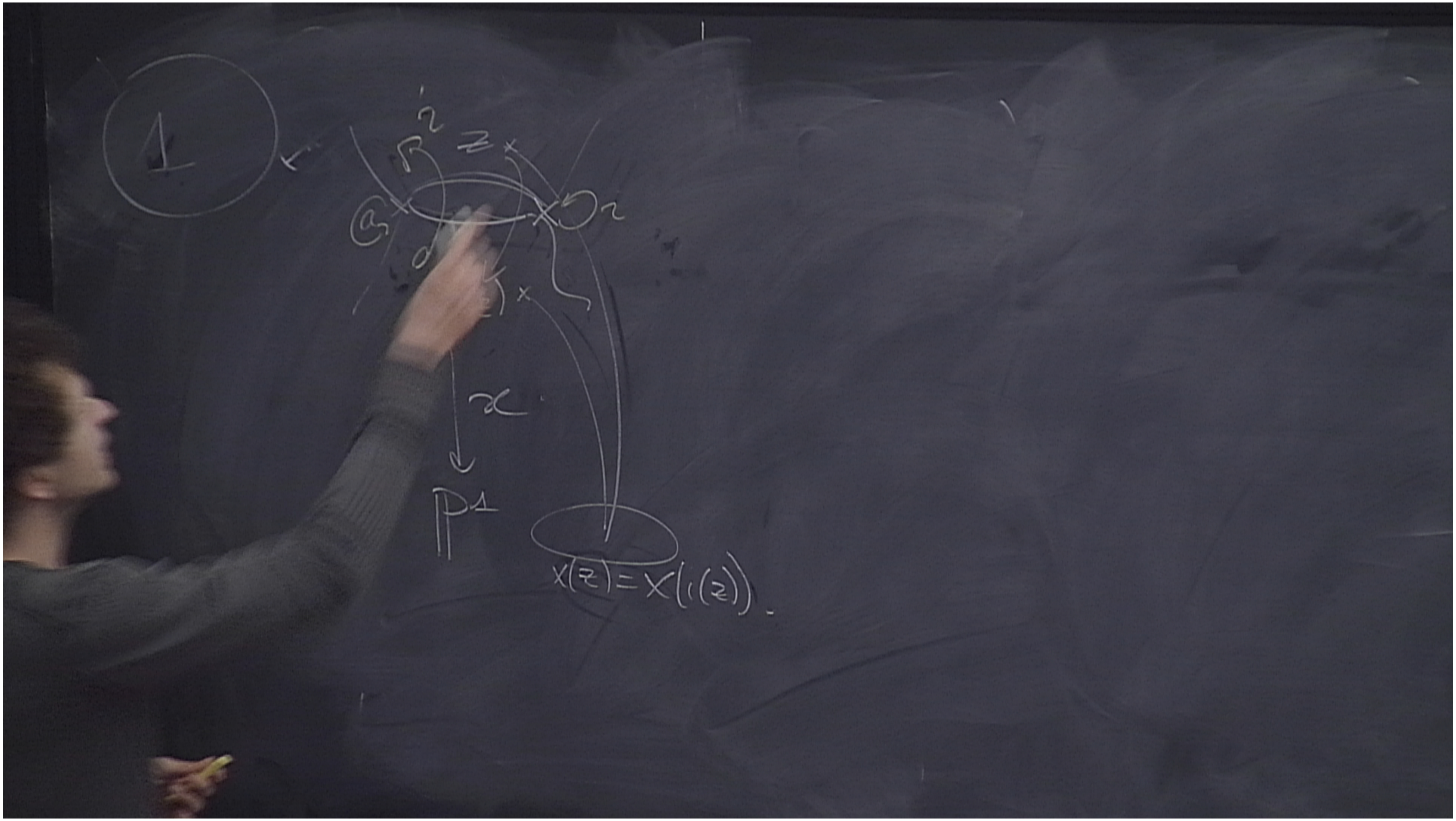
Title: Blobbed topological recursion

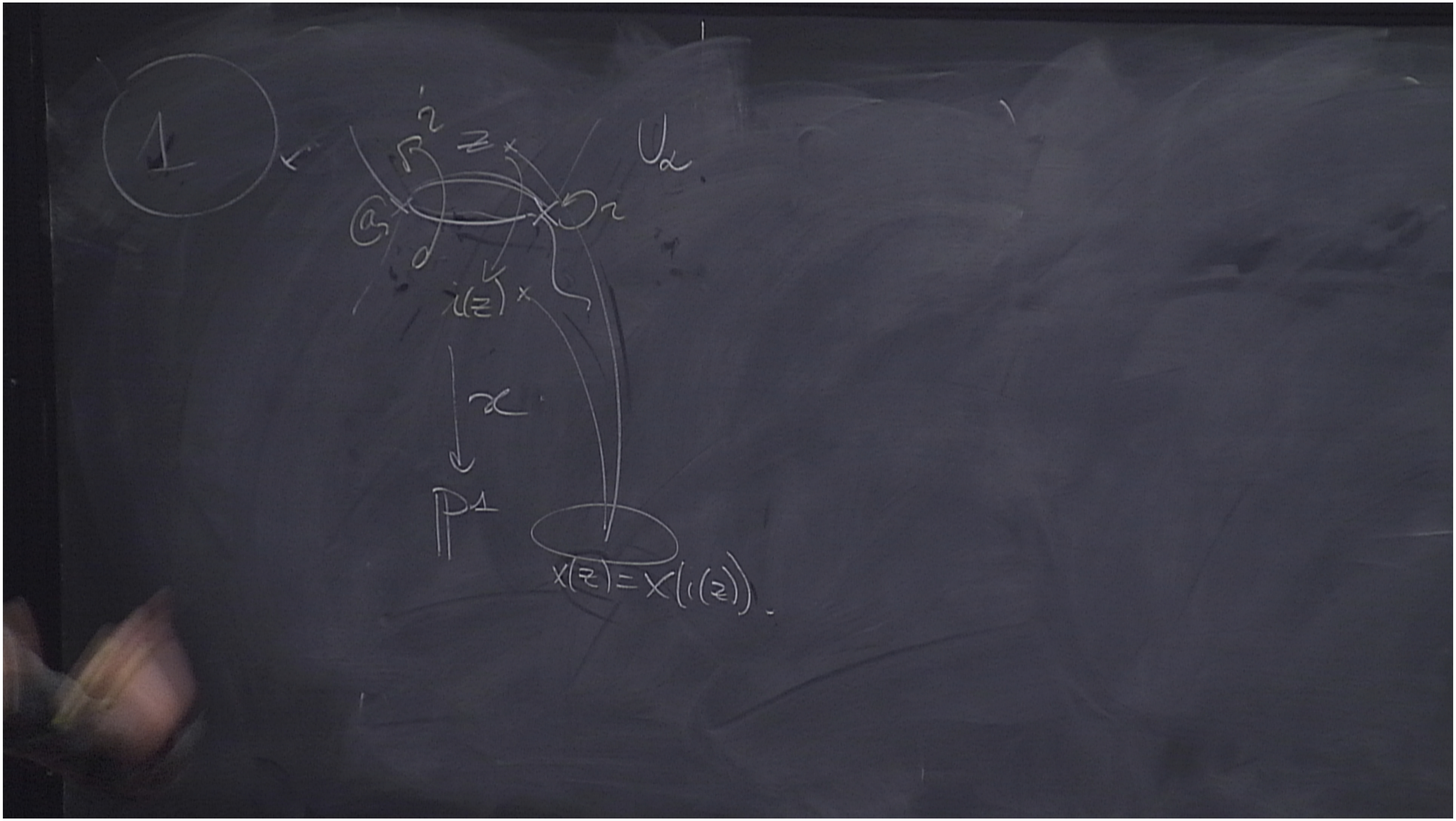
Date: Oct 22, 2013 04:15 PM

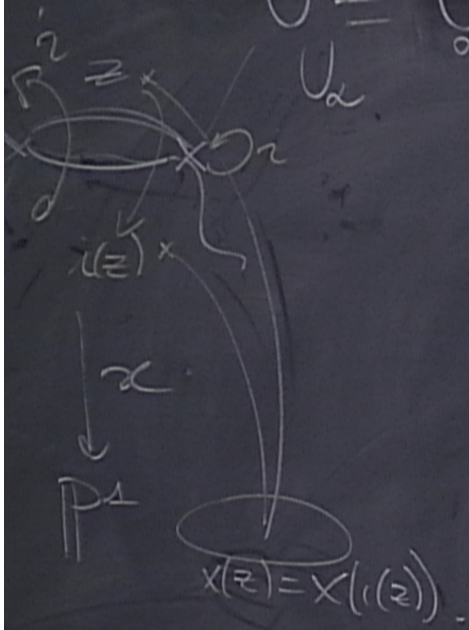
URL: http://pirsa.org/13100118

Abstract: <div>Hermitian matrix models have been used since the early days of 2d quantum gravity, as generating series of discrete surfaces, and sometimes toy models for string theory. The single trace matrix models (with measure dM exp(- N Tr V(M)) have been solved in a 1/N expansion in the 90s by the moment method of Ambjorn et al. Later, Eynard showed that it can be rewritten more intrinsically in terms of algebraic geometry of the spectral curve, and formulated the so-called topological recursion.</div>
<div>In a similar way, we will show that double hermitian matrix models are solved by the same topological recursion, and more generally, that arbitrary hermitian matrix models are solved by a "blobbed topological recursion", whose properties still have to be investigated.</div>



1
$\mathbb{R}^2$
z
x
$\mathbb{P}^1$
x(z) = X(ι(z))


1
z
U_α
ι(z)
x
P1
X(z) = X(ι(z))



$U = \bigcup_{\alpha} U_\alpha$.

$\omega^0_1(z)$: 1-form on U_α, simple zeroes at Fix(ι)

$\omega^0_2(z_1, z_2)$ 1-form z_1, 1-form z_2, $\sim \dfrac{dz_1\, dz_2}{(z_1 - z_2)^2} + \text{reg}$.

$U = \bigcup_\alpha U_\alpha$.

U_α

$\omega^0_{1}(z)$ 1-form on U_α, simple zeroes at $Fix(\iota)$

$\omega^0_{2}(z_1, z_2)$ 1-form z_1 / 1-form z_2, $\sim \frac{dz_1 dz_2}{(z_1-z_2)^2} + \text{reg}$.

$\omega^g_n(z_1, \ldots, z_n) \in \mathrm{Sym}^n \Omega^1(U)$.

$)=x(\iota(z))$.

$\bigcup_{\alpha} U_{\alpha}.$

$\omega^0_{1}(z)$: 1-form on U_{α}, simple zeroes at $\mathrm{Fix}(\iota)$

$\omega^0_{2}(z_1, z_2)$ 1-form z_1, 1-form z_2, $\sim \dfrac{dz_1\, dz_2}{(z_1 - z_2)^2}$ + reg.

