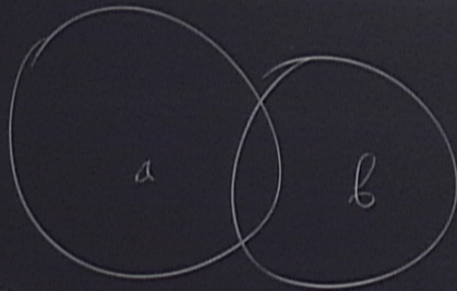


Title: Wall-crossing structures

Date: Oct 23, 2013 10:00 AM

URL: <http://pirsa.org/13100119>

Abstract: The concept of wall-crossing structure (WCS for short) was introduced recently in my joint work with Maxim Kontsevich. WCS appear in different disguises in the theory of Donaldson-Thomas invariants of Calabi-Yau 3-folds, quiver representations, integrable systems of Hitchin type, cluster algebras, Mirror Symmetry, etc.
I plan to discuss the definition of WCS and illustrate it in several well-known examples. If time permits I will speak about a special class of WCS called rational WCS. It gives rise to wall-crossing formulas with factors which are algebraic functions. Conjecturally such WCS appear in Hitchin integrable systems with singularities.



Motivation.

- WCF ~ 2004
M.K.+Y.S. (math.AG/0406564)
 - version WCF: theory of $\mathcal{DT}$ -invariants (0811.2435)
- $\mathcal{DT}$ -invariants = BPS-invariants

$$\mathcal{J} = \bigoplus_{\gamma \in \Gamma} \mathcal{J}_\gamma, \subset V \quad \mathcal{J}_\gamma = \mathcal{J}_\gamma(h)$$

67.

$$\Gamma \cong \mathbb{Z}^n, \langle \cdot, \cdot \rangle : \Lambda^2 \Gamma \rightarrow \mathbb{Z}, \text{ Poisson}$$

$$\leadsto \mathcal{J} = \mathcal{J}_\Gamma = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot e_\gamma$$

(tous Lie alg.)

$$\mathcal{O}(\text{Hom}(\Gamma, \mathbb{C}^*))$$

$$[e_{\gamma_1}, e_{\gamma_2}] = \pm \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}$$

$$\forall \gamma \in \Gamma \leadsto T_\gamma : \ell_M \mapsto \underbrace{(1 - e_\gamma)^{\langle \gamma, M \rangle}}_{\exp[L_{\mathcal{J}_\gamma}(e_\gamma), \cdot]} e_M \leftarrow \text{previous } [\cdot, \cdot]$$

$$\mathcal{J} = \bigoplus_{\gamma \in \Gamma} \mathcal{J}_\gamma; \subset V \quad \mathcal{J}_\gamma = \mathcal{J}_\gamma(h)$$

ex.

$$\Gamma (\cong \mathbb{Z}^n), \quad \langle \cdot, \cdot \rangle : \Lambda^2 \Gamma \rightarrow \mathbb{Z}, \quad \text{Poisson}$$

$$\leadsto \mathcal{J} = \mathcal{J}_\Gamma = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot e_\gamma$$

(tower Lie alg.)

$$\mathcal{O}(\text{Hom}(\Gamma, \mathbb{C}^*))$$

$$[e_{\gamma_1}, e_{\gamma_2}] = \pm \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}$$

$$\forall \gamma \in \Gamma \leadsto \mathcal{P}_\gamma : \mathcal{L}_M \mapsto (1 - e_\gamma)^{\langle \gamma, M \rangle} e_M \leftarrow \text{previous } [\cdot, \cdot]$$

$$\forall \gamma \in \Gamma \leadsto \exists \text{ "wall" } W_\gamma \text{ in } \text{Hom}(\Gamma, \mathbb{C}), \quad W_\gamma = \{ Z \mid \text{Im}(Z(\gamma)) = 0 \}$$

$$\mathcal{J} = \bigoplus_{\gamma \in \Gamma} \mathcal{J}_\gamma; \subset V \quad \mathcal{J}_\gamma = \mathcal{J}_\gamma(h)$$

67.

$$\Gamma \cong \mathbb{Z}^n, \quad \langle \cdot, \cdot \rangle : \Lambda^2 \Gamma \rightarrow \mathbb{Z}, \quad \text{Poisson}$$

$$\leadsto \mathcal{J} = \mathcal{J}_\Gamma = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot e_\gamma$$

(torsion-free Lie alg.)

