Title: A symplectic approach to generalized complex geometry

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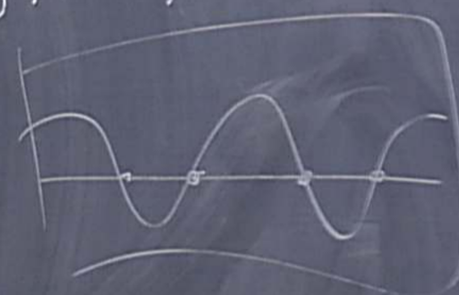
Abstract: I
will describe a new method for understanding a large class of
generalized complex manifolds, in which we view them as usual
symplectic structures on a manifold with a kind of log structure. I
will explain this structure in detail and explain how it can be used
to prove a Tian-Todorov unobstructedness theorem as well as
topological obstructions for existence of nondegenerate generalized
complex structures.

1. Log symplectic structure.

Def^n M $2n$-mfld $\pi \in \Gamma(M, \Lambda^2 T)$ Poisson st

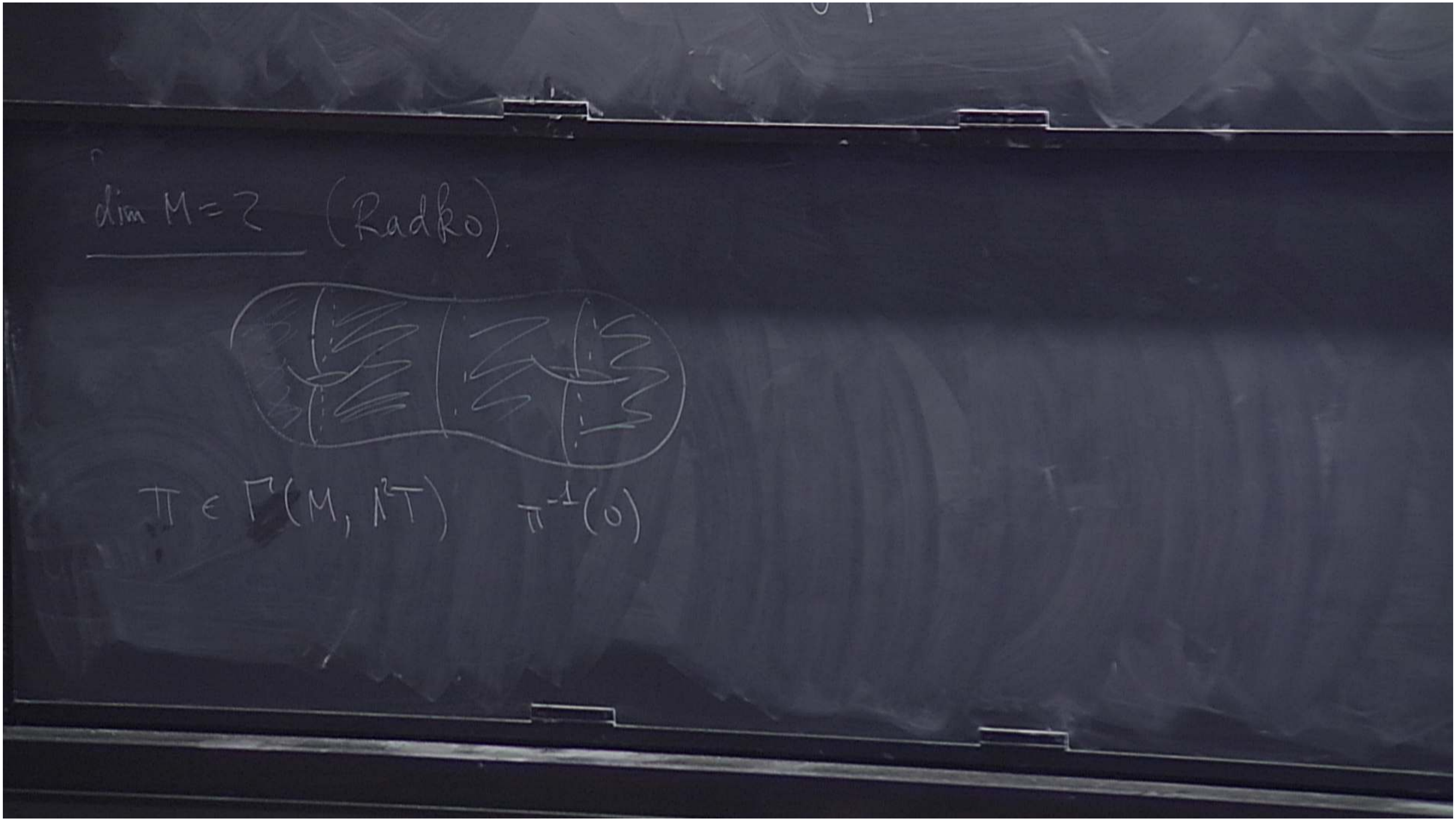
$$\pi^n \pitchfork 0.$$



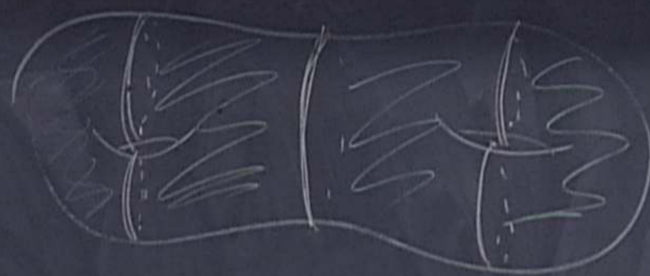
$\det TM$.

(If $\pi^n \cap \{0\} = \phi$ then $\pi^{-1} = \omega$ symplectic)

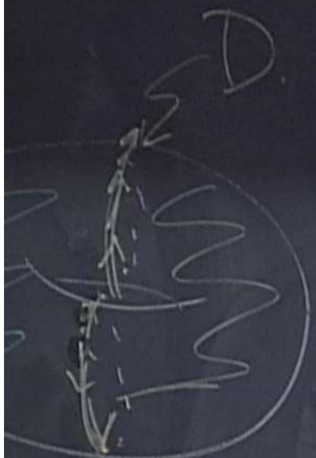
dim $M = 2$


dim M = 2 (Radko)
$\pi \in \Gamma(M, \Lambda^2 T)$
$\pi^{-1}(0)$

dim $M = 2$ (Radko)



$\pi \in \Gamma(M, \Lambda^2 T)$ $\quad \pi^{-1}(0) =$ collect. of embedded S^1.



$$d\pi|_D \in N_D^* \otimes \Lambda^2 T_M|_D.$$

$$\Big\downarrow \mathrm{Tr.}$$

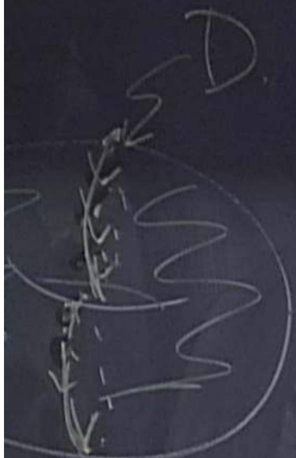
$$T_D$$

v. field along D
nonzero.