$\omega^g_n(z_1, \ldots, z_n) \in \mathrm{Sym}^n\, \Omega^1_{\iota}(U).$

- lin loop eqn $\forall n, g$ $\omega^g_n(z, \overbrace{z_1, \ldots, z_n}^{z_I}) + \omega^g_n(\iota(z), z_I)$ has simple zero at $\mathrm{Fix}(\iota)$

$\ldots x(\iota(z))$. lin loop eqn $\forall n, g$ $\omega_n^g(z, z_2, \ldots, z_n) + \omega_n^g(\iota(z), z_I)$ has simple zero at $\mathrm{Fix}(\iota)$

quadr loop eqn

$\forall n, g.$ $I = \{2, \ldots, n\}.$

$$\left[\omega^{g-1}_{n+1}(z, \iota(z), z_I) + \sum_{\substack{J \subseteq I \\ 0 \le h \le g}} \omega^{h}_{|J|+1}(z, z_J)\, \omega^{g-h}_{n-|J|}(\iota(z), z_{I \setminus J}) \right]$$

has double zero at $\mathrm{Fix}\ \iota$

$n, g.$

$I = \{2, \dots, n\}.$

$$\omega^{g-1}_{n+1}(z, i(z), z_I) + \sum_{\substack{J \subseteq I \\ 0 \leq h \leq g}} \omega^{h}_{|J|+\dots}$$

Blobbed T. R

$\omega^0_1, \omega^0_2, \quad 2g-2+n > 0$

$$\omega^g_n(z_1, \underbrace{z_2, \dots, z_n}_{z_I}) = \phi^g_n(z_1, z_2, \dots, z_n) + \sum_i \operatorname*{Res}_{z \to Fix(i)}$$

fix z

$$-\frac{\frac{1}{2}\int_{\iota(z)}^{z}\omega_2^0(z_1,\cdot)}{\omega_1^0(z)-\omega_1^0(\iota(z))}\left\{\omega^{g-1}_{n+1}(z,\iota(z),z_I) + \sum'_{\substack{J\\h}} \omega^{h}_{|J|+1}(z,z_J)\,\omega^{g-h}_{n-|J|}(\iota(z),z_{I\setminus J})\right\}\Bigg]$$

Blobbed T. R

$\omega_1^0, \omega_2^0, \quad 2g-2+n>0$ blobs

$$\omega_n^g(z_1, \overbrace{z_2, \dots, z_n}^{z_I}) = \phi_n^g(z_1, z_2, \dots, z_n) + \sum$$

- $\phi^g_n \in \mathrm{Sym}^n H^1(U)$.

- If $\phi^g_n \equiv 0$: Eynard–Orantin (2007)

- If $\phi^g_n \not\equiv 0$.

$\longrightarrow$ topological exp in hermitian matrix models. (blobbed T

$\longrightarrow$ BKMP 2007 :

- $\phi^g_n \in \mathrm{Sym}^n H^1(U)$.

- If $\phi^g_n \equiv 0$: Eynard–Orantin (2007)

- If $\phi^g_n \not\equiv 0$.

$\rightarrow$ topological exp in hermitian matrix models. (blobbed TR).

$\rightarrow$ BKMP 2007 : $\begin{cases} \omega_1^0 = \ln y \, d\ln x \\ \omega_2^0 \end{cases} \longrightarrow \omega_n^g$ GW inv of toric CY3 folds.

EO 2012

EO 2012

II APPLICATIONS : MATRIX MODELS

M hermitian $N \times N$

$$Z_N = \int dM \exp\Big(\sum_{\ell \geq 1} N t_\ell \operatorname{Tr} M^\ell\Big)$$

$\Rightarrow$ ... $\begin{cases} \omega_1 = \ldots \\ \omega_2^0 \end{cases}$ EO 2012

II APPLICATIONS : MATRIX MODELS

M hermitian $N \times N$.

- $Z_N = \int dM \exp\left(\sum_{\ell \geq 1} N t_\ell \operatorname{Tr} M^\ell\right)$

generating series for discrete surfaces made of polygons.

A_ℓ per ℓ-gon.

N^χ χ = Euler char.

$\frac{1}{\mathrm{Aut}}$ sym. factor.

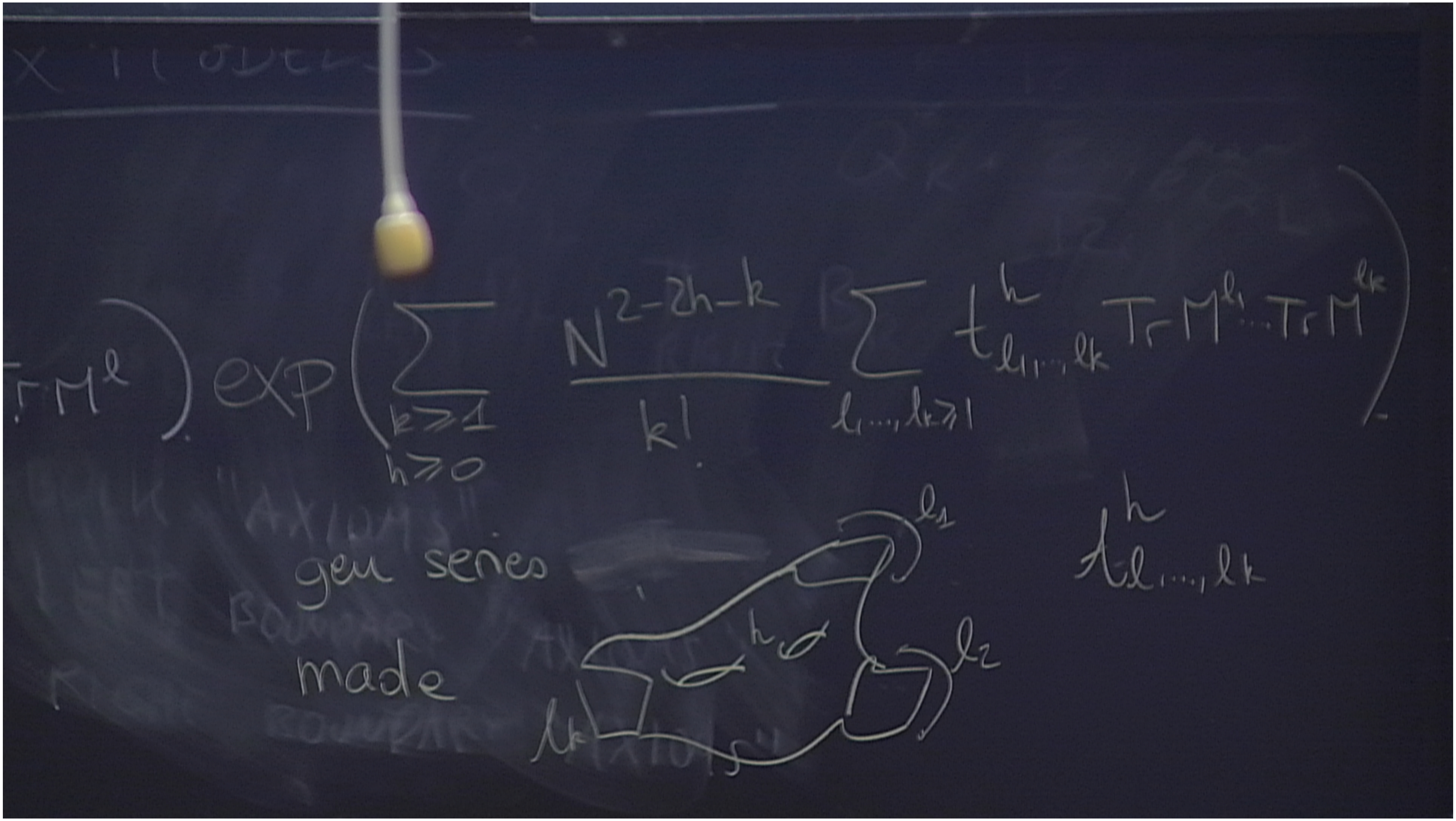
A_ℓ per ℓ-gon.

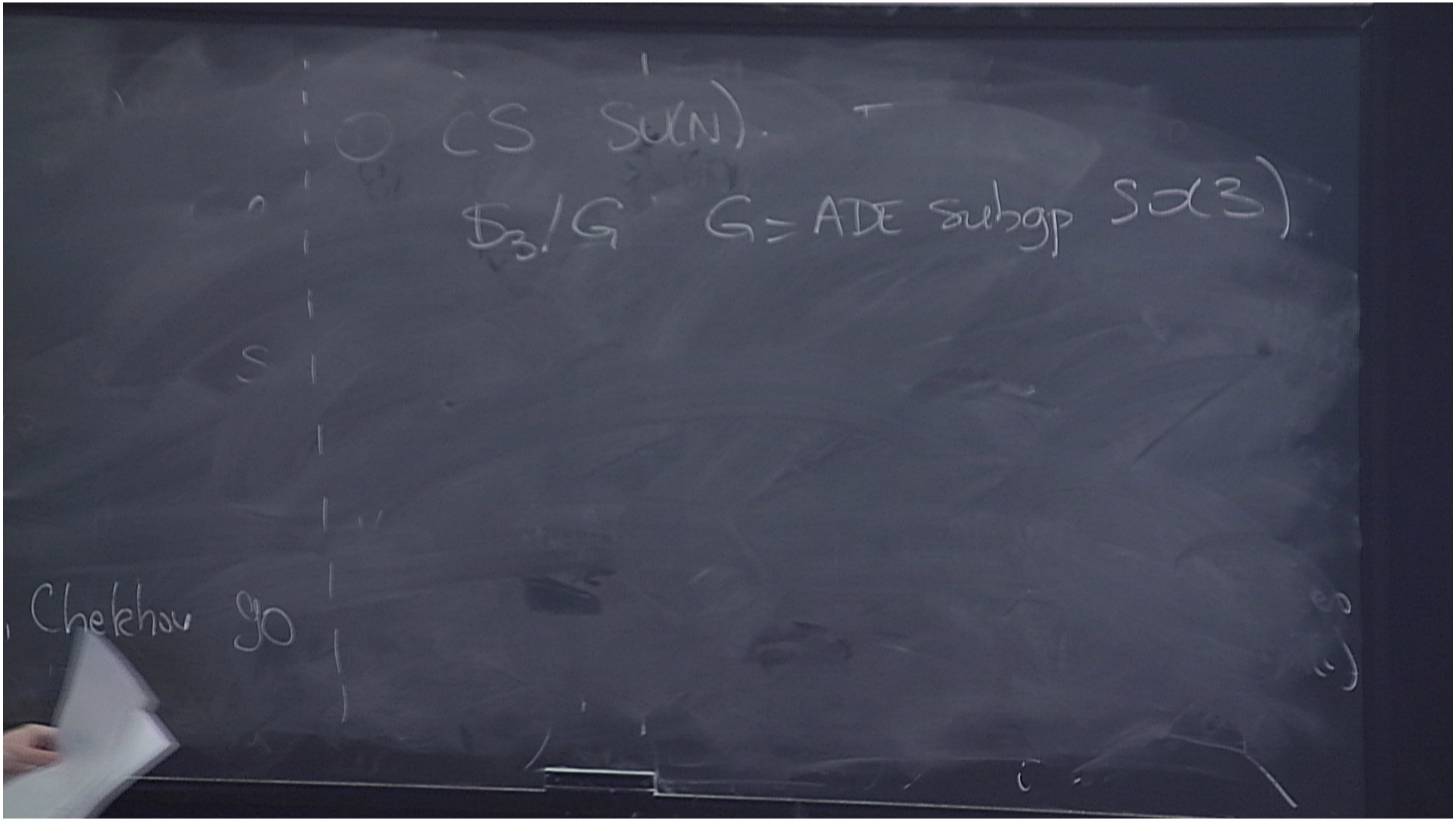
N^{χ} χ = Euler char.

