Title: Some simple extensions of Mathieu Moonshine

Date: Oct 21, 2013 10:10 AM

URL: http://pirsa.org/13100112

Abstract: Mathieu Moonshine is a striking and unexpected relationship between the sporadic simple finite group M24 and a special Jacobi form, the elliptic genus, which arises naturally in studies of nonlinear sigma models with K3 target. In this talk, we first discuss its predecessor (Monstrous Moonshine), then discuss the current evidence in favor of Mathieu Moonshine. We also discuss extensions of this story involving `second quantized mirror symmetry,' relating heterotic strings on K3 to type II strings on Calabi-Yau threefolds.

The subject of this talk: mysterious relations between three beautiful and enigmatic classes of objects.

To be self-contained, I will have to "summarize" many things. My outline is as follows:

I. Introduction to Moonshine

II. Enter K3 and BPS States

III. String duality, Calabi-Yau threefolds, and modular forms

1. Introduction to Moonshine

By the end of the twentieth century, mathematicians were closing in on a classification of simple finite groups.

In addition to 18 infinite classes (e.g. cyclic groups), there are 26 sporadic simple finite groups.

The two heroes of our story today will be two rather large sporadics: the Fischer-Griess Monster, and Mathieu 24.

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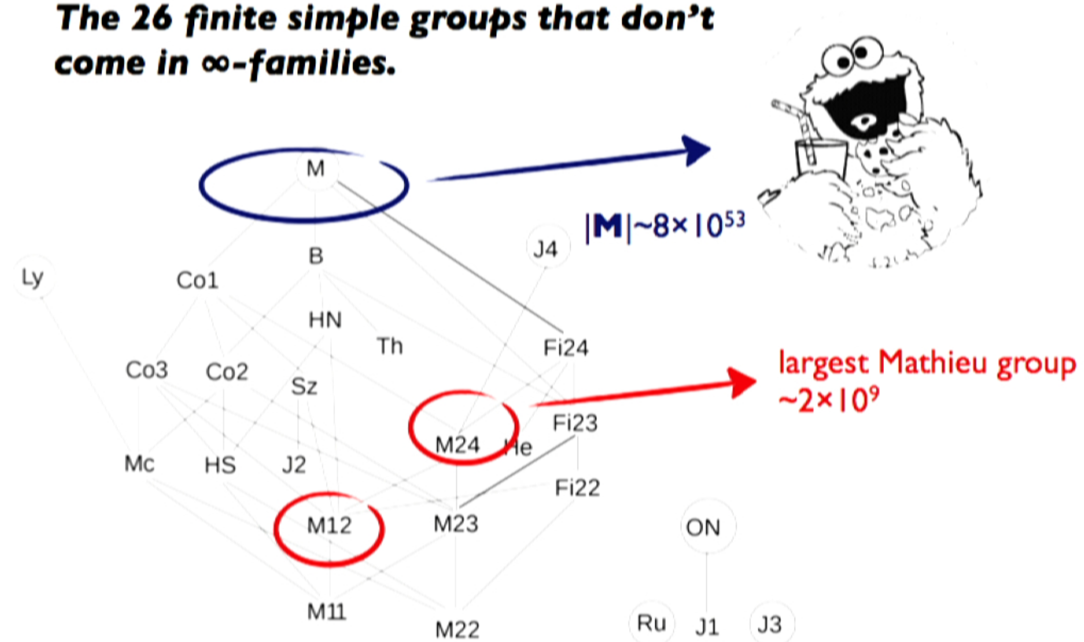
Sporadic Groups

The 26 finite simple groups that don't come in ∞-families.



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It has been speculated that these groups, as special sets of symmetries, may play a role in physics. E.g.

"I have a sneaking hope, a hope unsupported by any facts or any evidence, that sometime in the twenty-first century physicists will stumble upon the Monster group, built in some unsuspected way into the structure of the Universe."

- F. Dyson, Unfashionable pursuits, Mathematical Intelligencer (1983).

In fact before the Monster was even fully constructed, unfolding developments began to tie it to "physics," though of an esoteric sort.

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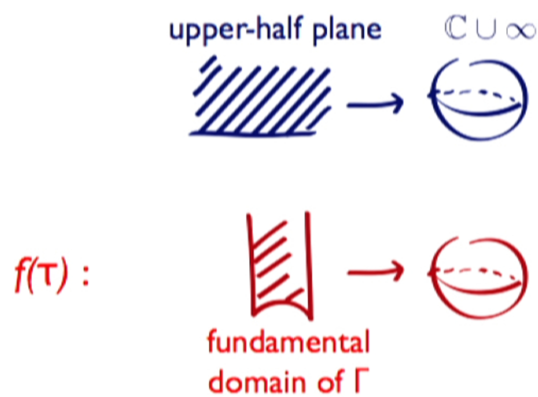
To understand this we need to introduce a second class of natural mathematical objects: modular forms.

