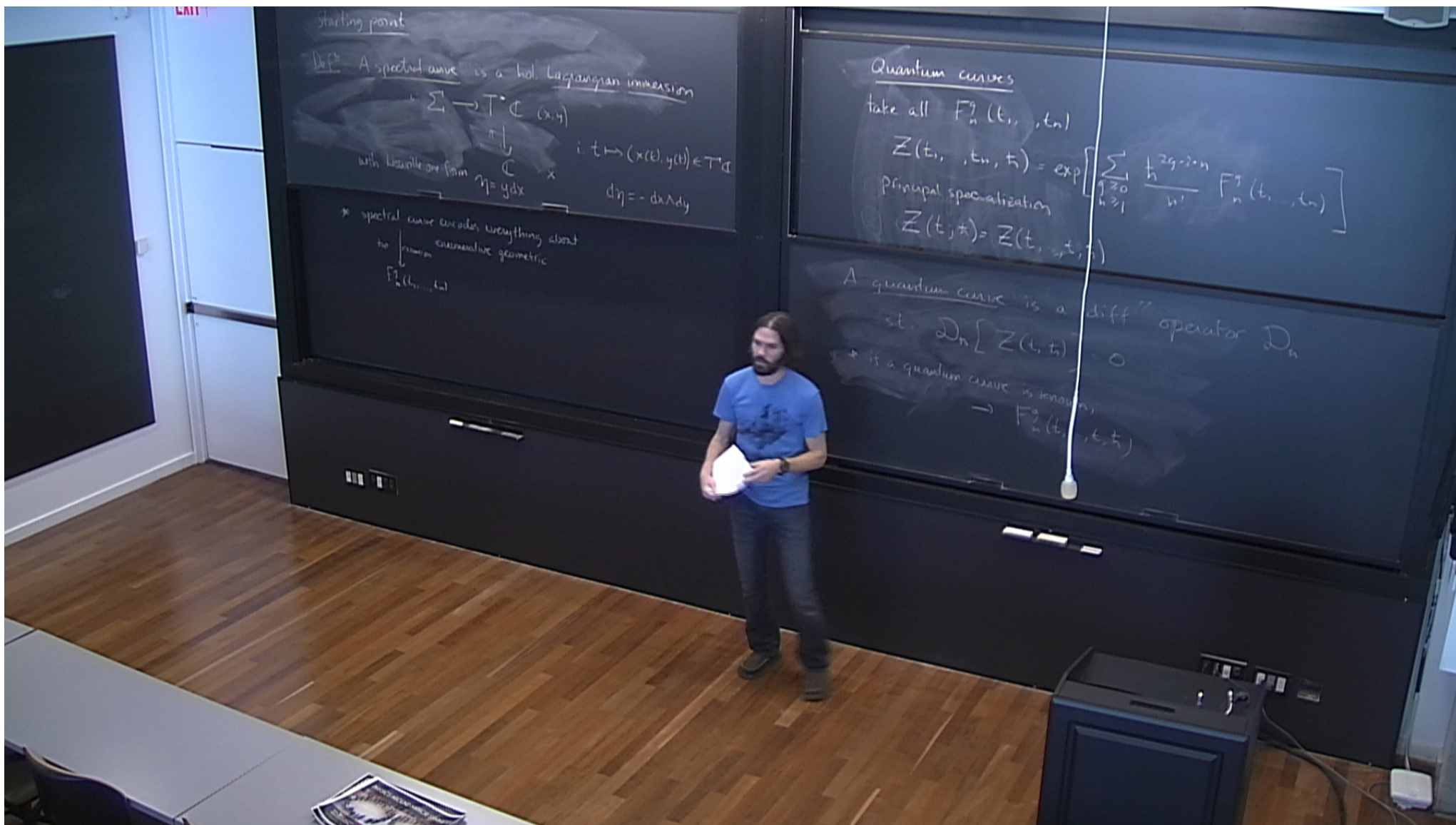


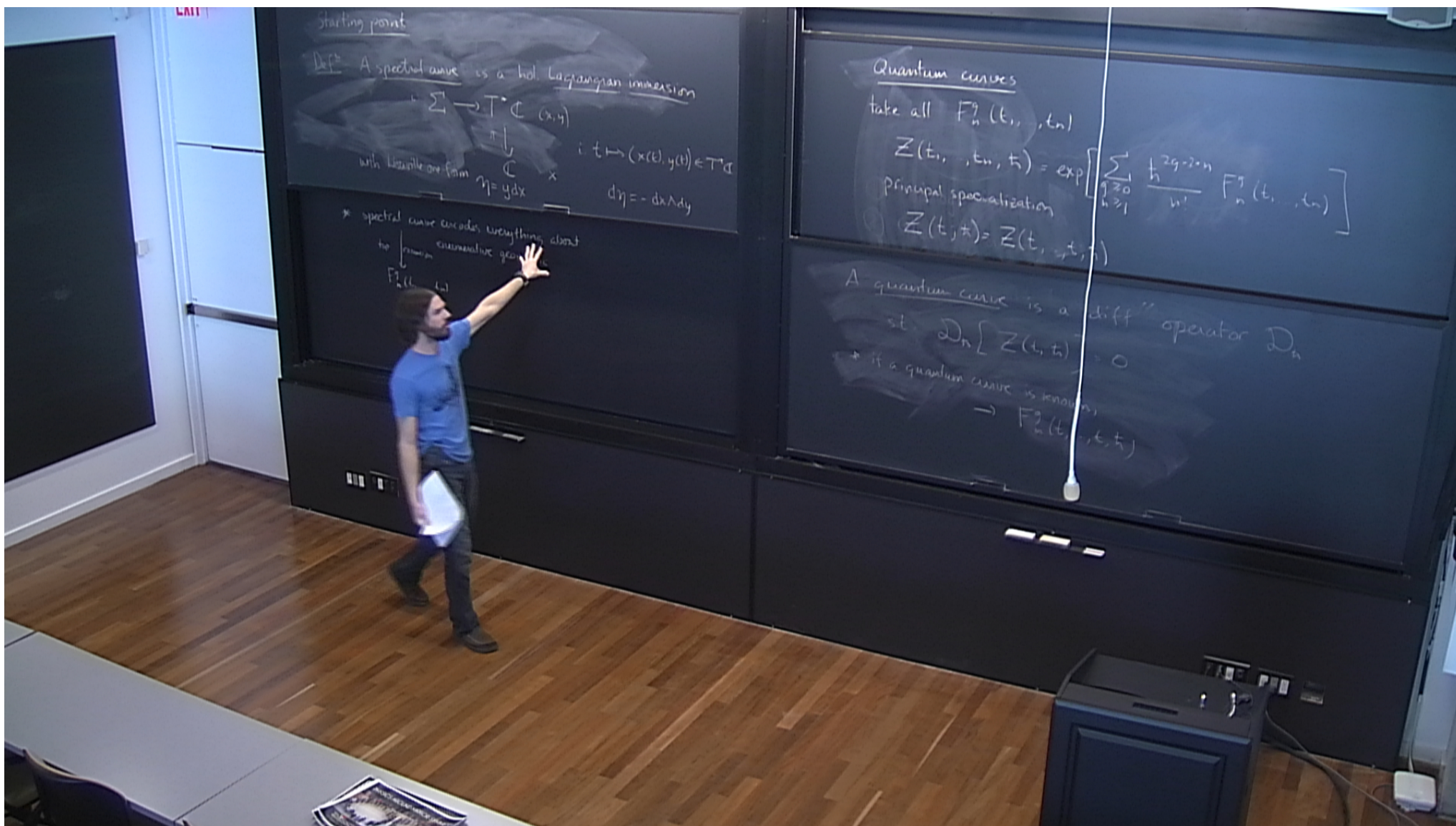


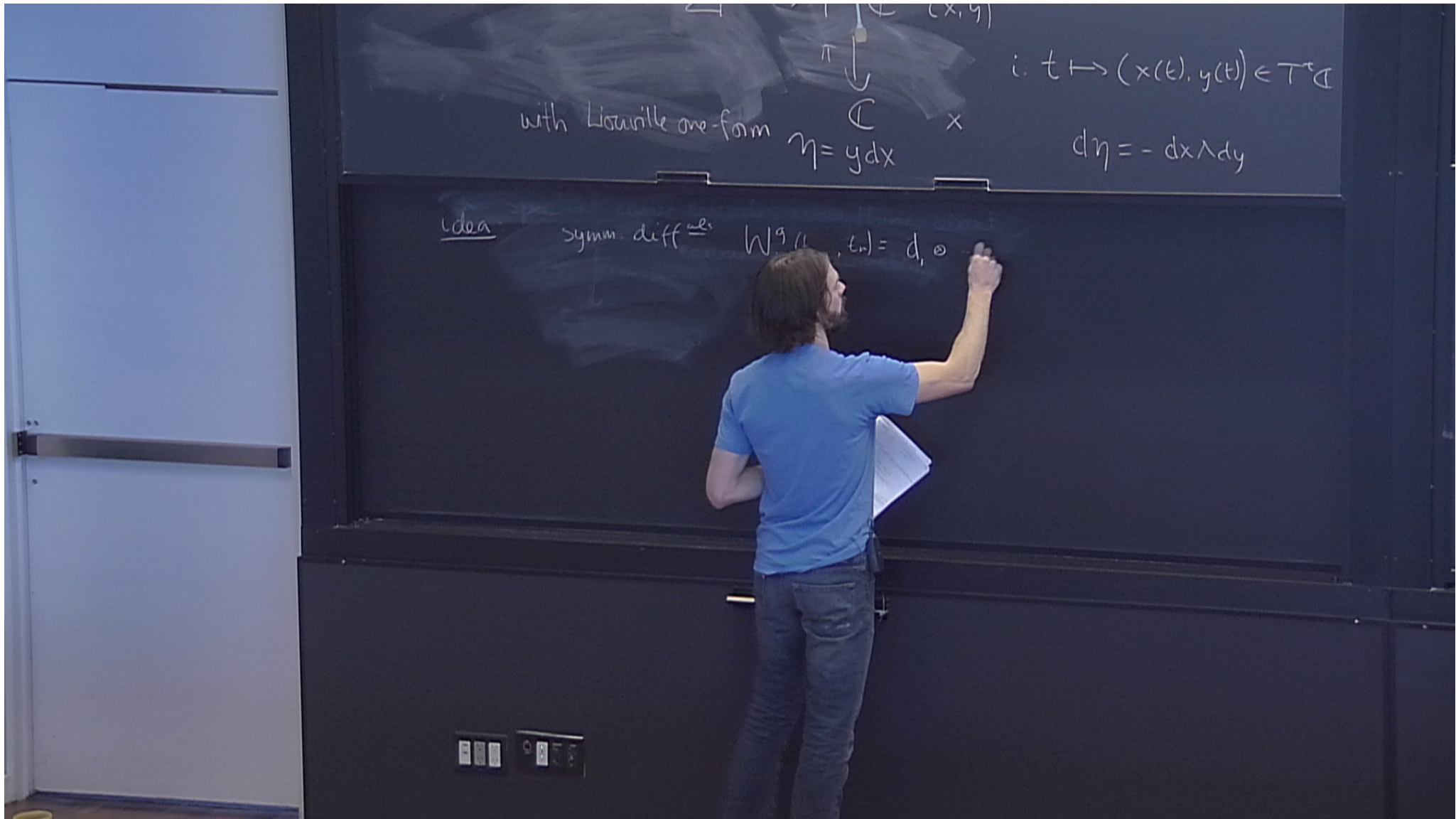
$$Z(t; \hbar) = Z(t, \dots, t, \hbar)$$

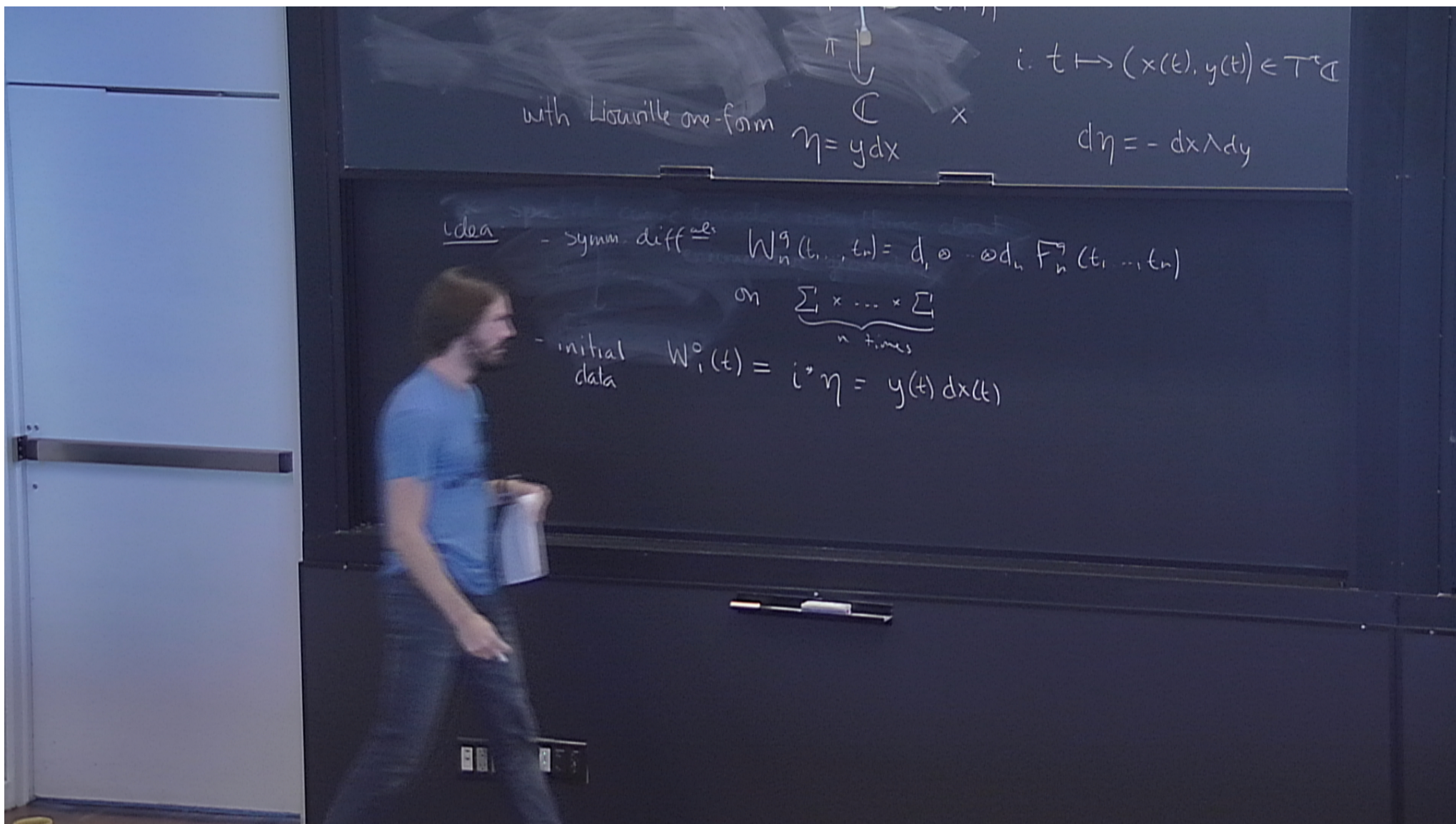
A quantum curve is a diffⁿ operator $\mathcal{D}_\hbar$

st. $\mathcal{D}_\hbar [Z(t, \hbar)] = 0$









$$Z(t; h) = Z(t, \dots, t, h)$$

$W_2^0(t_1, t_2)$: canonical bilinear di

$$Z(t;h) = Z(t, \dots, t, h)$$

$W_2^0(t_1, t_2)$: canonical bilinear diff^{al}

- symm. on $\Sigma \times \Sigma$

- double pole normal $(t_1 = t_2)$
w/ no $\dots$ & no $\dots$

$$U_{2,n}^g(q_1, q_2; \vec{p}) = W_{n+2}^{g-1}(q_1, \tau_2, \vec{p}) + \sum_{\substack{\tilde{q}_1 + \tilde{q}_2 = g \\ \tilde{J}_1 + \tilde{J}_2 = n}} \left[W_{\tilde{J}_1+1,1}^{\tilde{q}_1}(q_1, \tilde{J}_1) W_{\tilde{J}_2+1}^{\tilde{q}_2}(q_2, \tilde{J}_2) \right]$$