$\frac{1}{\mathrm{Aut}}$ sym. factor.

$$\ln Z_N = \sum N^{2-2g} F_g$$

Ambjørn-Makeenko-Kristjansen, Chekhov


$\mathrm{Tr}\,M^{\ell}\big)\ \exp\Big(\sum_{\substack{k\geq 1\\ h\geq 0}} \frac{N^{2-2h-k}}{k!} \sum_{l_1,\dots,l_k\geq 1} t^{h}_{l_1,\dots,l_k}\, \mathrm{Tr}\,M^{l_1}\cdots\mathrm{Tr}\,M^{l_k}\Big)$
gen series
made
l_1
l_2
l_k
h
$t^{h}_{l_1,\dots,l_k}$


CS SU(N).
S3/G G = ADE subgp
Chekhov go

① CS $SU(N)$.

S^3/G $G =$ ADE subgp $SU(2)$)

contribution of trivial flat connection

$$Z = \int dM \, \exp(N \operatorname{Tr} V(M)) \exp\left(\sum_{i=1}^{r} c_i \operatorname{Tr} \psi\left(\frac{M\otimes 1}{p_i}, \frac{1\otimes M}{p_i}\right)\right)$$

$$\psi(x,y) = \ln\left(\frac{\operatorname{sh}\left(\frac{x-y}{2}\right)}{\frac{x-y}{2}}\right)$$

S

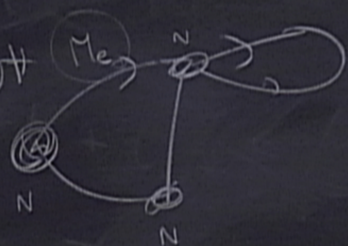
Chekhov go

Ambjørn–Makeenko–Kristjansen, Chekhov 90

$\varphi(x,y) =$

② Quiver matrix models

$M_e \to U M_e U^\dagger$



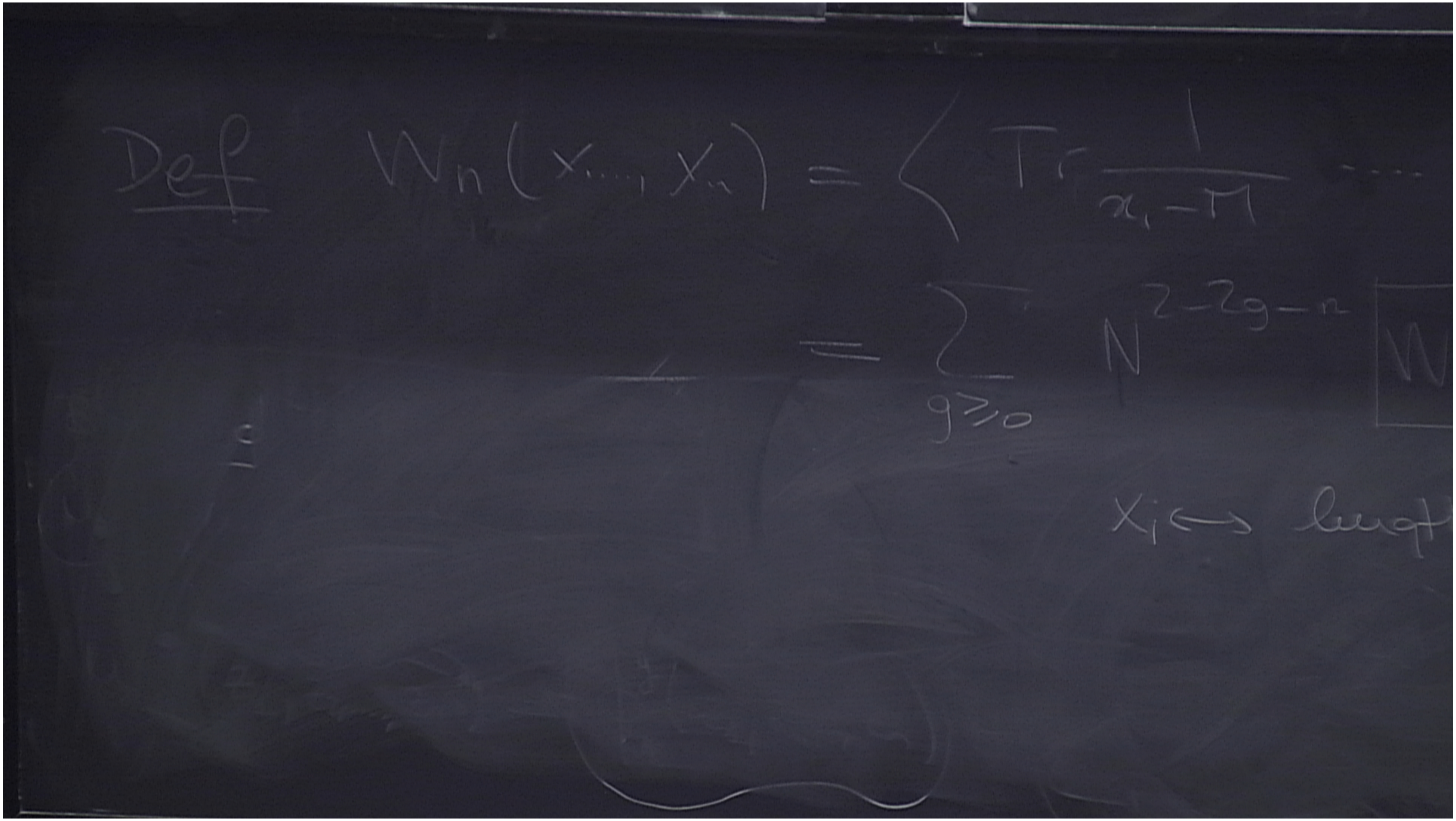
$$\int \prod dM_e \exp\left(N \,\mathrm{Tr}\, V_e(M_e)\right) \prod_{\substack{v \\ e' \nearrow \nwarrow e}} \exp\left(C_v \,\mathrm{Tr}\, M_e M_{e'}\right)$$

$$= \int dM_e$$

$$\ldots, x_n) = \left\langle \mathrm{Tr}\frac{1}{x_1 - M} \cdots \mathrm{Tr}\frac{1}{x_n - M} \right\rangle_{\text{cumulant}}$$

$$= \sum_{g \geq 0} N^{2-2g-n} \boxed{W_n^g(x_1, \ldots, x_n)}$$

$x_i \leftrightarrow$ length i^{th} boundary


Def
$W_n(x_1, \dots, x_n) = \left\langle \mathrm{Tr} \frac{1}{x_1 - M} \cdots \right.$
$= \sum_{g \geq 0} N^{2-2g-n}$
$x_i \leftrightarrow$ lengt

$$\left\langle \mathrm{Tr}\frac{1}{x_1 - M} \cdots \mathrm{Tr}\frac{1}{x_n - M} \right\rangle_{\text{cumulant}}$$

$$= \sum_{g \geq 0} N^{2-2g-n} \boxed{W_n^g(x_1 \ldots x_n)}$$

$x_i \leftrightarrow$ length i^{th} boundary.

• Inv by reparametrisation (⟺ integrat by parts)

⟺ $L_m \cdot Z_N = 0$, $[L_m, L_n] = (m-n)L_{m+n}$. $m \geq -1$, $n \geq -1$.

(SD equat)

• Analysing SD eqn

— W_n^g have

EO

made of polygons.

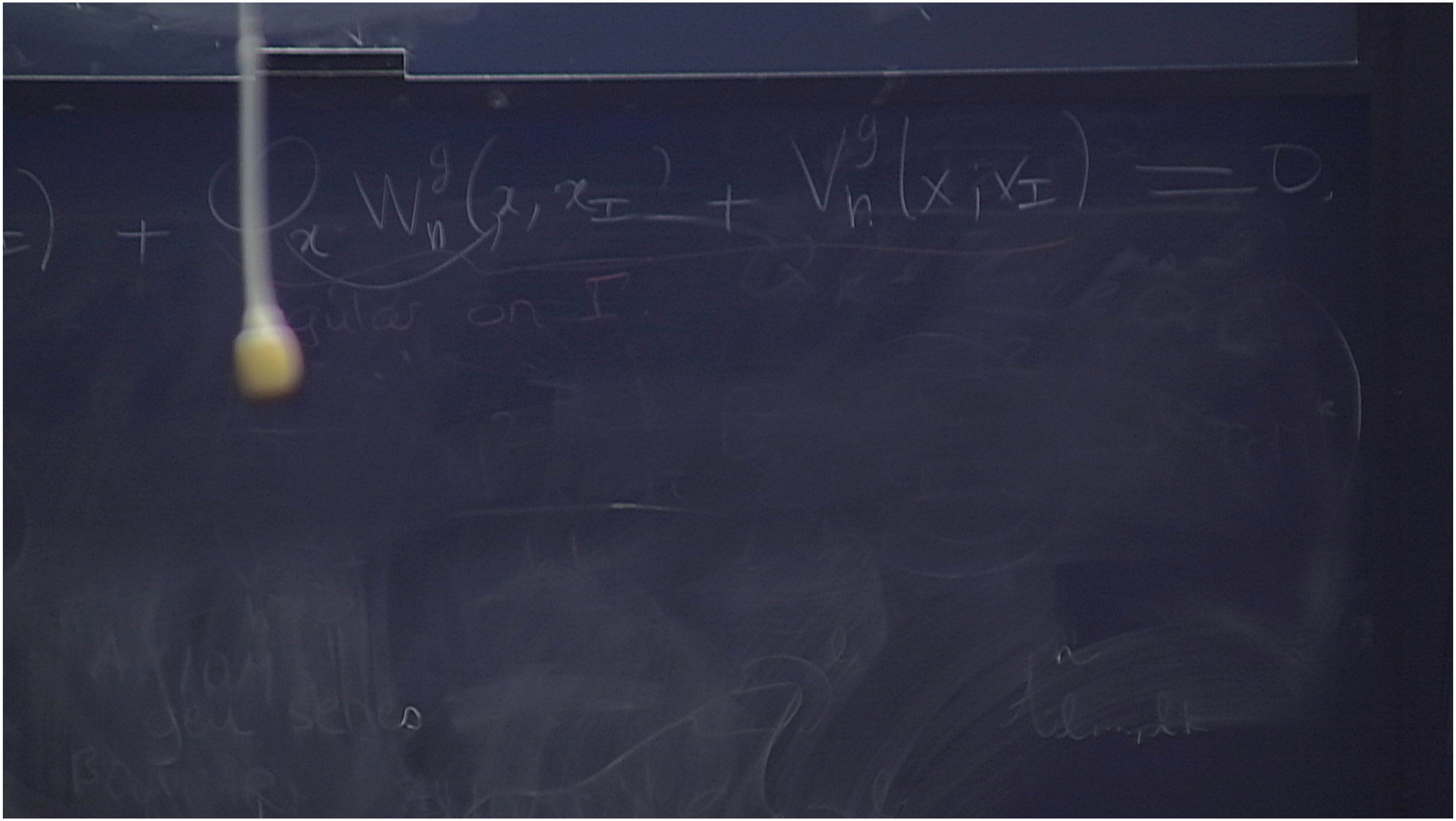
- Inv by reparametrisation ($\Leftrightarrow$ integrat by parts).