$$\subset \text{Hom}(\Gamma, \mathbb{C}^*)$$

$$[e_{\gamma_1}, e_{\gamma_2}] = \pm \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}$$

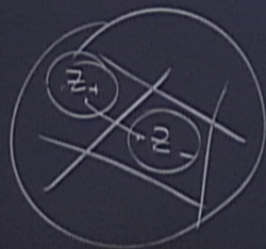
$$\forall \gamma \in \Gamma \leadsto T_\gamma : \mathcal{L}_M \mapsto (1 - e_\gamma)^{\langle \gamma, M \rangle} e_M \leftarrow \text{previous } [\cdot, \cdot]$$

$$\forall \gamma \in \Gamma \leadsto \exists \text{ "wall" } W_\gamma \text{ in } \text{Hom}(\Gamma, \mathbb{C}), \quad W_\gamma = \{ Z \mid \text{Im}(Z(\gamma)) = 0 \}$$

$$\begin{aligned}
 & \text{DP-theory} \leadsto \left\{ \Omega_z(\gamma) \right\}_{\gamma \in \mathbb{F} - \{0\}} \\
 & \quad \uparrow \\
 & \quad \text{integers} \\
 & -c \cancel{0} 0 c
 \end{aligned}$$

D) $\mathcal{P}$ -theory $\leadsto \left\{ \Omega_{\mathbb{Z}}(\gamma) \right\}_{\substack{\gamma \in \mathbb{I} - \{0\} \\ \text{integers}}}$

$-\mathbb{C} \bigcirc \bigcirc \mathbb{C}$



$$\prod_{\gamma \in \mathbb{C}} \mathbb{P}_{\gamma} \Omega_{\mathbb{Z}_+}(\gamma) = \prod_{\gamma \in \mathbb{C}} \mathbb{P}_{\gamma} \Omega_{\mathbb{Z}_-}(\gamma)$$

clockwise order of $\text{Arg } \mathbb{Z}(\gamma)$

clockwise order of $A_2 \mathbb{Z}(\gamma)$

(WCF) : works for any graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma} \mathfrak{g}_{\gamma}$ endowed with stability data, $\underline{d(\gamma)} \in \mathfrak{g}_{\gamma}$

What is underlying structure? 1303, 3253

clockwise order of $A_2 \mathfrak{g} \mathbb{Z}(\gamma)$

(WCF) : works for any graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma} \mathfrak{g}_{\gamma}$ endowed with stability data, $\underline{d(\gamma)} \in \mathfrak{g}_{\gamma}$

What is underlying structure? 1303, 3253

WCS in a vector space (nilpotent case)

- $\Gamma (\cong \mathbb{Z}^n)$ - free abelian gp
- $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_{\gamma}$ - graded Lie algebra / $\mathbb{Q}$

clockwise order of $A_2 \mathfrak{g} \mathbb{Z}(\gamma)$

(WCF) : works for any graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma} \mathfrak{g}_{\gamma}$ endowed with stability data, $\lambda(\gamma) \in \mathfrak{g}_{\gamma}$

What is underlying structure? 1303, 3253

WCS in a vector space (nilpotent case)

• $\Gamma (\cong \mathbb{Z}^n)$ - free abelian gp

• $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_{\gamma}$ - graded Lie algebra / $\mathbb{Q}$

$\text{Supp}(\mathfrak{g}) = \{\gamma \mid \mathfrak{g}_{\gamma} \neq 0\}$. Assume: $\text{Supp}(\mathfrak{g})$ is a ... contained in an open

clockwise order of $A_2 \mathfrak{g} \mathbb{Z}(\gamma)$

(WCF) : works for any graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma} \mathfrak{g}_{\gamma}$ endowed with stability data, $\lambda(\gamma) \in \mathfrak{g}_{\gamma}$

What is underlying structure? 1303, 3253

WCS in a vector space (nilpotent case)

- Γ - free abelian gp
- $\mathfrak{g} = \bigoplus_{\gamma} \mathfrak{g}_{\gamma}$ - graded Lie algebra / $\mathbb{Q}$

$= \{ \gamma \in \Gamma \mid \mathfrak{g}_{\gamma} \neq 0 \}$. Assume: $\text{Supp}(\mathfrak{g})$ is a finite set contained in an open half-space in $\Gamma_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$ //// $\Rightarrow \mathfrak{g}_0 = 0$.