Starting point

Defⁿ A spectral curve is a hol. Lagrangian immersion

$$i: \Sigma \rightarrow T^*\mathbb{C} \quad (x, y)$$

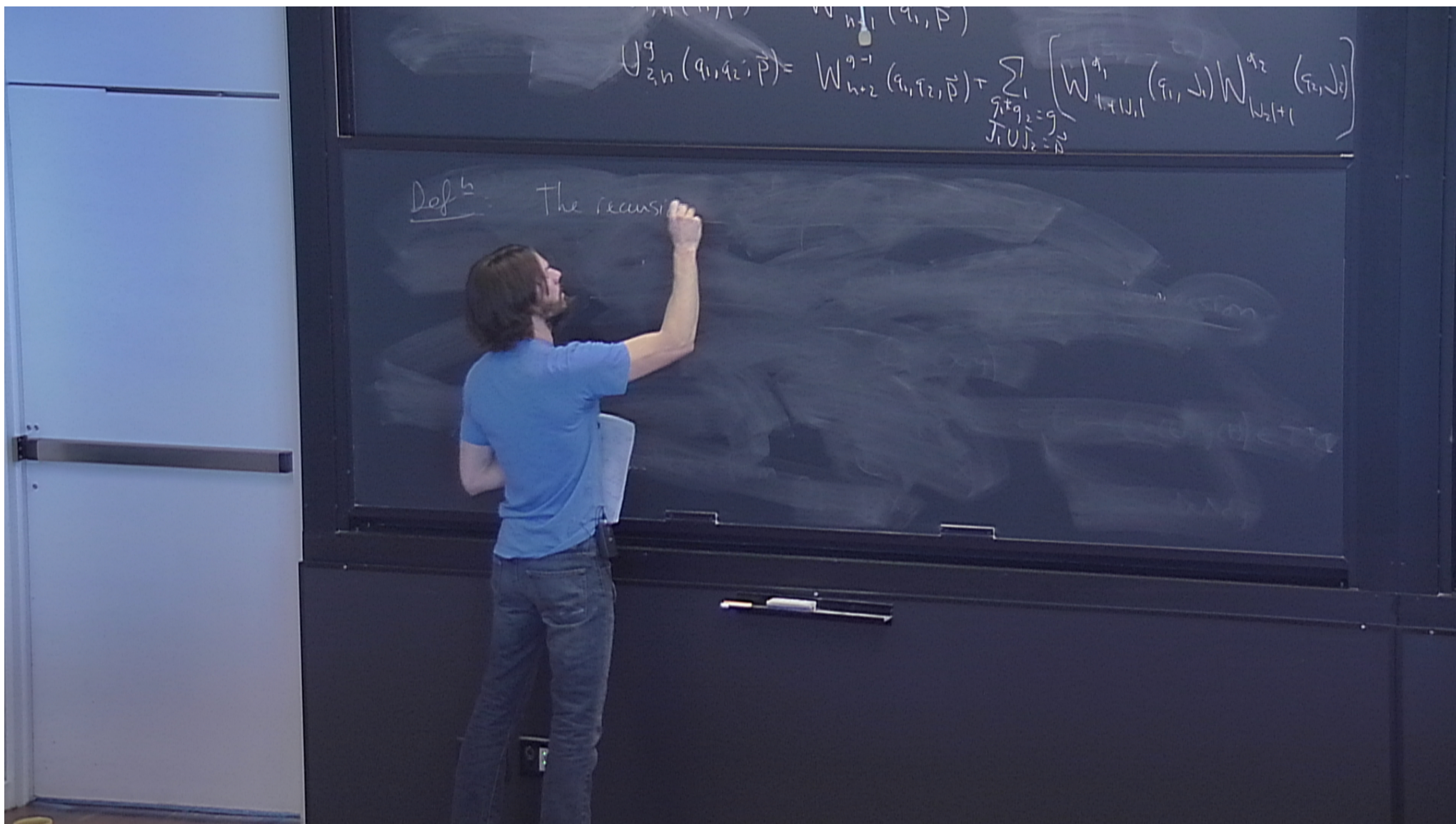
$$\pi \downarrow$$

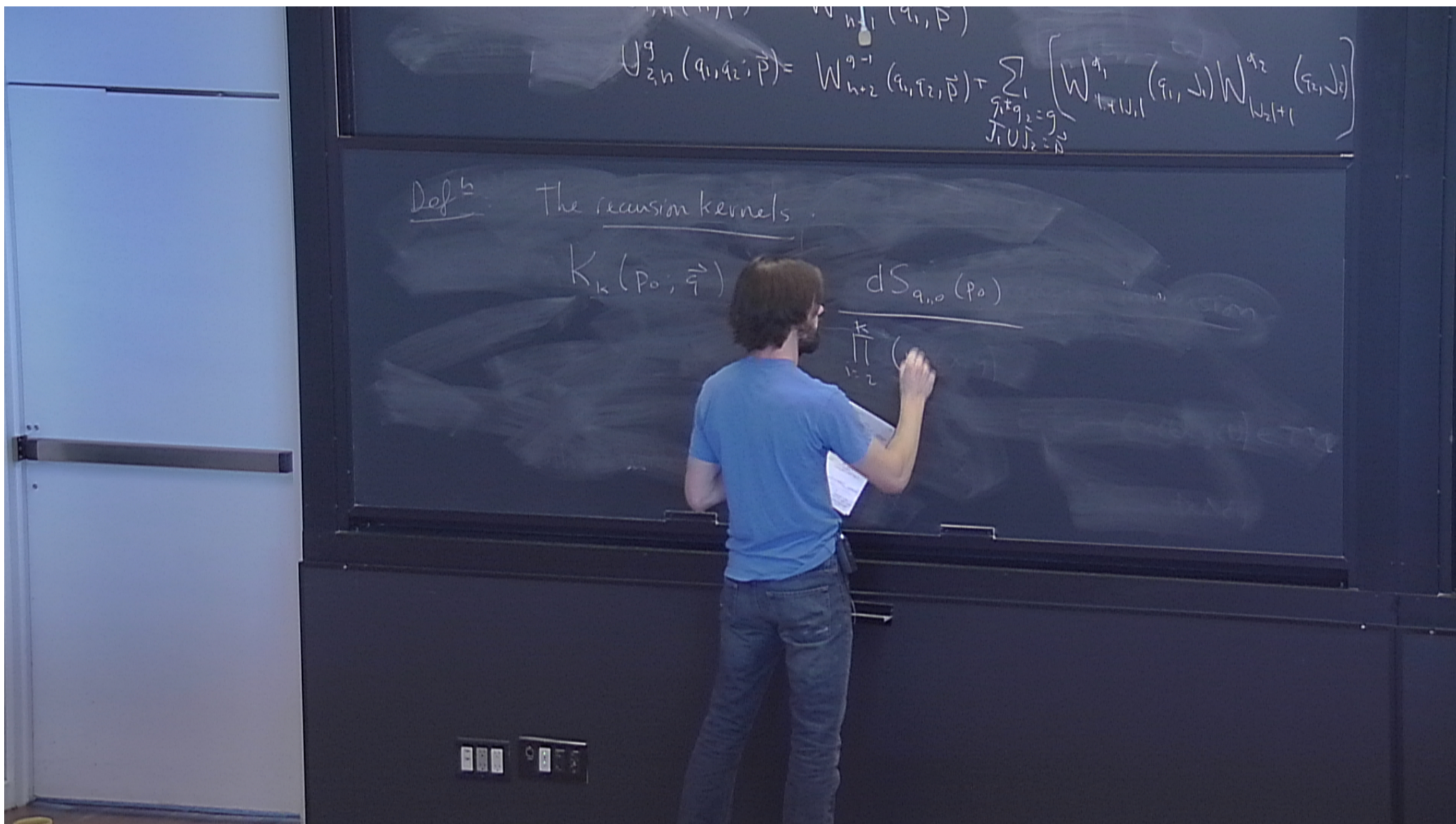
$$\mathbb{C} \quad x$$

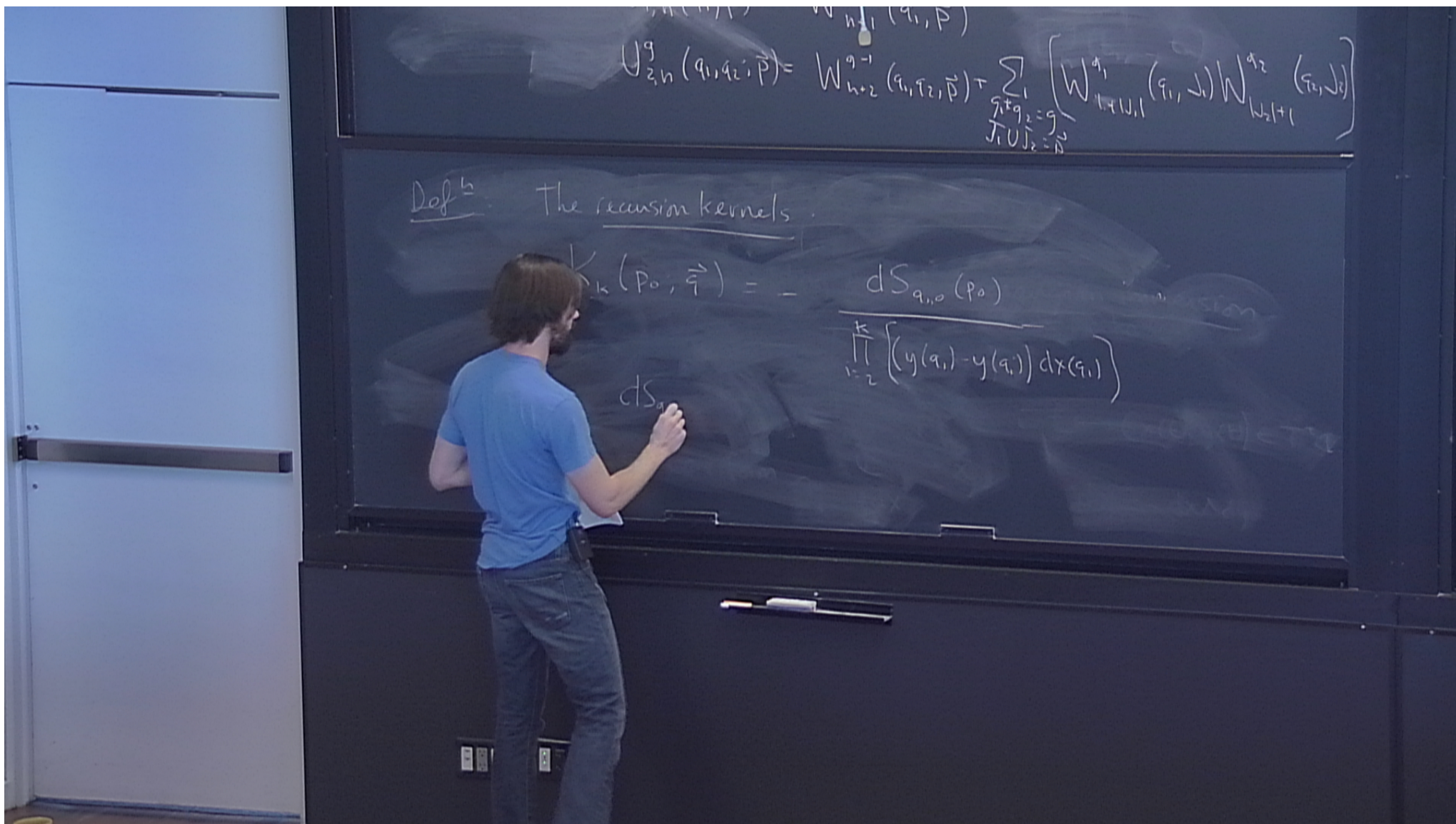
$$\eta = y dx$$

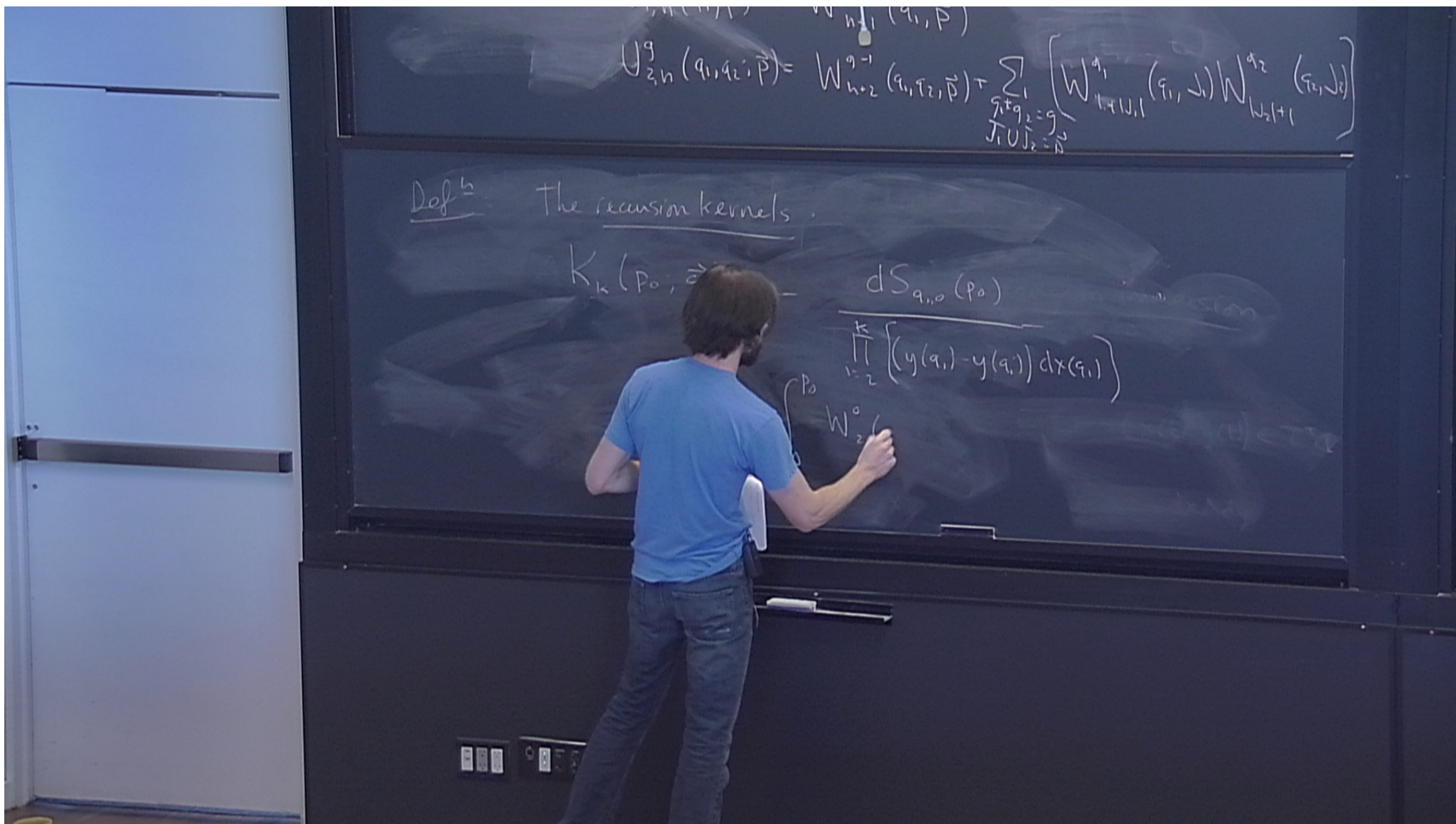
$$i: t \mapsto (x(t), y(t)) \in T^*\mathbb{C}$$

