Title: Blobbed topological recursion

Date: Oct 22, 2013 04:15 PM

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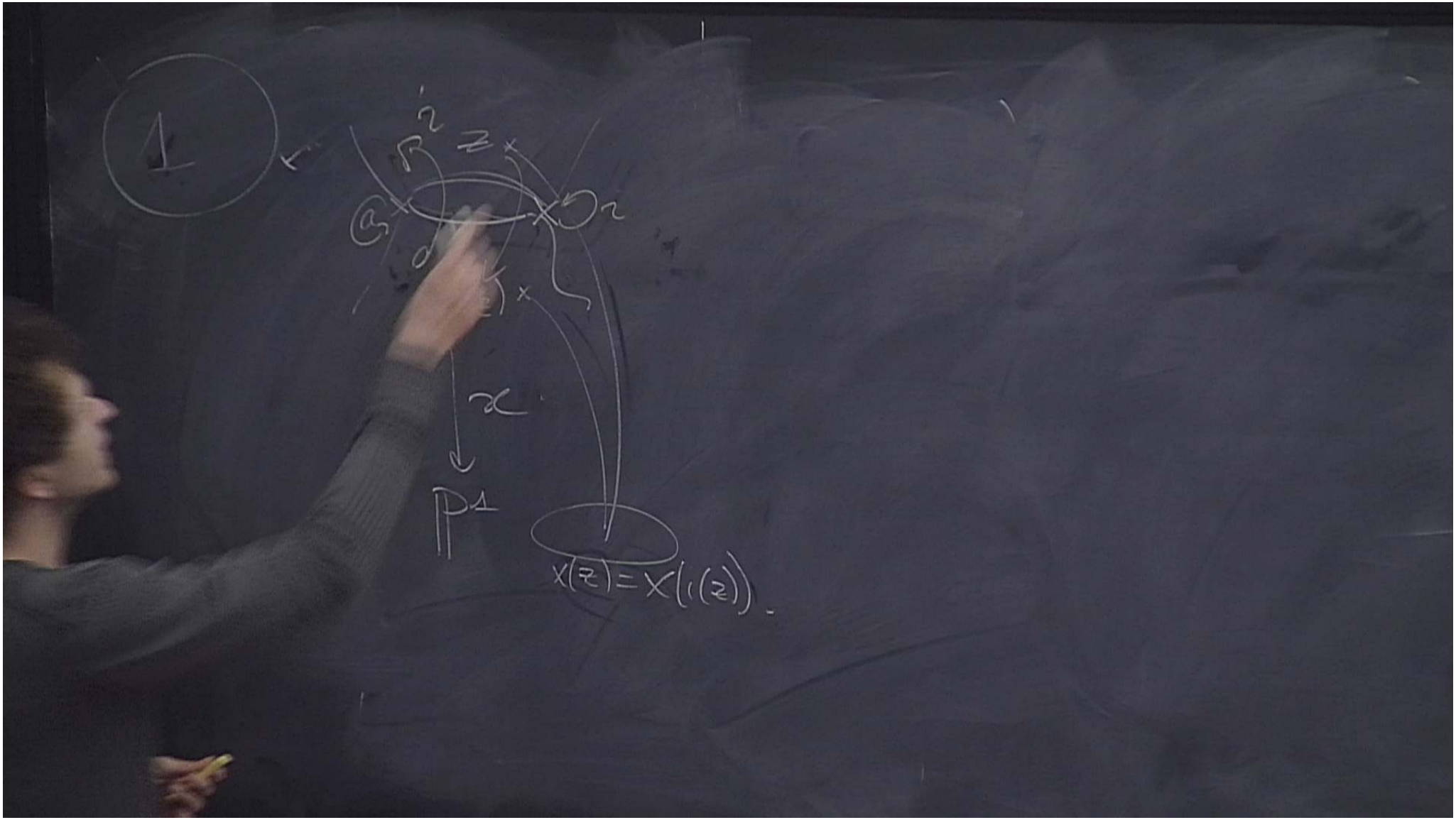
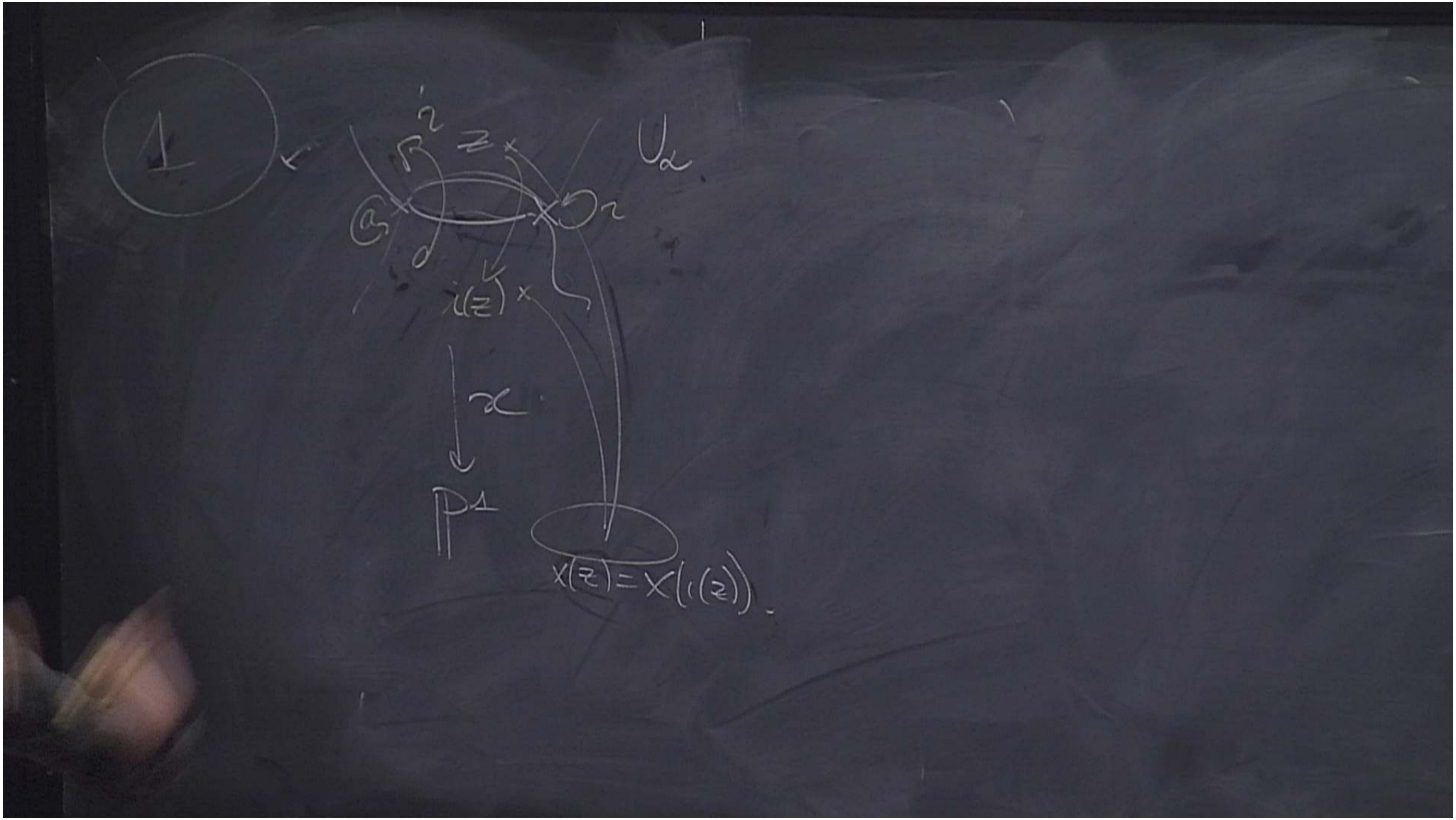
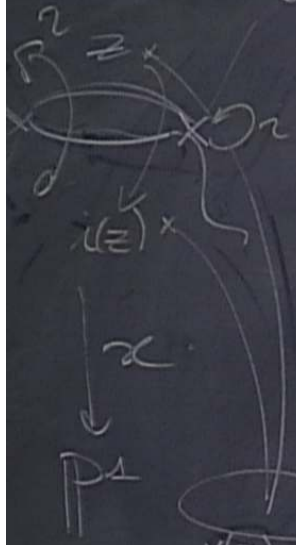
Abstract: Hermitian matrix models have been used since the early days of 2d quantum gravity, as generating series of discrete surfaces, and sometimes toy models for string theory. The single trace matrix models (with measure dM exp(- N Tr V(M)) have been solved in a 1/N expansion in the 90s by the moment method of Ambjorn et al. Later, Eynard showed that it can be rewritten more intrinsically in terms of algebraic geometry of the spectral curve, and formulated the so-called topological recursion.
In a similar way, we will show that double hermitian matrix models are solved by the same topological recursion, and more generally, that arbitrary hermitian matrix models are solved by a "blobbed topological recursion", whose properties still have to be investigated.


1
$\mathbb{R}^2$
z
x
α
$\mathbb{P}^1$
$\chi(z) = \chi(\iota(z))$


1
$\mathbb{R}^2$
z
U_2
$i(z)$
π
$\mathbb{P}^1$
$X(z) = X(i(z))$

$U = \bigcup_\alpha U_\alpha$

U_α



$\omega^{(0,1)}_{1\,(z)}$ 1-form on U_α, simple zeroes at $Fix(\alpha)$

$\omega^{(0,1)}_{2\,(z_1,z_2)}$ 1-form z_1, 1-form z_2, $\sim \frac{dz_1 dz_2}{(z_1 - z_2)^2}$ + reg.

$U = \bigcup_\alpha U_\alpha$

U_α

$\omega^0_{(z)^1}$ 1-form on U_α, simple zeroes at $Fix(n)$

$\omega^0_{(z_1,z_2)^2}$ 1-form z_1, 1-form z_2, $\sim \frac{dz_1 dz_2}{(z_1-z_2)^2} + reg.$

$\omega^g_{(z_1,\dots,z_n)^n} \in Sym^n \Omega^1_i(U).$

$= X(i(z))$

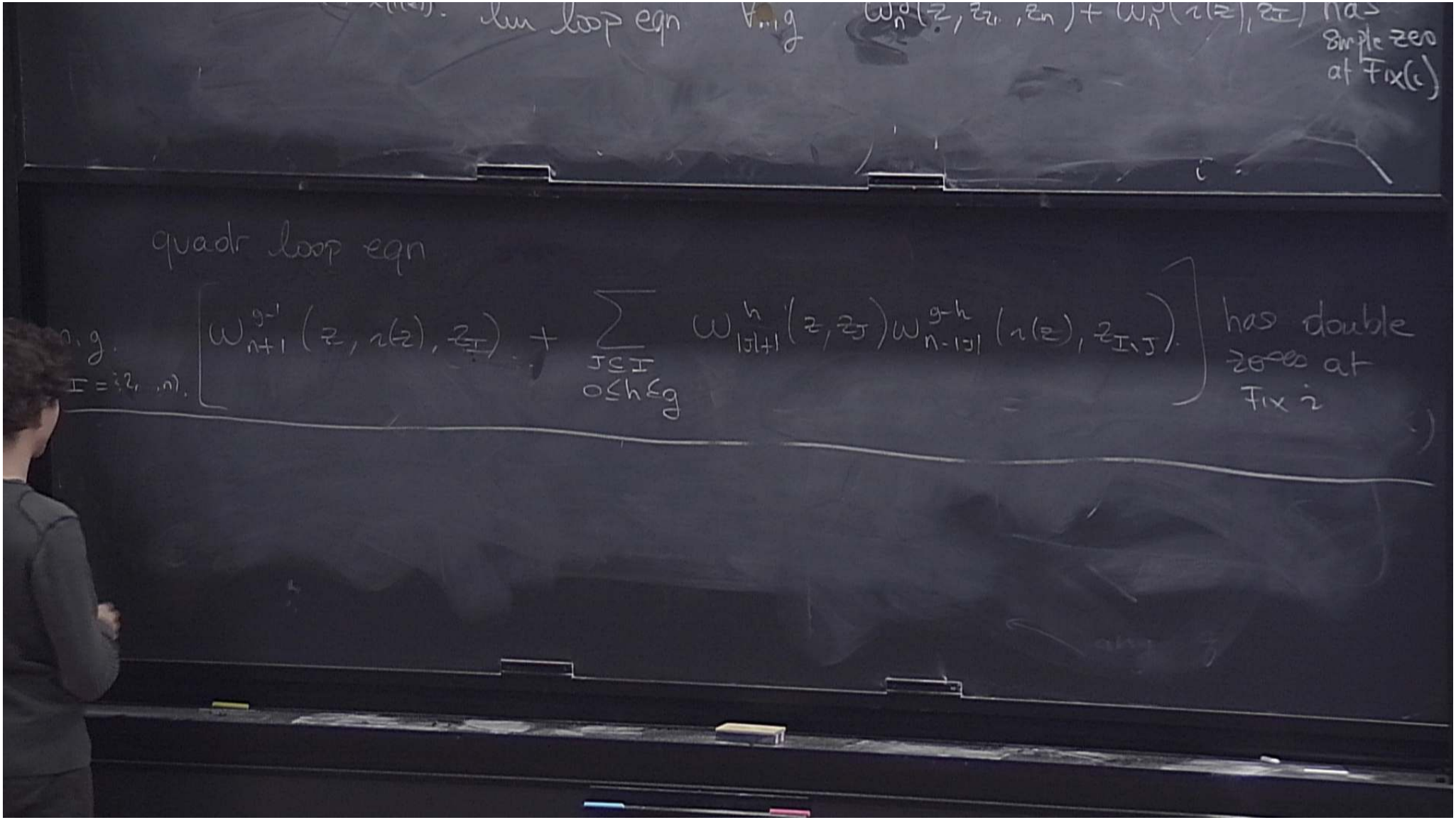
$\bigcup_\alpha U_\alpha$

ω_{1}^{0} : 1-form on U_α, simple zeroes at $Fix(\iota)$

ω_{2}^{0} : 1-form z_1, 1-form z_2, $\sim \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{reg.}$

$\omega_n^g \in \mathrm{Sym}^n \Omega^1_{\Sigma}(U)$.

- lm loop eqn $\forall n, g$ $\omega_n^g(z, \overbrace{z_1, \dots, z_n}^{z_=}) + \omega_n^g(\iota(z), z_=)$ has simple zero at $Fix(\iota)$


simple zero at Fix(i)
quadr loop eqn
has double zeroes at Fix i

n.g.

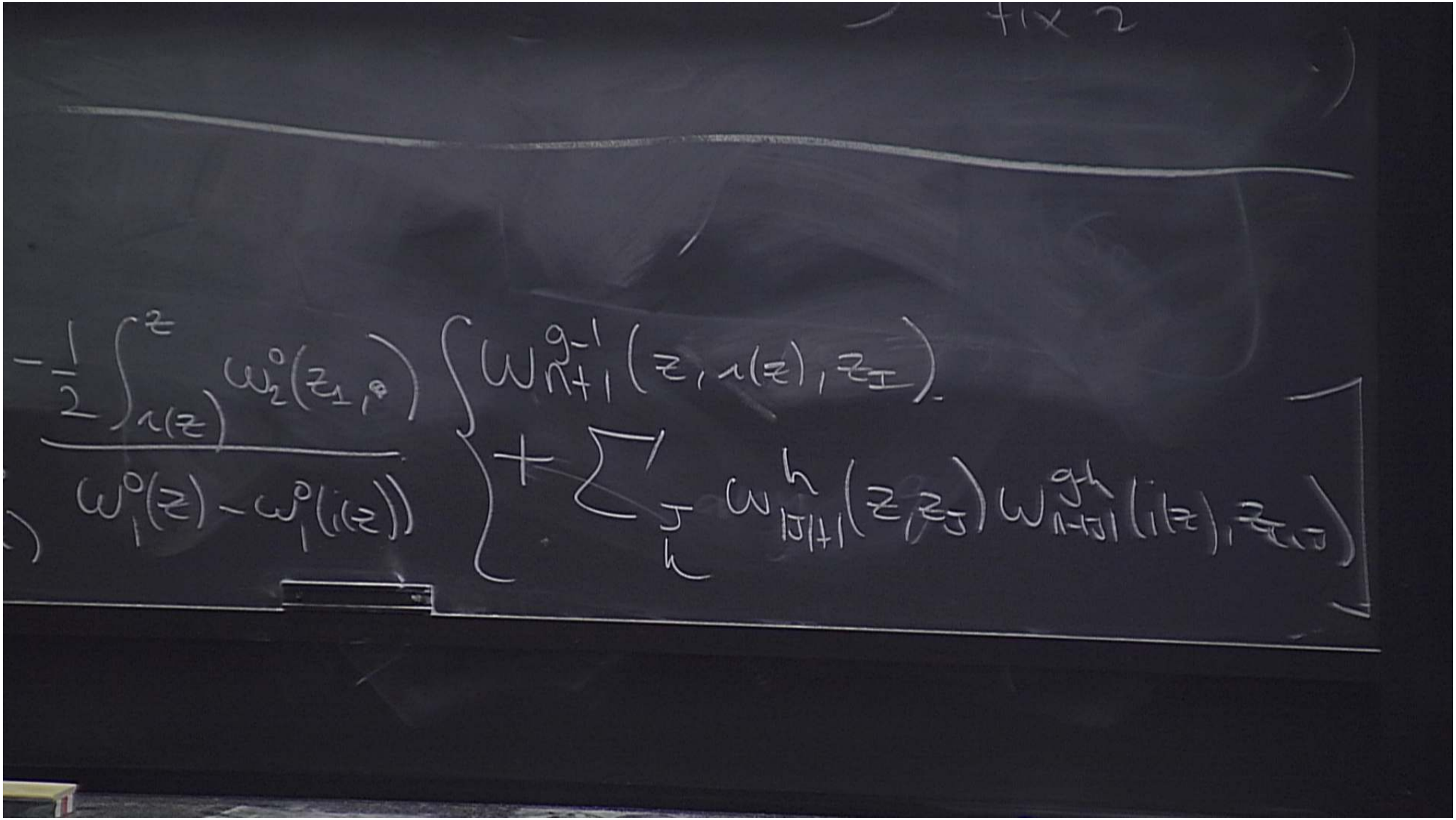
$I = \{2, \dots, n\}$.