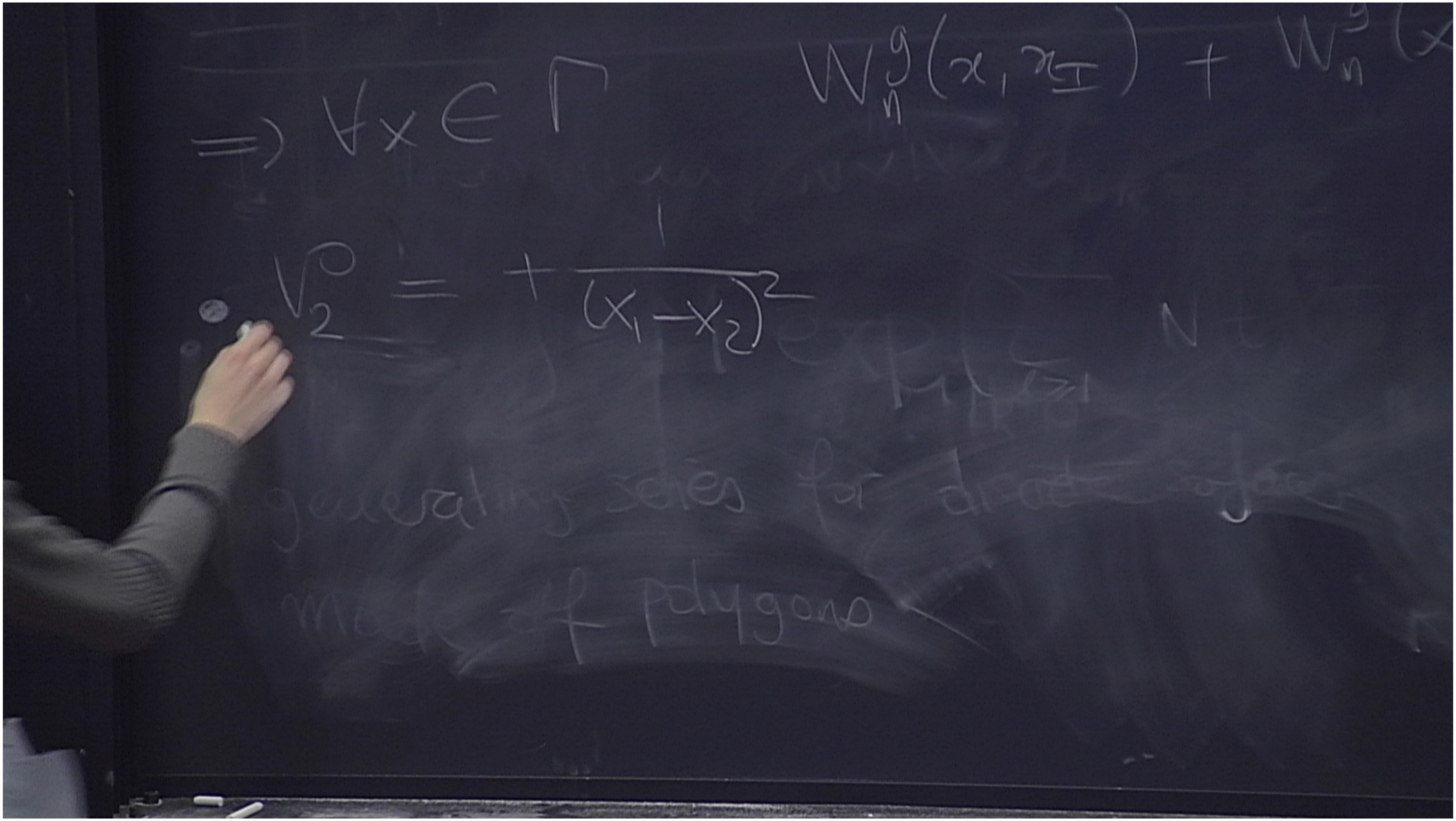
$\Leftrightarrow L_m \cdot Z_N = 0$, $[L_m, L_n] = (m-n)L_{m+n}$. $m \geq -1$, $n \geq -1$.

(SD equat)

- Analysing SD eqn
  - W_n^g have cuts Γ does not dpd on n, g
  - have squareroot (generically) at $\partial\Gamma$

$$\Rightarrow \forall x \in \Gamma \qquad W^g_n(x, x_I) + W^g_n(x^-, x_I)$$


) + ∂x Wn^g(x, xI) + Vn^g(x; xI) = 0.
gular on I.

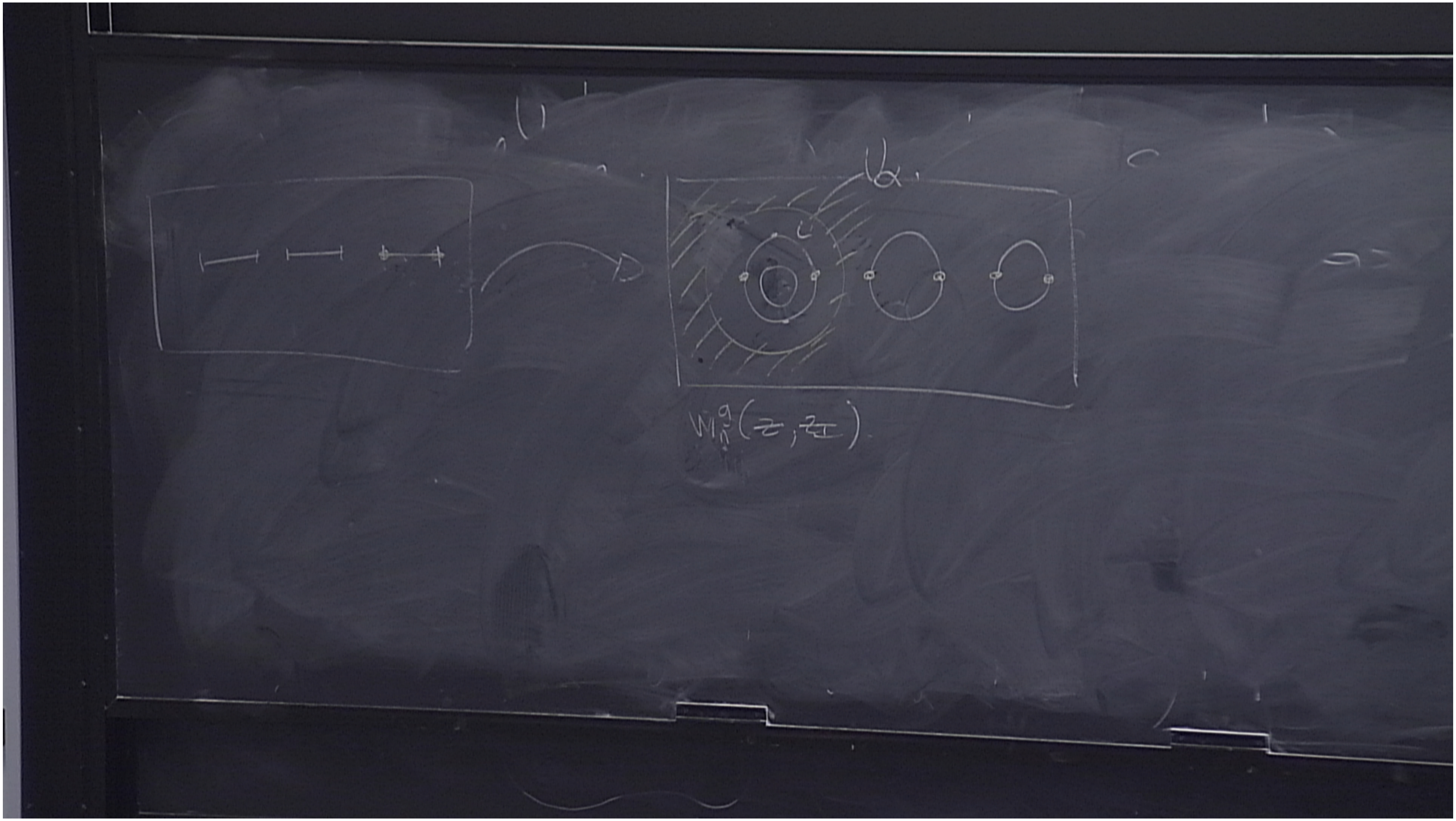

⇒ ∀x ∈ Γ
W_n^g(x, x_I) + W_n^g(x
V_2^0 = + 1/(x_1 − x_2)^2
generating series for
polygons

$\Longrightarrow \forall x \in \Gamma \qquad W_n^g(x, x_I) + W_n^g(x^-, x_I) + \oint_x W_n^g(x, \ldots$

- $V^0_2(x_1, x_2) = + \frac{1}{(x_1 - x_2)^2}$
- V_n^g = expression in terms of $W_{n'}^{g'}$ $\quad 2g' - 2 + n' < 2g - 2 + n$.