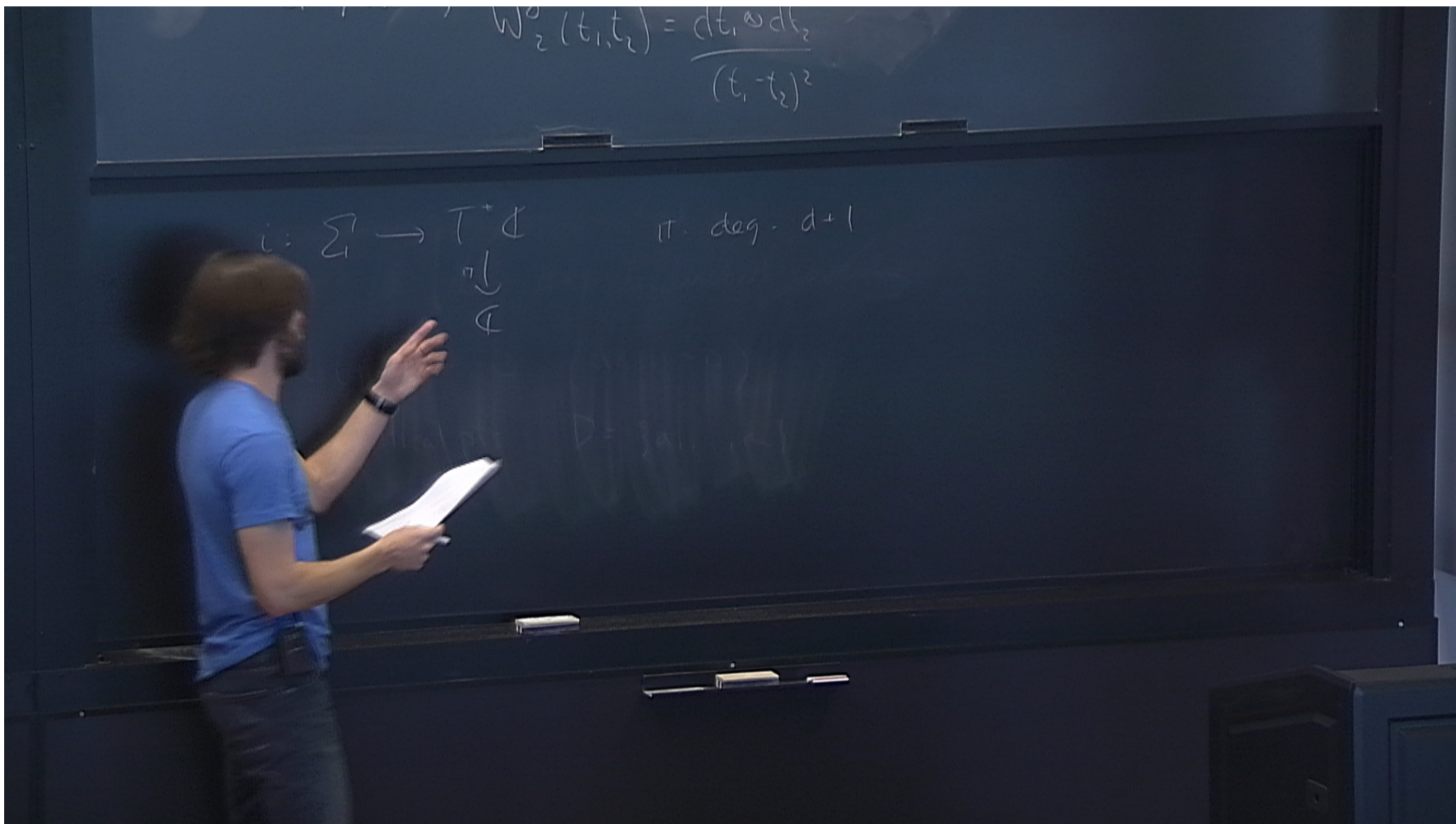
$$d\eta = -dx \wedge dy$$

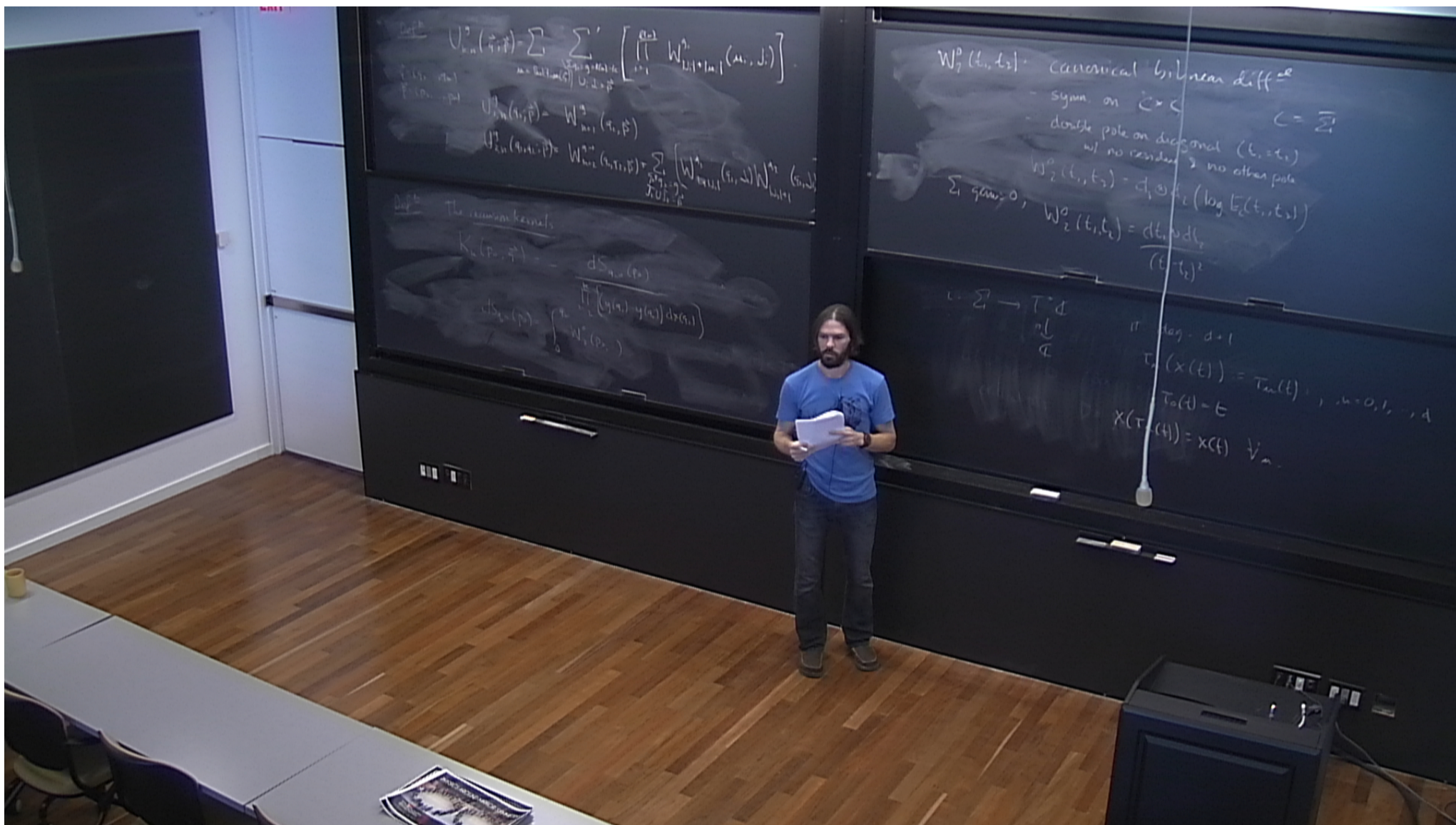


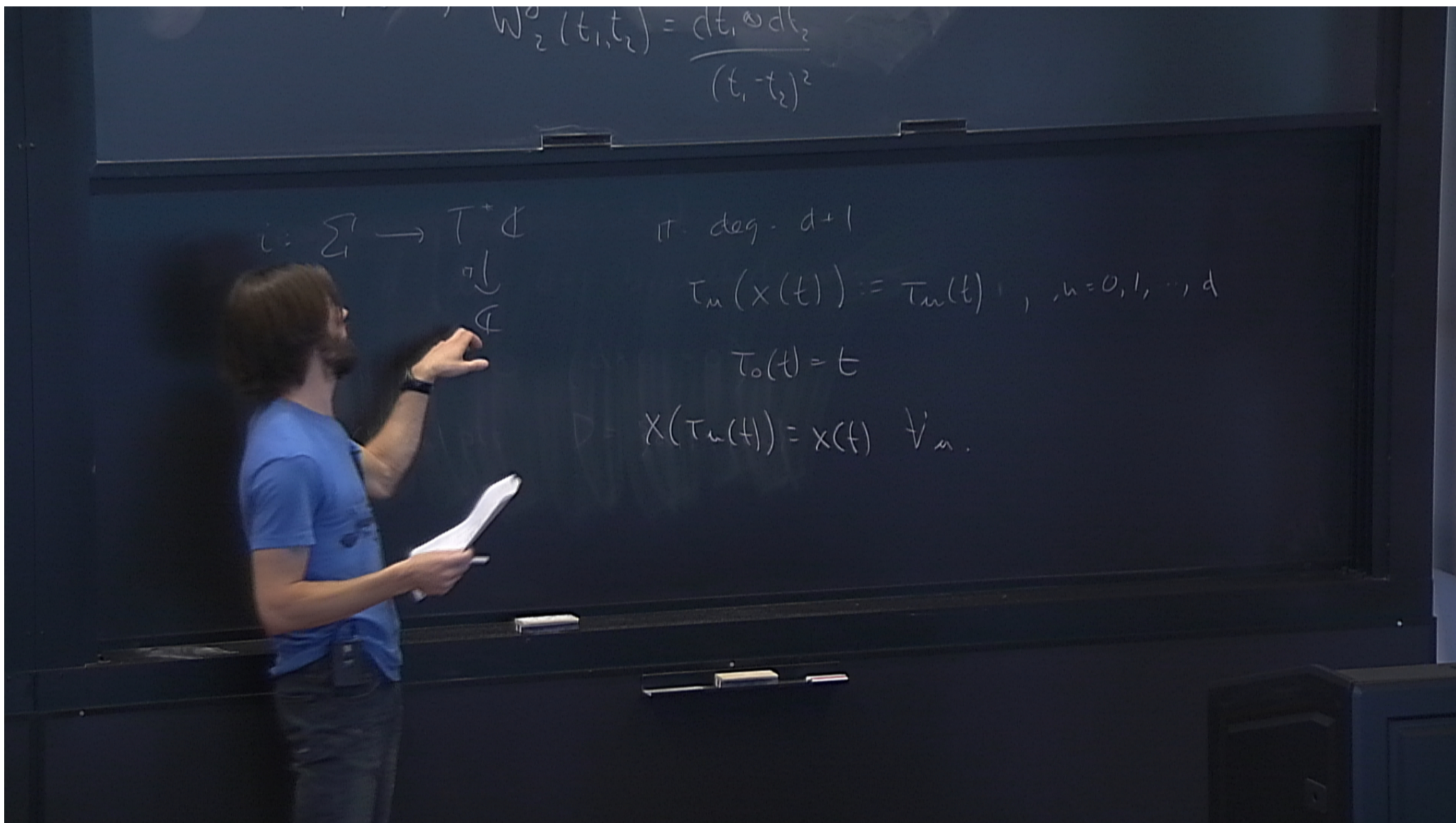


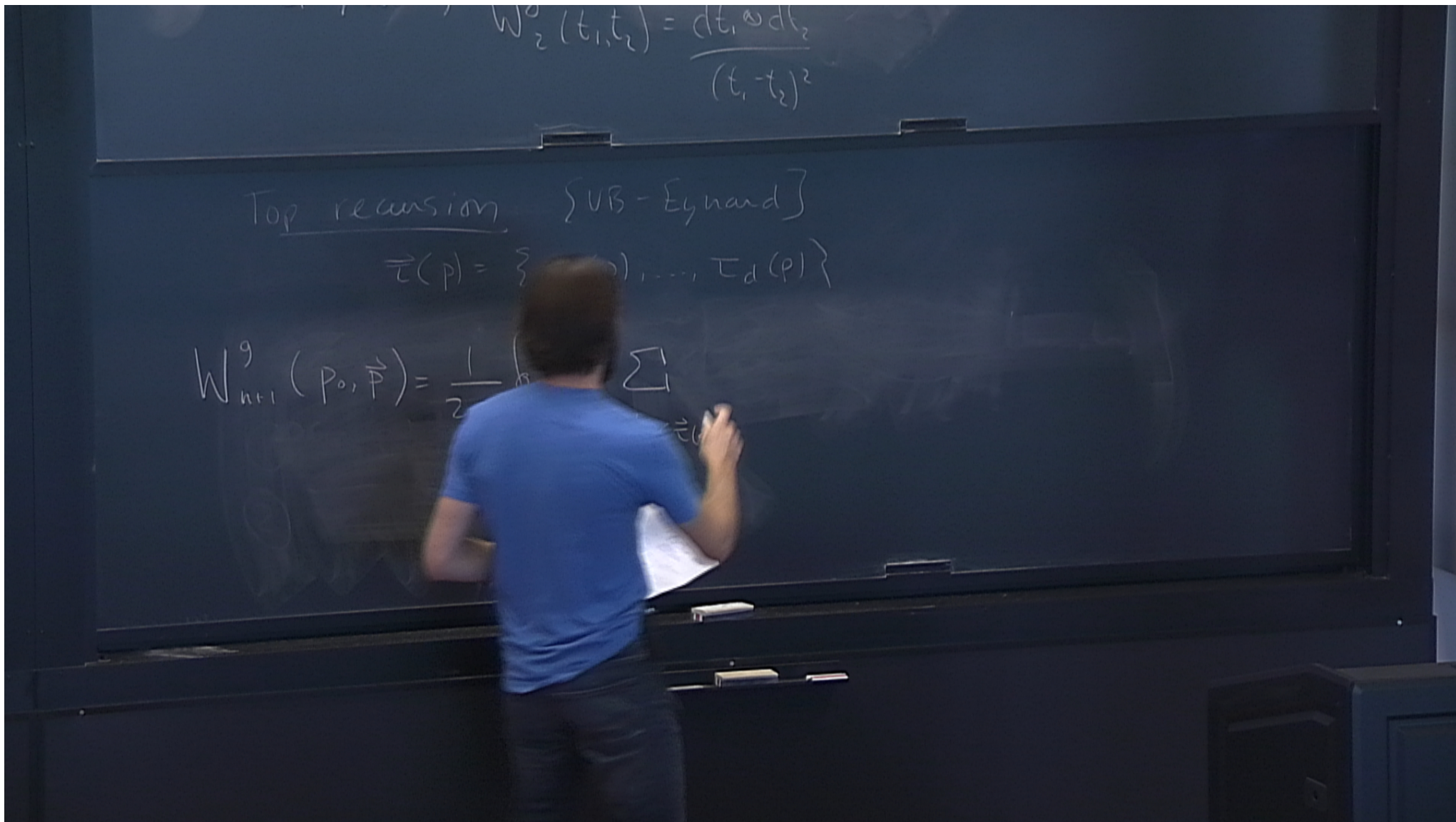


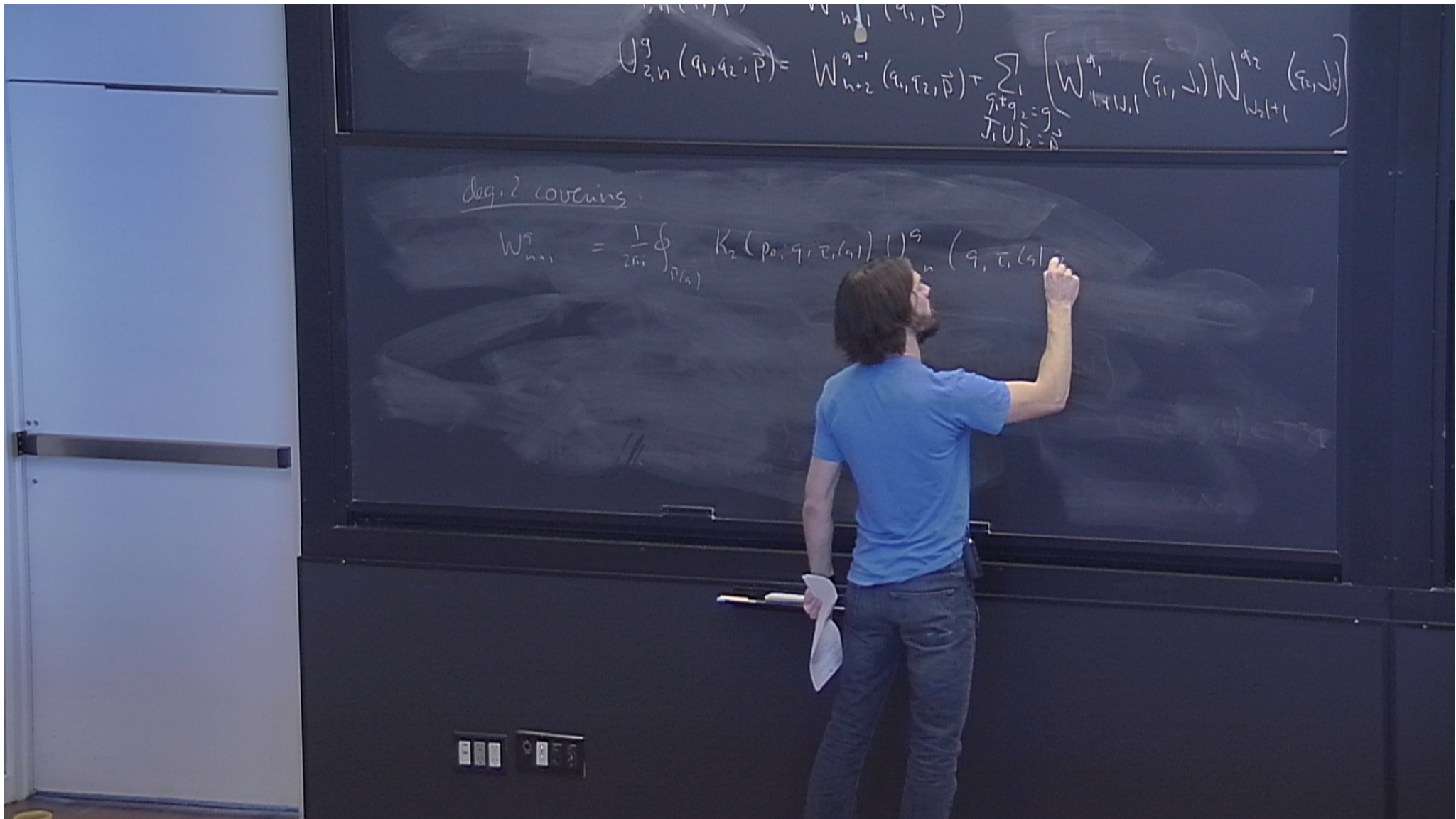












$$U_{2,n}^g(q_1, q_2, \vec{p}) = W_{n+2}^{g-1}(q_1, \tau_2, \vec{p}) + \sum_{\substack{\vec{q}_1 + \vec{q}_2 = \vec{g} \\ J_1 \cup J_2 = \vec{n}}} \left[W_{|J_1|+1}^{q_1}(q_1, J_1) W_{|J_2|+1}^{q_2}(q_2, J_2) \right]$$

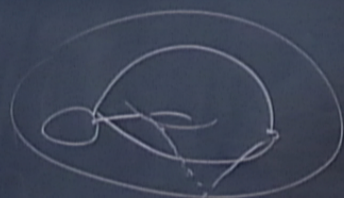
deg. 2 covering

$$W_{n+1}^g = \frac{1}{2\pi i} \oint_{\Gamma(g)} K_2(p_0, q, \tau, (g)) U_n^g(q, \tau, (g))$$