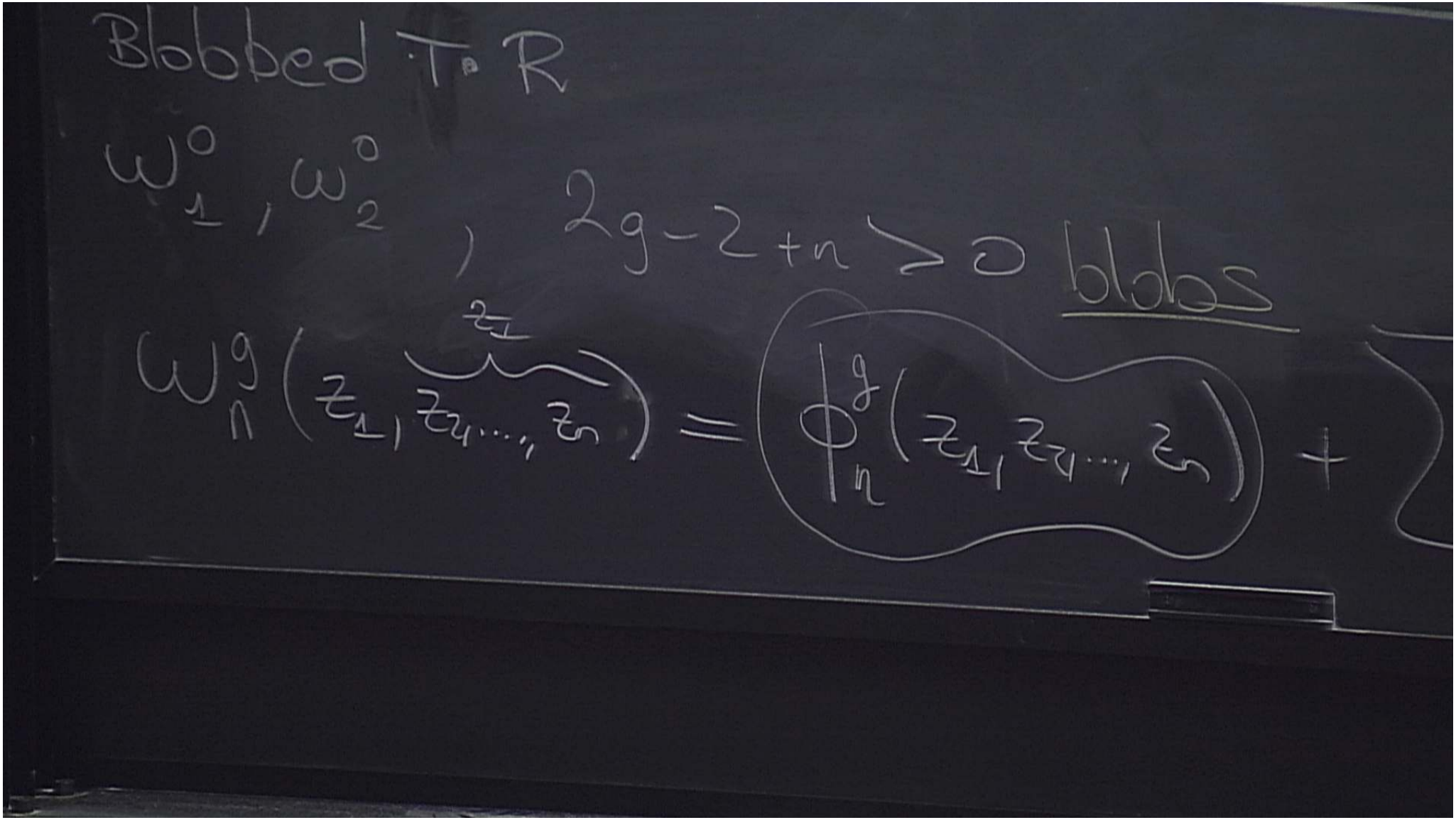
$$\omega_{n+1}^{g-1}(z, \bar{z}(z), z_I) + \sum_{\substack{J \subseteq I \\ 0 \le h \le g}} \omega_{|J|+1}^{h}$$

Blobbed T.R

ω_1^0, ω_2^0, $2g-2+n > 0$

$$\omega_n^g(\overbrace{z_1, z_2, \dots, z_n}^{z_I}) = \phi_n^g(z_1, z_2, \dots, z_n) + \sum_i \operatorname*{Res}_{z \to F_i(z)}$$


+ (1 x 2


Blobbed T R
ω_1^0, ω_2^0 , $2g-2+n > 0$ blobs
$W_n^g(z_1, z_2, \ldots, z_n) = \left(\phi_n^g(z_1, z_2, \ldots, z_n) + \right.$

- $\phi_n^g \in \mathrm{Sym}^n H^1(U)$.

- If $\phi_n^g \equiv 0$: Eynard–Orantin (2007)

- If $\phi_n^g \not\equiv 0$.

→ topological exp in hermitian matrix models. (blobbed T

→ BKMP 2007 :

- $\phi_n^g \in \mathrm{Sym}^n H^4(U)$.

- If $\phi_n^g \equiv 0$: Eynard–Orantin (2007)

- If $\phi_n^g \not\equiv 0$.

→ topological exp in hermitian matrix models. (blobbed TR)

→ BKMP 2007 : $\begin{cases} w_1^0 = \text{Lyd-Lux} \\ w_2^0 \end{cases}$ → w_n^g GW inv of toric CY3 folds.

EO 2012

ω_2^0 EO 2012

II: APPLICATIONS : MATRIX MODELS

M hermitian $N \times N$.

$$Z_N = \int dM \exp\left(\sum_{\ell \geq 1} N t_\ell \, \mathrm{Tr} M^\ell\right)$$

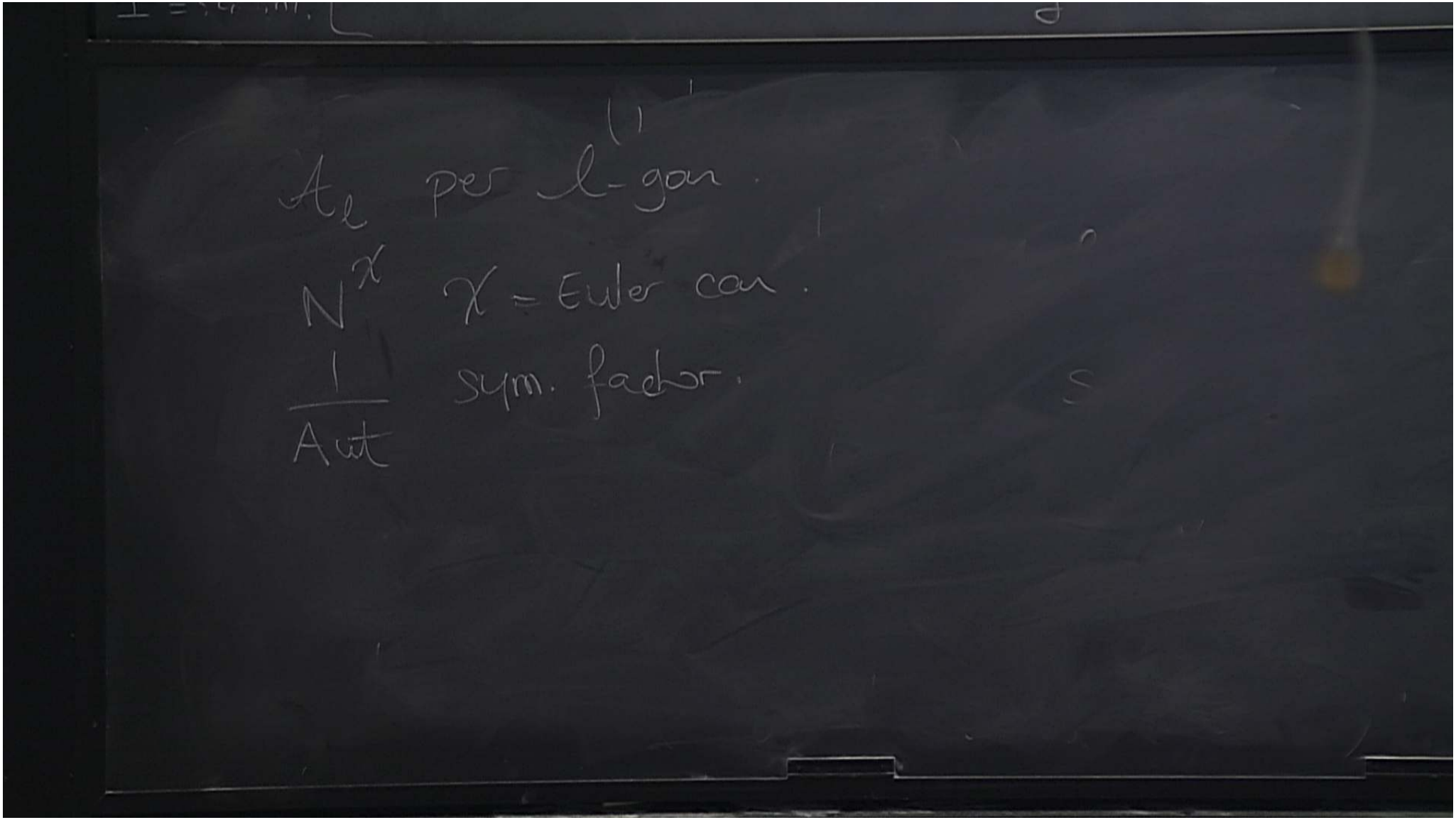
EO 2012

II APPLICATIONS : MATRIX MODELS

M hermitian $N \times N$.

- $Z_N = \int dM \exp\left(\sum_{\ell \geq 1} N t_\ell \operatorname{Tr} M^\ell\right)$

generating series for discrete surfaces made of polygons.


A_ℓ per ℓ-gon.
N^χ χ = Euler con.
$\frac{1}{\text{Aut}}$ sym. factor.

A_ℓ per ℓ-gon.

N^χ χ = Euler car.

$\frac{1}{\text{Aut}}$ sym. factor.

$$\ln Z_N = \sum N^{2-2g} F_g$$

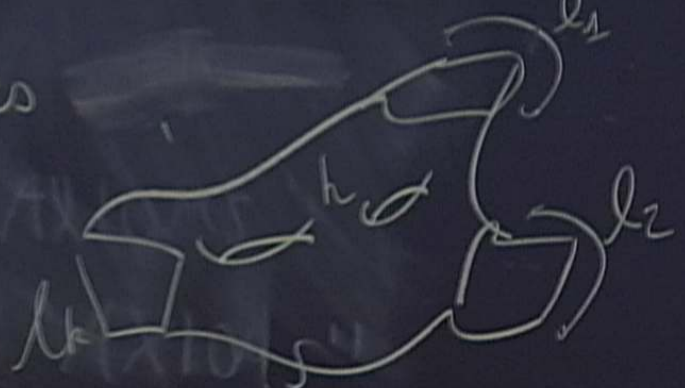
Ambjørn-Makeenko-Kristjansen, Chekhov

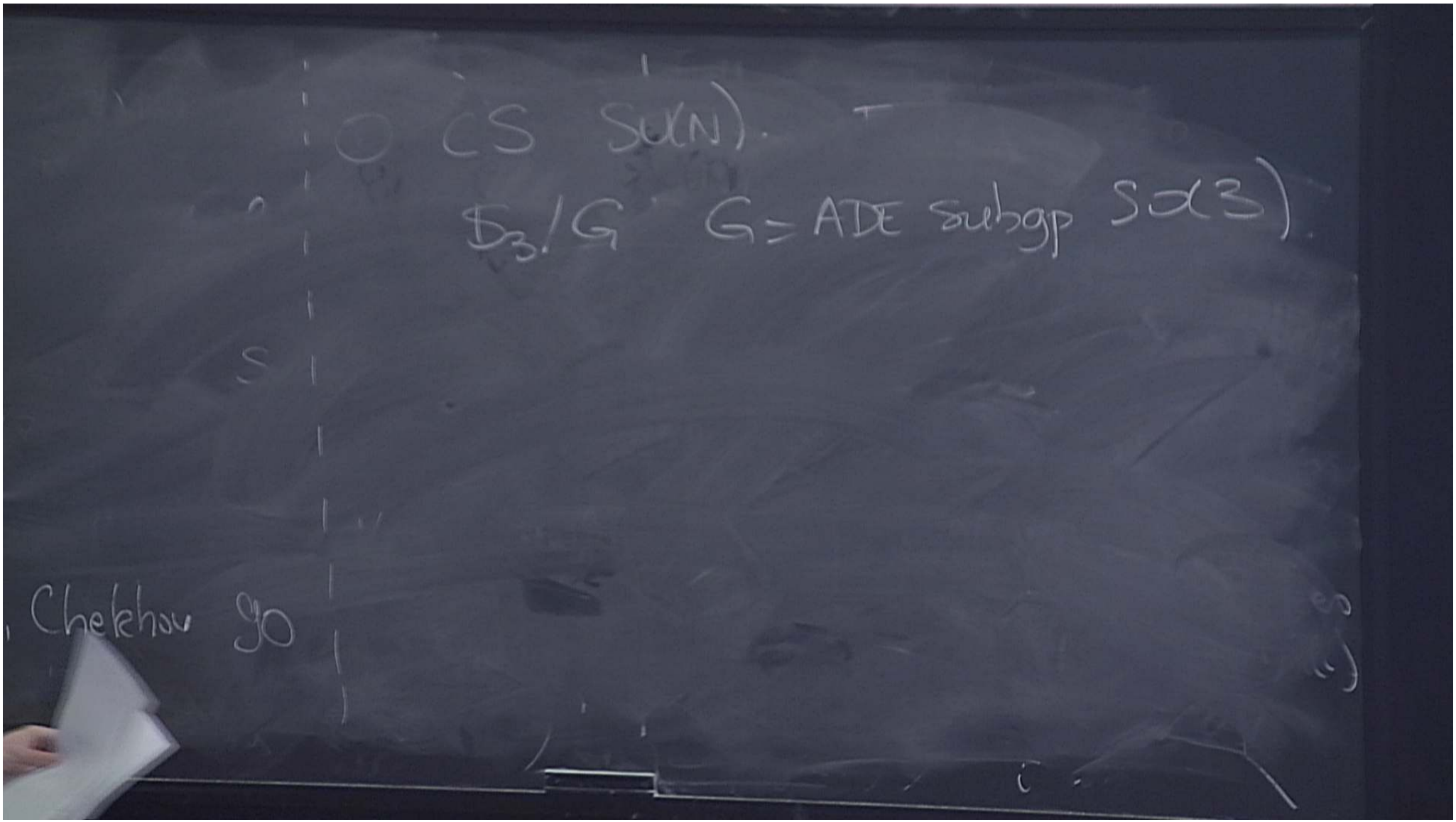
$$\left(\sum_{l} \operatorname{Tr} M^{l}\right) \exp\left(\sum_{\substack{k\geq 1\\ h\geq 0}} \frac{N^{2-2h-k}}{k!} \sum_{l_1,\dots,l_k\geq 1} t^{h}_{l_1,\dots,l_k} \operatorname{Tr} M^{l_1} \cdots \operatorname{Tr} M^{l_k}\right)$$