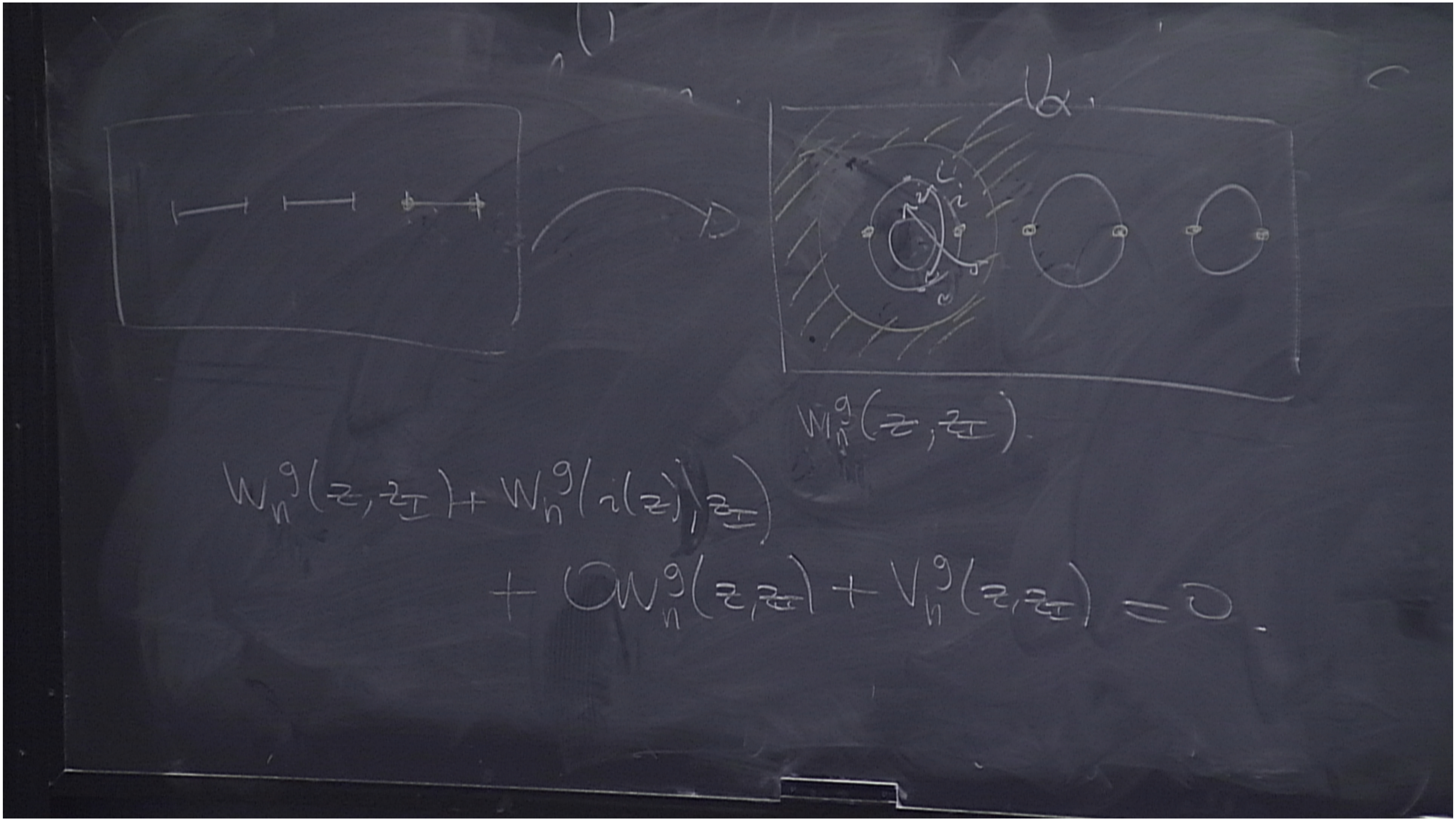
$2g - 2 + n \geq 0$. polygons

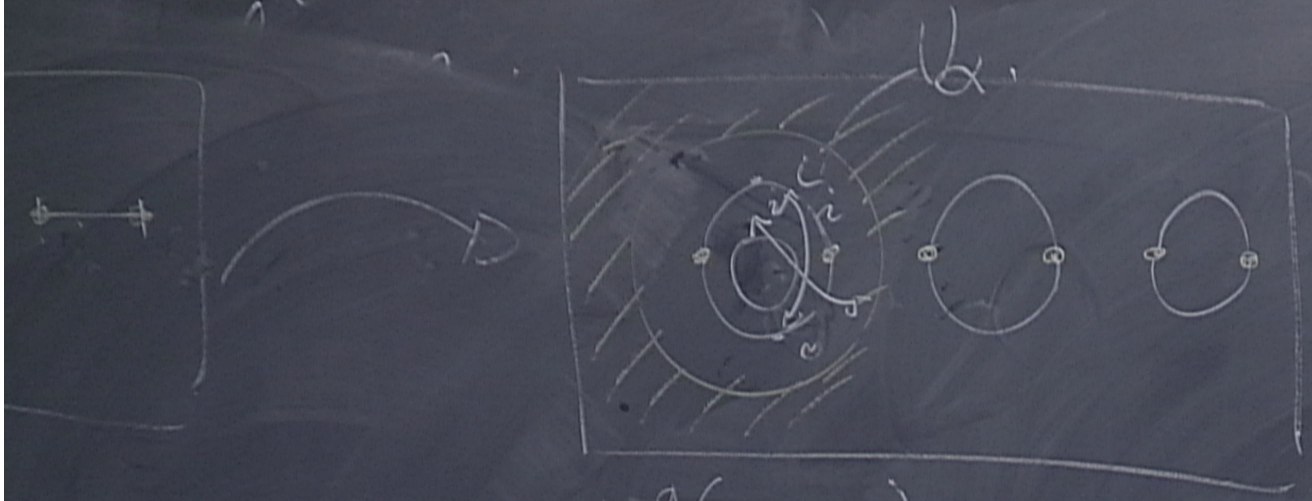
$t^h_{l_1, \ldots, l_k} \quad \begin{cases} 2h - 2 + k > 0 \\ h \leq g \end{cases}$


$W_n^g(z, z_I)$

$$W_n^g(z, z_I)$$

$$W_n^g(z, z_I) + W_n^g(i(z)) + O W_n^g(z, z_I) + V_n^g(z, z_I) = 0.$$


$W_n^g(z, z_I)$
$W_n^g(z, z_I) + W_n^g(\iota(z), z_I)$
$+ \mathcal{O} W_n^g(z, z_I) + V_n^g(z, z_I) = 0.$

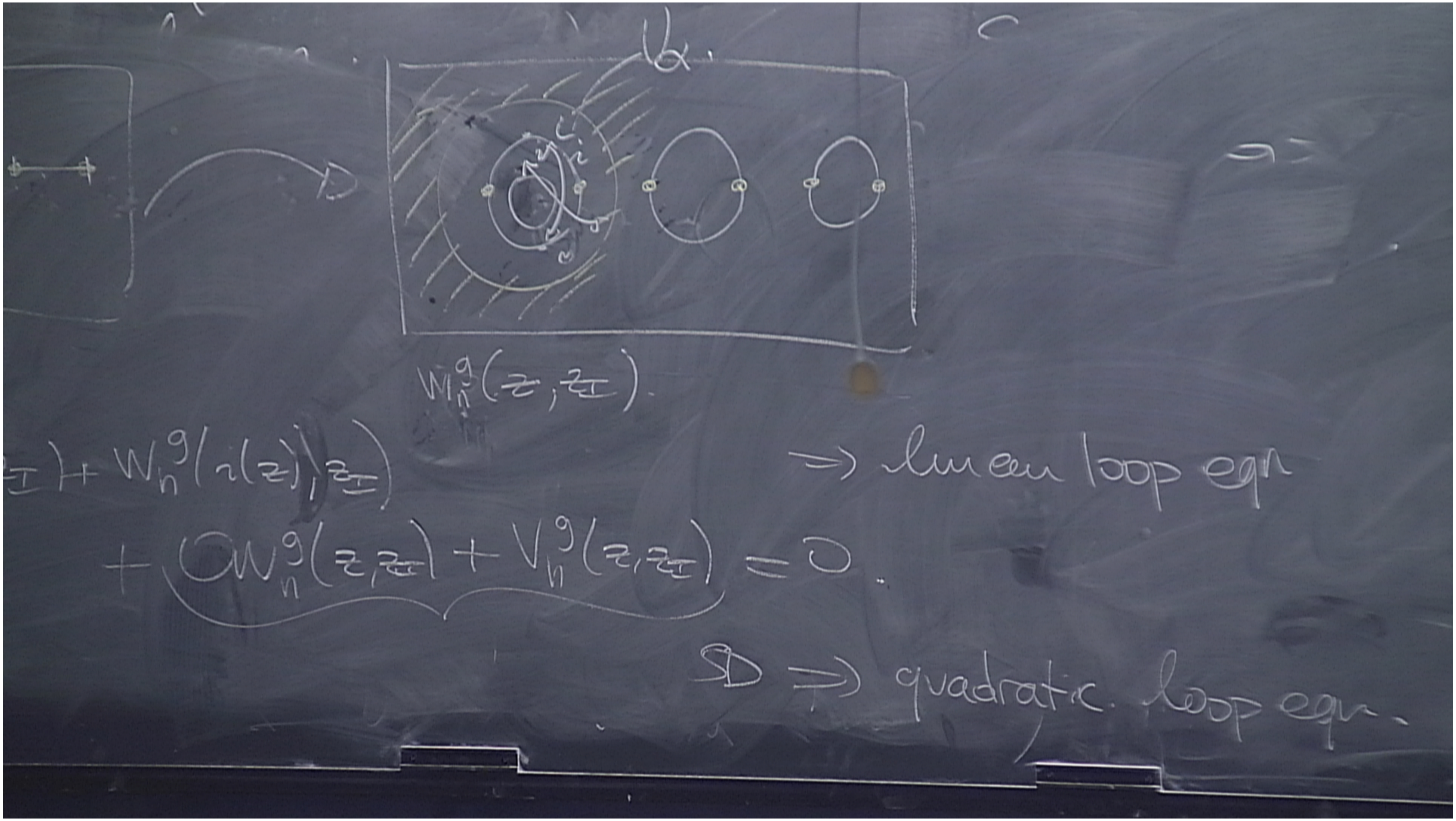


$W^g_n(z, z_I)$.

$z_I) + W^g_n(i(z), z_I)$

$+ \underbrace{\Omega W^g_n(z, z_I) + V^g_n(z, z_I)} = 0.$

$\Rightarrow$ linear loop eqn


$W_n^g(z, z_I)$.
$+ \underbrace{\Omega W_n^g(z, z_I) + V_n^g(z, z_I)} = 0$.
⇒ linear loop eqn
SD ⇒ quadratic loop eqn.

$\Rightarrow$ quad

Def $\quad W_n(x_1,\dots,x_n) = \left\langle \mathrm{Tr}\,\frac{1}{x_1 - M} \cdots\cdots \mathrm{Tr} \right.$

$= \sum_{g\geq 0} N^{2-2g-n}\, \boxed{W_n^g(}$

$$\boxed{\omega_n^g(z_1,\dots,z_n) = W_n^g(z_1,\dots,z_n)\,dx(z_1)\cdots dx(z_n) + \frac{dx(z_1)\,dx(z_2)}{(x(z_1)-x(z_2))^2}\,\delta_{n,2}\,\delta_{g,0}}$$

$x_i \leftrightarrow$ length

• ω_1^0, ω_2^0 : hard to compute.