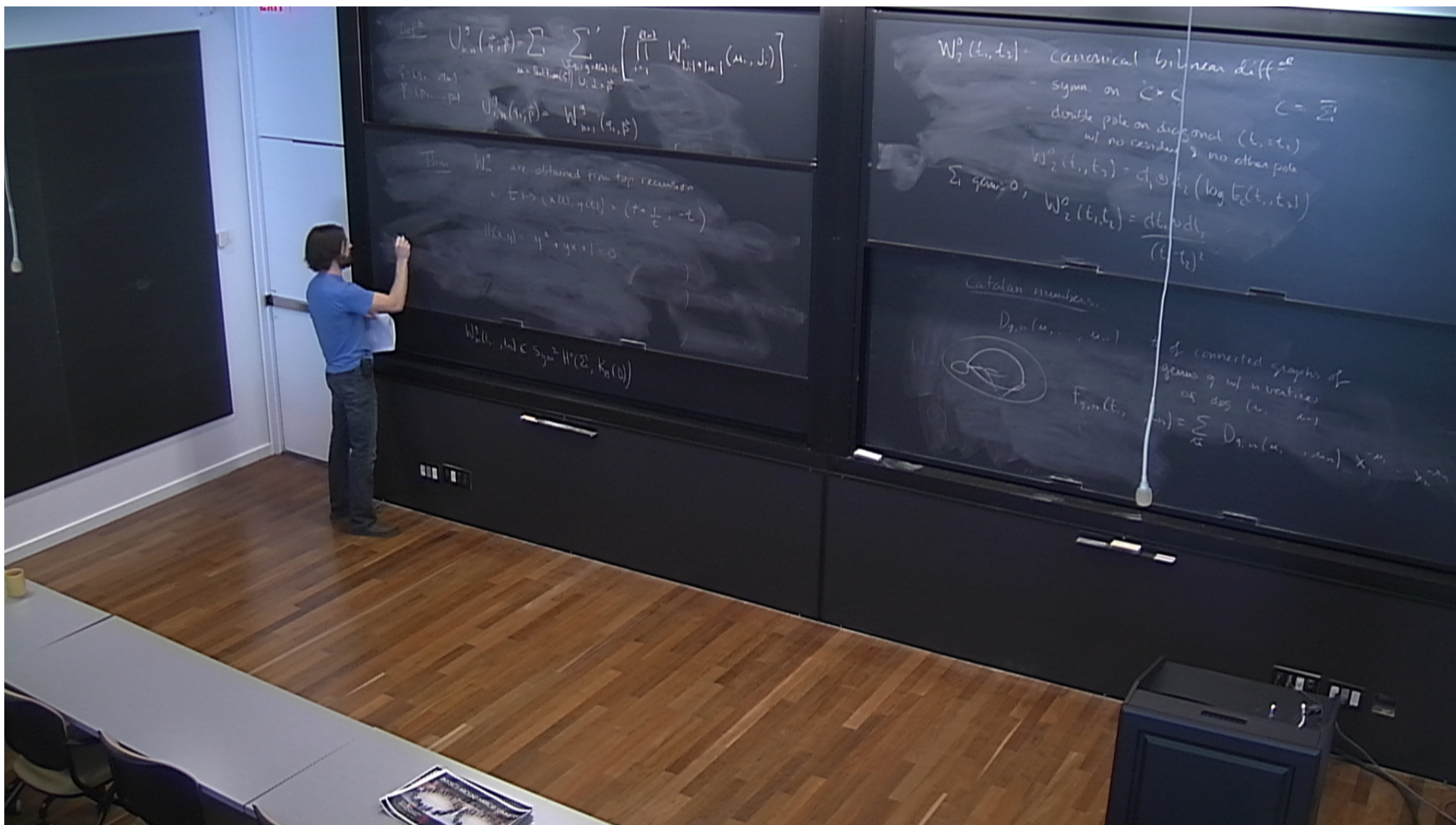
$$D_2(t_1, t_2) = \frac{t_1 - t_2}{(t_1 - t_2)^2}$$

Catalan numbers.

$D_{g,n}(u_1, \dots, u_n)$ = # of connected graphs of
 genus g w/ n vertices
 of deg $(u_1, \dots, u_n)$



$F_{g,n} = \sum$



$$U_{n,n}^0(\vec{q}, \vec{p}) = \sum_{m=0}^n \sum_{j=0}^m \left[\prod_{i=1}^m W_{|j_i|+1, |m_i|}^0(\mu_i, j_i) \right]$$

$$U_{n,n}^0(\vec{q}, \vec{p}) = W_{n+1, n}^0(\vec{q}, \vec{p})$$

Then W_n^0 are obtained from top recursion

$$t \mapsto (x(t), y(t)) = \left(t + \frac{1}{t}, -t\right)$$

$$H(x, y) = y^2 + yx + 1 = 0$$

$$W_{n,n}^0(\mu, \mu) \in \text{Sym}^2 H^0(\mathbb{P}^1, K_H(0))$$

$W_2^0(t_1, t_2)$ - canonical bilinear diff^{eq}

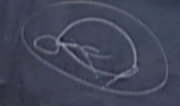
- symm. on $\mathbb{C} \times \mathbb{C}$
- double pole on diagonal (t_1, t_2) w/ no residues $\rightarrow$ no other poles

$$W_2^0(t_1, t_2) = c_1 \otimes c_2 (\log E_2(t_1, t_2))$$

$$\sum_i \text{gen}_i = 0, \quad W_2^0(t_1, t_2) = \frac{(dt_1 \otimes dt_2)}{(t_1 - t_2)^2}$$

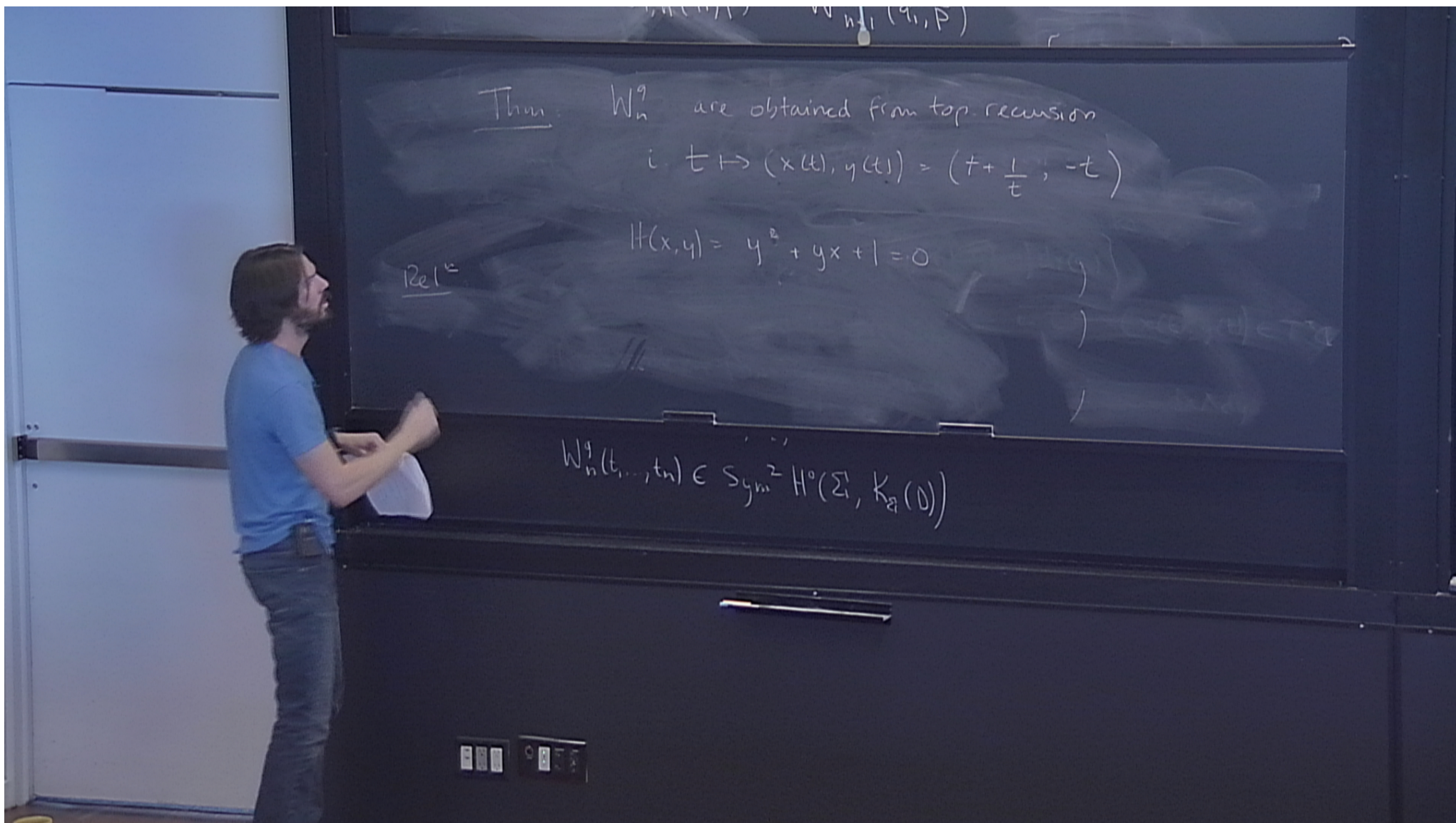
Catalan numbers:

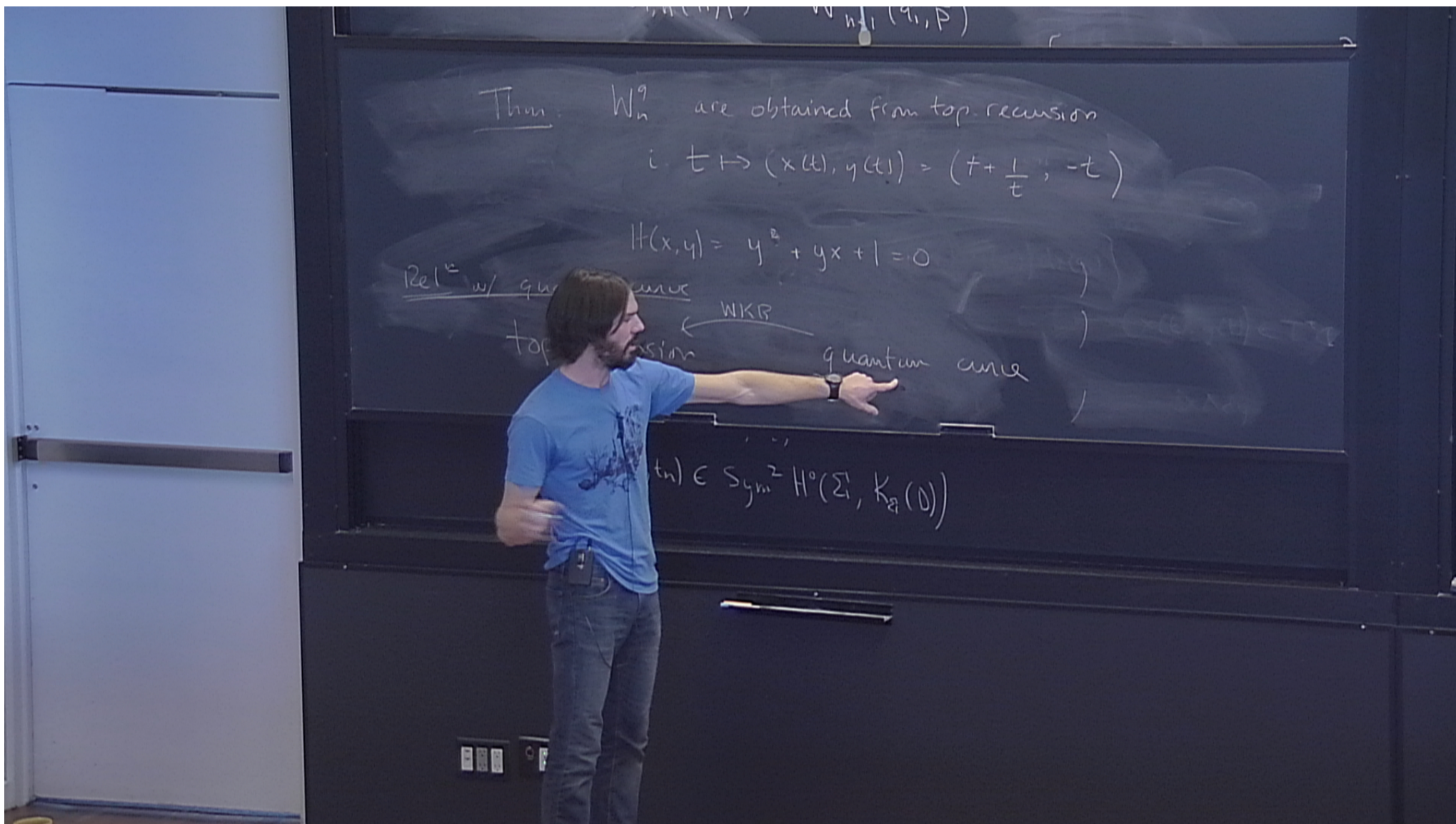
$$D_{g,n}(\mu_1, \dots, \mu_n)$$

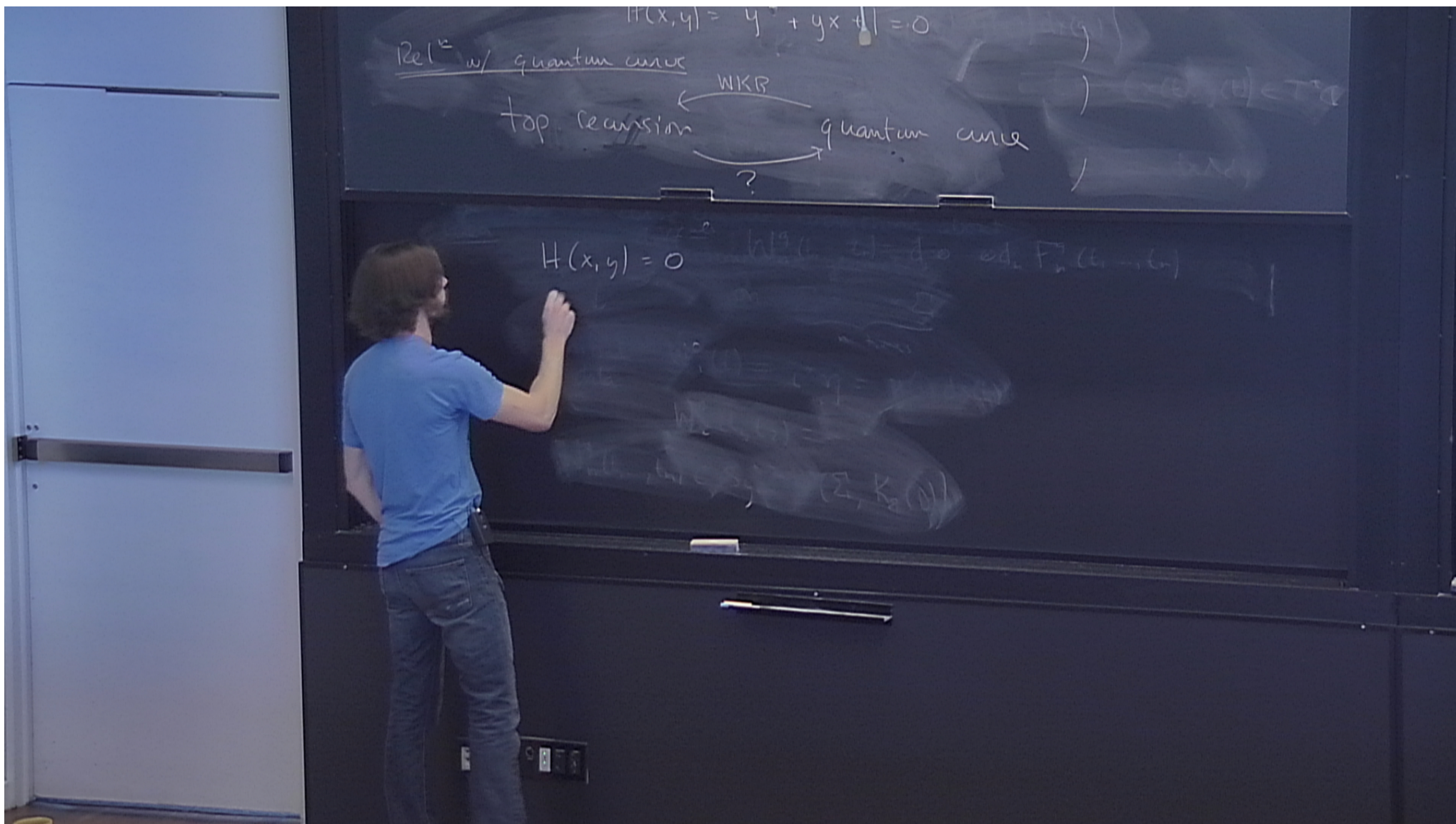


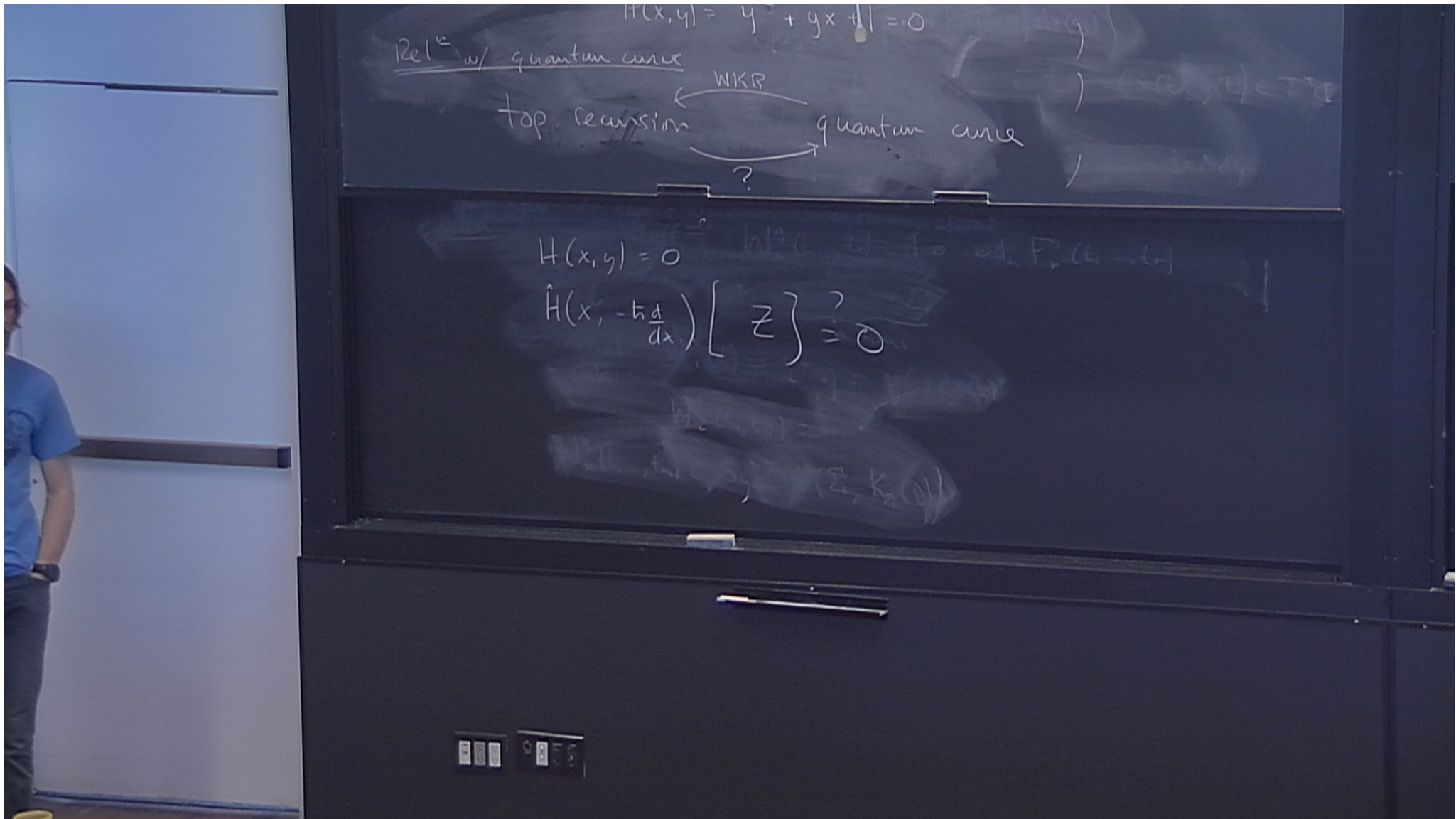
of connected graphs of genus g w/ n vertices of deg $(\mu_1, \dots, \mu_n)$

$$F_{g,n}(t_1, \dots, t_n) = \sum_{\mu} D_{g,n}(\mu_1, \dots, \mu_n) x_1^{\mu_1} \dots x_n^{\mu_n}$$









$$H(x, y) = y^2 + yx + 1 = 0$$

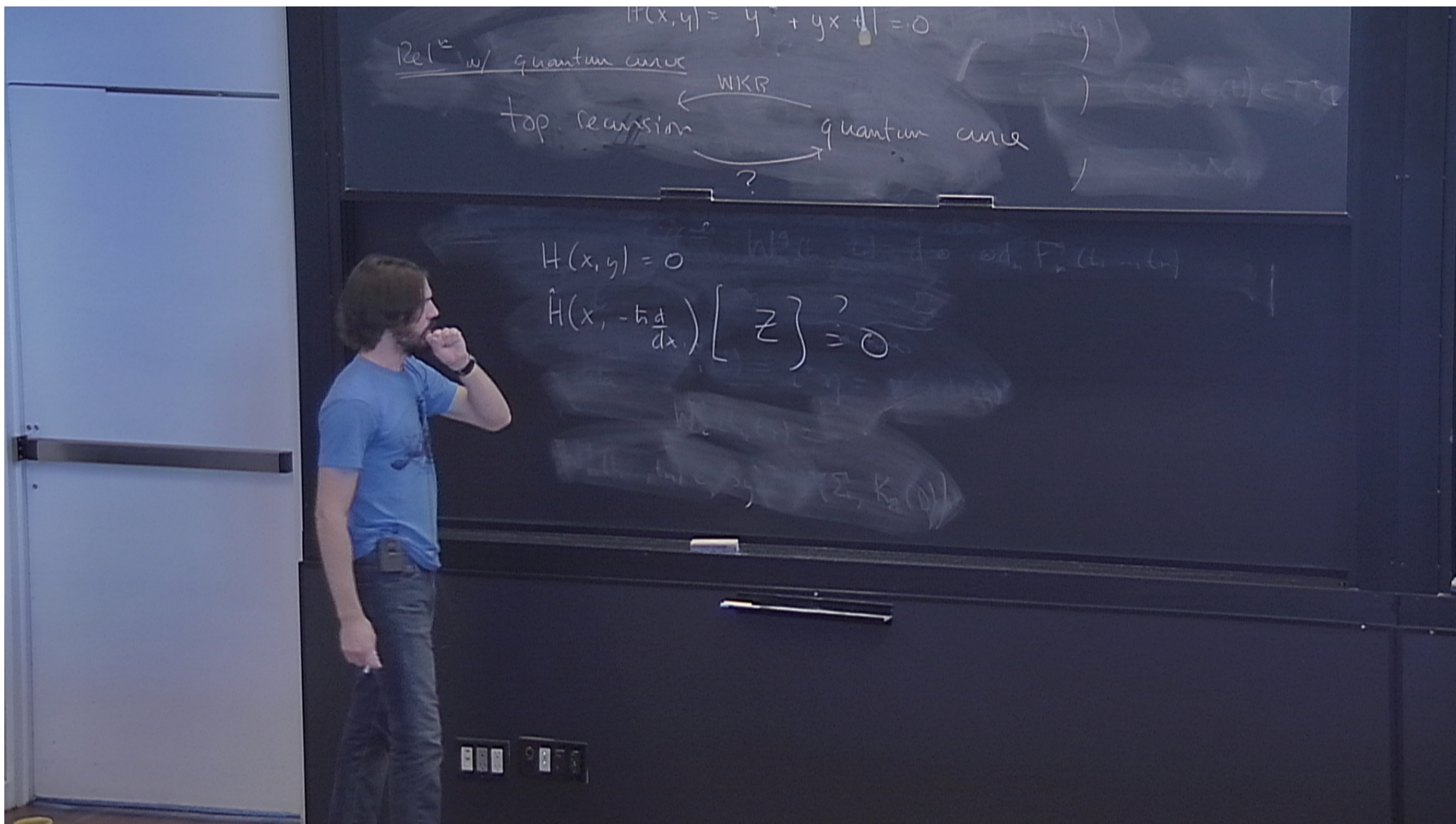
Rel^w w/ quantum curve

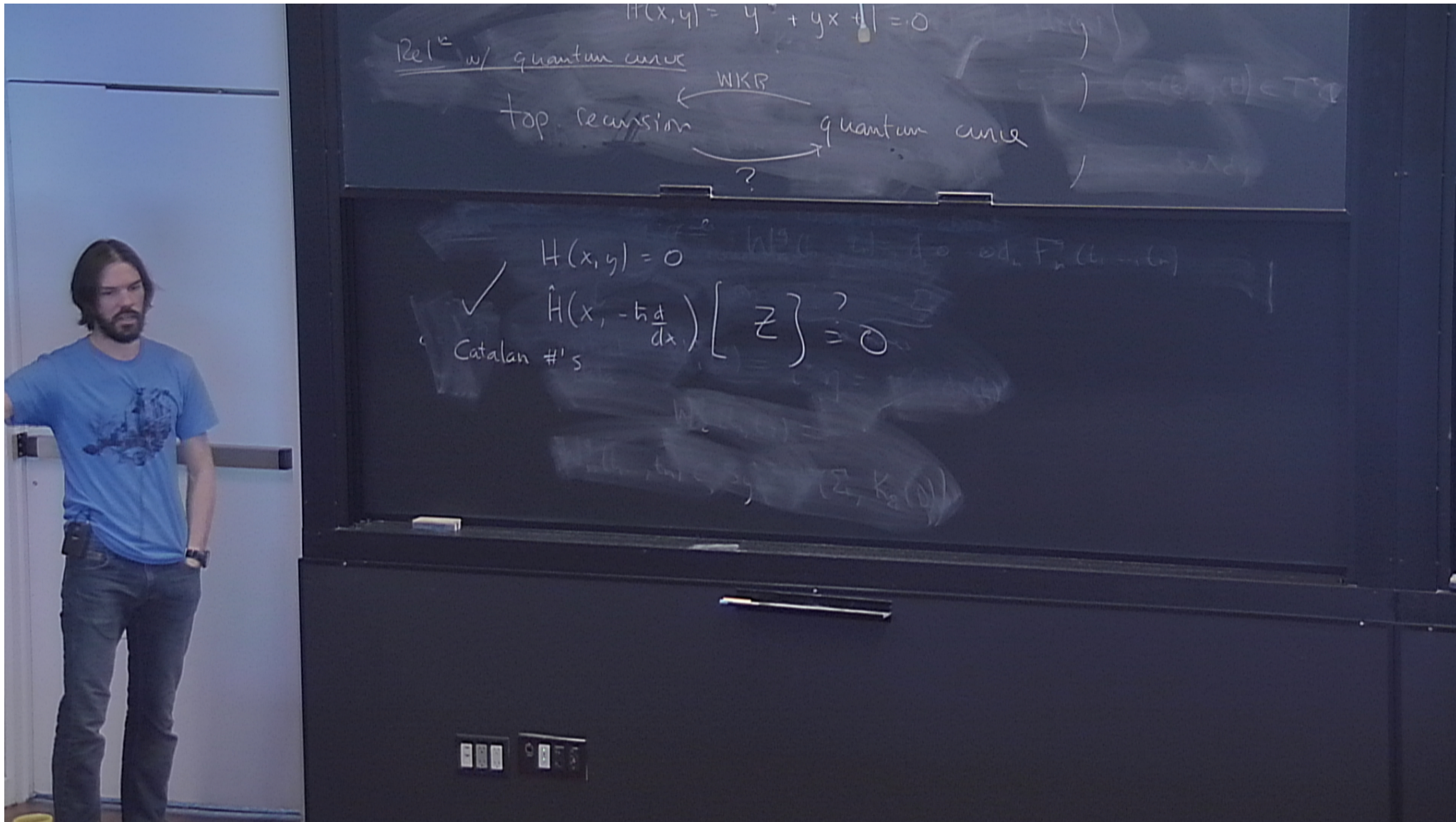
top recursion $\xleftarrow{\text{WKB}}$ quantum curve