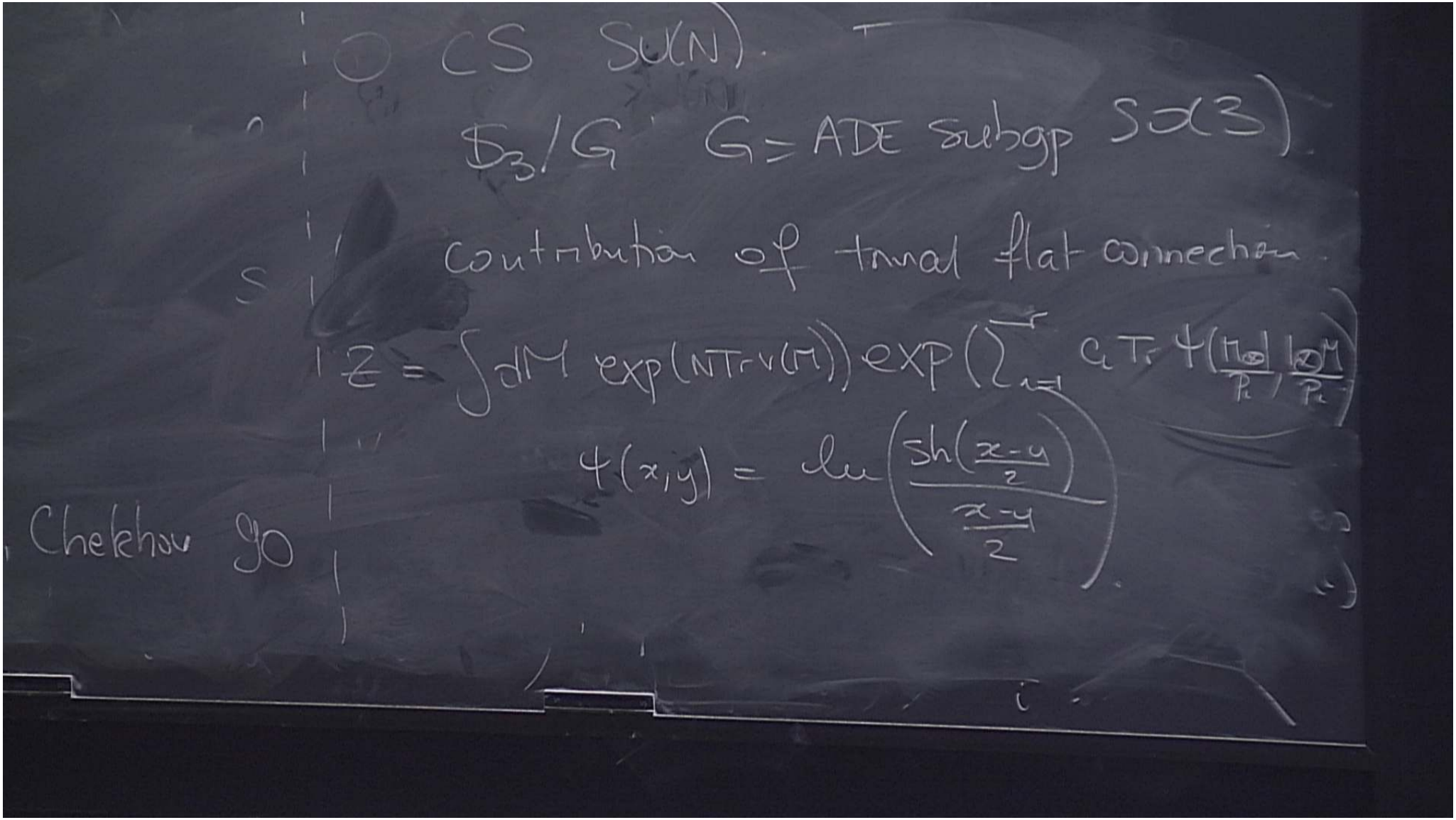
gen series

made

$t^{h}_{l_1,\dots,l_k}$



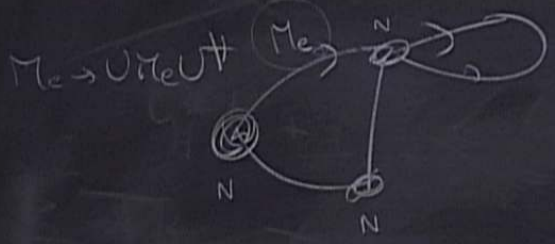

CS SU(N).
$\mathbb{C}^3/G$ G = ADE subgp SU(3)
Chekhov go


CS SU(N).
S_3/G $G \subset ADE$ subgp $SO(3)$
contribution of trivial flat connection.
$Z = \int dM \exp(NTr\,V(M)) \exp\left(\sum_{i=1}^{r} c_i T_i \psi\left(\frac{\Pi_\infty}{P_i}, \frac{\Omega^M}{P_i}\right)\right)$
$\psi(x,y) = \ln\left(\frac{\sh\left(\frac{x-y}{2}\right)}{\frac{x-y}{2}}\right)$.
Chekhov 90

Ambjørn-Makeenko-Kristjansen, Chekhov 90

$$\phi(x,y) = \ln\left(\frac{x-y}{2}\right)$$

② fluxes matrix models



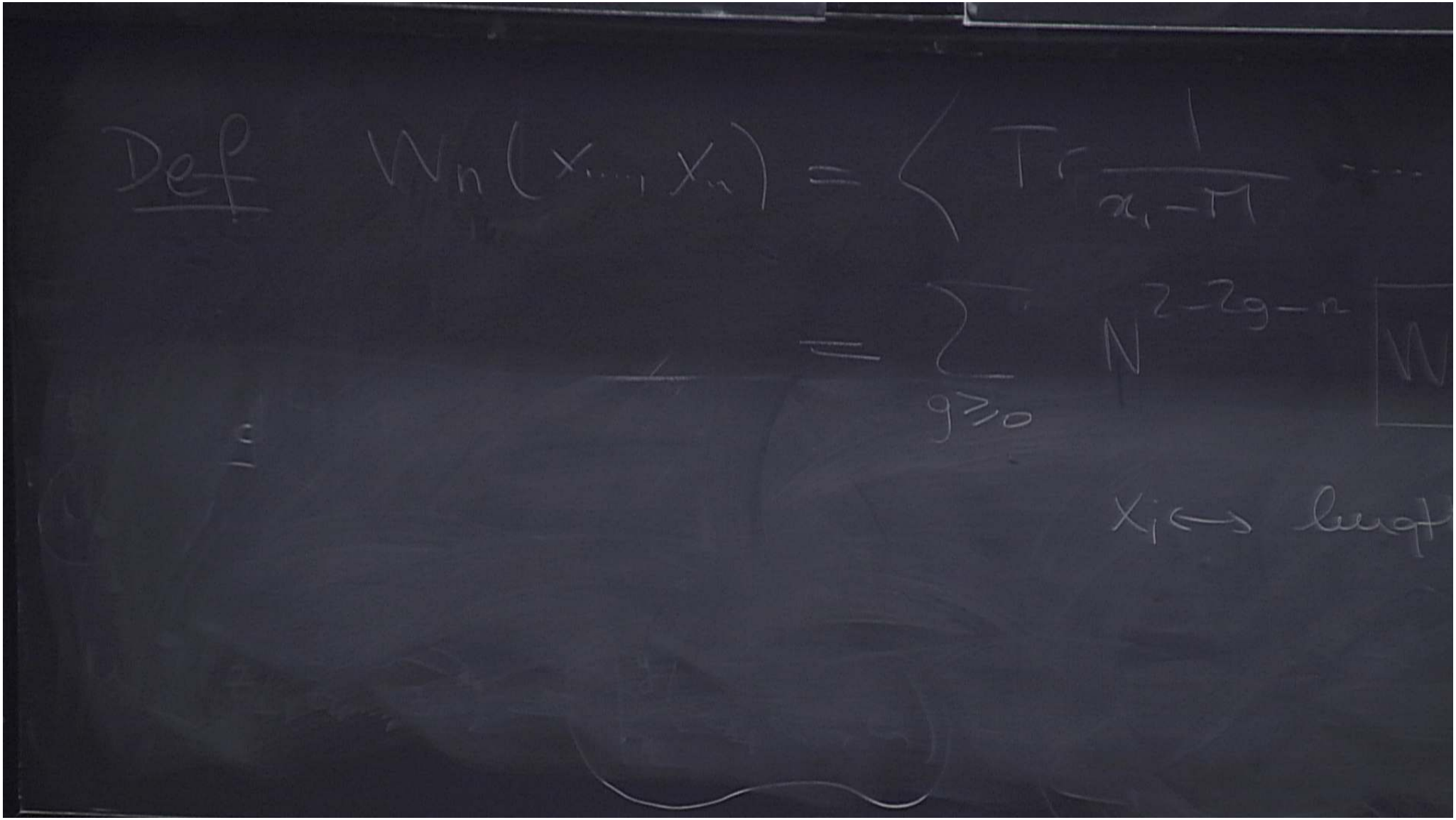
$$\int \prod_e dM_e \exp\left(N \mathrm{Tr}\, V_e(M_e)\right) \prod_{v} \exp\left(c_v \mathrm{Tr}\, M_e M_{e'}\right)$$

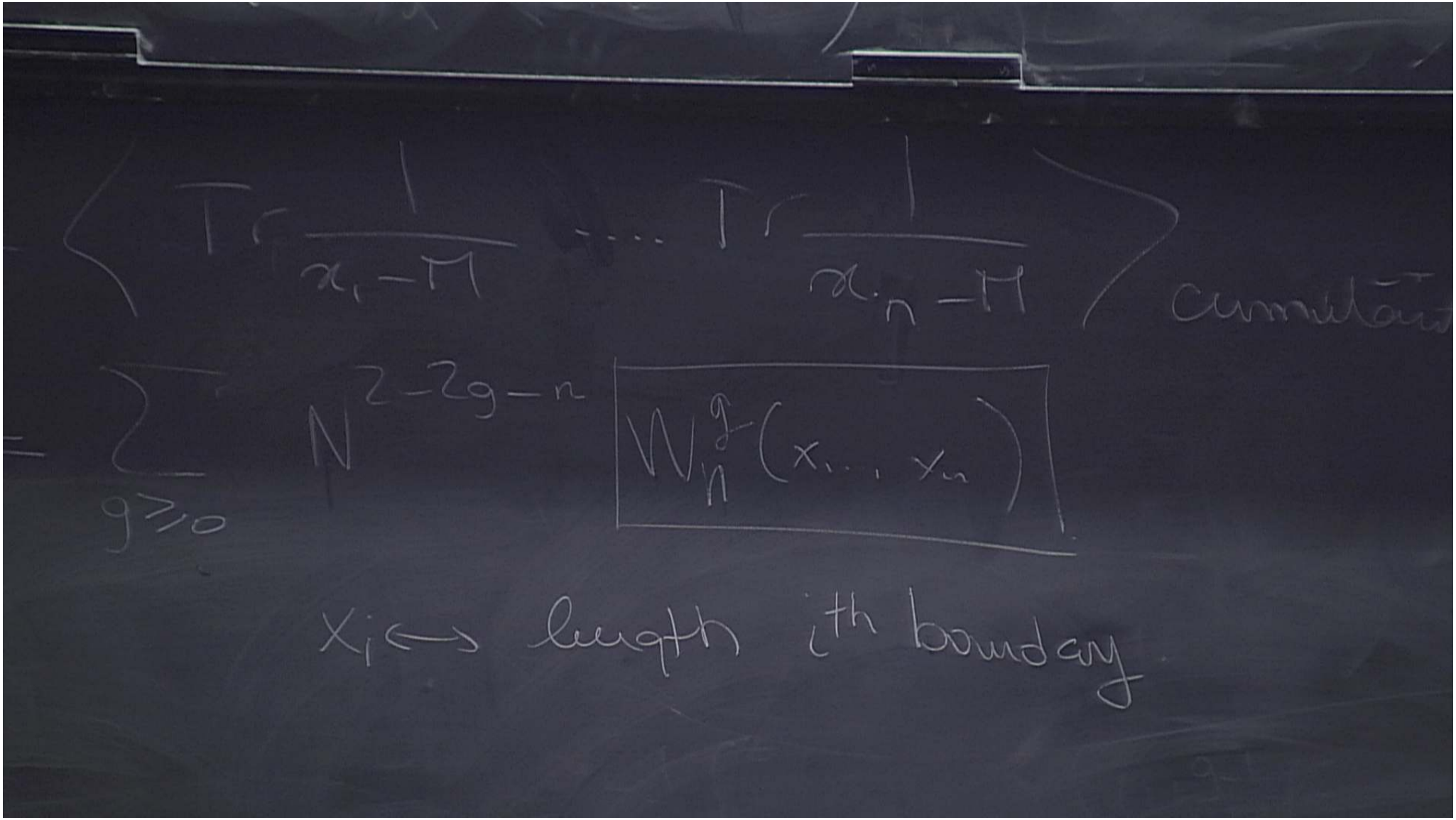
$$= \int dM_{e_0}$$

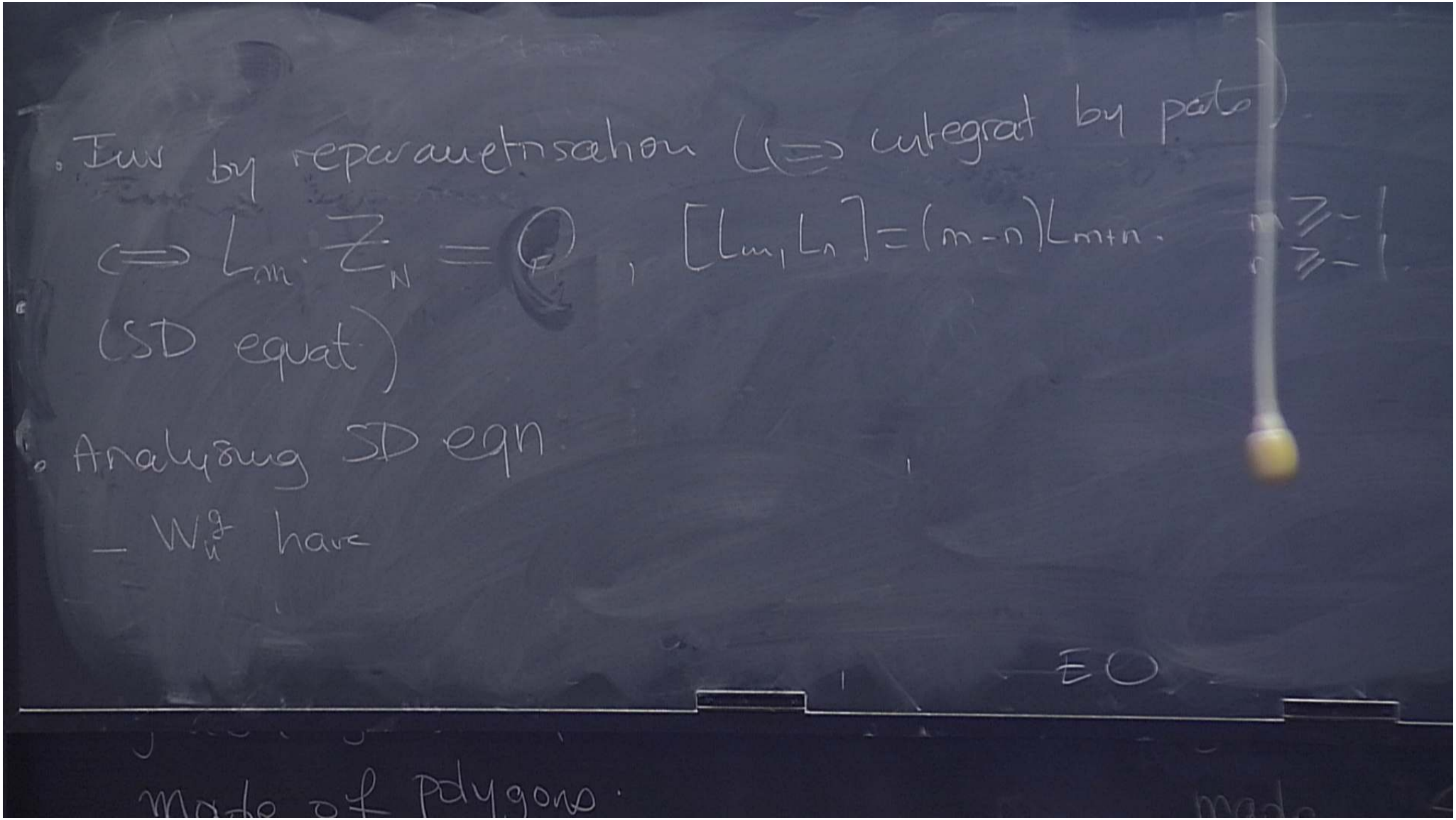
$$W_n(x_1,\dots,x_n) = \left\langle \mathrm{Tr}\frac{1}{x_1 - M} \cdots \mathrm{Tr}\frac{1}{x_n - M} \right\rangle_{\text{cumulant}}$$

$$= \sum_{g\geq 0} N^{2-2g-n} \boxed{W_n^g(x_1,\dots,x_n)}$$

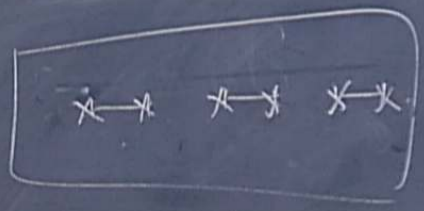
$x_i \leftrightarrow$ length i^{th} boundary.