$\Rightarrow \mathfrak{g} = \mathfrak{g}_{\neq 0}$

clockwise order of $A_2 \mathbb{Z}(\gamma)$

(WCF) : works for any graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma} \mathfrak{g}_{\gamma}$ endowed with stability data, $\lambda(\gamma) \in \mathfrak{g}_{\gamma}$

$\Gamma (\cong \mathbb{Z}^n)$ - free abelian gp

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$\implies \mathfrak{g}$ is nilpotent $\implies G = \exp(\mathfrak{g})$.

$\forall \gamma \in \Gamma \leadsto \text{Wall}_{\gamma} = \gamma^{\perp} \cap \Gamma_{\mathbb{R}}^*$

clockwise order of $A_2 \mathbb{Z}(\gamma)$

(WCF) : works for any graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma} \mathfrak{g}_{\gamma}$ endowed with stability data, $\lambda(\gamma) \in \mathfrak{g}_{\gamma}$

$\Gamma (\cong \mathbb{Z}^n)$ - free abelian gp

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$\Rightarrow \mathfrak{g}$ is nilpotent $\Rightarrow G = \exp(\mathfrak{g})$.

$\forall \gamma \in \Gamma \leadsto \text{Wall}_{\gamma} = \gamma^{\perp} \Rightarrow \bigcup_{\gamma \in \Gamma} \text{Wall}_{\gamma} := \text{Wall}_{\mathfrak{g}}$
 $\bigcap_{\gamma \in \Gamma} \gamma^{\perp} \Rightarrow \text{stratification of } \Gamma_{\mathbb{R}}^*$



$$\gamma_1^\perp = \gamma_2^\perp \Leftrightarrow \gamma_1 \parallel \gamma_2$$

Def. A (global) WCS for y is an assignment
 $(y_1, y_2) \mapsto g_{y_1, y_2} \in G$ s.t.

outside
 σ_y Wall y

a) cocycle: $g_{y_1, y_2} \cdot g_{y_2, y_3} = g_{y_1, y_3}$

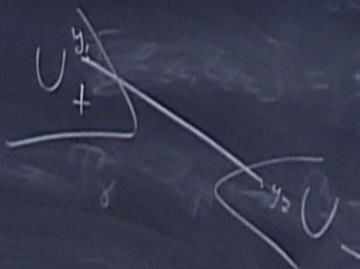
b) $\log(g_{y_1, y_2}) \in \bigoplus_{\gamma' \parallel \gamma} \gamma'$



$$b) \log(g_{y_1, y_2}) \in \bigoplus_{y \in \mathcal{Y}} y$$

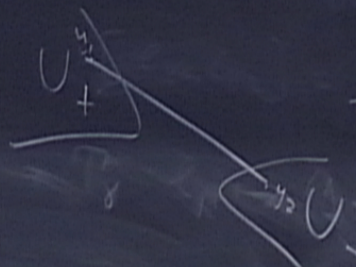
Description in terms of sheaves of sets.

Ex. $\exists$ components $U_+, U_- \subset \Gamma_{\mathbb{R}}^*$ s.t. if $y \in U_+$ then $y|_{\text{Supp}(y)} > 0$
 $y \in U_-$ $y|_{\text{Supp}(y)} < 0$



$$b) \log(g_{y_1, y_2}) \in \bigoplus_{y_1 \parallel y_2} y_1 y_2$$

components $U_+, U_- \subset \mathbb{R}$ s.t. if $y \in U_+$ then $y|_{\text{Supp}(g)} > 0$
 $y \in U_-$ $y|_{\text{Supp}(g)} < 0$



$$\Rightarrow g_{+, -} := g_{y_1, y_2} \in G$$

$$\text{WCS} \mapsto g \in G$$

Th. In this way we obtain a 1:1 between WCSs and G .

What is underlying structure? 1303, 3253

$\forall y \in \Gamma_{\mathbb{R}}^*$ we have a decomposition

$$y = \underbrace{y_{-}^{(y)}}_{\substack{\oplus y_x \\ y(x) < 0}} \oplus \underbrace{y_0^{(y)}}_{y(x)=0} \oplus \underbrace{y_{+}^{(y)}}_{y(x) > 0} \Rightarrow G = G_{-}^{(y)} G_0^{(y)} G_{+}^{(y)}$$