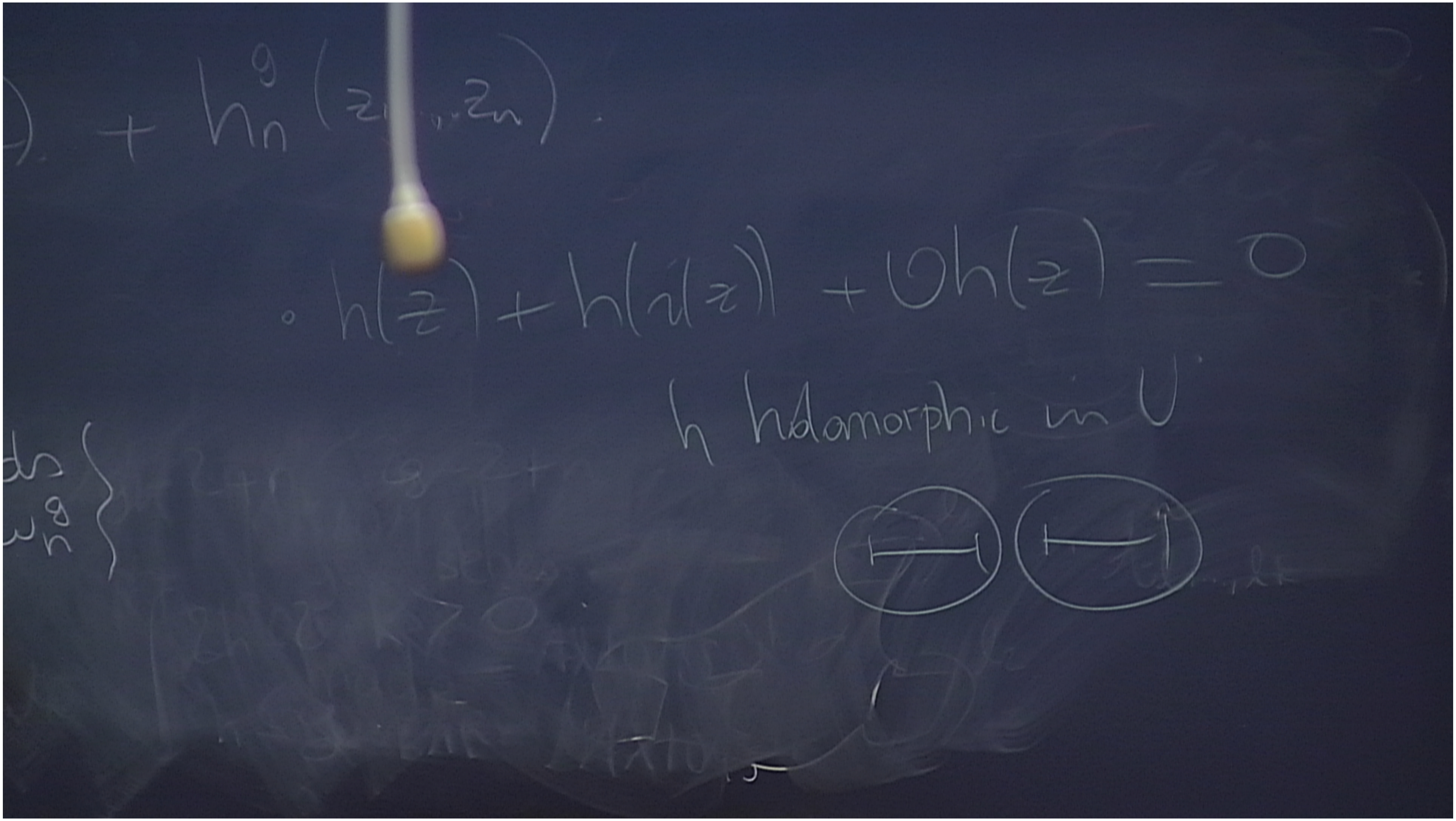
$$\phi_n^g(z_1, z_I) = \frac{1}{4i\pi} \oint_{z\in\Gamma} \omega_2^0(z_1, z)\, V_n^g(z, z_I) + h_n^g(z_1, \dots, z_n)$$

$$h(z) + h($$

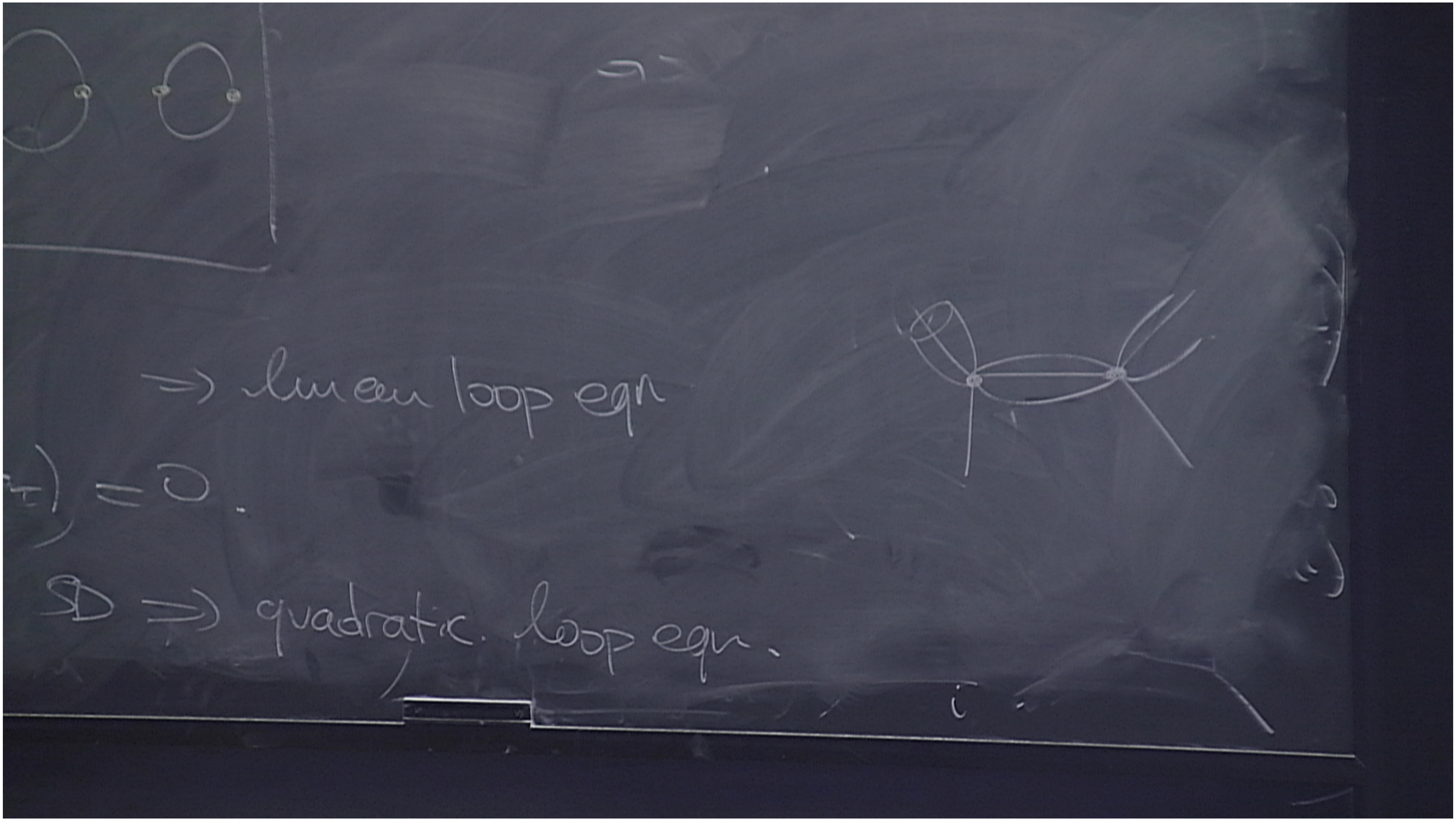
if $\Gamma = 1$-segment $\quad h_n^g \equiv 0$

$\Gamma = \quad \{h_n^g\} \longleftrightarrow \{\text{periods of } \omega_n^g\}$

$n \geq$ polygons



$+ h_n^g(z_1,\dots,z_n)$
$h(z) + h(\iota(z)) + \upsilon h(z) = 0$
h holomorphic in U
ω_n^g


⇒ linear loop eqn
= 0.
SD ⇒ quadratic. loop eqn.

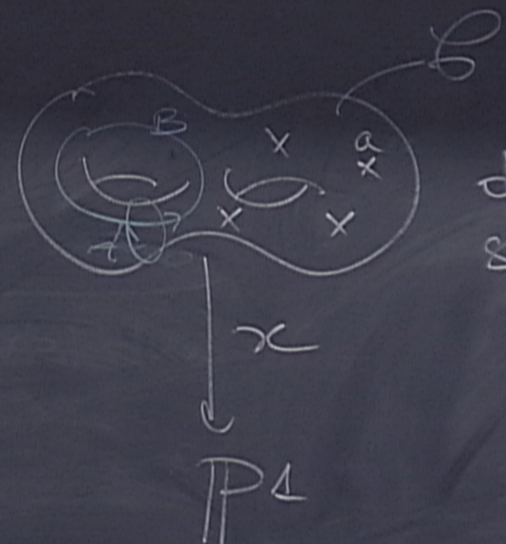
III PPT

Global setting:

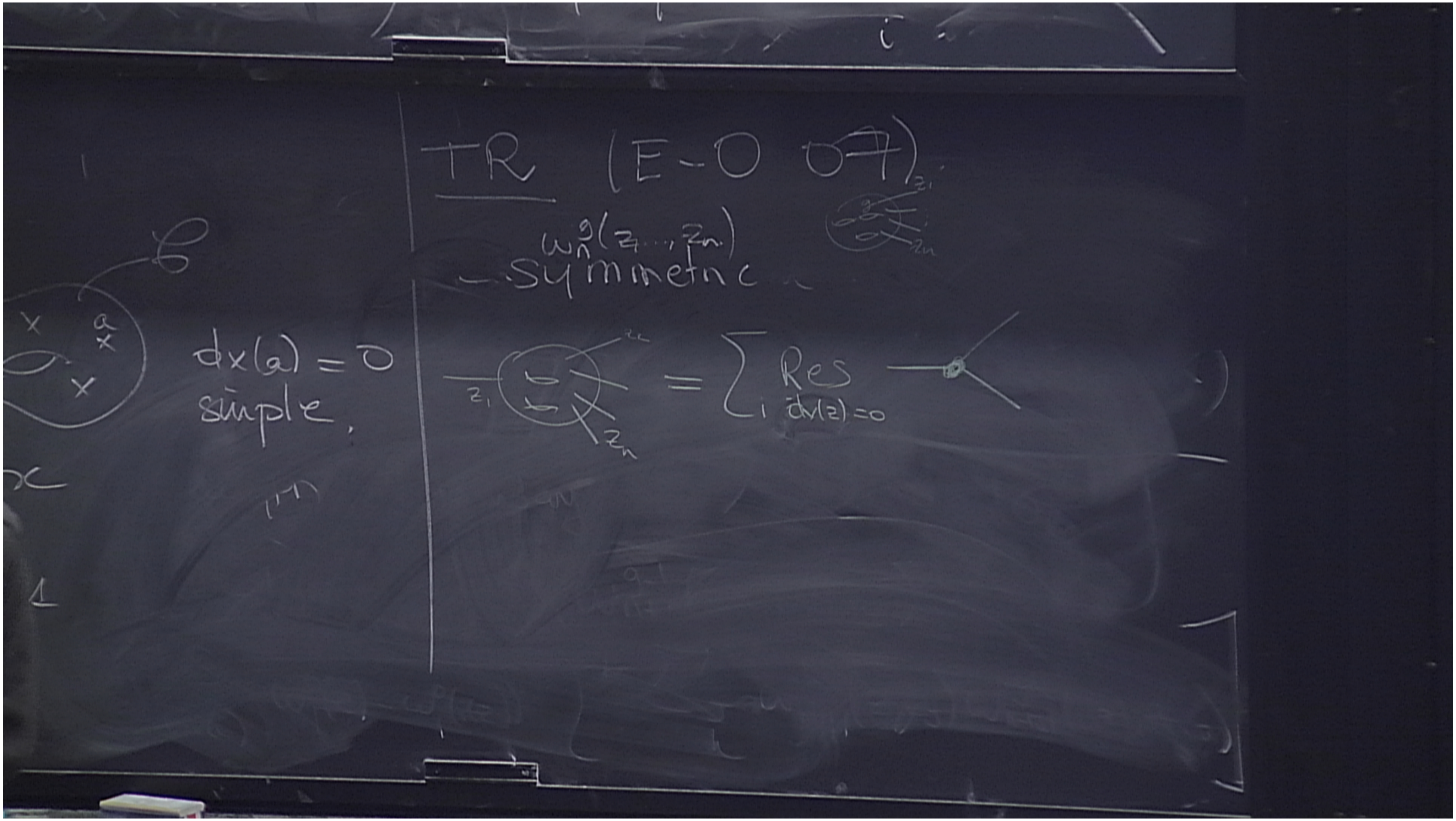
$\mathcal{C}$ compact curve.