Def
$W_n(x_1, \ldots, x_n) = \sum_{g \geq 0} N^{2-2g-n}$
$x_i \leftrightarrow$

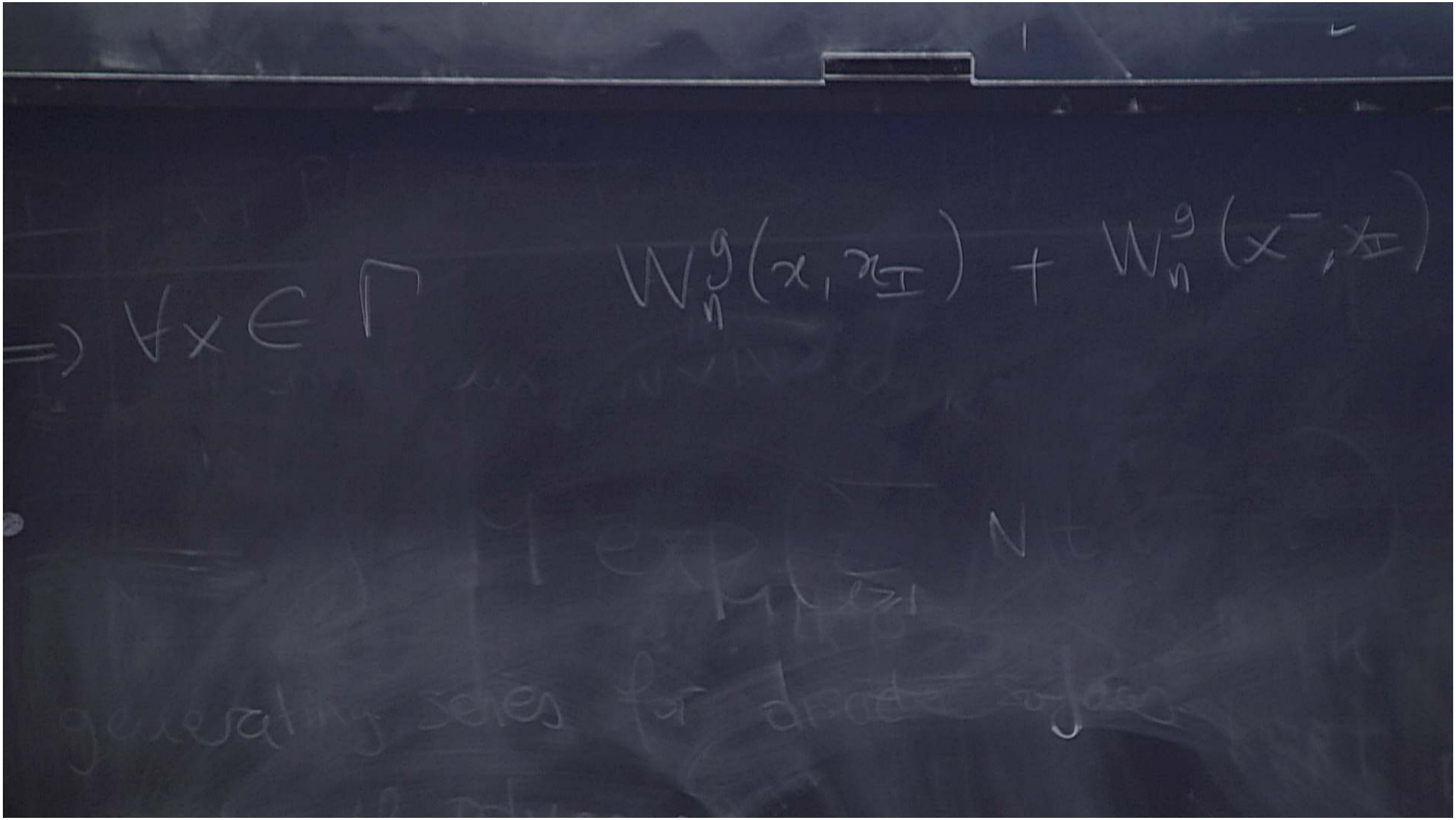

$\left\{ \frac{1}{x_1 - H} \cdots \frac{1}{x_n - H} \right\}$ cumulant
$= \sum_{g \geq 0} N^{2-2g-n} \boxed{W_n^g(x_1 \ldots x_n)}$
$x_i \leftrightarrow$ length i^{th} boundary

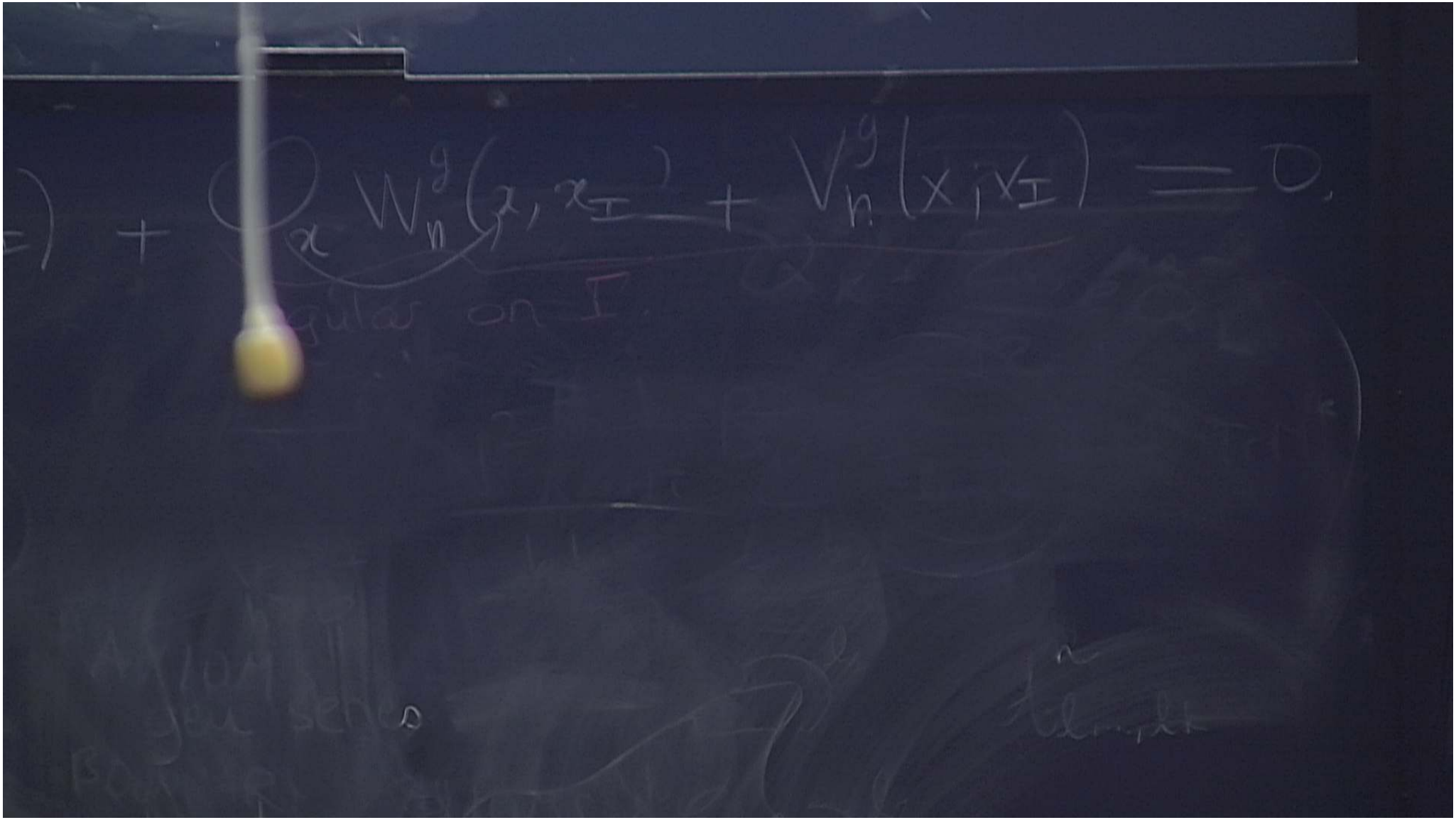

. Inv by reparametrisation (⟺ integrat by parts)
⟺ $L_m \cdot Z_N = 0$, $[L_m, L_n] = (m-n)L_{m+n}$.
(SD equat)
. Analysing SD eqn
– We have
EO
made of polygons.

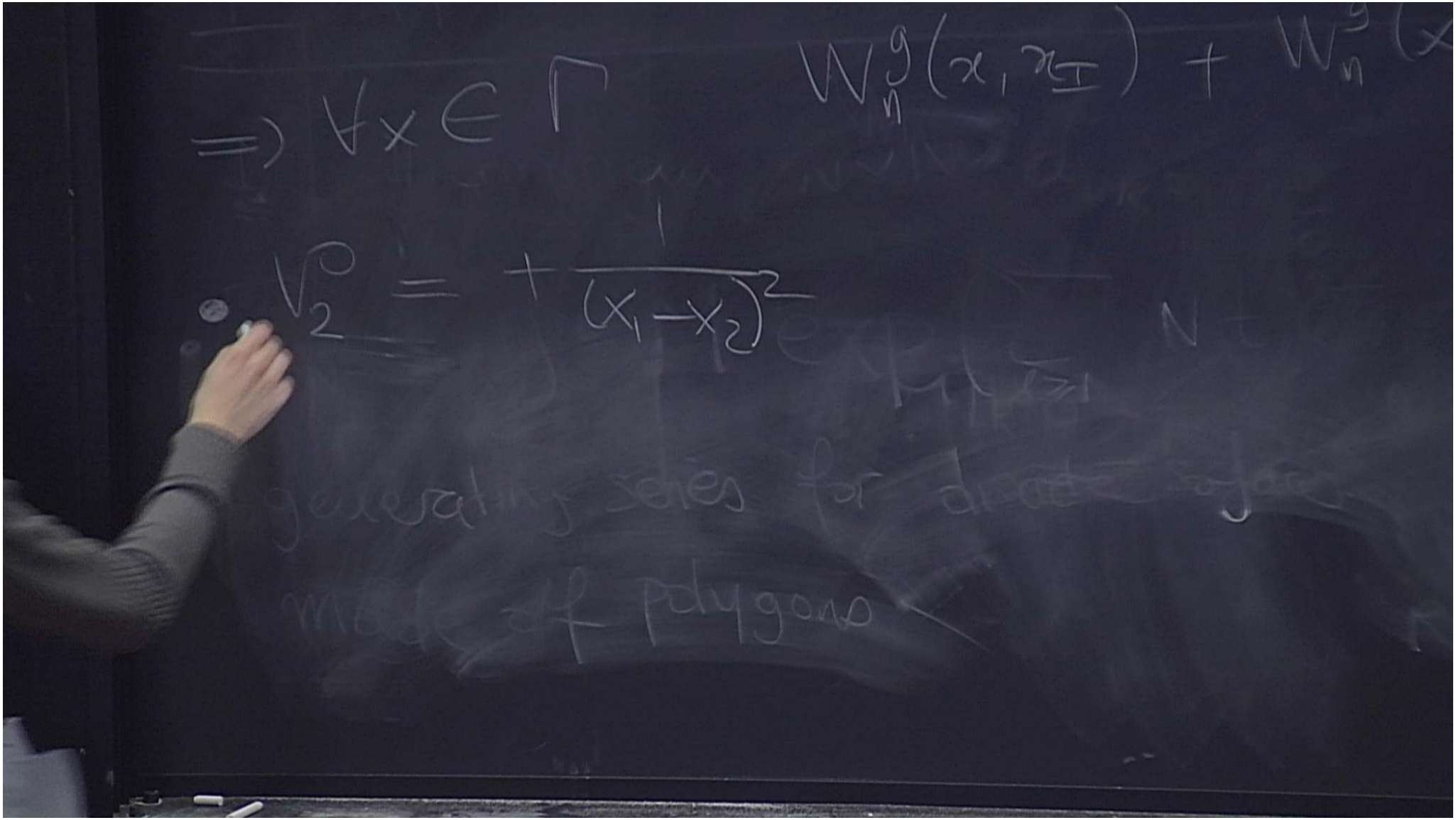
- Inv by reparametrisation ($\Leftrightarrow$ integrat by parts)
  $\Leftrightarrow L_m \cdot Z_N = 0$, $[L_m, L_n] = (m-n)L_{m+n}$. $n \geq -1$, $n \geq -1$.
  (SD equat)



- Analysing SD eqn
  - W_n^g have cuts Γ does not dpd on n,g
  - have squareroot (generically) at $\partial\Gamma$


$\Rightarrow \forall x \in \Gamma \quad W_n^g(x, x_I) + W_n^g(x^-, x_I)$


$W_n^g(x, x_I) + V_n^g(x, x_I) = 0$

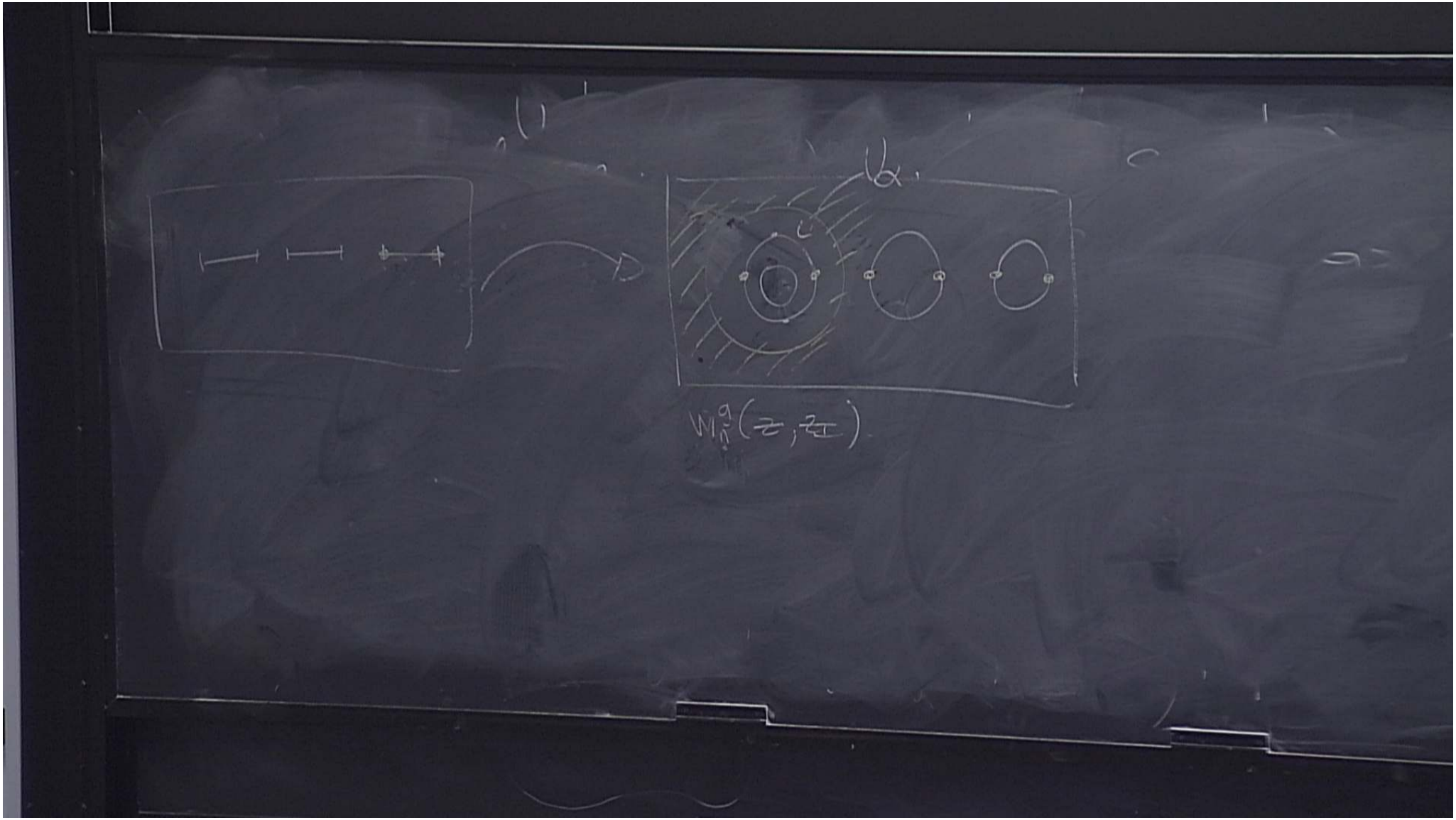

$\Rightarrow \forall x \in \Gamma$
$W_n^g(x, x_I) + W_n^g$
$V_2 = + \frac{1}{(x_1 - x_2)^2}$
generating series for
of polygons