WCF

What is underlying structure? 1303, 3253

$\forall y \in \Gamma_{\mathbb{R}}^*$ we have a decomposition

$$y = \underbrace{y_{-}^{(y)}}_{\substack{\oplus y_{-} \\ y(s) < 0}} \oplus \underbrace{y_0^{(y)}}_{\substack{\oplus y_0 \\ y(s) = 0}} \oplus \underbrace{y_{+}^{(y)}}_{\substack{\oplus y_{+} \\ y(s) > 0}}$$

$$\Rightarrow G = G_{-}^{(y)} G_0^{(y)} G_{+}^{(y)}$$

WCS

$$g_0^{(y)} \in G_0^{(y)} = \frac{G}{G_{-}^{(y)} G_{+}^{(y)}}$$

Claim In this way we get a heap of sets (st)

WCF

What is underlying structure? 1303, 3253

$\forall y \in \Gamma_{\mathbb{R}}^*$ we have a decomposition

$$y = \underbrace{y|_{-}}_{\substack{\oplus y|_{-} \\ y(x) < 0}} \oplus \underbrace{y|_0}_{y(x)=0} \oplus \underbrace{y|_+}_{y(x) > 0}$$

$$\Rightarrow G = G_-^{(y)} \cdot G_0^{(y)} \cdot G_+^{(y)}$$

WCS

Claim In this way we obtain a sheaf of sets (stalks are $G_0^{(y)}$)
Global WCS

$$g_0^{(y)} \in G_0^{(y)} = \frac{G}{G_-^{(y)} \cdot G_+^{(y)}}$$

WCF

What is underlining etc. 1 2 3 4 5 6

$\forall \gamma \in \Gamma_R^*$ we have a decomposition

$$\gamma = \gamma_{-}^{(\gamma)} \oplus \gamma_0^{(\gamma)} \oplus \gamma_{+}^{(\gamma)} \rightarrow \gamma = \gamma_{-}^{(\gamma)} \gamma_0^{(\gamma)} \gamma_{+}^{(\gamma)}$$

$\oplus \gamma_{\gamma}$ $\gamma(\gamma) < 0$ $\gamma(\gamma) = 0$ $\gamma(\gamma) > 0$

WC

Claim In this way we obtain a sheaf of sets (stalks are $G_0^{(\gamma)}$)
 Global WCSs $\leftrightarrow$ sections of this sheaf, WCS_{γ}

WCF : ... Γ_R^*

$$c) \log(g_{y_1, y_2}) \in \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$$

Generalization

If $\text{Supp}(y)$ is infinite then assume $\exists$ strict convex Cone $C \subset \Gamma_{\mathbb{R}}^*$ s.t.
 $\text{Supp}(y) \subset C$

$$\Rightarrow y_C = \prod_{\gamma \in C} y_\gamma - \text{pronilpotent L.A.} \mapsto G_C = \exp(-y_C)$$



$$b) \log(g_{y_1, y_2}) \in \bigoplus_{y' \parallel y} y_{y'}$$

Generalization

If $\text{Supp}(y)$ is infinite then assume $\exists$ strict convex Cone $C \subset \Gamma_{\mathbb{R}}^*$ s.t.
 $\text{Supp}(y) \subset C$

$$\Rightarrow y_C = \prod_{y \in C} y_y - \text{pronilpotent L.A.} \mapsto G_C = \exp(-y_C)$$



we allow C to vary: $\{C_y \subset \Gamma_{\mathbb{R}}^*\}$

WCS in top. spaces.

condition

• M a Hausdorff top. space

• $\Gamma \rightarrow M$ - local system of lattices

• $\underline{\mathfrak{g}} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ - local syst. of Γ -graded Lie algebras

• $\gamma : \Gamma \rightarrow \text{Cont}_M$ - homom. of sheaves of abelian gps

($\Leftrightarrow$ locally γ is a map from a domain in M to Γ^*_R)

Having a $WCS_\gamma \rightsquigarrow WCS_{\gamma, \gamma} = \gamma^*(WCS_\gamma)$

(WCF)

WCS in top. space.

condition

• M a Hausdorff top. space

• $\underline{\Gamma} \rightarrow M$ - local system of lattices

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• $\gamma : \underline{\Gamma} \rightarrow \text{Cont}_M$ - homom. of sheaves of abelian gps

($\Leftrightarrow$ locally γ is a map from a domain in M to $\Gamma_{\mathbb{R}}^*$)

Having a $WCS_{\underline{\gamma}} \leadsto WCS_{\underline{\gamma}, \gamma} = \gamma^*(WCS_{\underline{\gamma}})$

closed strict convex cone

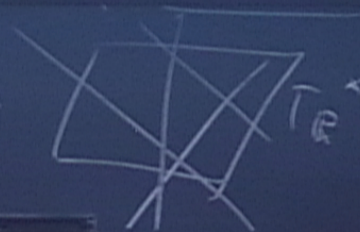
Prop. Can define support of WCS on M is a family