$\pi^{-1}(0)$ = collect. of embedded S^1

$\cup$

D component



$$d\pi|_D \in N_D^* \otimes \Lambda^2 T_M|_D$$

$$\downarrow \mathrm{Tr.}$$

$$T_D$$

$\pi^{-1}(0)$ = collect. of embedded S^1

$\cup$ periods. $\lambda \in \mathbb{R}_{>0}$

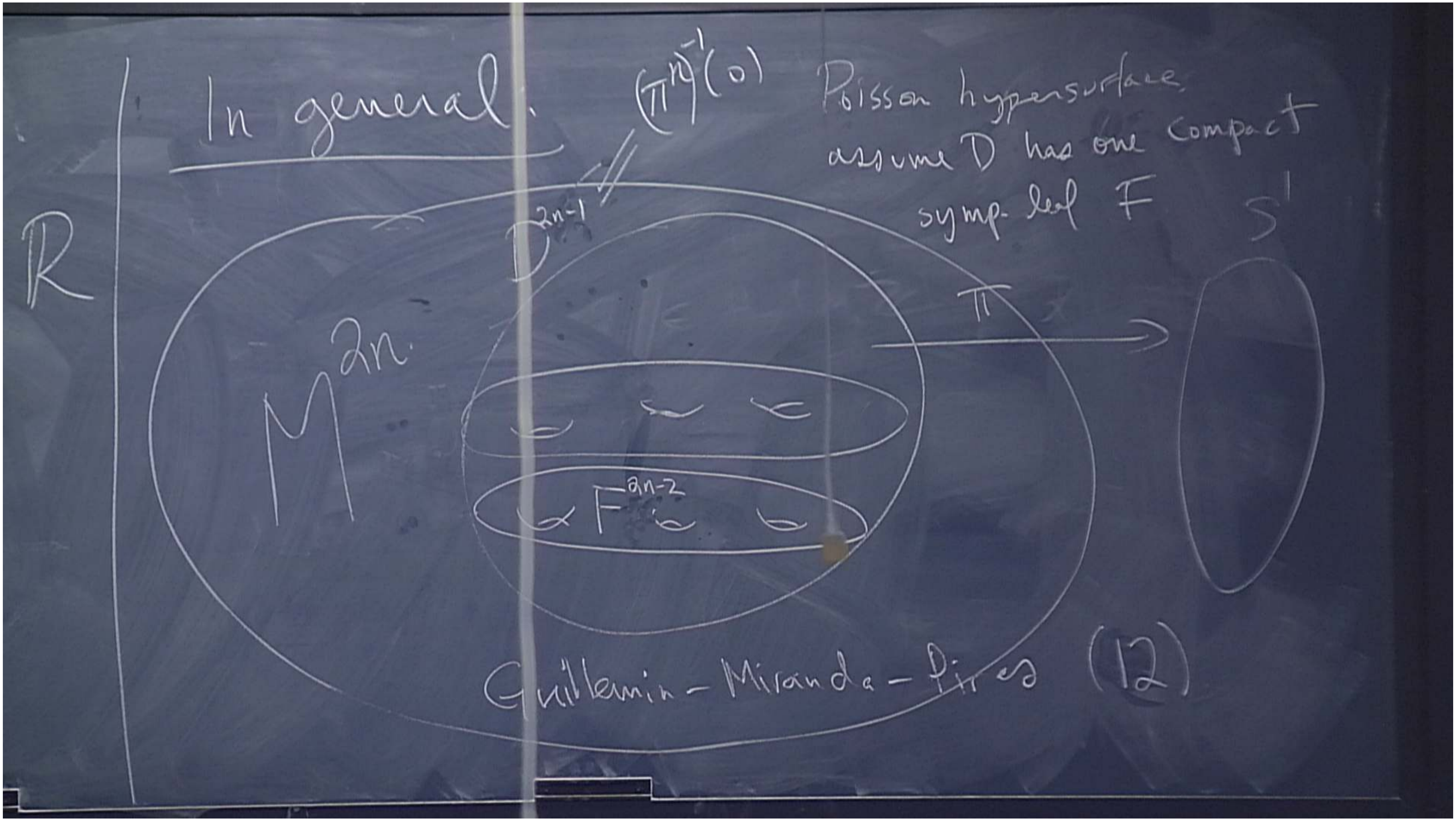
D component

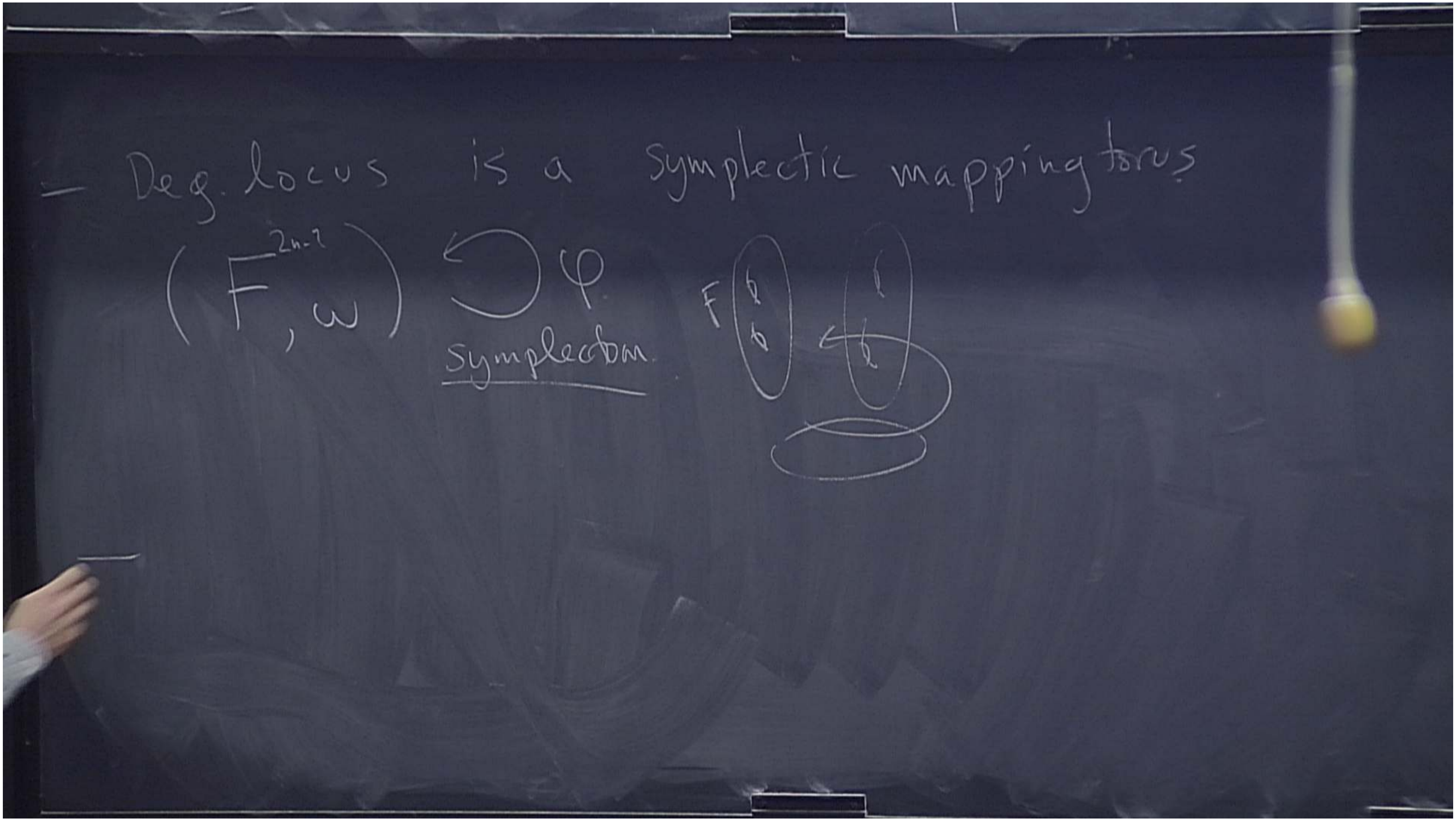
Integ. v. field along D

nonzero.

- collect. of S^1
- periods $\in \mathbb{R}_{>0}$.
- well-defined volume $\int_M \pi^{-1} \in R$

Radko: this classifies.


In general.
$(\pi^{n})^{-1}(0)$
Poisson hypersurface
assume D has one compact
symp. leaf F
R
D^{2n-1}
M^{2n}
π
S^1
F^{2n-2}
Guillemin - Miranda - Pires (12)


– Deg. locus is a symplectic mapping torus
$(F^{2n-2}, \omega) \circlearrowleft \varphi$
symplectom.
F

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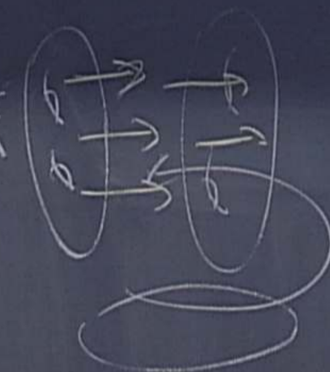
F



— Same v.f. from earlier defines transverse Poiss. v.f.