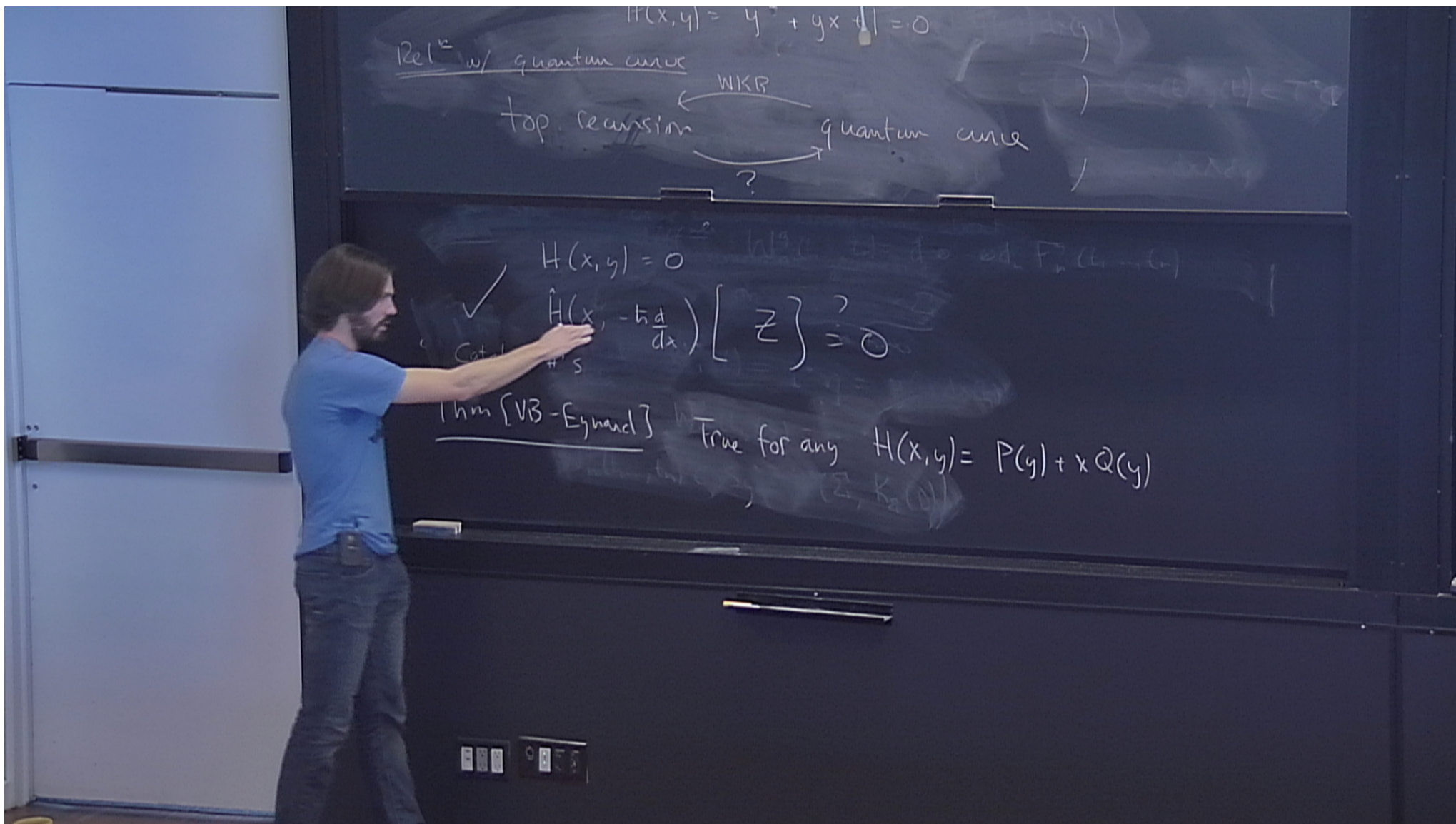
$\xrightarrow{?}$

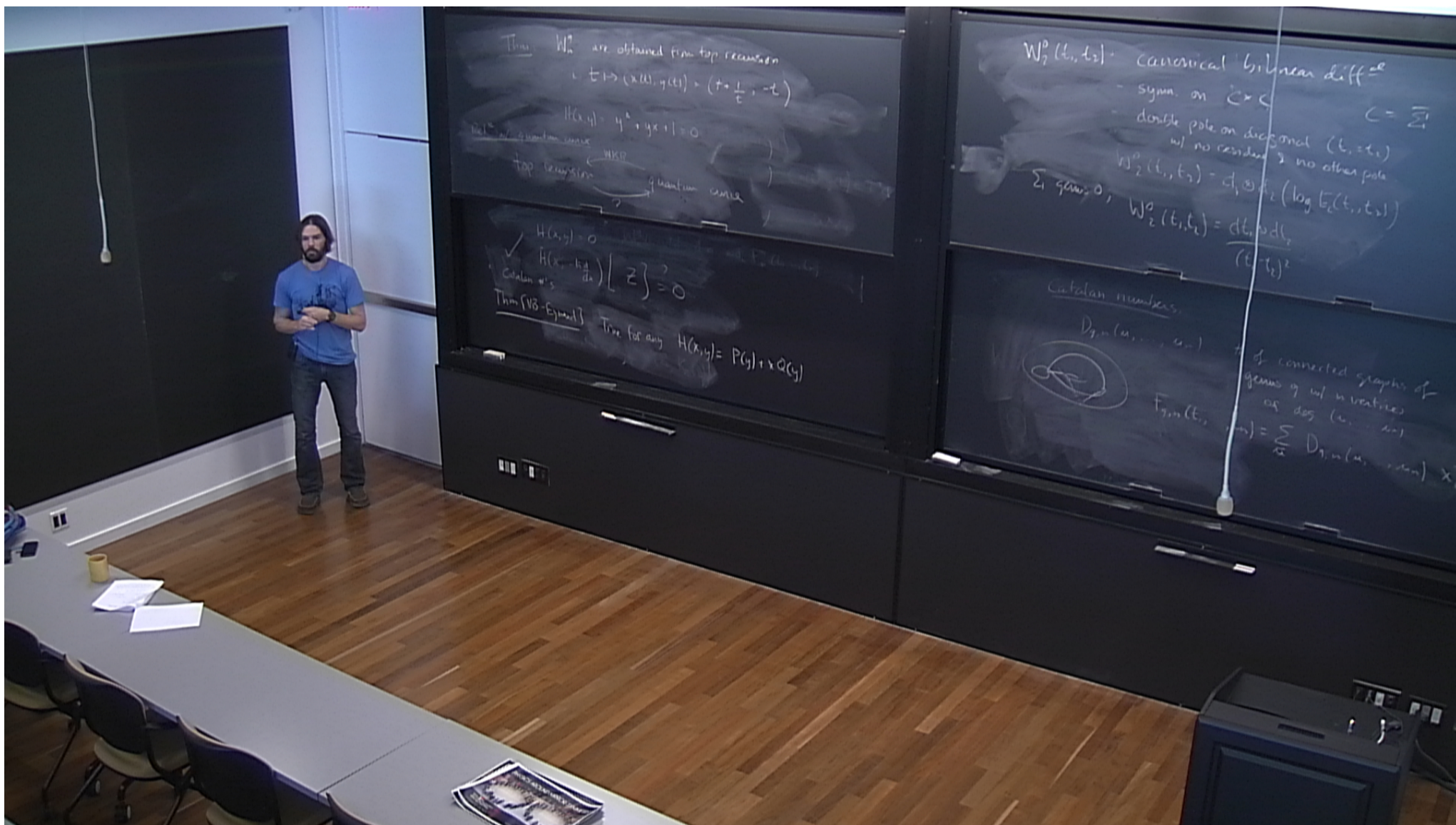
$$H(x, y) = 0 \quad \text{with } H(x, y) = 0 \text{ on } F_n(A, b_n)$$