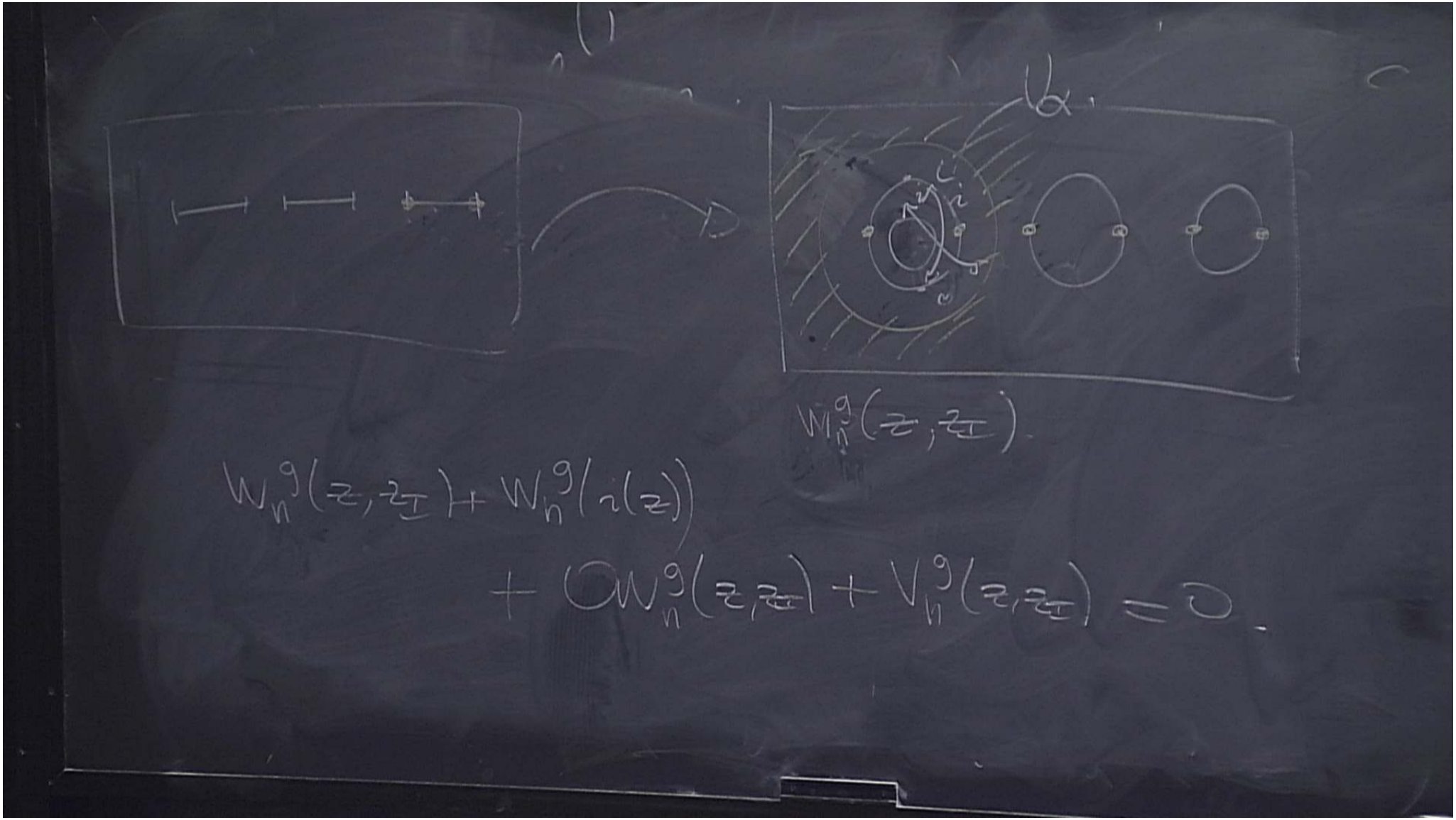
$$\Rightarrow \forall x \in \Gamma \qquad W_n^g(x, x_I) + W_n^g(x^-, x_I) + \sum_\alpha \operatorname{Res}_{z=\alpha} W_n^g(z, x_I)$$

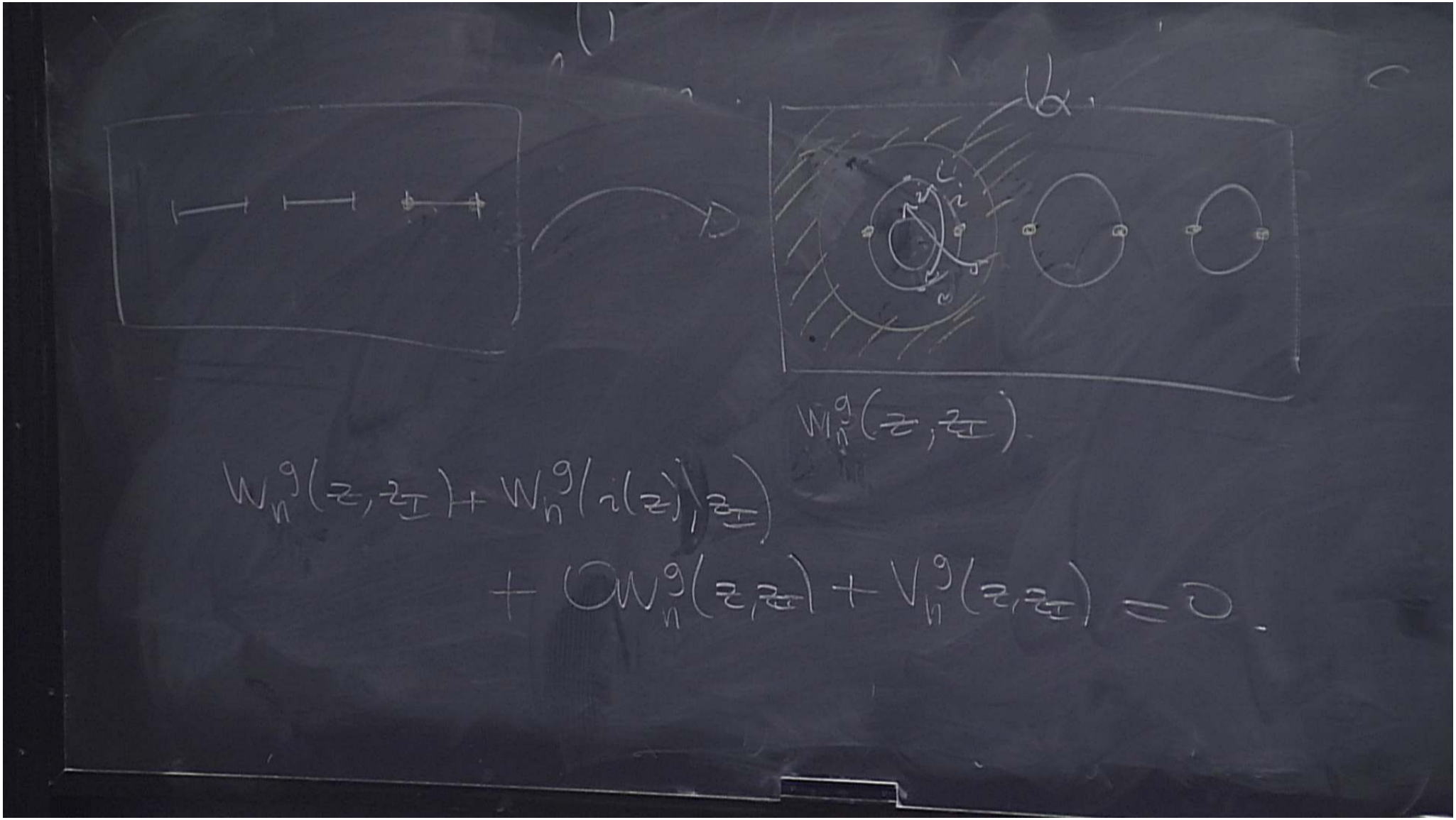
- $V_{(x_1,x_2)}^0 = + \dfrac{1}{(x_1 - x_2)^2}$
- $V_n^g =$ expression in term of $W_{n'}^{g'}$ $\quad 2g'-2+n' < 2g-2+n$.

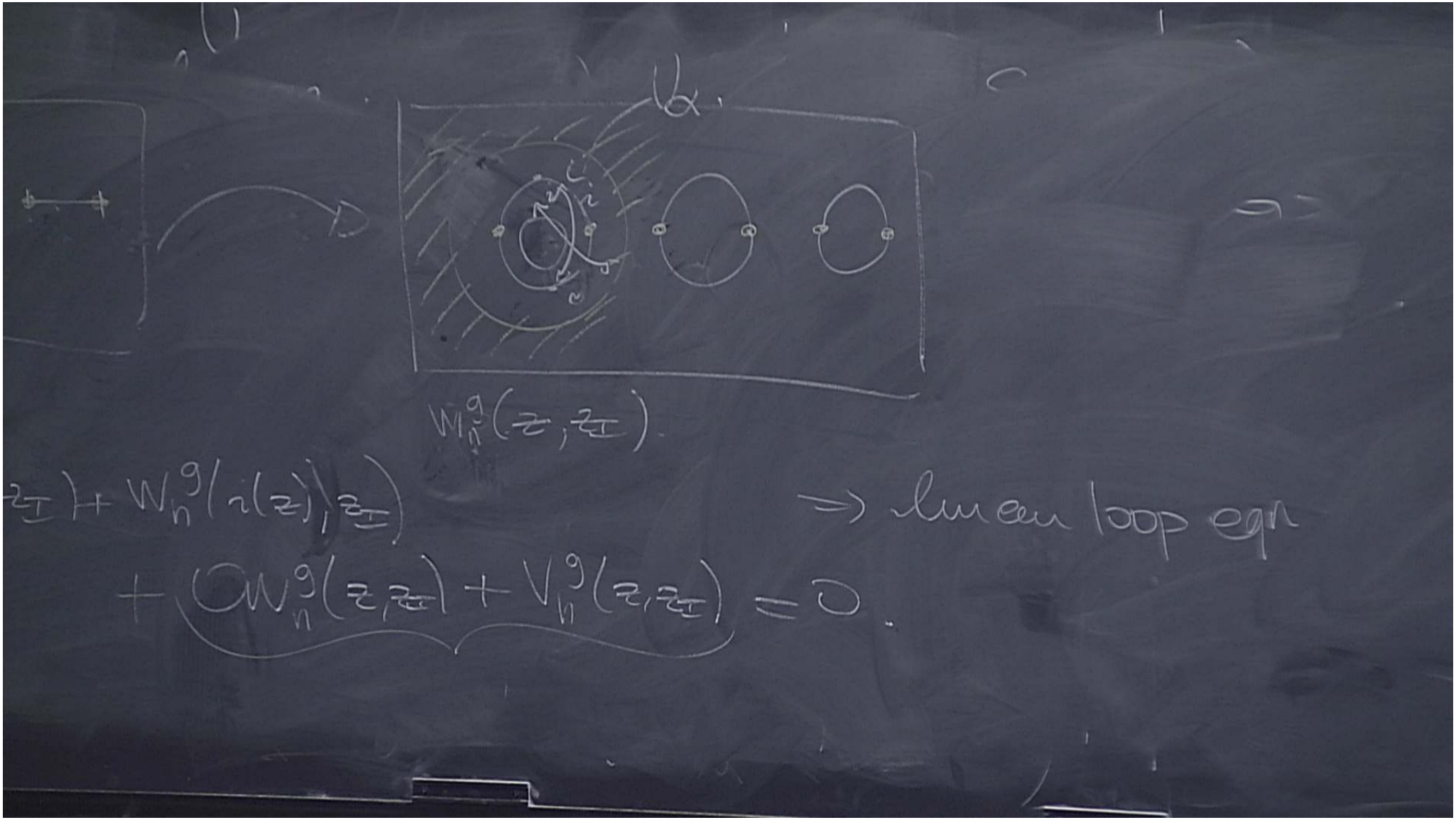
$2g-2+n \geq 0$: polygon

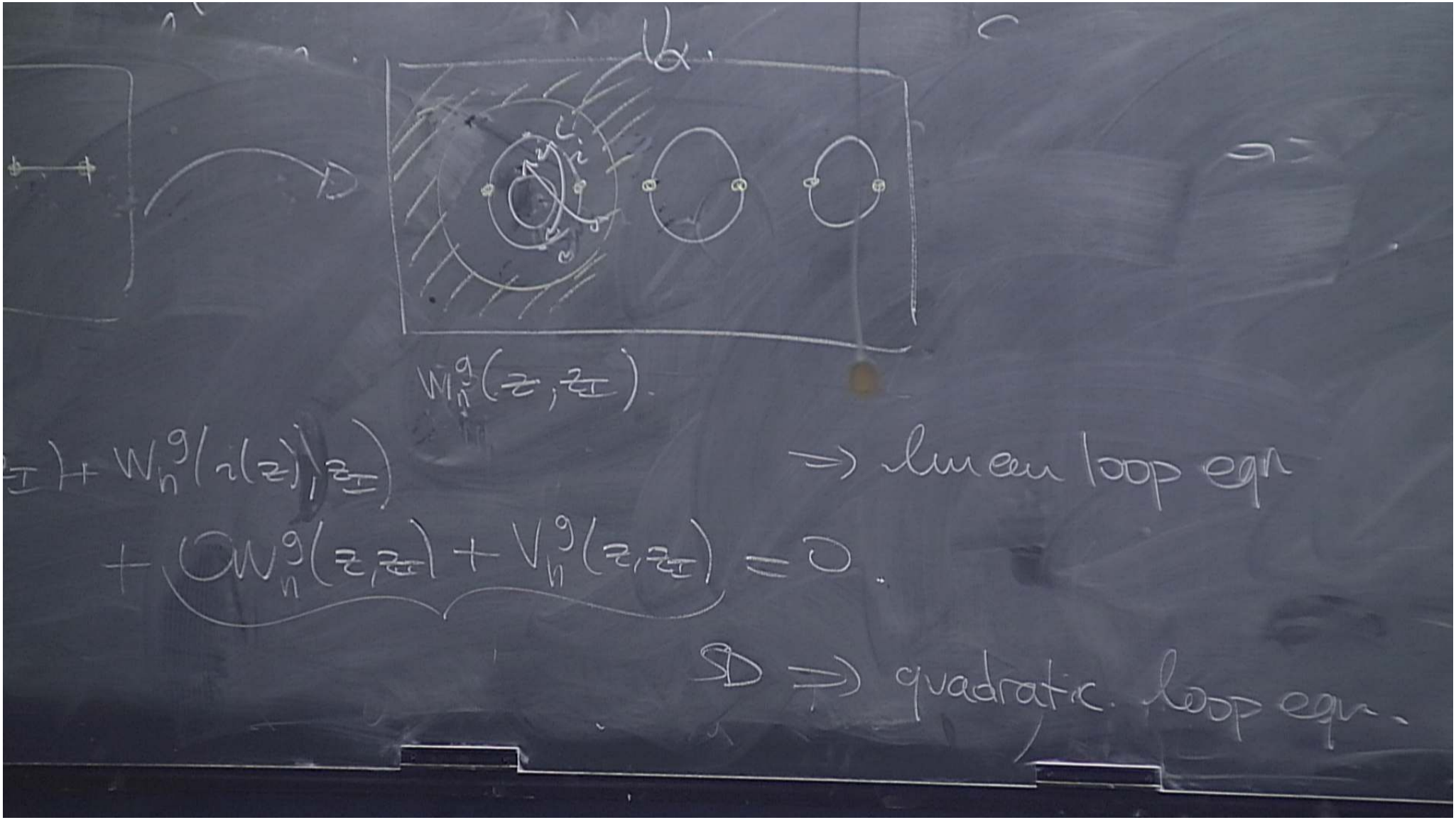
$t_{l_1,\dots,l_k}^h$ $\quad \begin{cases} 2h-2+k \geq 0 \\ h \leq g \end{cases}$


$W_n^q(z, z_1)$


$W_n^g(z, z_I)$
$W_n^g(z, z_I) + W_n^g(\gamma(z))$
$+ \mathcal{O} W_n^g(z, z_I) + V_n^g(z, z_I) = 0.$


$W_n^g(z, z_I)$
$W_n^g(z, z_I) + W_n^g(\gamma(z), z_I)$
$+ C W_n^g(z, z_I) + V_n^g(z, z_I) = 0.$


$W_n^g(z, z_I)$
$z_I) + W_n^g(z(z), z_I)$
$\Rightarrow$ linear loop eqn
$+ \underbrace{\omega W_n^g(z, z_I) + V_n^g(z, z_I)} = 0.$


$W_n^g(z, z_I)$
$+ W_n^g(z(z), z_I)$
$\Rightarrow$ linear loop eqn
$+ \underbrace{\mathcal{O} W_n^g(z, z_I) + V_n^g(z, z_I)} = 0.$
SD $\Rightarrow$ quadratic loop eqn.