mapping torus

③ char. class of ND

elt in

$$H^1(F, \mathbb{Z}_2)^{\varphi} \times \mathbb{Z}_2.$$

$\longrightarrow$ period λ

② $\uparrow$

$\mathbb{R}_{>0}$

mapping torus

③ char. class of ND elt in

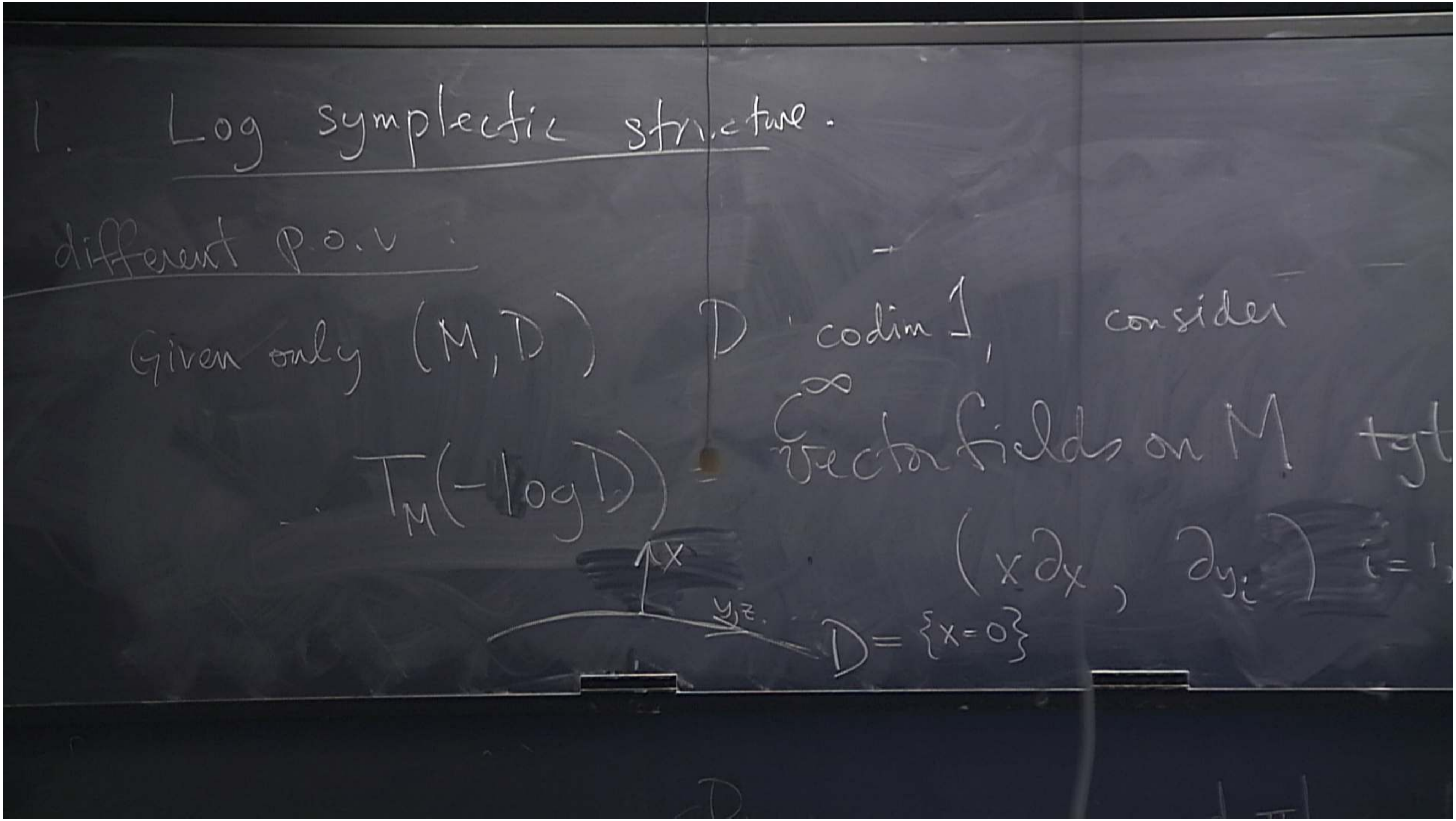
$$H^1(F, \mathbb{Z}_2)^{\varphi} \times \mathbb{Z}_2.$$