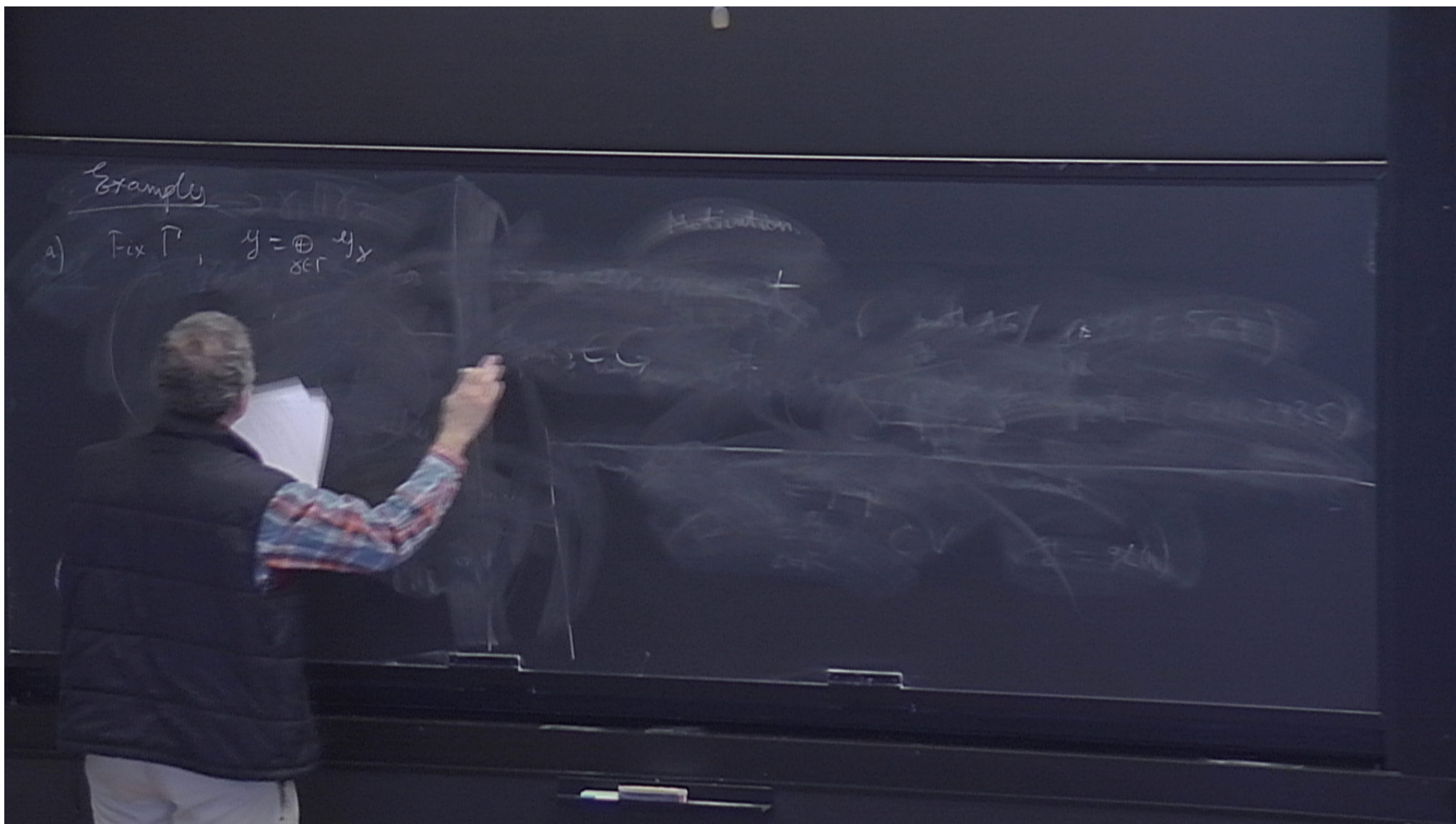
$$C_m \subset \Gamma_m \otimes \mathbb{R}$$

- δ is important $\Rightarrow \zeta = \exp(-\gamma)$

$\forall \gamma \in \underline{\Gamma} \leadsto \text{Wall}_{\gamma} = \gamma^{\perp} \Rightarrow \bigcup_{\gamma \in \underline{\Gamma}^*} \text{Wall}_{\gamma} := \text{Wall}_{\underline{\gamma}}$

$\Rightarrow$ stratification of $\Gamma_{\mathbb{R}}^*$





Example

a) Fix Γ , $y = \bigoplus_{\gamma \in \Gamma} y_\gamma$, $z: \Gamma \rightarrow \mathbb{C}$

Take $M = S^1 = \mathbb{R} / 2\pi\mathbb{Z}$

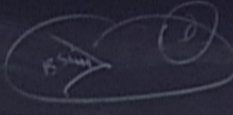
$y \in \Gamma$ are const. local systems on M .