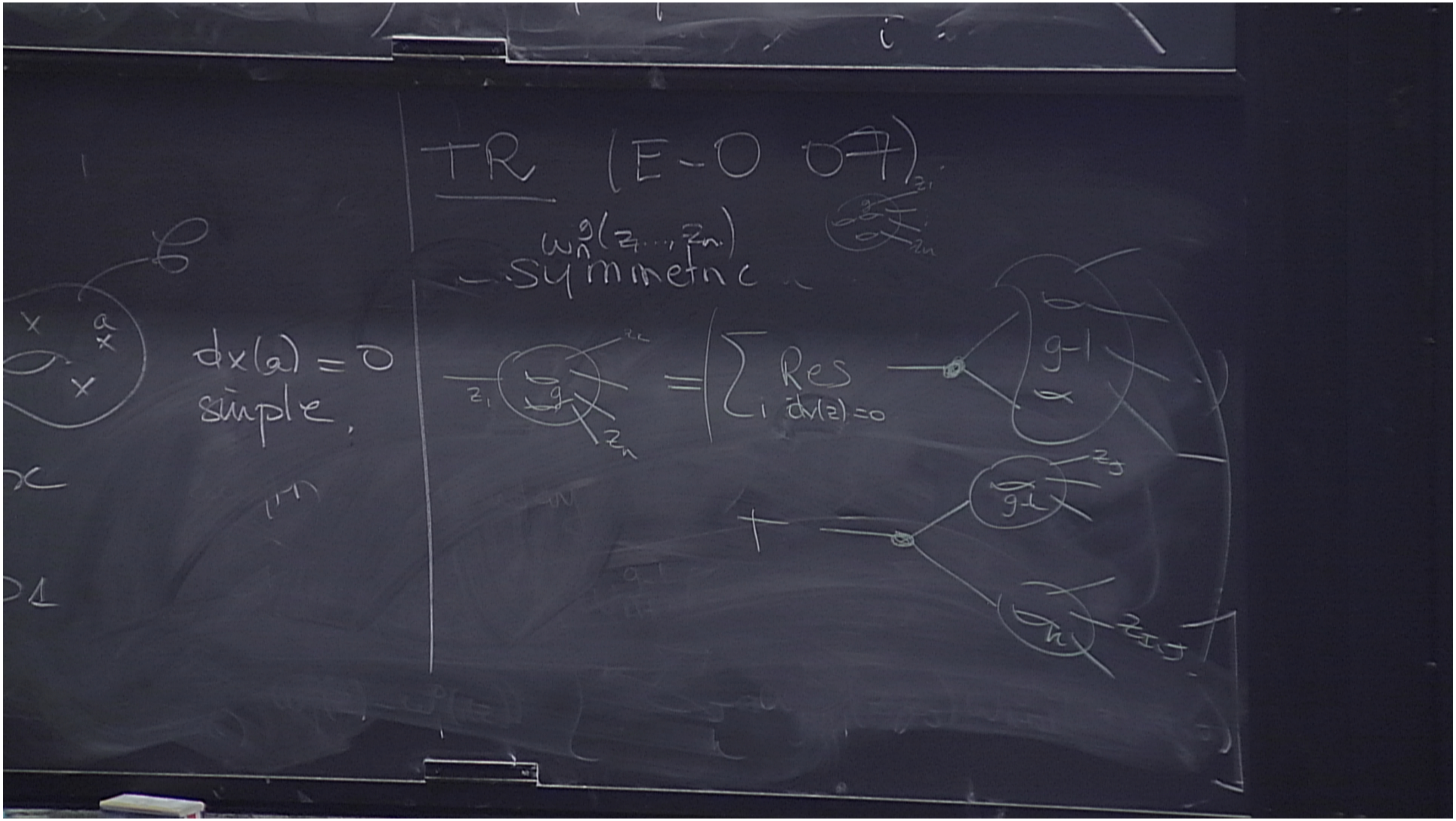
x, y f^n

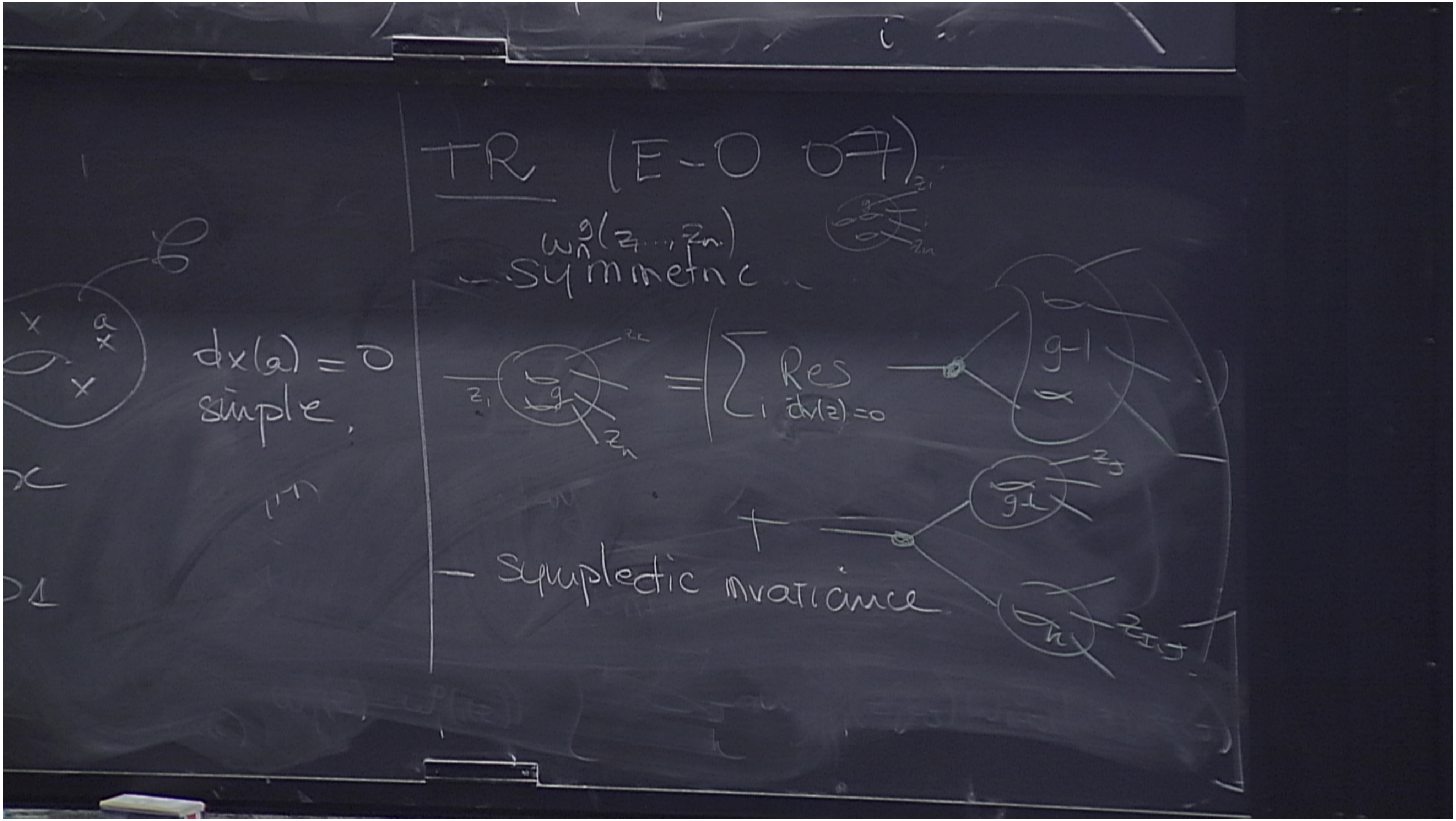
(A, B) symplectic basis of cycles

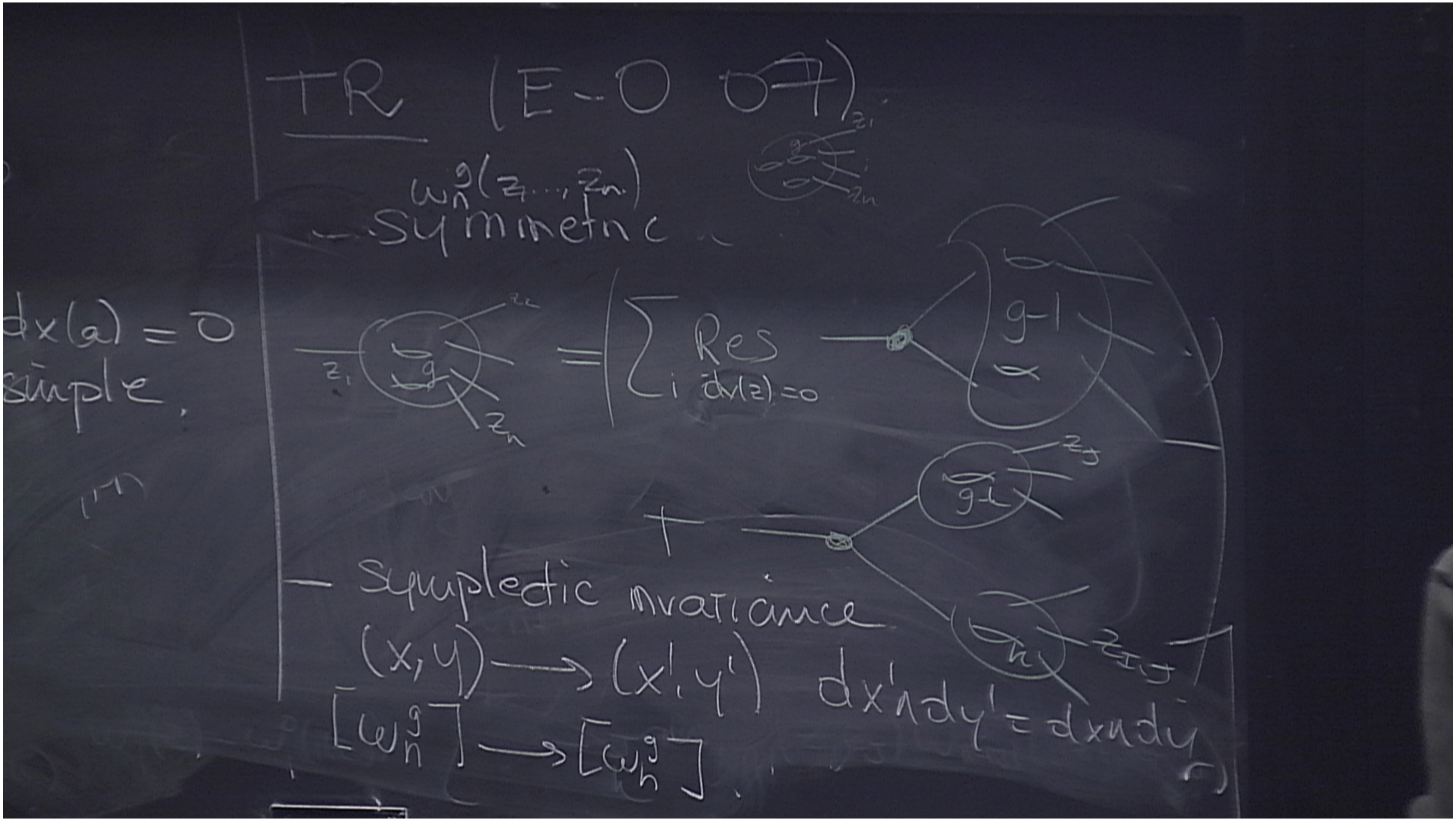


$dx(a) = 0$ simple.


TR (E-O 07)
$\omega_n^g(z_1, \ldots, z_n)$
symmetric
$dx(a) = 0$
simple
$= \sum_i \operatorname{Res}_{dx(z)=0}$


TR (E-O 07)
ω_n^g(z_1,...,z_n)
symmetric
dx(a) = 0
simple.
Res
dx(z)=0
g-1
g-1
h
z_1
z_n


TR (E-O 07)
$\omega_n^g(z_1,\dots,z_n)$
— symmetric
$dx(a)=0$
simple.
$= \sum_i \operatorname{Res}_{dx(z)=0}$
g-1
— symplectic invariance



TR (E-O 07)
$\omega_n^g(z_1, \dots, z_n)$
— symmetric
dx(a) = 0
simple.
$= \left(\sum_i \mathrm{Res}_{dx(z)=0} \right.$
g
g-1
z_1
z_n
z_J
— symplectic invariance
$(x, y) \longrightarrow (x', y')$ $dx' \wedge dy' = dx \wedge dy$
$[\omega_n^g] \longrightarrow [\omega_n^g]$

– have squareroot (generically) at $\partial\Gamma$

– infinitesimal deformation

If $\begin{cases} \partial_t \omega_1^0(z_1) = \int_{\Lambda^*} \omega_2^0(z_1, \cdot) \\ \partial_t \omega_2^0(z_1, z_2) = \int_{\Lambda^*} \omega_3^0(z_1, z_2, \cdot) \end{cases} \Longrightarrow \partial_t \omega_n^g(z_1 \ldots, z_n) = \int \omega_{n+1}^g(z_1 \ldots, z_n, \cdot)$

— infinitesimal deformation

If $\begin{cases} \partial_t \omega_1^0(z_1) = \int_{\Lambda^*} \omega_2^0(z_1, \cdot) \\ \partial_t \omega_2^0(z_1, z_2) = \int_{\Lambda^*} \omega_3^0(z_1, z_2, \cdot) \end{cases} \Longrightarrow \quad \partial_t$

$\Longleftarrow$

— Eynard 11

$$\omega_n^g = \int_{\overline{\mathcal{M}}_{g,n}^{|\mathrm{Fix}(\sigma)|}}$$

$$\Rightarrow \quad \partial_t \, \omega^g_n(z_1,\dots,z_n) = \int \omega^g_{n+1}(z_1,\dots,z_n,\cdot)$$

Blobbed TR (in progress with Shadrin)

* ϕ_n^g symmetric $\Longleftrightarrow$ ω_n^g symmetric

Blobbed TR (in progress with Shadrin)

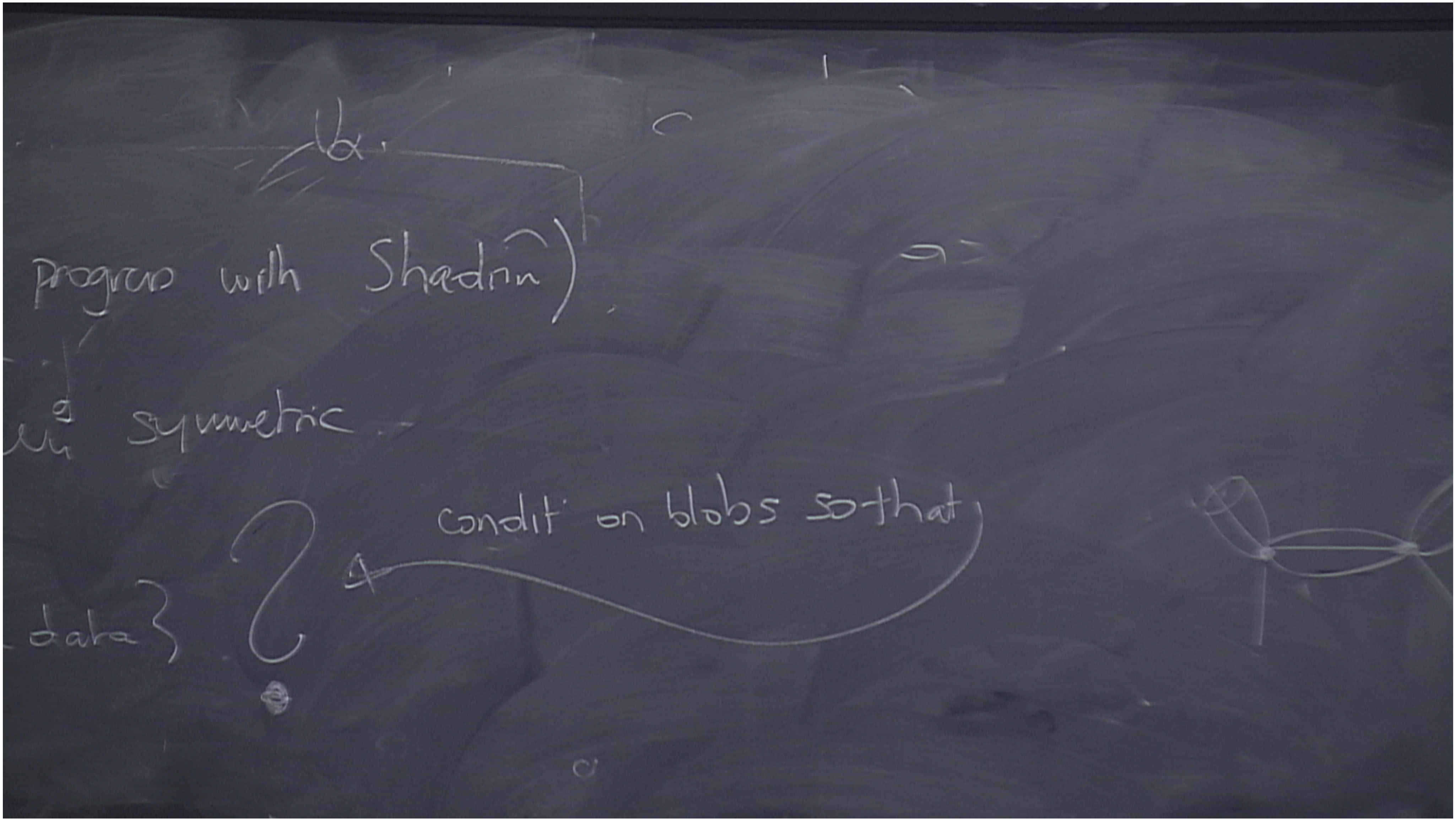
ω_1^0, ω_2 = initial data.

* ϕ_n^g symmetric $\iff$ ω_n^g symmetric

* symp inv

* natural flow on {initial data} ?

* $\omega_n^g = \int_{\overline{\mathcal{M}}_{g,n}}$


program with Shadrin)
symmetric
condit on blobs so that
data}

- infinitesimal deformation

If $\begin{cases} \partial_t \omega_1^0(z_1) = \int_{\Lambda^*} \omega_2^0(z_1, \cdot) \\ \partial_t \omega_2^0(z_1, z_2) = \int_{\Lambda^*} \omega_3^0(z_1, z_2, \cdot) \end{cases} \Longrightarrow \partial_t \omega_n^g(z_1, \dots, z_n) =$

V_n^g

$$\Phi_n = \left(\nabla^\Lambda \ln \Theta\right) \cdot du(z_1) \cdots du(z_n)$$

- Eynard 11

$$\omega_n^g = \int_{\overline{\mathcal{M}}_{g,n}^{|Fix(\sigma)|}} \Lambda$$

- infinitesimal deformation

If
$$\begin{cases} \partial_t \omega_1^0(z_1) = \int_{\Lambda^*} \omega_2^0(z_1, \cdot) \\ \partial_t \omega_2^0(z_1, z_2) = \int_{\Lambda^*} \omega_3^0(z_1, z_2, \cdot) \end{cases} \Longrightarrow \partial_t \omega_n^g(z_1, \dots, z_n) = \int_{\Lambda^*} \omega_{n+1}^g$$

$$\boxed{\Phi_n = \left(\nabla^n \ln \Theta\begin{bmatrix}\mu \\ \nu\end{bmatrix}\right) du(z_1) \cdots du(z_n)}$$

- Eynard 11

$$\omega_n^g = \int_{\overline{\mathcal{M}}_{g,n}^{|Fix(\sigma)|}}$$