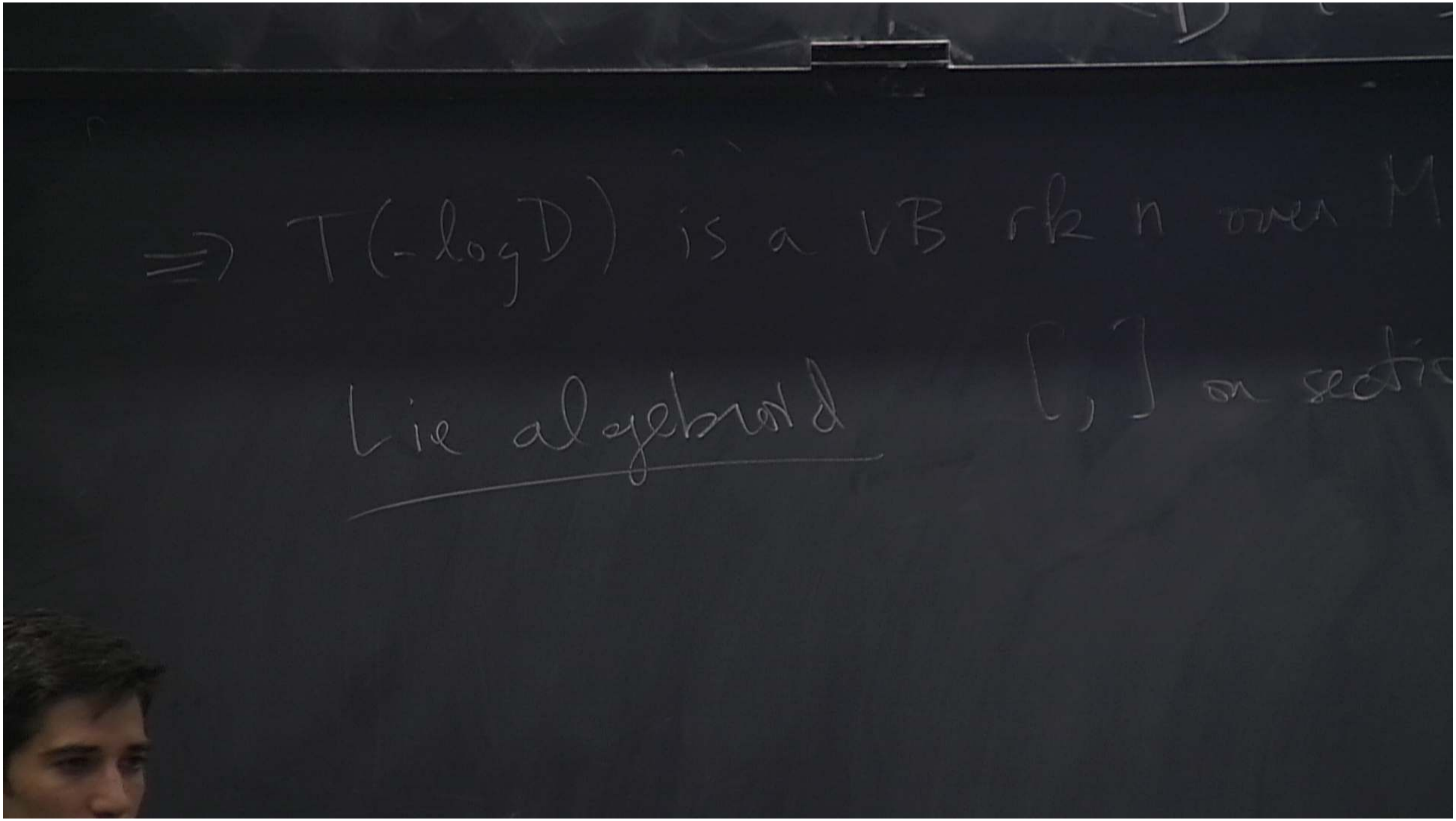
$\rightarrow$ period λ

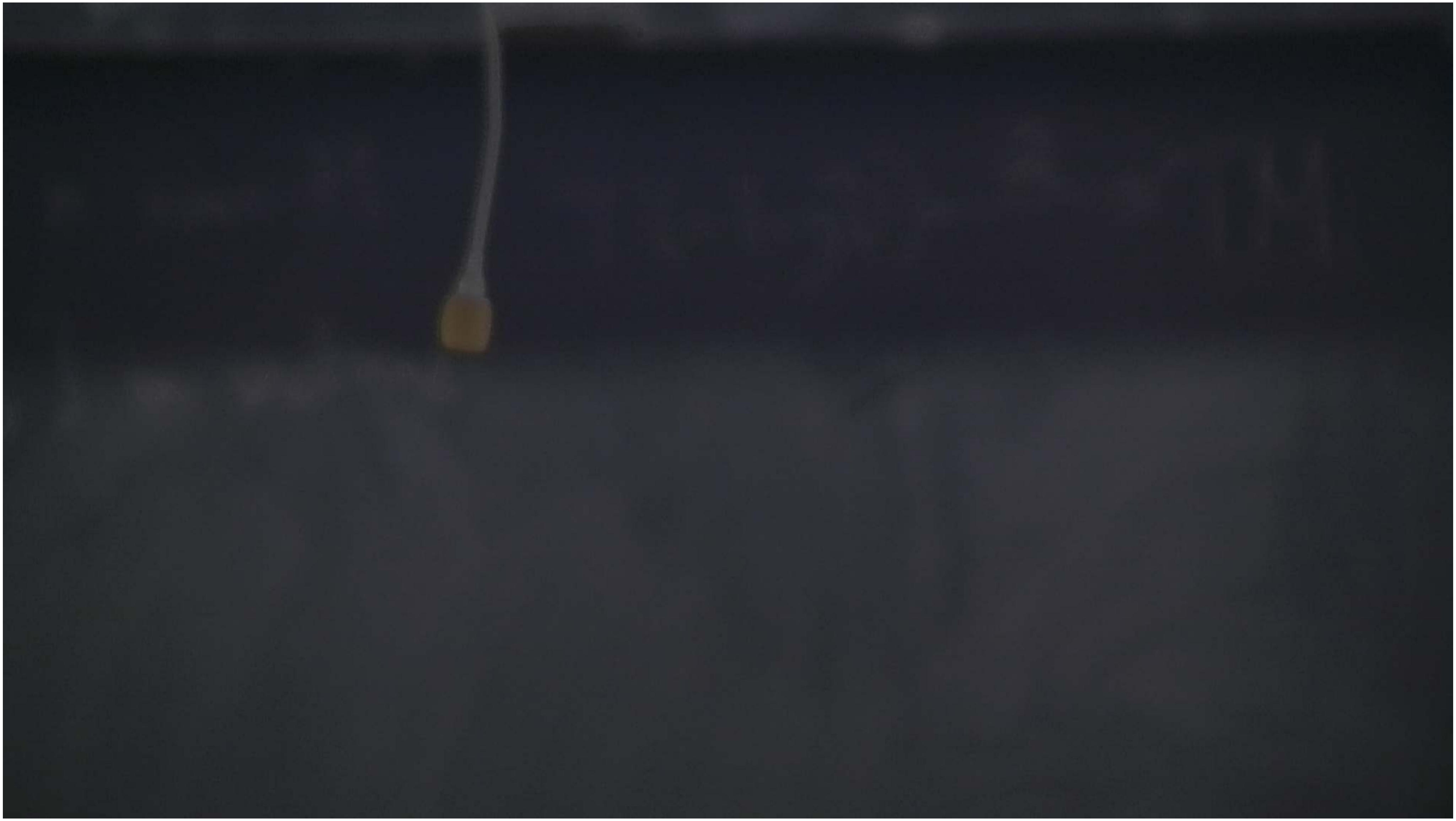
② $\uparrow$

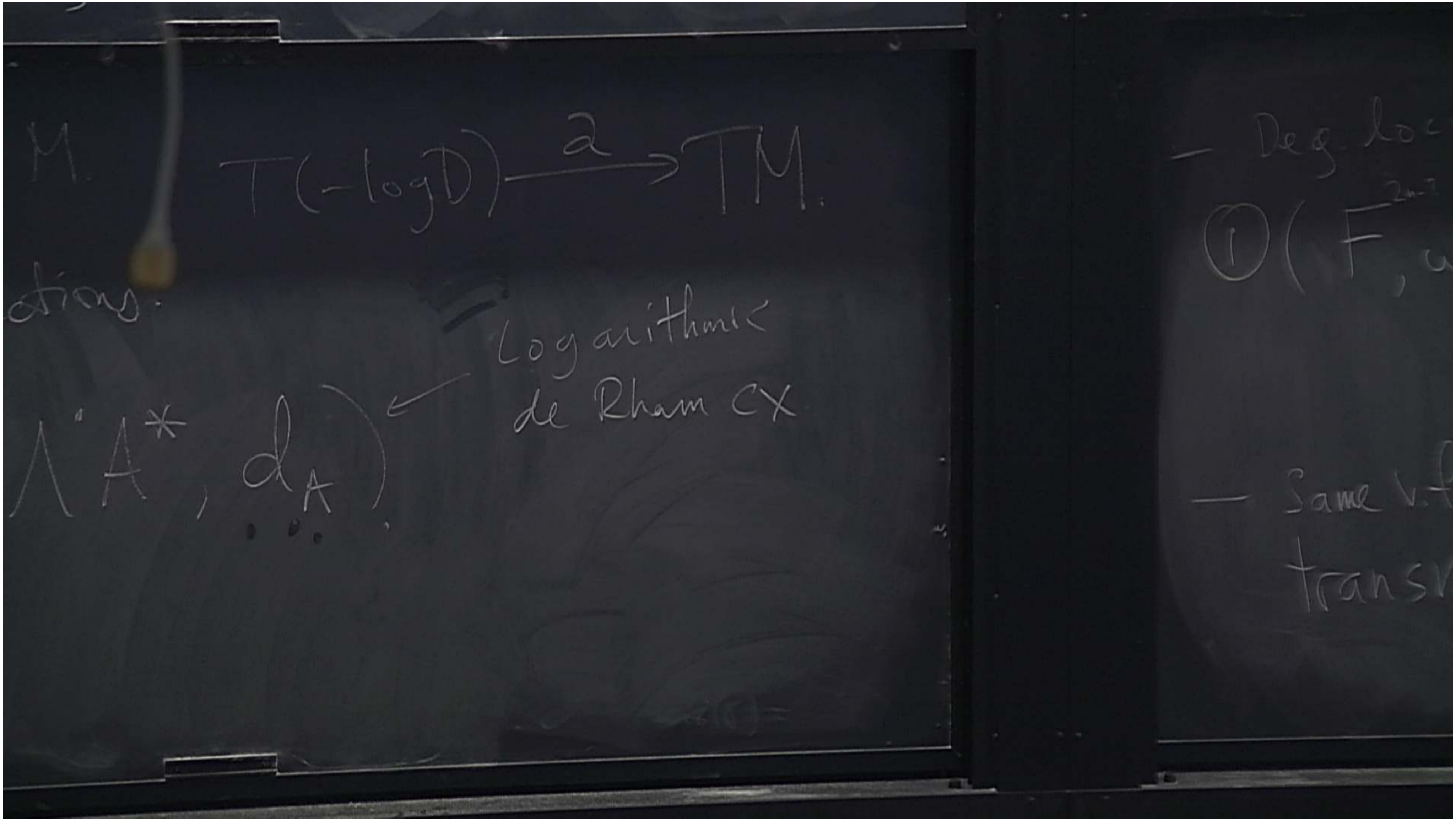
$R \to 0$

① + ② + ③ Classifies.


1. Log symplectic structure.
different p.o.v :
Given only (M, D)
D : codim 1, consider
$T_M(-\log D)$ = C^∞ vector fields on M tgt
$(x\partial_x, \partial_{y_i})$ i = 1,
x
y, z
$D = \{x = 0\}$


⇒ T(-log D) is a VB rk n over M
Lie algebroid
[,] on sectio




M.
$T(-\log D) \xrightarrow{\partial} TM.$
Logarithmic
de Rham cx
$(\Lambda^{\bullet} A^{*}, d_A).$
— Deg. loc
— Same v.f

... over M. $\quad T(-\log D) \longrightarrow TM$.

$[\,,\,]$ on sections.

$$(\Lambda^\bullet A^*, d_A) \longleftarrow \text{Logarithmic de Rham cx}$$

not elliptic.

$$\text{(Melrose)} \Rightarrow H^k_{\log D} = H^k(M) \oplus H^{k-1}(D)$$

$\Rightarrow T(-\log D)$ is a VB ...

Lie algebroid $\quad [\,,\,]$ on sect...

Q...

$$A = T(-\log D)$$

Cohomology? (Mazzeo-Melrose) $\Rightarrow$

Pf: $0 \to \Omega^k_M \longrightarrow \Omega^k_M(\log D) \xrightarrow{\text{Res}} \Omega^{k-1}_D \to 0$

$$\Omega^k_M(\log D) \ni \alpha \wedge \frac{dx}{x} + \beta$$

smooth

in cohom, the l.e.s splits (use bump) $\Rightarrow H^k_{\log D} = H^k_M \oplus H^{k-1}_D$

$$
\begin{tikzcd}
T^*(\log D) \arrow[r, "\tilde{\pi}", "\cong"'] & T(-\log D) \arrow[d, "\partial"] \\
T^*M \arrow[u, "\partial^*"] \arrow[r, "\pi"'] & TM.
\end{tikzcd}
$$

Poisson s.t. π^{r} vanish $\pitchfork D$.

$$\det \pi : \Lambda^{2n} T^* \longrightarrow \Lambda^{2n} T$$

$$= (\pi^{n}) \otimes (\pi^{n})$$

$$\pi = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

$$\pi^{-1} = \frac{dx}{x} \wedge dy$$

nondeg

$$
\begin{tikzcd}
T^*(\log D) \arrow[r, "\tilde{\pi}", "\cong"'] & T(-\log D) \arrow[d, "\partial"] \\
T^*M \arrow[u, "\partial^*"] \arrow[r, "\pi"'] & TM.
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$$\pi^{-1} = \frac{dx}{x} \wedge dy$$

nondeg

Log symplectic structure.

…erent p.o.v :

π may be viewed as a usual symp. str $\omega \in \Gamma(\Lambda^2 A^*)$.

(nondeg, $d_A \omega = 0$) (for $A = T(-\log D)$).

is a VB rk n over M. $T(-\log D) \xrightarrow{a} TM$

Output: Moser's thm: unobstructed deform space

given by

$$H^2_{\log D} = H^2(M) \oplus H^1(D)$$

(2) Symplectic $[\omega] \in H^2$. $[\omega]^n \neq 0$.

Marcut - OsornoTorres (13): on a log symp. mfld

$\exists$ $c \in H^2(M)$ s.t. $c^{n-1} \neq 0$.

(2) Symplectic $[\omega] \in H^2$ $[\omega]^n \neq 0$.