Set $Y: M \rightarrow \Gamma_{\mathbb{R}}^*$

$\theta \mapsto Y_\theta(x) = \text{Im}(e^{-i\theta} z(x))$

6)

Global WCS $\Leftrightarrow$ stability data on $\mathcal{Y}$ ($\Leftrightarrow (a(x), \epsilon) \in \mathcal{Y}_x \mid \exists$ quad. form Q on $\mathbb{R}^n$ s.t. $\downarrow \begin{smallmatrix} \vdots \\ \times \\ \vdots \end{smallmatrix} \leftarrow \epsilon \pi_*$)
 $Q|_{\ker Z} < 0$ and $Q|_{\text{Supp } a} \geq 0$

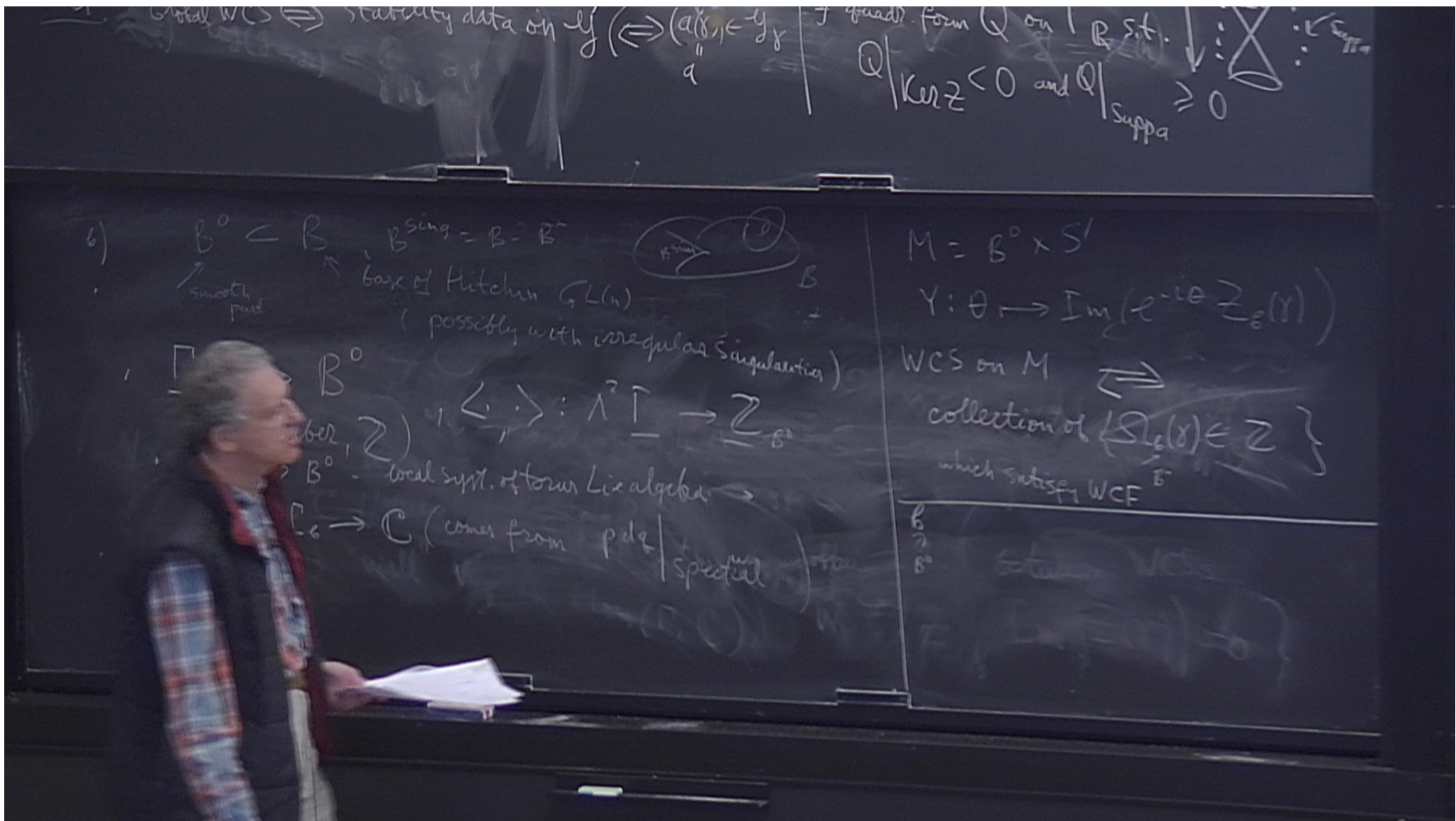
6) $B^0 \subset B$ $B^{\text{sing}} = B - B^0$ 
 $\nwarrow$ base of Hitchin $G/L(n)$
 (possibly with irregular singularities)