$$\hat{H}(x, -\hbar \frac{d}{dx}) [Z] = 0$$

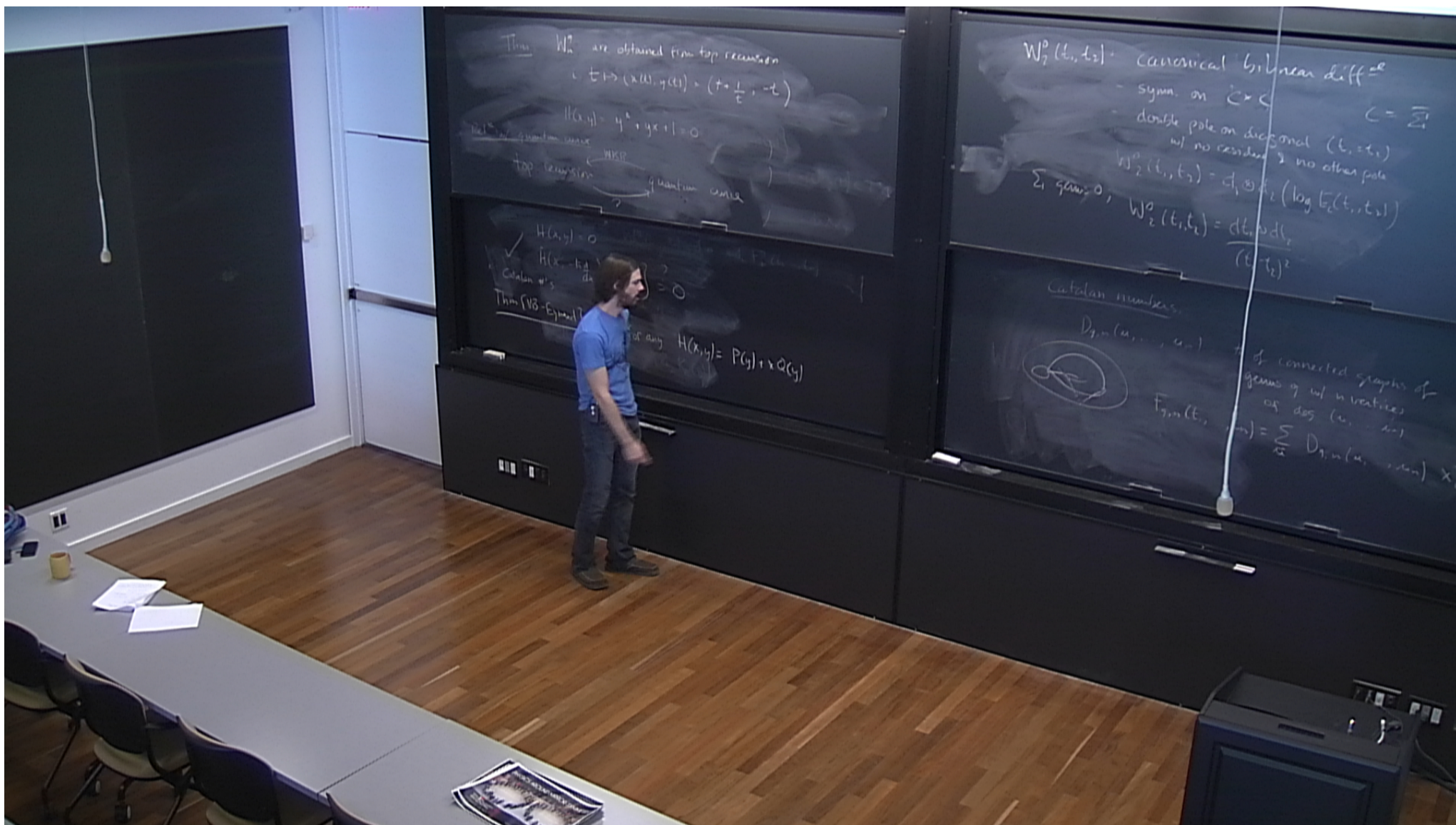


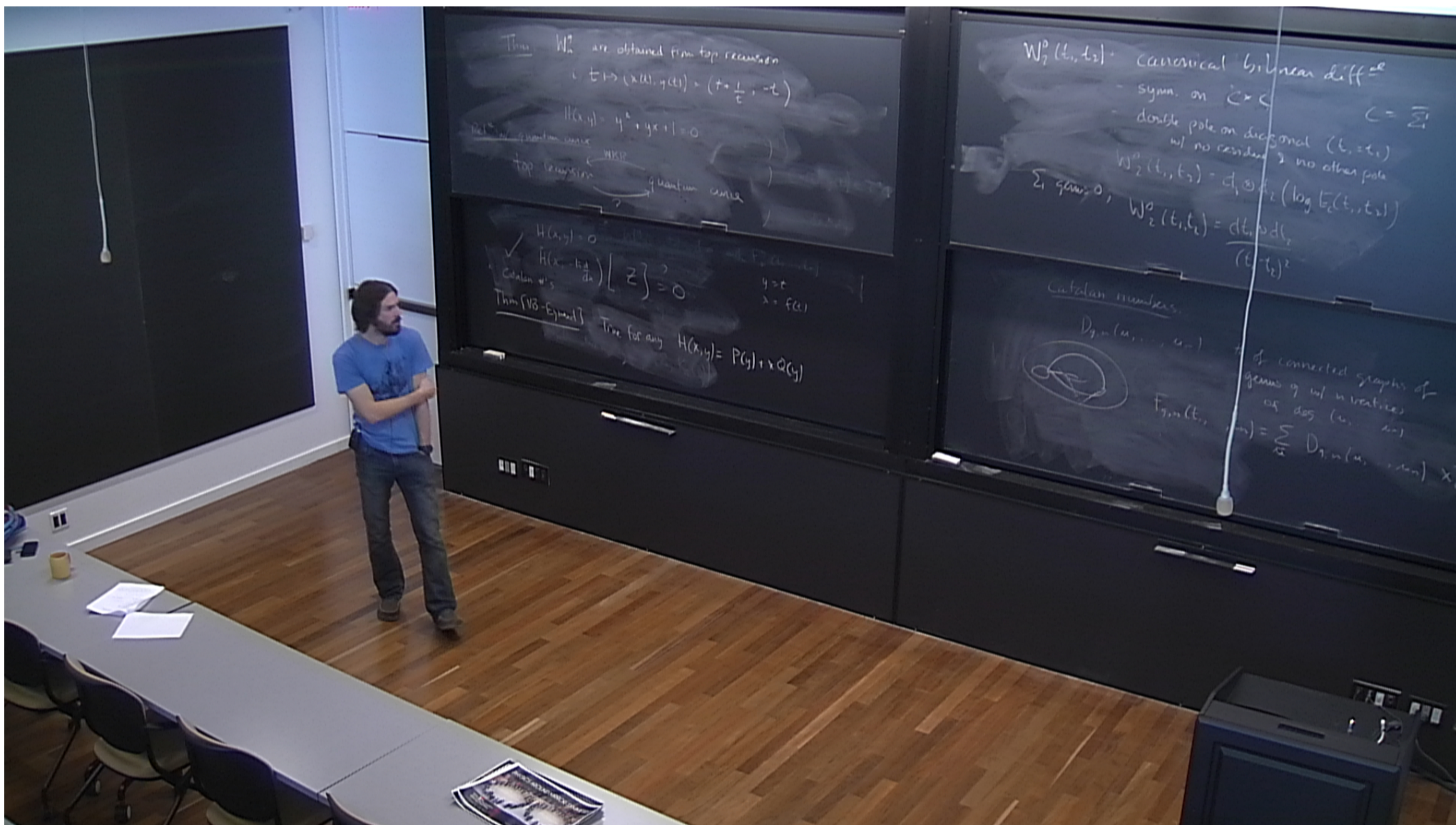












Then W_2^0 are obtained from top recursion
 $t \mapsto (x(t), y(t)) = (t + \frac{1}{t}, -t)$

$$H(x, y) = y^2 + yx + 1 = 0$$

Rel. top quantum curve $\xrightarrow{\text{WKB}}$ quantum curve
 top recursion $\xrightarrow{\text{WKB}}$ quantum curve

$$H(x, y) = 0$$

$$H(x, -\frac{t}{x}) \Big|_{Z=0} = 0$$

Thm (VSE-Exponent)

True for any $H(x, y) = P(y) + xQ(y)$

$W_2^0(t_1, t_2)$ canonical bilinear diff^{eq}

- symm. on $\mathbb{C} \times \mathbb{C}$

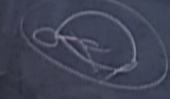
- double pole on diagonal (t_1, t_2)
 w/ no residues & no other poles

$$W_2^0(t_1, t_2) = \int_{\gamma} \omega_{t_2}(\log t_1(t, t_2))$$

$$W_2^0(t_1, t_2) = \frac{dt_1 dt_2}{(t_1 - t_2)^2}$$

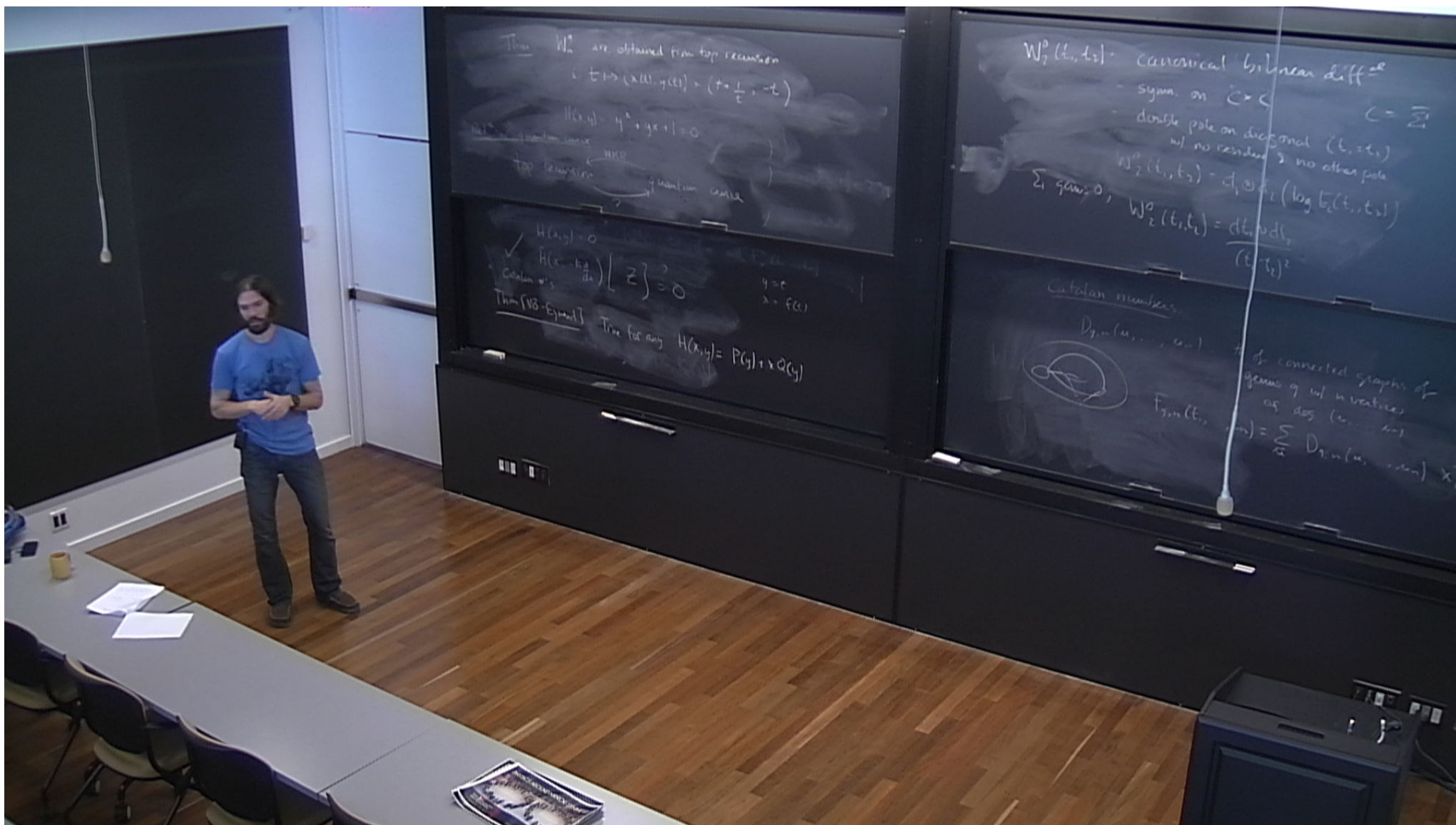
Catalan numbers:

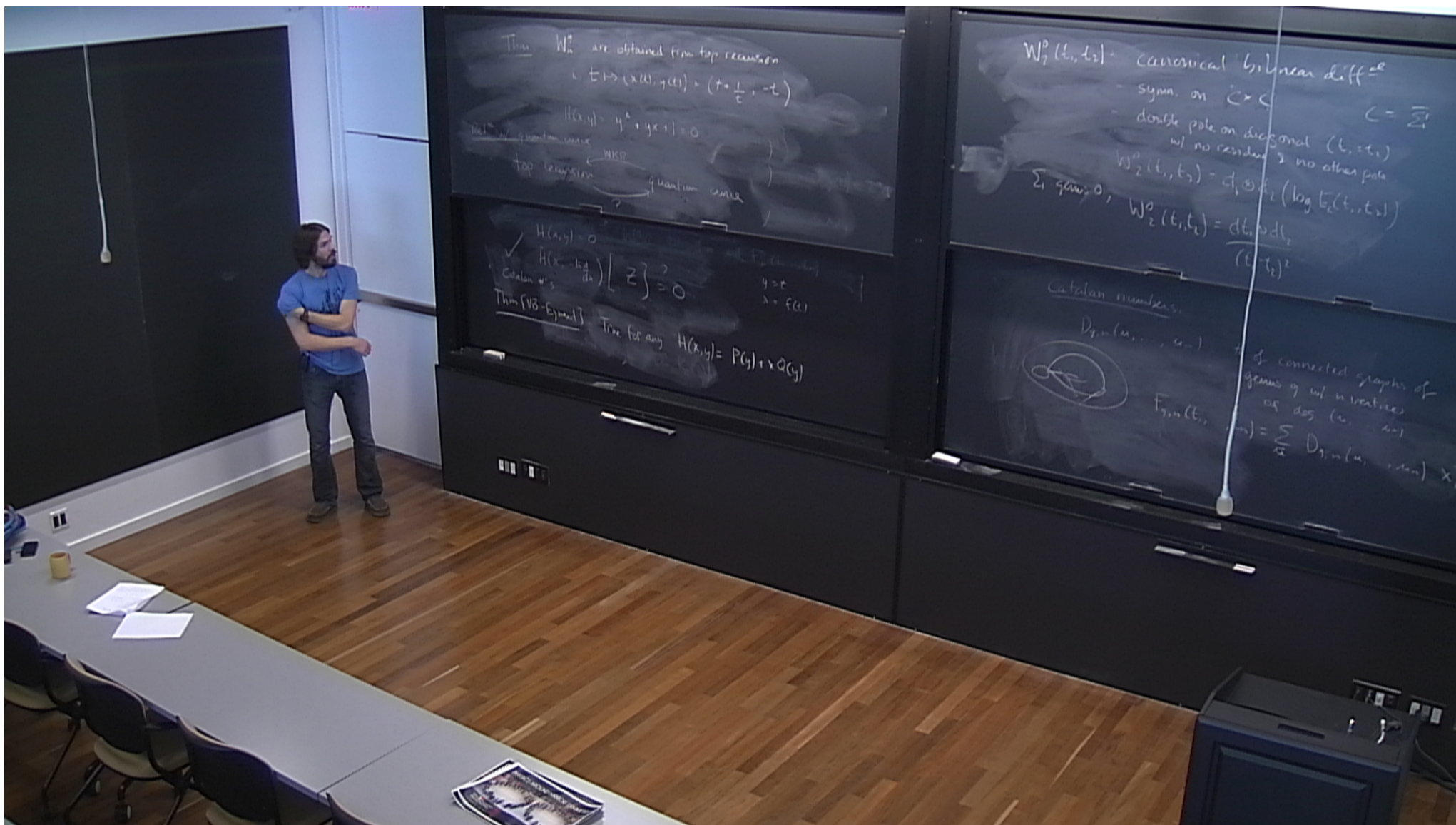
$$D_{g,n}(u_1, \dots, u_n)$$



of connected graphs of genus g w/ n vertices
 as deg $(u_i) = d_i$

$$F_{g,n}(t_1, \dots, t_n) = \sum_{\alpha} D_{g,n}(u_1, \dots, u_n) x_1^{\alpha_1} \dots x_n^{\alpha_n}$$





Thm W_n^0 are obtained from top recursion
 $t \mapsto (x(t), y(t)) = (t + \frac{1}{t}, -t)$

$$H(x, y) = y^2 + xy + 1 = 0$$

Rel. of quantum curve
 top recursion
 quantum curve

$$H(x, y) = 0$$

$$\checkmark H(x, -\frac{1}{x}) \Big|_Z = 0$$

Catalan n's

Thm (VSE-Eymal) True for any $H(x, y) = P(y) + xQ(y)$

$y = t$
 $x = f(t)$

$W_2^0(t_1, t_2)$ canonical bilinear diff^{eq}

- symm. on $\mathbb{C} \times \mathbb{C}$
 - double pole on diagonal (t_1, t_2)
 w/ no residue & no other pole

$$W_2^0(t_1, t_2) = \frac{1}{(t_1 - t_2)^2} \frac{d}{dt_1} \frac{d}{dt_2} (\log L_2(t_1, t_2))$$

Catalan numbers:

$$D_{g,n}(u_1, \dots, u_n)$$



of connected graphs of genus g w/ n vertices
 as deg $(u_i) = d_i$

$$F_{g,n}(t_1, \dots, t_n) = \sum_{u_1, \dots, u_n} D_{g,n}(u_1, \dots, u_n) x_1^{d_1} \dots x_n^{d_n}$$