Marcut - OsornoTorres (13): On a log symp. mfld $\exists$ $c \in H^2(M)$ s.t. $c^{n-1} \neq 0$.

$\Longrightarrow$ e.g. S^6 has <u>no</u> log symp. str.

GC geometry (nondeg. case)

a g.c. str is $\mathbb{J} : T \oplus T^* \circlearrowleft$ cx str.

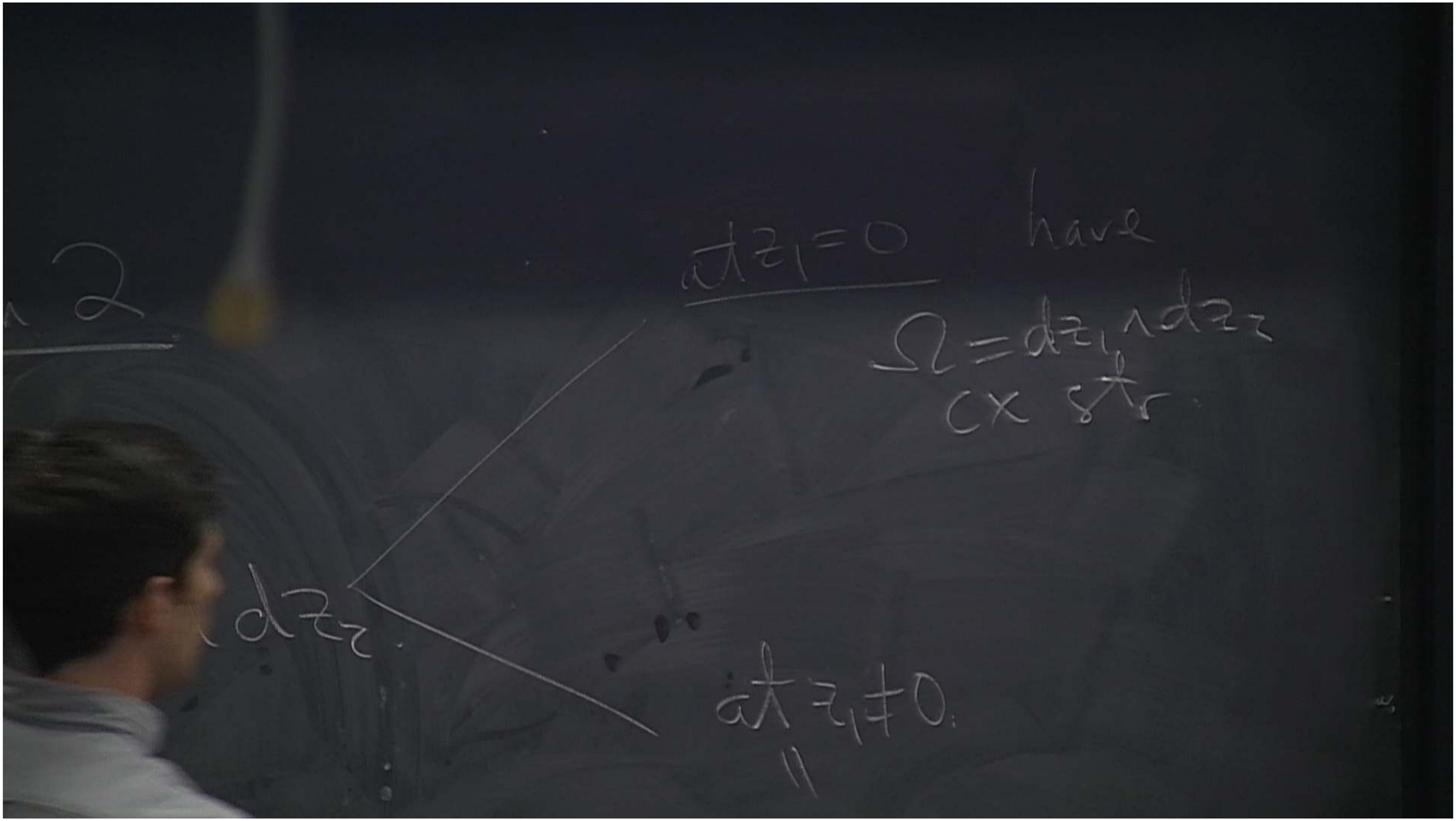
e.g. $\begin{array}{cc} \begin{pmatrix} I & \\ & -I^* \end{pmatrix} & \begin{array}{c} T \\ \oplus \\ T^* \end{array} \circlearrowleft \end{array}$

CX case

$$\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{array}{c} T \\ \oplus \\ T^* \end{array} \circlearrowleft$$

$\omega : T \to T^*$

symp. case.


at z1=0
have
Ω = dz1 ∧ dz2
dz2
at z1 ≠ 0

$\wedge 2$

at $z_1 = 0$

have

$\Omega = dz_1 \wedge dz_2$

cx str.

$dz_1 \wedge dz_2$.

at $z_1 \neq 0$:

$$= z_1 \left(e^{\frac{dz_1}{z_1} \wedge dz_2} \right)$$

$$= z_1 (e$$

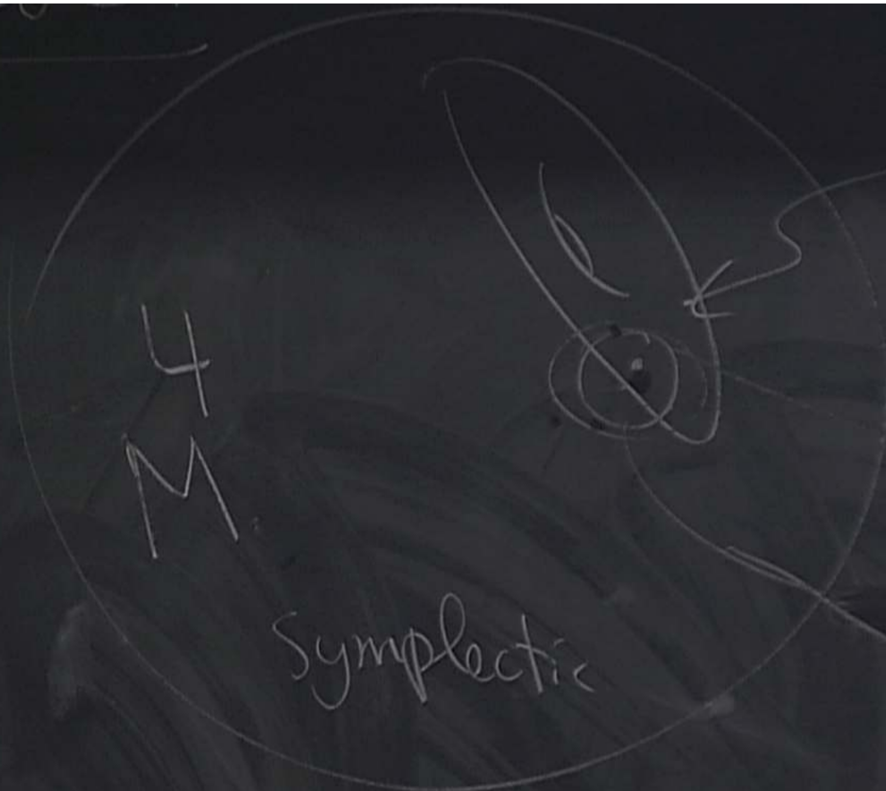
B-field.

$B + i\omega$

symp.