$\Gamma \rightarrow B^0$
 $H_1(\text{fiber}, \mathbb{Z})$, $\langle \cdot, \cdot \rangle : \Lambda^2 \Gamma \rightarrow \mathbb{Z}_{B^0}$

$\mathcal{Y} \rightarrow B^0$ - local syst. of torus Lie algebra $\rightarrow$

$Z_\epsilon : \Gamma_\epsilon \rightarrow \mathbb{C}$ (comes from $\text{pdg} \mid \text{spectral}$)

$M = B^0 \times S^1$
 $Y : \theta \mapsto \text{Im}(e^{-i\theta} Z_\epsilon(Y))$
 WCS on $M \Leftrightarrow$
 collection of $\{Z_\epsilon(x) \in \mathbb{Z}\}$
 which satisfy WCF $\mathbb{R}$



Global WCS $\Leftrightarrow$ stability data on $\mathcal{Y}$ ($\Leftrightarrow (a(x), \epsilon - y_x) \mid \exists$ quad. form Q on Γ s.t. $Q|_{\ker Z} < 0$ and $Q|_{\text{Supp } a} \geq 0$)

$$\Gamma \rightarrow B, \quad H_1(\text{fiber}, \mathbb{Z}), \quad \langle \cdot, \cdot \rangle : \Lambda^2 \Gamma \rightarrow \mathbb{Z}_B$$

$\mathcal{Y} \rightarrow B^0$ - local syst. of torus Lie algebra $\rightarrow$

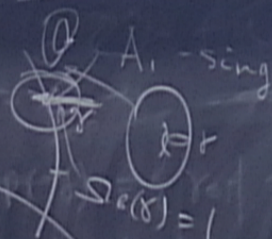
$Z_e : \Gamma_e \rightarrow \mathbb{C}$ (comes from $p.d.g.$ spectral)

collection of $\{\Omega_e(x) \in \mathbb{Z}\}$
which satisfy WCF B

$$\begin{aligned} \Gamma_e &\rightarrow CY_3, \quad X_e = \{uv = f(u, v)\} \\ \mathbb{R}^0 &\quad \mathcal{F}(X_e) \rightarrow (\Omega^{\mathcal{F}(X_e)}(\mathcal{F})) \xrightarrow{DF_{\text{Ferm}}} H^2(X_e, \mathbb{Z}) \\ \text{Orig. } \Omega^{\mathcal{F}(X_e)}(x) &= \Omega_e(x) \end{aligned}$$

$(\Rightarrow)$ locally γ is a map from a domain in M to $T^*\mathbb{R}$
 Having a WCS $\gamma \leadsto WCS_{\gamma} = \gamma^*(WCS_{\mathbb{R}})$
Prob Can define support of WCS on M is γ^* continuous family $C_m \subset T_m \otimes \mathbb{R}$
 closed strict convex cone

Initial data



B_{sing}

$\gamma \neq 0$ Assume $\text{supp}(\gamma)$ is a family of oriented open half-line
 $\Rightarrow \gamma$ is not a point $\Rightarrow G = \text{cap}(\gamma)$
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($\Leftrightarrow$) locally γ is a map from a domain in M to $\Gamma_{\mathbb{R}}^*$