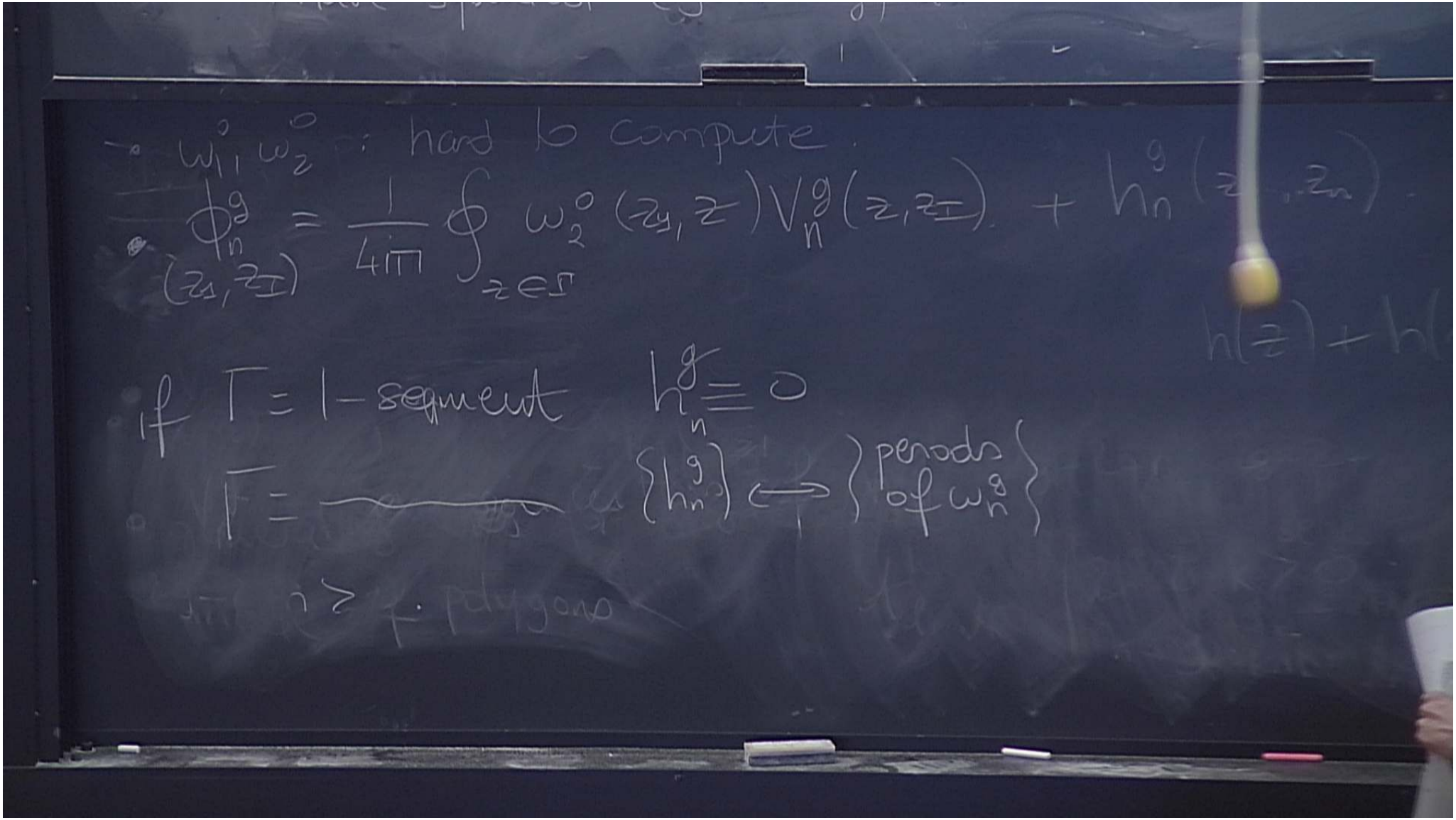
$\Rightarrow$ quad

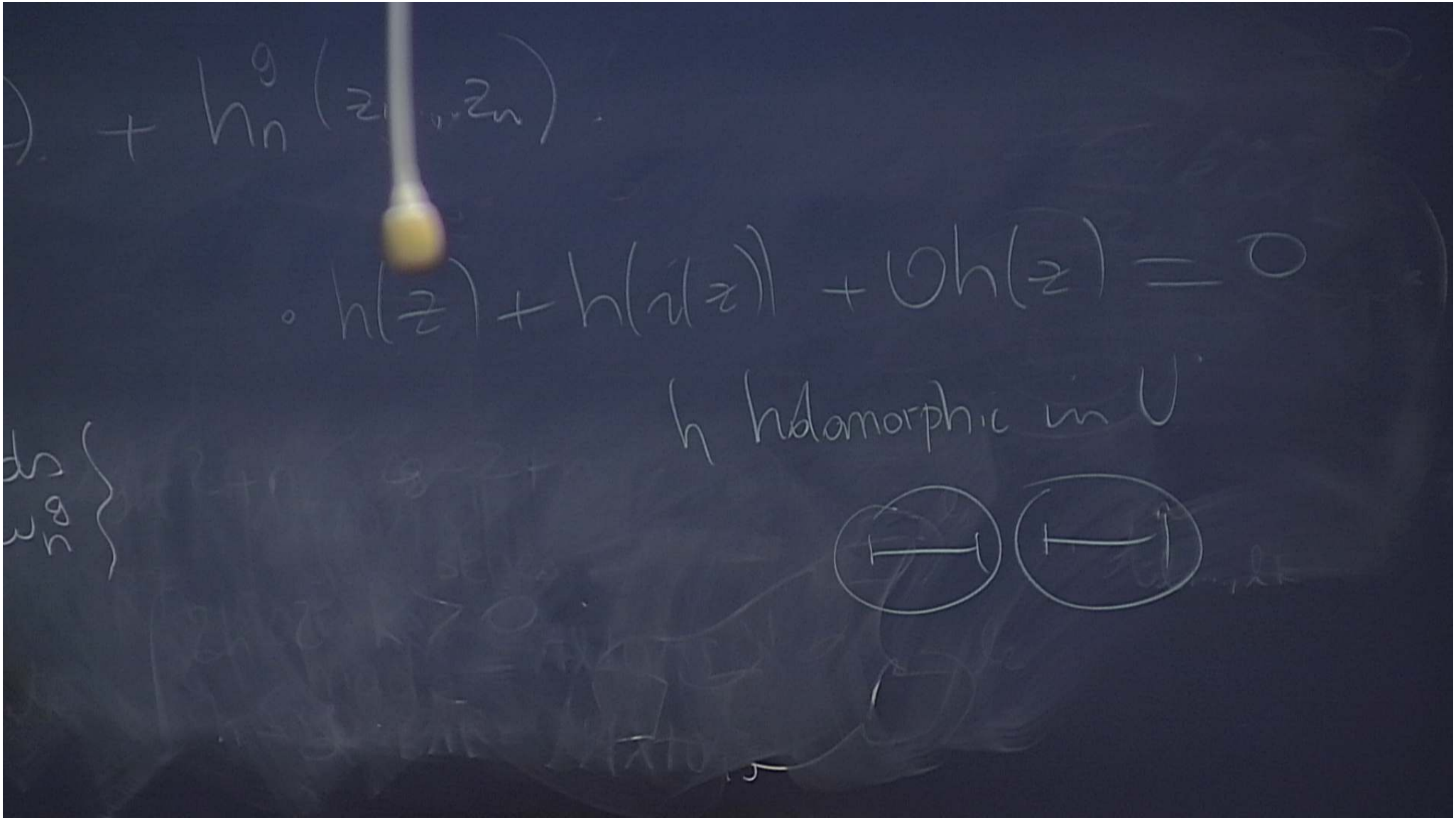
Def $W_n(x_1, \dots, x_n) = \left\langle \mathrm{Tr} \frac{1}{x_1 - M} \cdots \mathrm{Tr} \right.$

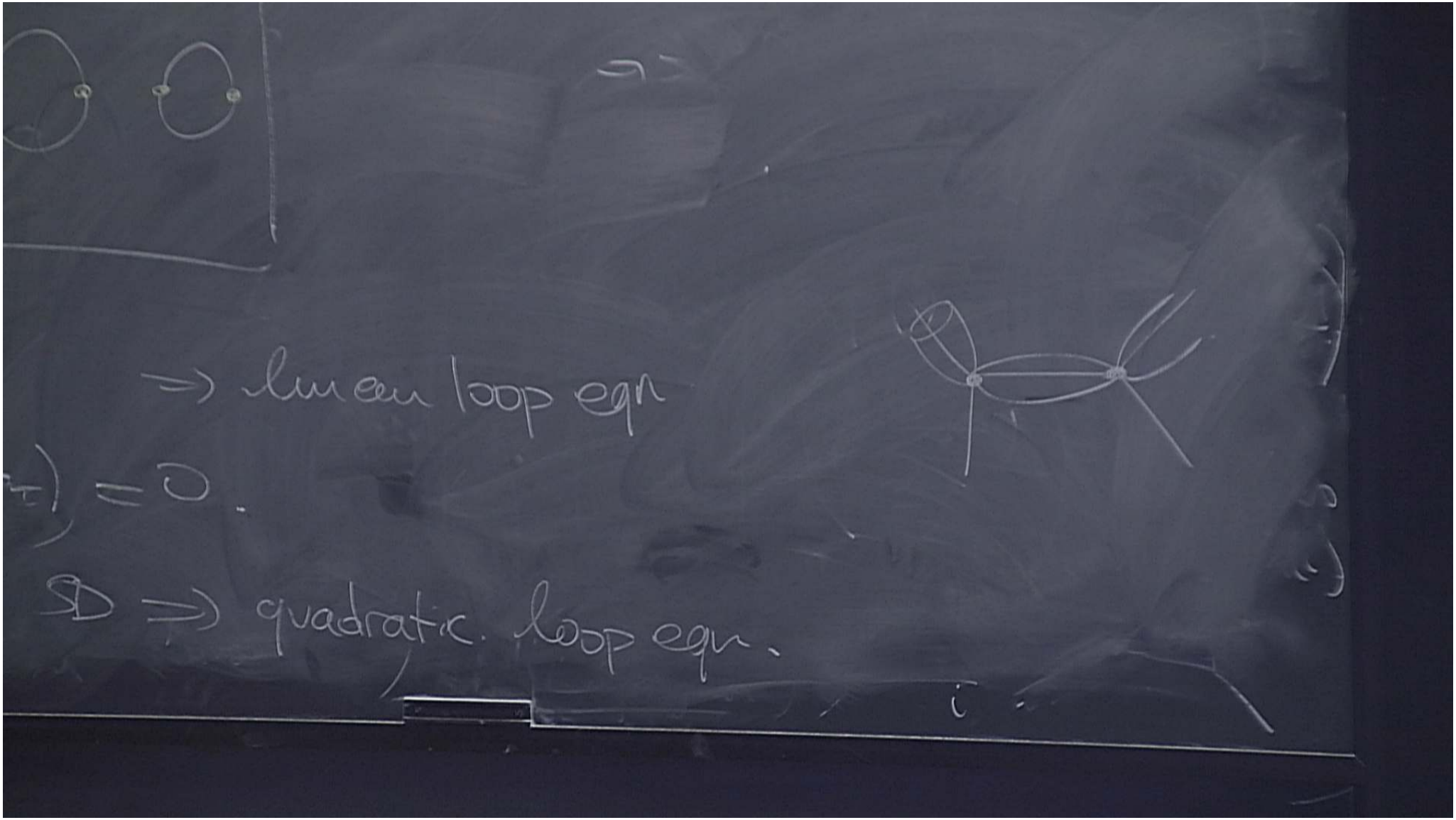
$$= \sum_{g \geq 0} N^{2-2g-n} \boxed{W_n^g(}$$

$$\boxed{\omega_n^g(z_1, \dots, z_n) = W_n^g(z_1, \dots, z_n)\, dx(z_1) \cdots dx(z_n) + \frac{dx(z_1)\, dx(z_n)}{(x(z_1) - x(z_n))^2} \delta_{n,2} \delta_{g,0}}$$

$x_i \hookrightarrow$ length


ω_1^0, ω_2^0 : hard to compute.
$\phi_n^g(z_1, z_I) = \frac{1}{4\pi} \oint_{z \in \Gamma} \omega_2^0(z_1, z) V_n^g(z, z_I) + h_n^g(z_1, \ldots z_n)$
$h(z) + h($
if Γ = 1-segment $h_n^g \equiv 0$
$\Gamma =$ $\{h_n^g\} \longleftrightarrow$ periods of ω_n^g
$n \geq$ polygon


$+ h_n^g(z_1, \ldots, z_n)$
$h(z) + h(\tau(z)) + \vartheta h(z) = 0$
h holomorphic in U


⇒ linear loop eqn
=0.
SD ⇒ quadratic. loop eqn.

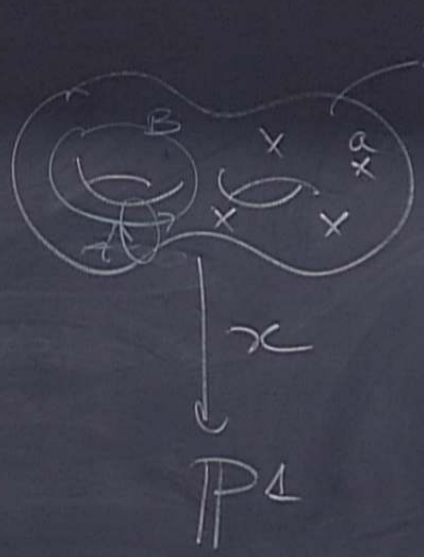
II PPT

Global setting:

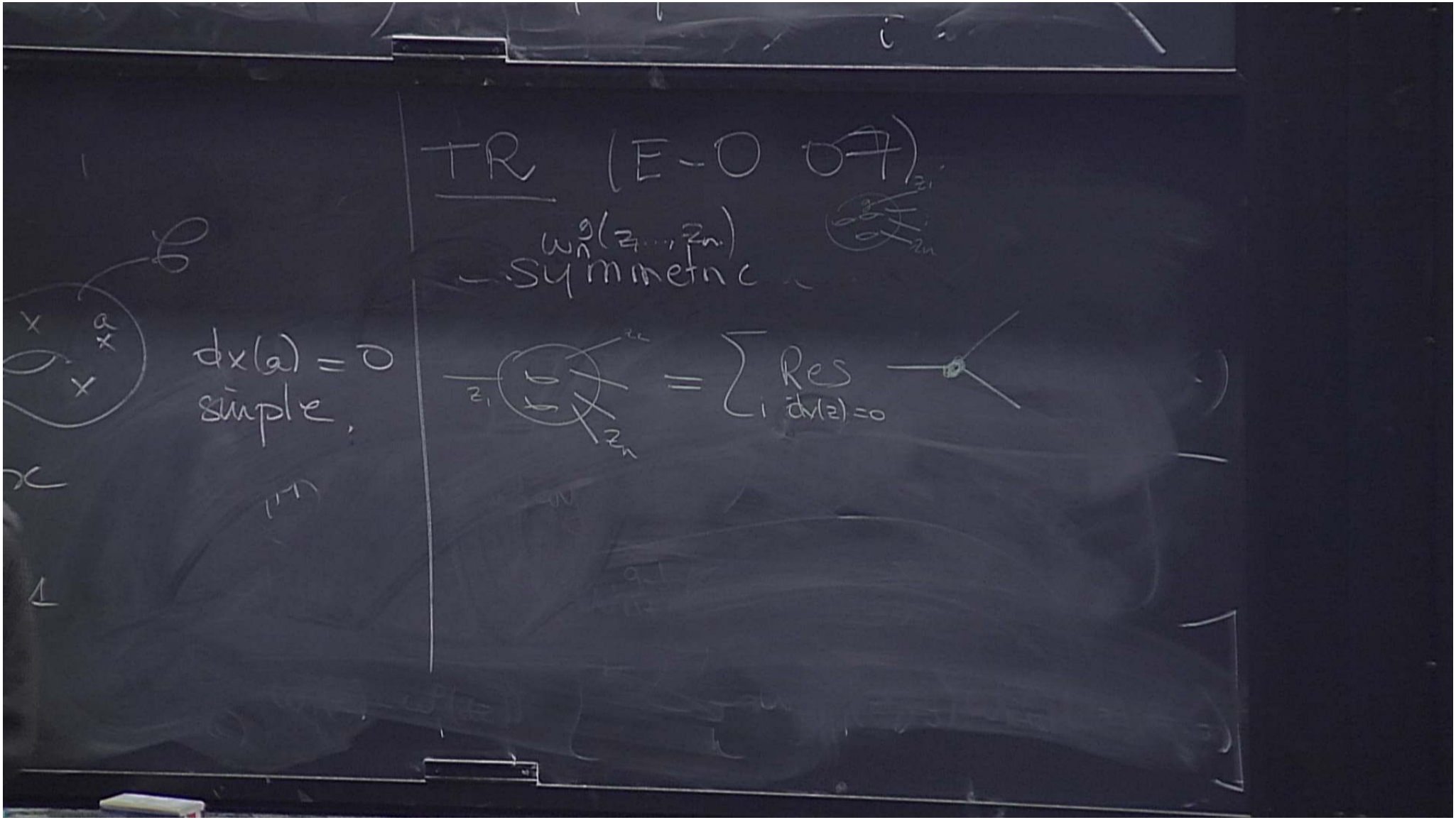
$\mathcal{C}$ compact curve.