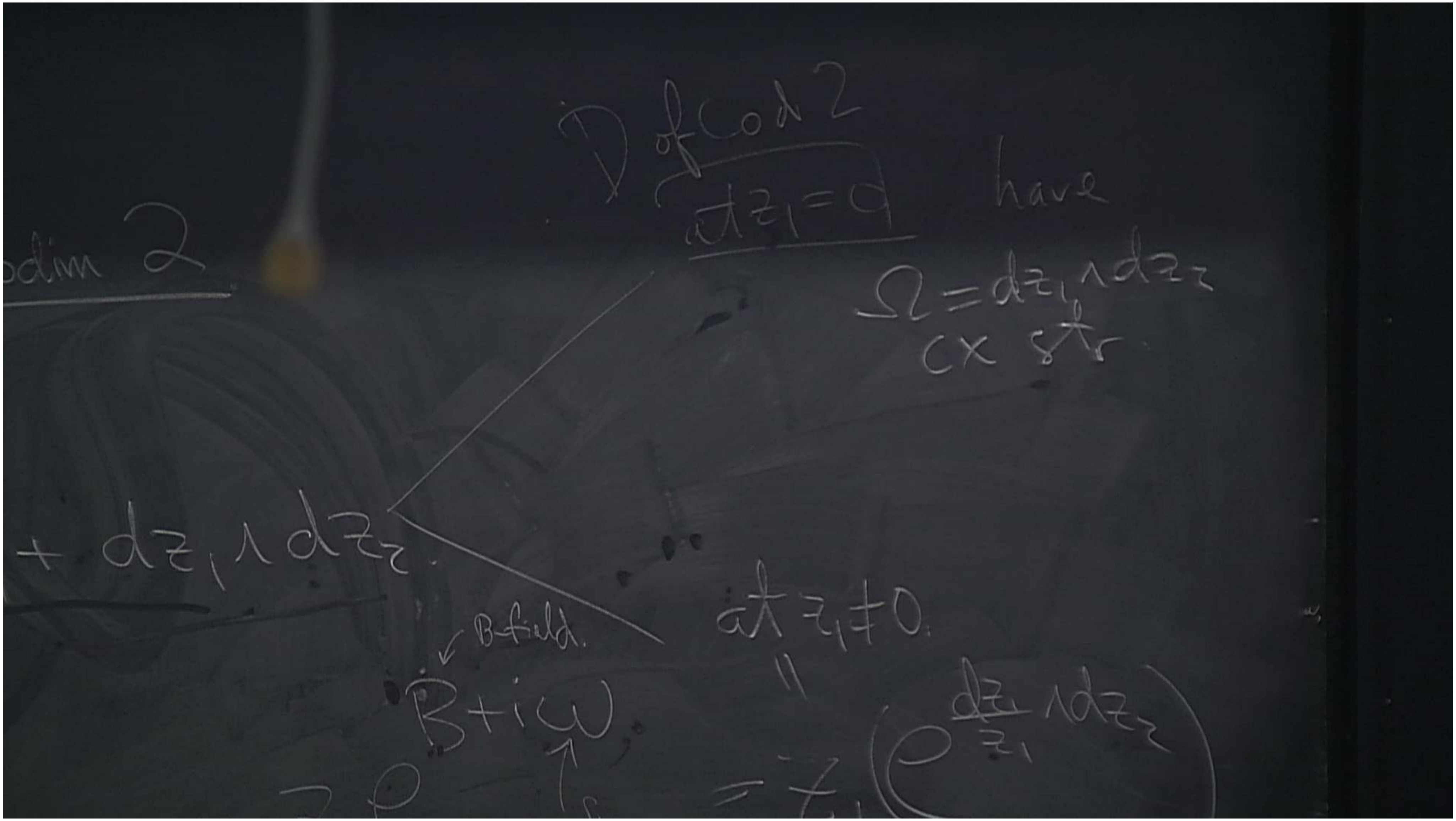
$z_1 e$

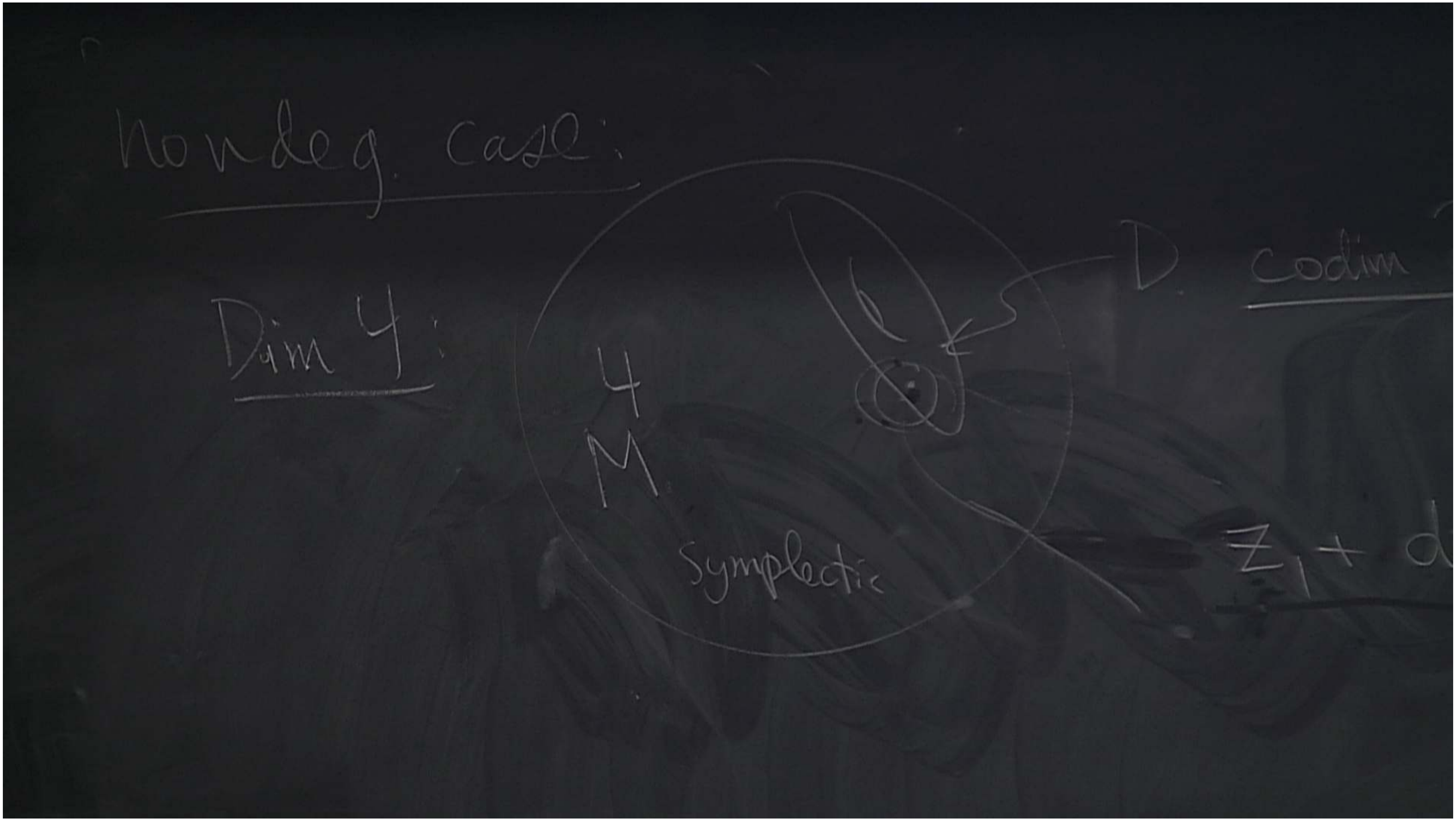
im 4:



codim 2

$z_1 + dz_1$


D of cod 2
at $z_1 = 0$
have
codim 2
$\Omega = dz_1 \wedge dz_2$
cx str
$+ \, dz_1 \wedge dz_2$
at $z_1 \neq 0$
B-field.
$B + i\omega$
$= z_1 \left(e^{\frac{dz_1}{z_1} \wedge dz_2} \right)$


nondeg. case:
Dim 4:
4
M
symplectic
D. codim
$z_1 + d$

$\to$ D has a cx str (Elliptic curve)

ND is a hol. line bundle :

→ - D has a cx str (Elliptic curve)

- N\D is a hol. line bundle

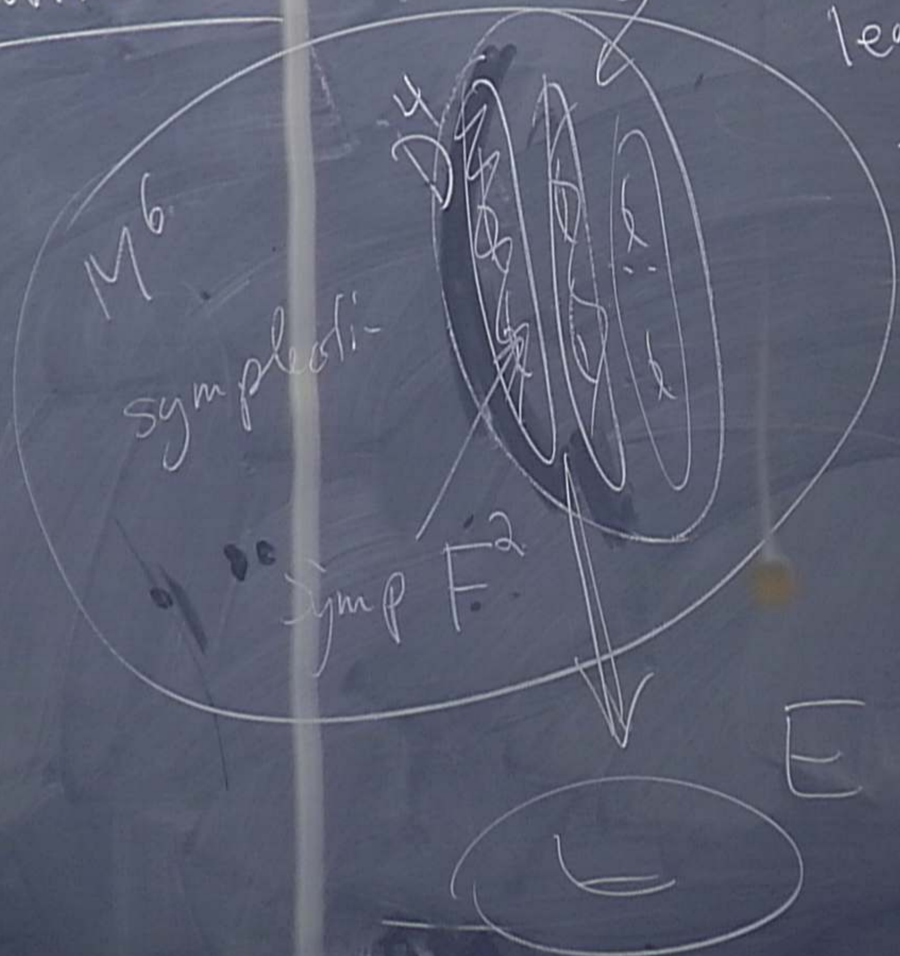
- M\D symplectic

(dim 4 !!)