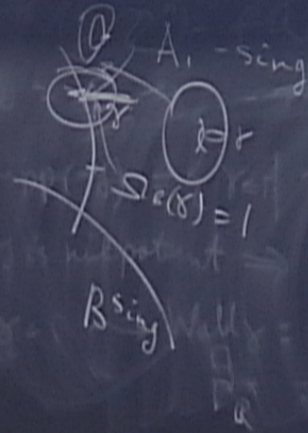
Having a $WCS_{\gamma} \rightsquigarrow WCS_{\gamma, \gamma} = \gamma^*(WCS_{\gamma})$

Prob Can define support of WCS on M is continuous family

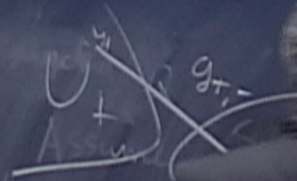
closed strict convex cone

$$C_m \subset T_m^* \otimes \mathbb{R}$$

Initial data



Rational WCS



$$G \subset \text{Aut}(\text{Hom}(\Gamma, \mathbb{C}^*))$$

$\pm$ WCS is rational if

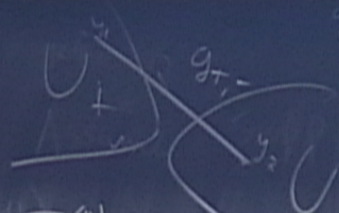
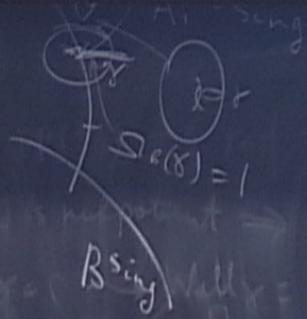
($\Rightarrow$) locally γ is a map from a domain in M to $\Gamma_{\mathbb{R}}^*$

Having a $WCS_{\gamma} \rightsquigarrow WCS_{\gamma} = \gamma^*(WCS_{\gamma})$

Prob Can define support of WCS on M is γ^* continuous family

closed strict convex cone

$$C_m \subset T_m \otimes \mathbb{R}$$



$g_{+,-} \in \text{Aut}(\text{Hom}(T, \mathbb{C}^*))$

Def. WCS is rational if γ_{can} is a rational funct.

$$g \in \mathbb{R} - \text{Wall} \Rightarrow \gamma_{\text{can}} = g_{\text{can}}^{-1} \cdot g_{\text{can},0} \cdot g_{\text{can},0}^{-1} \cdot g_{\text{can},0} \cdot g_{\text{can},0}^{-1}$$

Th If WCS rational then all factors are algebraic funct.

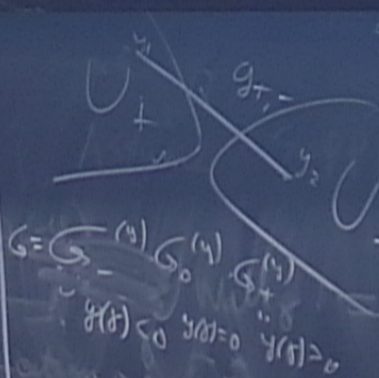
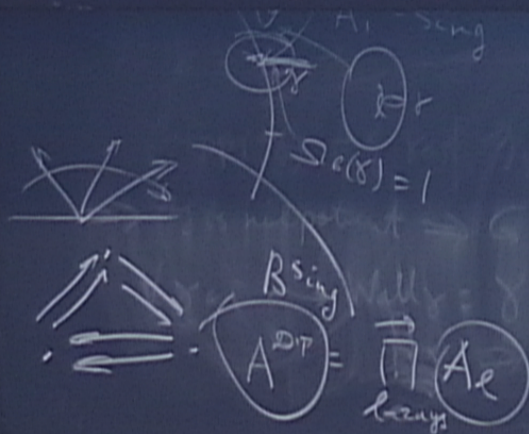
($\Rightarrow$) locally Y is a map from a domain in M to $\Gamma_{\mathbb{R}}^*$

Having a $WCS_y \leadsto WCS_y, Y = Y^*(WCS_y)$

Prob Can define support of WCS on M is $\overset{\text{continuous}}{\text{family}}$

closed strict convex cone

$$C_m \subset T_m \otimes \mathbb{R}$$



$g_{+,-} \in G \subset Aut(Mom(T, \mathbb{C}^*))$

Def. WCS is rational if g_{can} is a rational fnc.

$$y \in \Gamma_{\mathbb{R}} - Wall \Rightarrow g_{can} = g_{can}^{(y)} = g_{can,0}^{(y)} \cdot g_{can,0}^{(y)}$$

Th If WCS is rational then all factors are algebraic fncs.