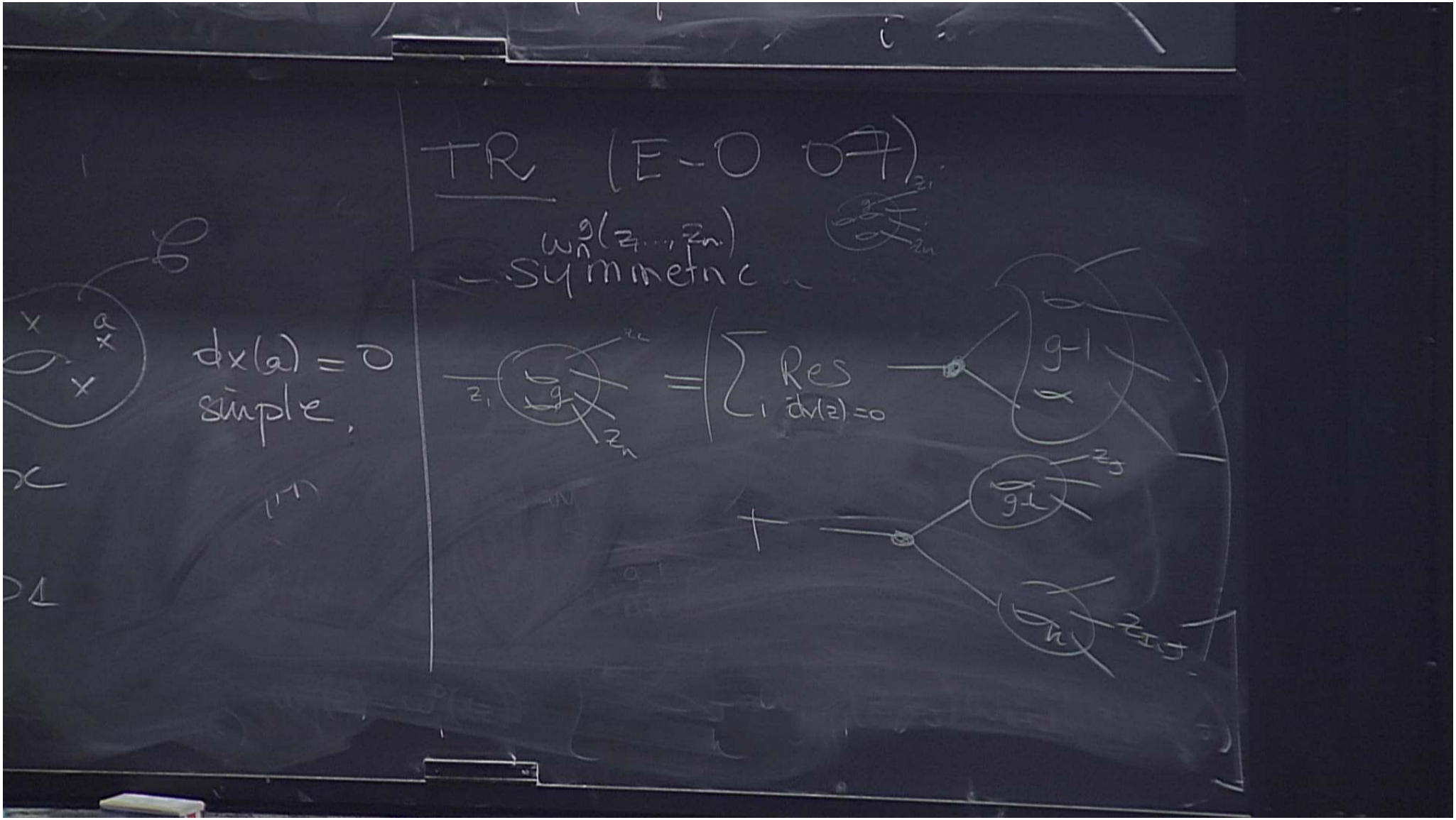
x, y f^n

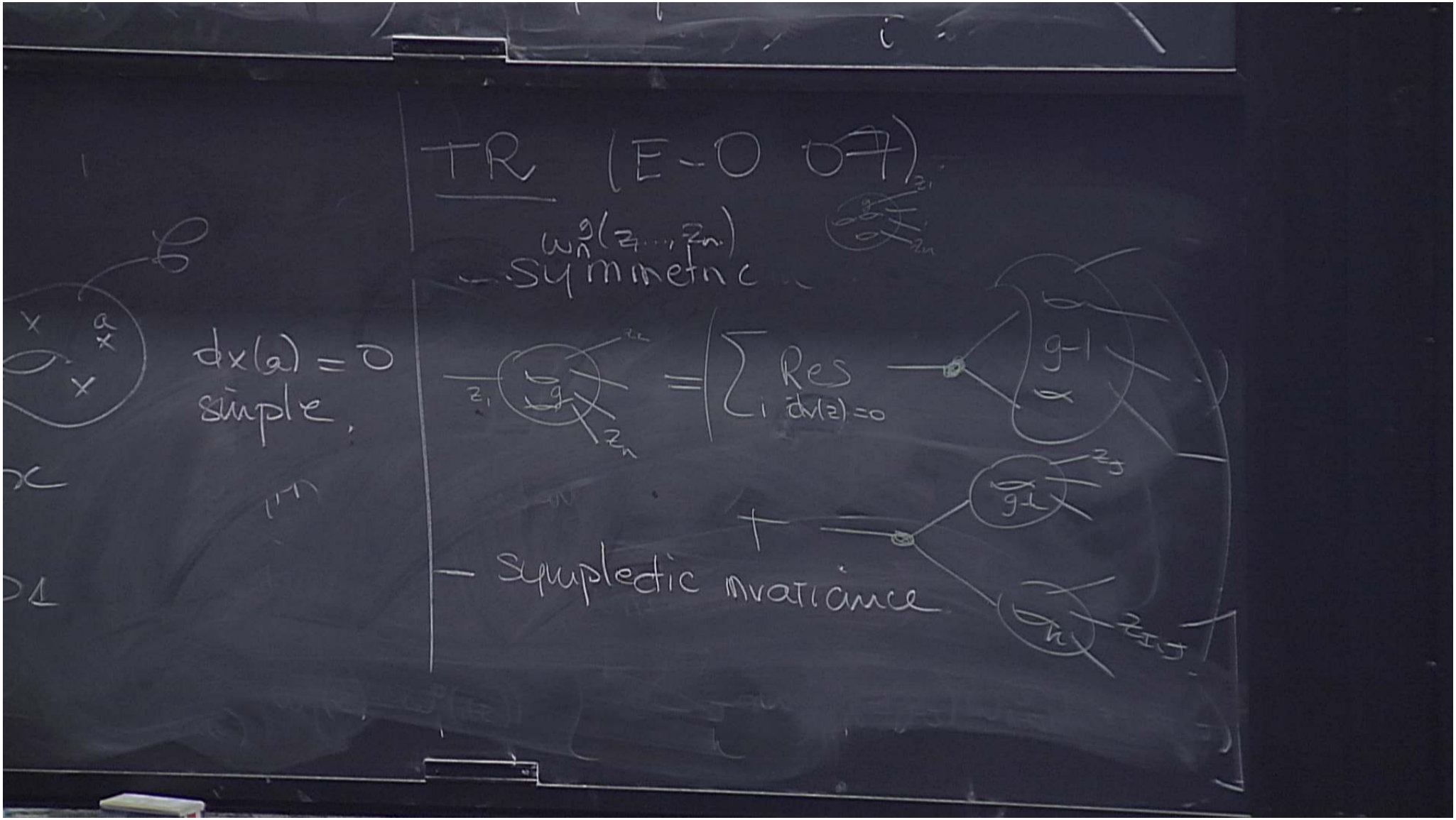
(A, B) symplectic basis of cycles



$dx(a) = 0$ simple.


TR (E-O O-7)
$\omega_n^g(z_1 \dots z_n)$
- symmetric
$dx(a) = 0$
simple.
$= \sum_{dx(z)=0} \mathrm{Res}$


TR (E-O 07)
$\omega_n^g(z_1,\dots,z_n)$
-symmetric
$dx(a)=0$
simple.
$= \sum_{dv(z)=0} \mathrm{Res}$
g-1
g1
h


TR (E-O-07)
$\omega_n^g(z_1,\dots,z_n)$
- symmetric
$dx(a)=0$
simple.
$= \sum_{i\ dx(z)=0} \mathrm{Res}$
g-1
g_1
g_2
- symplectic invariance

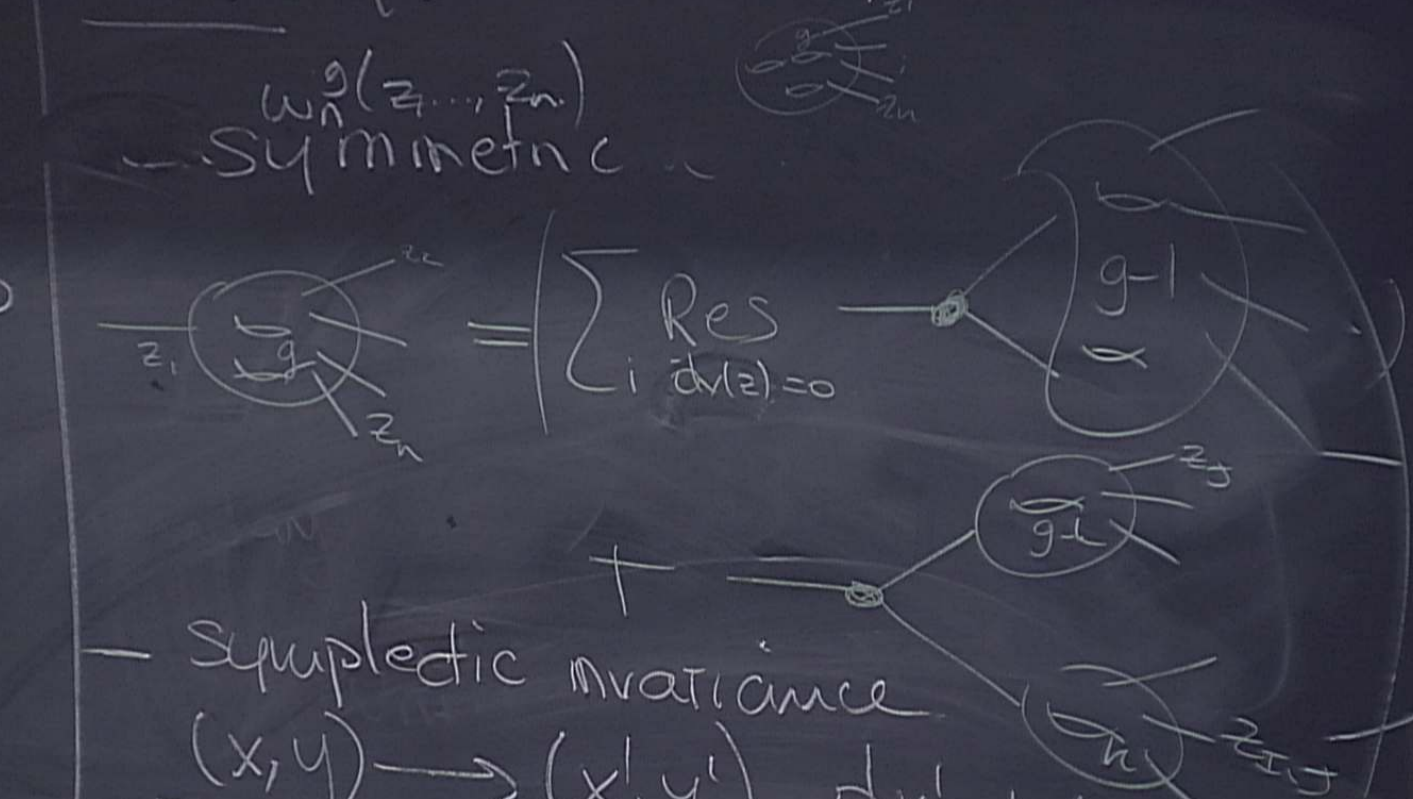
TR (E-O 07)

$\omega_n^g(z_1,\dots,z_n)$

- symmetric

$dx(a)=0$ simple.

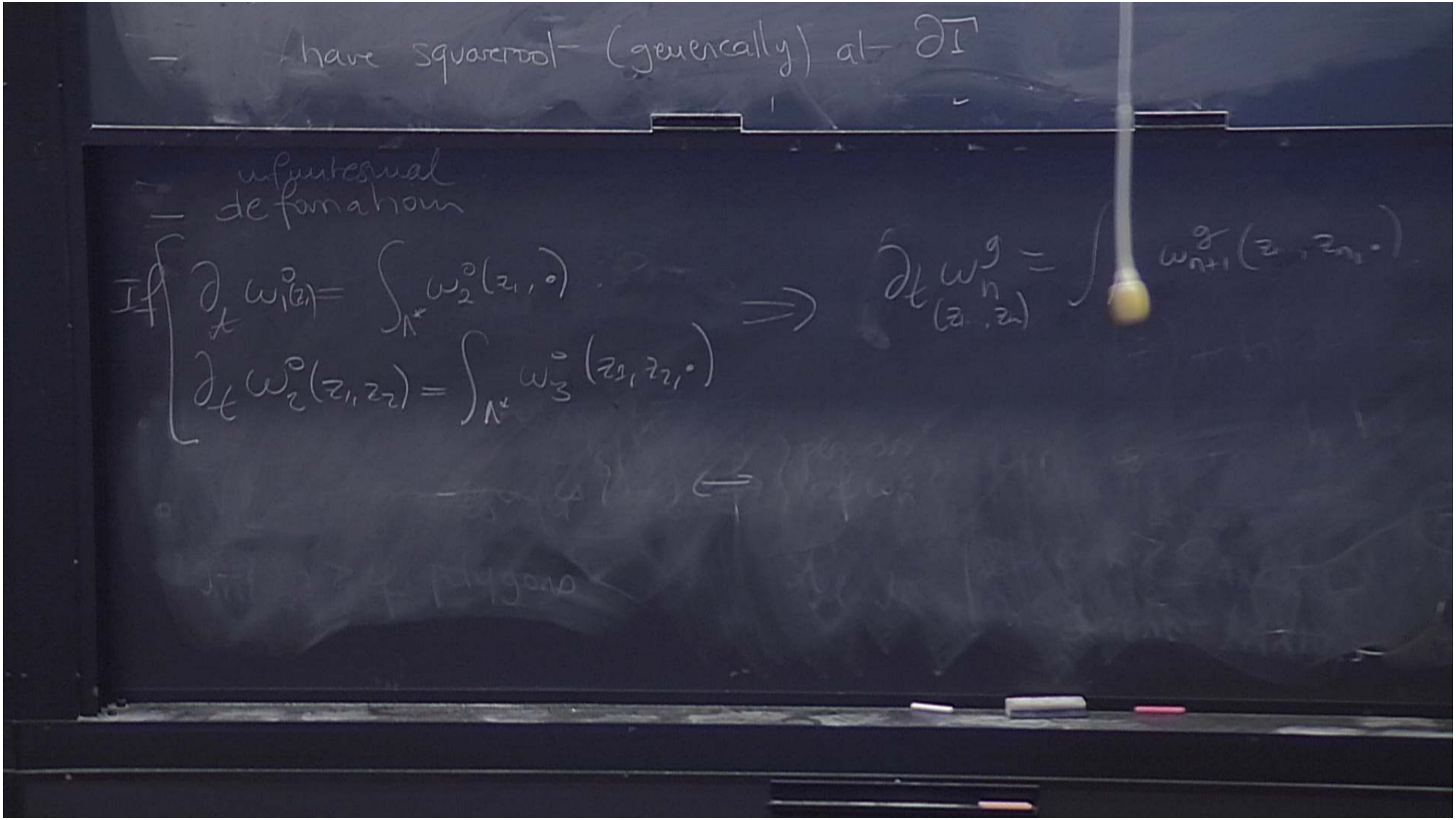
$$\omega_{n+1}^g(z_0,z_1,\dots,z_n) = \sum_{i\ dx(z)=0} \operatorname{Res}_{z=a_i} \int_{z_0}^{z} B(z_0,z)\,\Big[\omega_{n+2}^{g-1}(z,\sigma(z),z_1,\dots,z_n) + \sum_{h=0}^{g}\sum_{I\subset\{1,\dots,n\}} \omega_{|I|+1}^{h}(z,z_I)\,\omega_{n-|I|+1}^{g-h}(\sigma(z),z_{I^c})\Big]$$