re).

dim 6

GCY str: Assume $\exists$ compact leaf (Poisson str) $\Rightarrow$ fibers over E ell. curve



E (hol str)

Funny example of such a 6-mfld

X 3-mfld

$\cup$

K knot,

choose a.cx str on N

$$\begin{array}{c} T^*X \leftarrow \text{6-mfld} \\ \downarrow \\ X \end{array}$$

symplectic $\xrightarrow{\text{twist}}$ transv. to knot

6 - mfld

...e a.cx str on N

$X \leftarrow$ 6 - mfld
Symplectic

$\xrightarrow[\text{transv. to knot}]{\text{twist}}$

A
$\downarrow$
X

$\leftarrow$ GCX
w/ type change
localized to knot

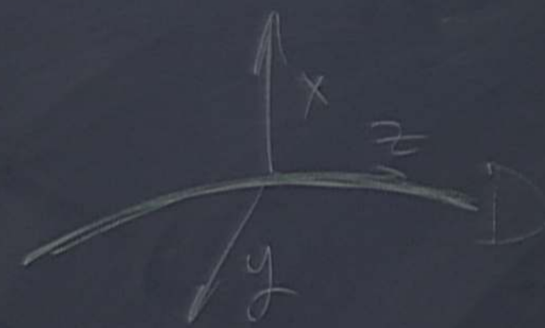
$A|_K \leftarrow$ 4 d.

ck applied to nondeg GCx str:

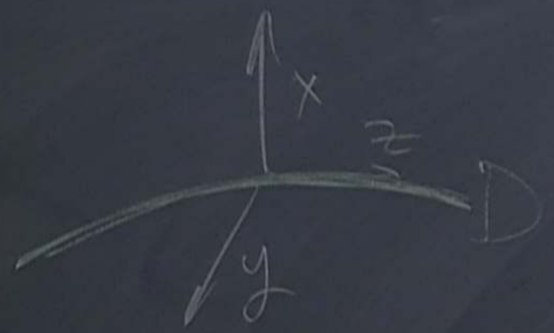
M $2n$-mfld

$\cup$

D $\mathrm{codim}_{\mathbb{R}} = 2.$



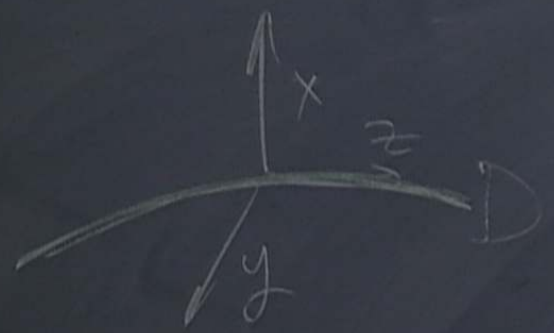
GCx str:



v.f. tgt to D

$$= \left\langle \frac{\partial}{\partial z},\ x\partial_x,\ y\partial_y,\ x\partial_y,\ y\partial_x \right\rangle$$

GCx sAr:



v.f. tgt to D $= \left\langle \frac{\partial}{\partial z}, x\partial_x, y\partial_y, x\partial_y, y\partial_x \right\rangle$

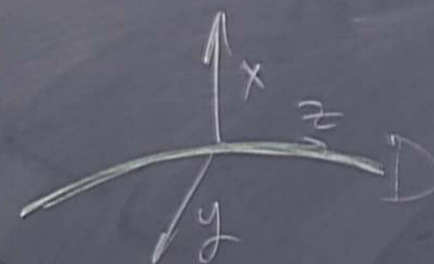
no good

Trick applied to nondeg GCx str:

M $2n$-mfld

$\cup$

D $\operatorname{codim}_{\mathbb{R}} = 2.$



v.f

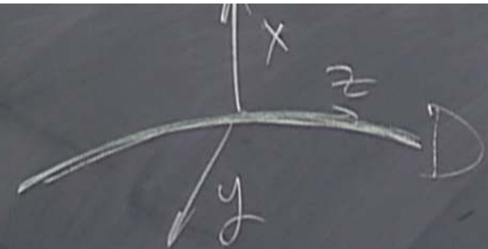
=

cutdown: choose an a.cx st. normal to D ; I

$) :=$ v. fields which are (1) tgt to D, and (2) preserve normal cx str.

cuts down to $\longrightarrow$ $\langle \partial_z, x\partial_x + y\partial_y, x\partial_y - y\partial_x \rangle$

locally free of rank $\dim M$



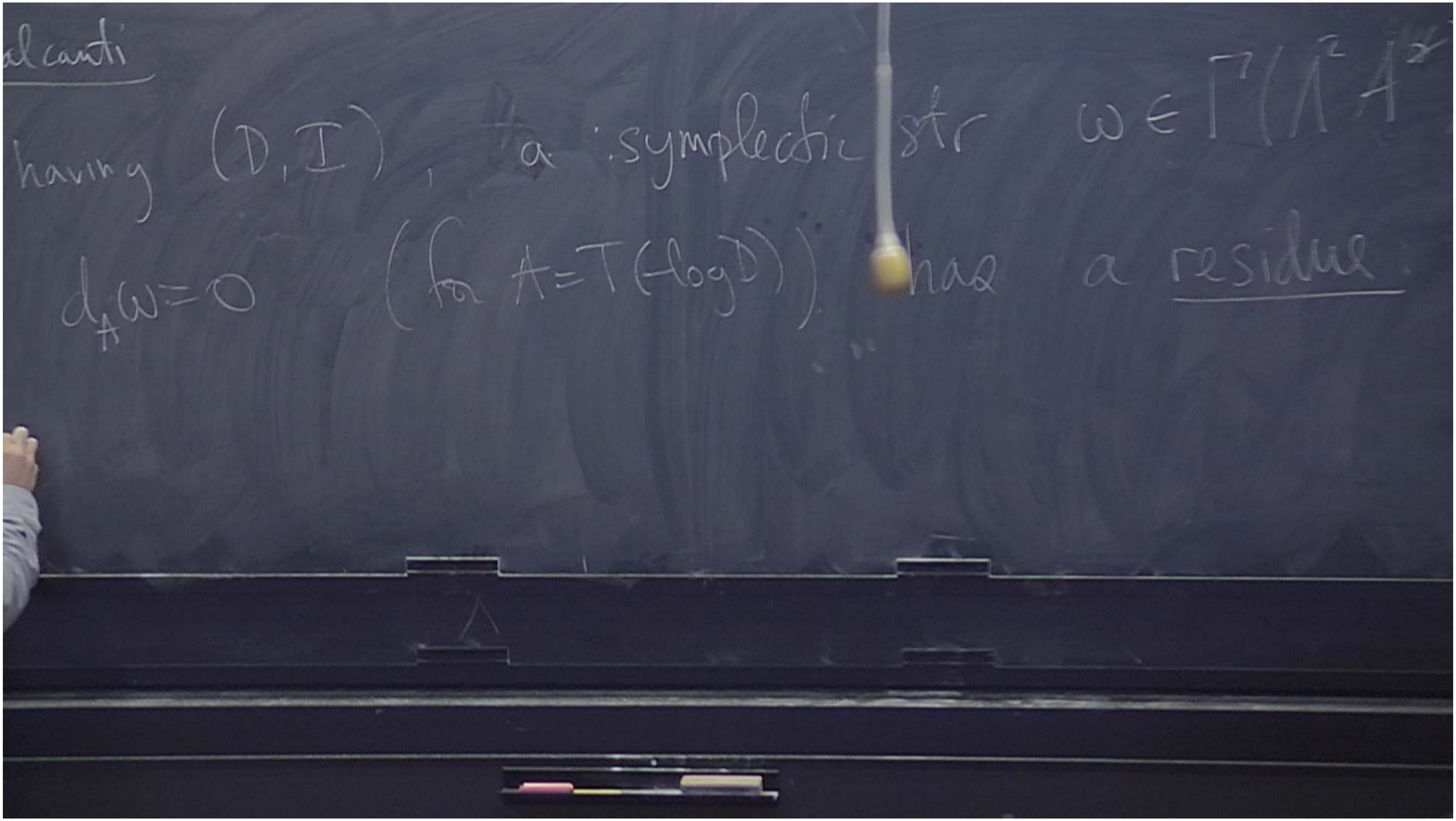
$$= \left\langle \frac{\partial}{\partial z},\ x\partial_x,\ y\partial_y,\ x\partial_y,\ y\partial_x \right\rangle$$

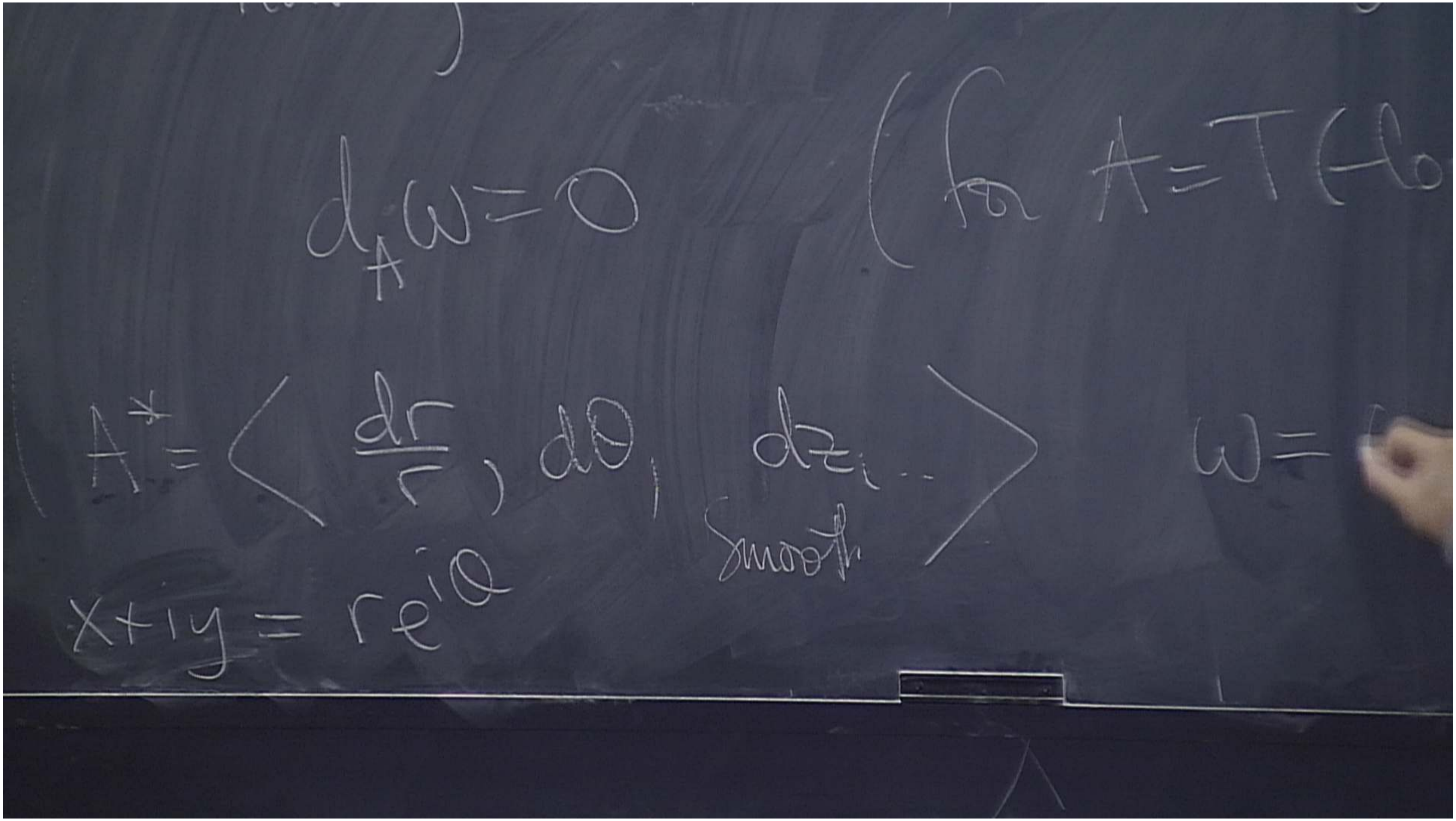
no good

…cx st. normal to D, I, (D, I) complex divisor.

v. fields which are ① tgt to D.

and ② preserve normal cx str.

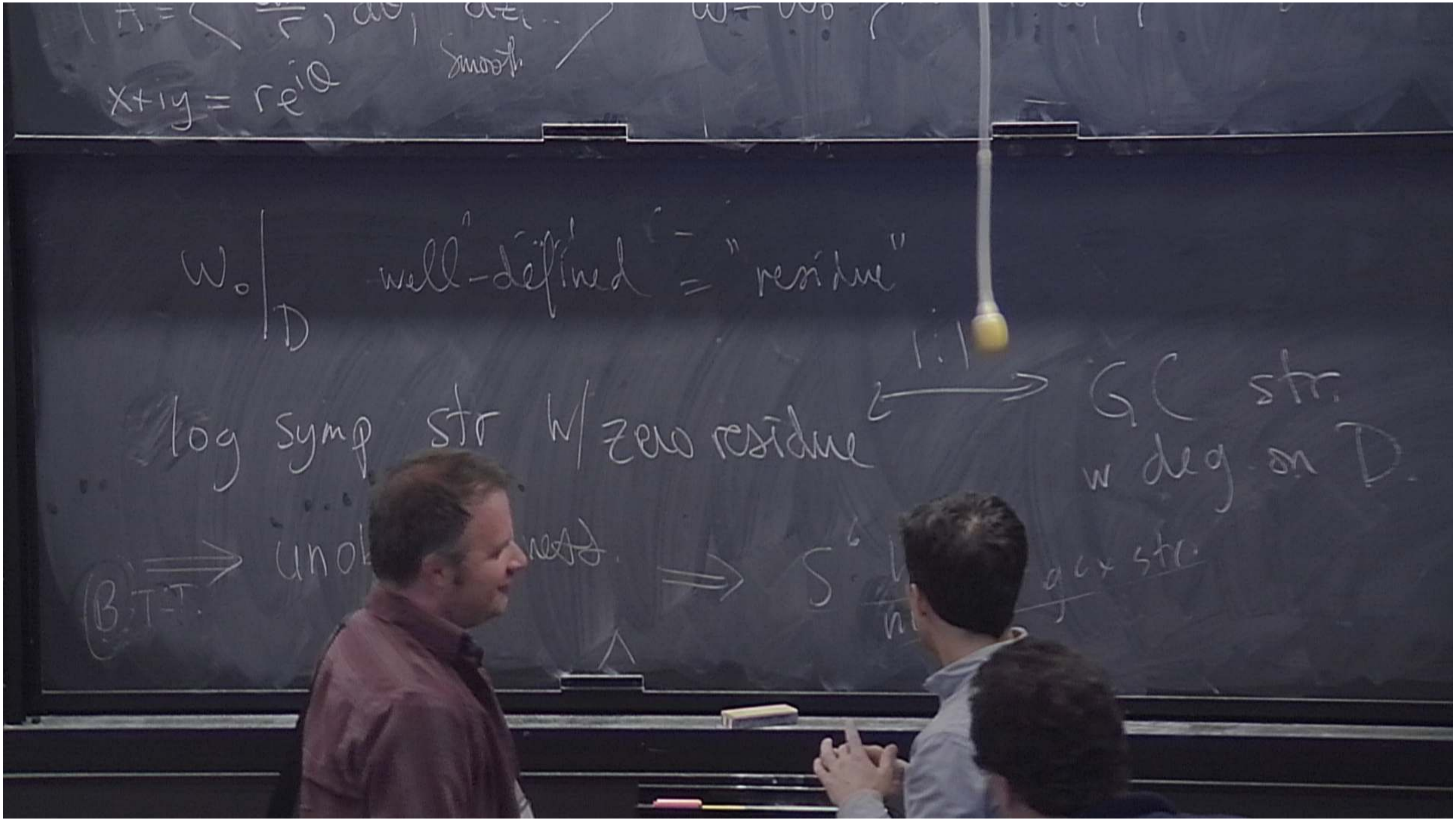

alcanti
having (D, I), a symplectic str $\omega \in \Gamma(\Lambda^2 A^*)$
$d_A \omega = 0$ (for $A = T(-\log D)$) has a residue



$d_A \omega = 0$
(for $A = T$ (
$A^* = \langle \frac{dr}{r}, d\theta, dz_i \ldots \rangle$
smooth
$\omega =$
$x + iy = re^{i\theta}$

$= T(\log D))$. has a residue:

$$\omega = \omega_0 \frac{dr}{r} \wedge d\theta + \omega_1 \frac{dr}{r} + \omega_2 \, d\theta + \omega_3$$


x+iy = re^{iθ}
smooth
w₀|D well-defined = "residue"
log symp str w/ zero residue
1:1
GC str.
w deg on D.