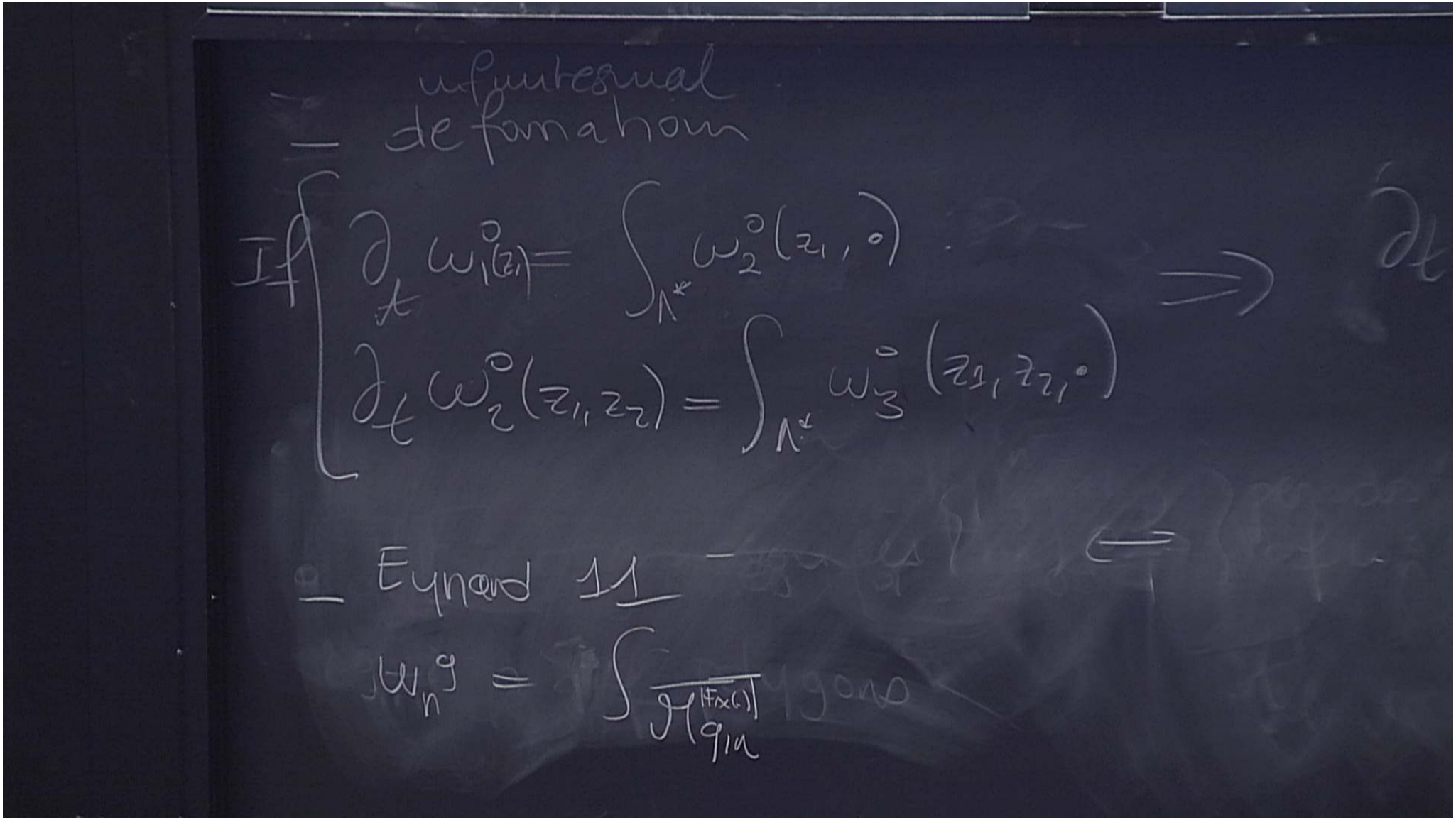


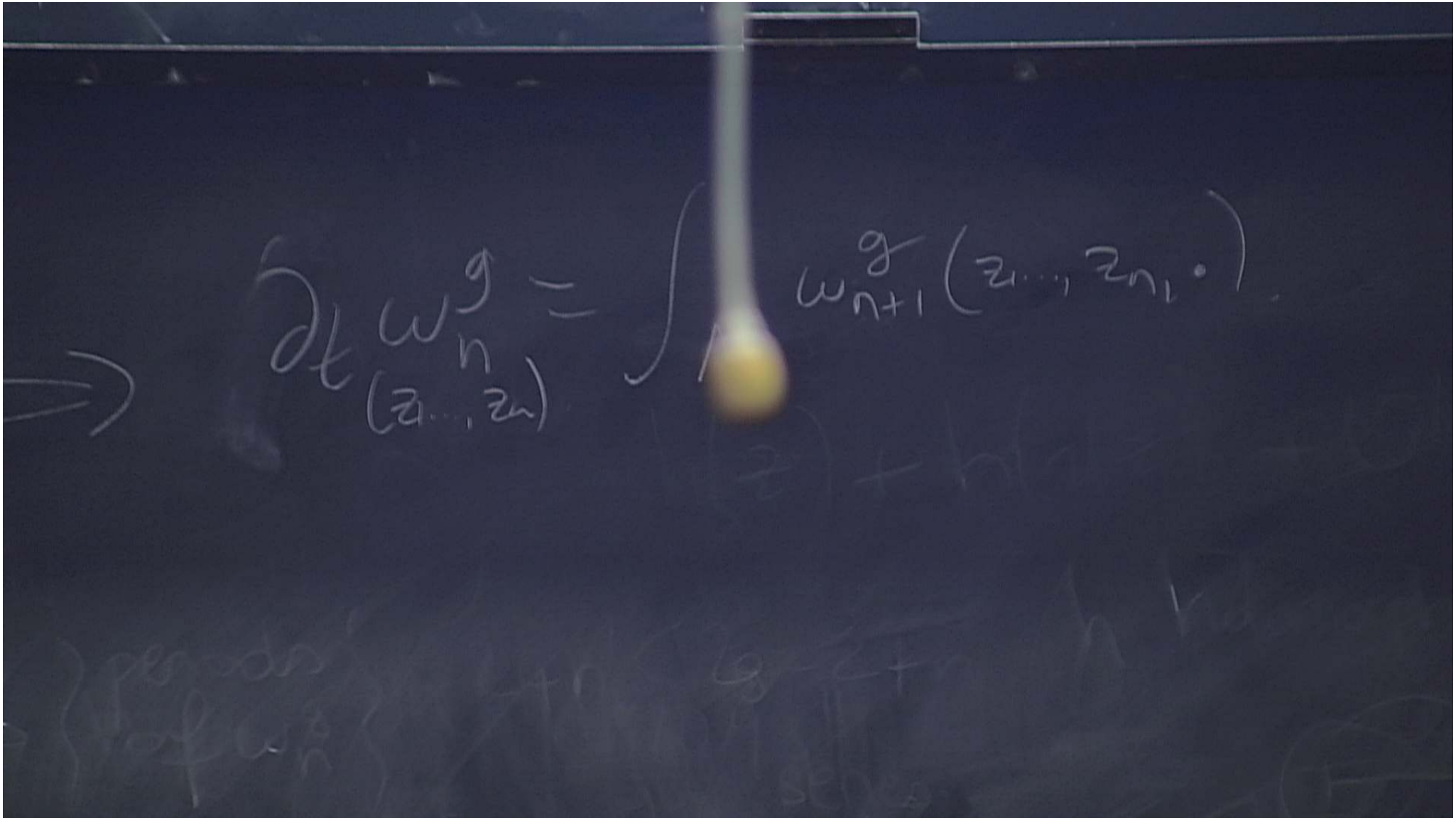
- symplectic invariance

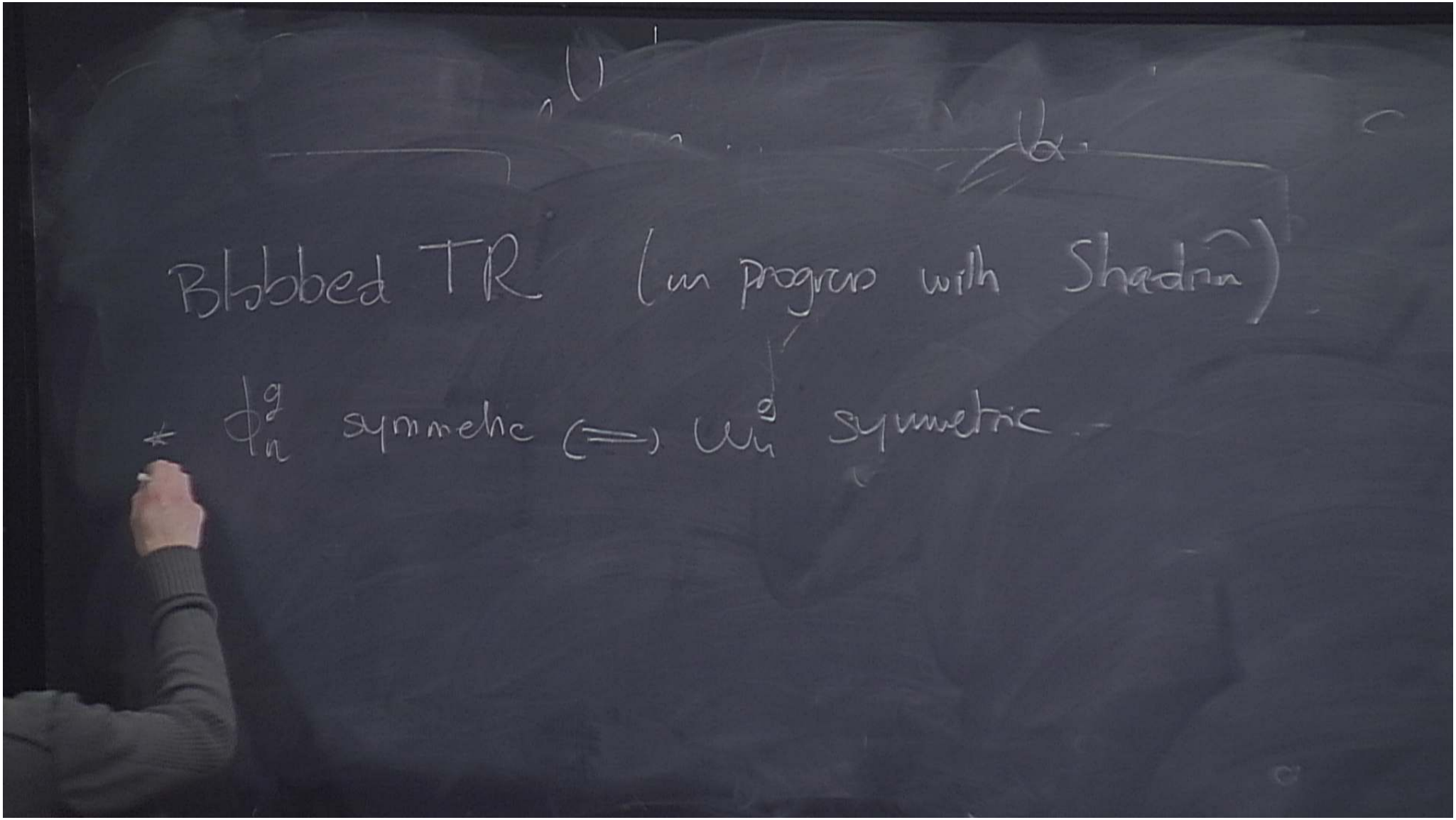
$(x,y)\longrightarrow(x',y')$ $dx'\wedge dy' = dx\wedge dy$

$[\omega_n^g]\longrightarrow[\omega_n^g]$


– have squareroot (generically) at ∂Γ
– infinitesimal deformations
If $\partial_t \omega_1^0(z_1) = \int_{\Lambda^*} \omega_2^0(z_1, \cdot)$
$\partial_t \omega_2^0(z_1, z_2) = \int_{\Lambda^*} \omega_3^0(z_1, z_2, \cdot)$
$\Rightarrow$
$\partial_t \omega_n^g(z_1, \ldots, z_n) = \int \omega_{n+1}^g(z, z_1, \cdot)$
$\Leftarrow$


infinitesimal
deformation
If
$\partial_t \omega_1^0(z_1) = \int_{\Lambda^*} \omega_2^0(z_1, 0)$
$\partial_t \omega_2^0(z_1, z_2) = \int_{\Lambda^*} \omega_3^0(z_1, z_2, 0)$
$\Rightarrow$
∂_t
$\Leftrightarrow$
Eynard 11
$\omega_n^g = \int \frac{}{g(q_{in})}$


$\partial_t \omega_n^g(z_1,\dots,z_n) = \int \omega_{n+1}^g(z_1,\dots,z_n,\cdot)$


Blobbed TR (on Progroup with Shadrin)
* ϕ_n^g symmetric $\Longleftrightarrow$ ω_n^g symmetric

Blobbed TR (in progress with Shadrin)

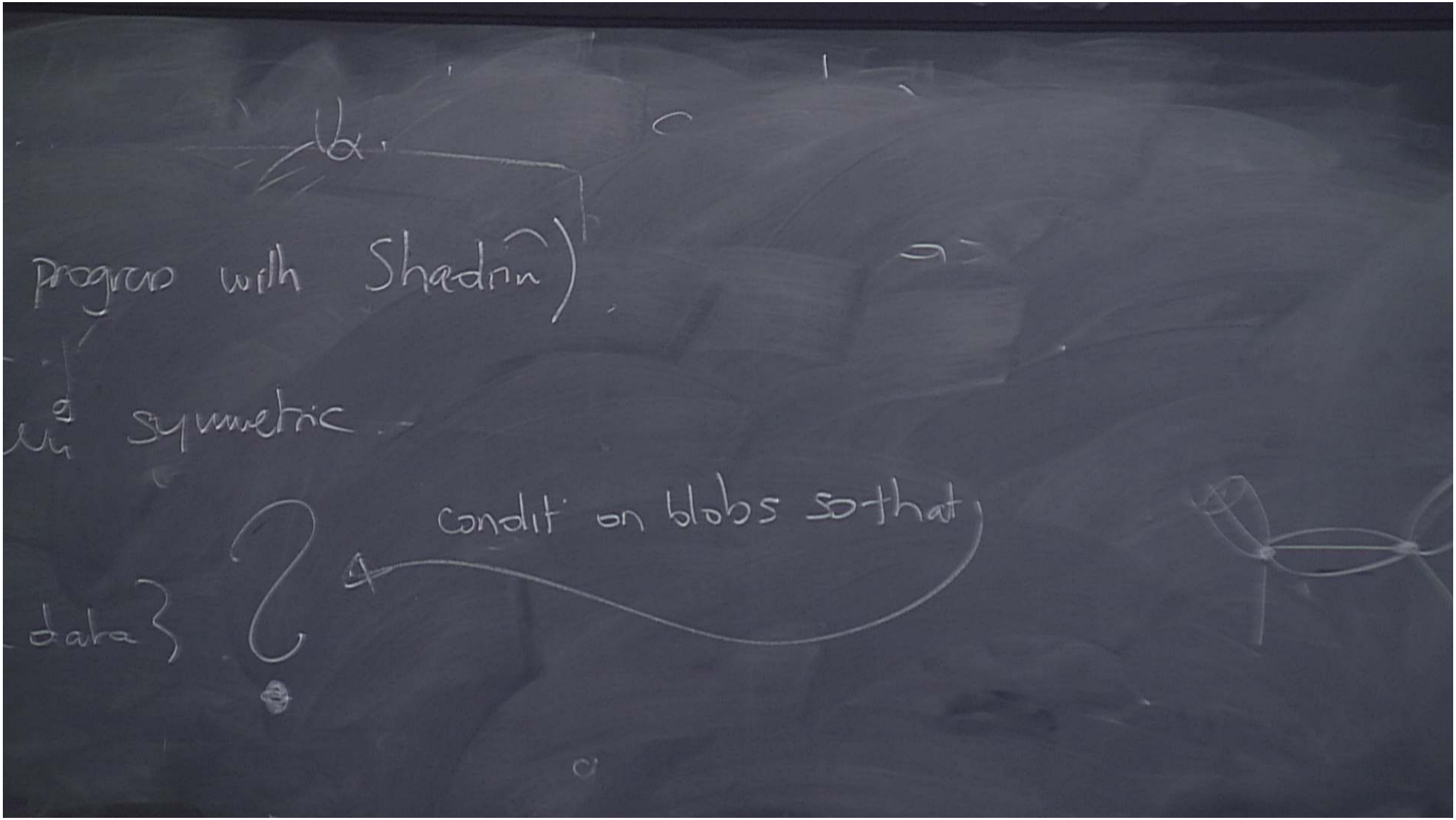
(ω_1^0, ω_2) = initial data

* ϕ_n^g symmetric $\Leftrightarrow$ ω_n^g symmetric

* symp inv

* natural flow on {initial data} ?

* $\omega_n^g = \int_{\overline{\mathcal{M}}_{g,n}}$


Progress with Shading
symmetric
condition blobs so that
data

infinitesimal
deformation

If $\begin{cases} \partial_t \omega_1^0(z_1) = \int_{\Lambda^*} \omega_2^0(z_1, \cdot) \\ \partial_t \omega_2^0(z_1, z_2) = \int_{\Lambda^*} \omega_3^0(z_1, z_2, \cdot) \end{cases} \Rightarrow \partial_t \omega_n^g(z_1 \dots z_n) =$

$$\int_{\Gamma^n} \left(\hat{\nabla}^n \ln \Theta \right) \cdot du(z_1) \cdots du(z_n)$$

Eynard 11

$$\omega_n^g = \int_{\overline{\gamma(|Fx(1)|}} \Lambda \omega_0$$

infinitesimal
deformation

If $\partial_t \omega_1^0(z_1) = \int_{\Lambda^*} \omega_2^0(z_1, 0)$ $\Rightarrow$ $\partial_t \omega_n^g(z_1 \dots z_n) =$

$\partial_t \omega_2^0(z_1, z_2) = \int_{\Lambda^*} \omega_3^0(z_1, z_2, 0)$ $\forall g$

$$\phi_n = \int \nabla^{\wedge} \ln \Theta[\cdot] \, du(z_1) \cdots du(z_n)$$

Eynard 11

$\omega_n^g =$ $\int$ $\frac{}{\gamma(q_{in})}$