In mathematics as well as in physics, it is natural to consider a special class of functions:

A modular form $f(\tau)$ transforms "covariantly" under a subgroup Γ of $SL(2,R)$.



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A common example: consider SL(2,Z) acting on the UHP via fractional linear transformations:

$$\tau \rightarrow \frac{a\tau+b}{c\tau+d} \equiv A \cdot \tau$$

Then a modular function is a meromorphic function which satisfies:

$$f(A \cdot \tau) = f(\tau)$$

while a modular form of weight k satisfies instead:

$$f(A \cdot \tau) = (c\tau + d)^k f(\tau)$$

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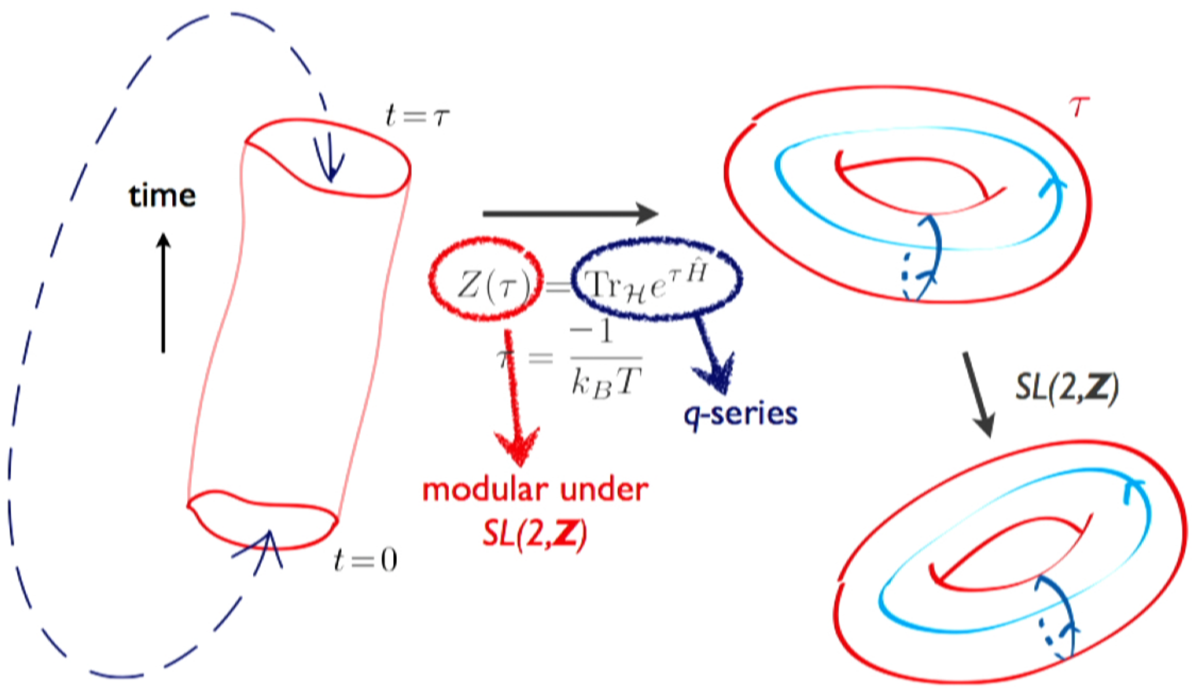
These beasts pop up everywhere in string theory, for two reasons: worldsheet modular invariance and space-time S-duality symmetries.

eg. The *SL(2,**Z**)* symmetry of the partition function.



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A basic result says that any modular function can be written as a rational function of

$$j(\tau) = \frac{1}{q} + 744 + 196,884\ q + 21,493,760\ q^2 + \cdots$$

$$q = e^{2\pi i \tau}$$

John McKay, taking a break from hard work on sporadic groups in the late 1970s, came across this expansion in a number theory paper. He noticed:

$$196,884 = 196,883 + 1$$

dims of irreps of Monster!

$$21,493,760 = 21,296,876 + 196,883 + 1$$

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Why should there be a relationship between the j-function and irreps of the Monster?

* A positive integer begs interpretation as dim(V) for some vector space V.

* So suppose there exists an infinite dimensional graded representation:

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus V_3 \oplus \dots$$

where

$$V_{-1} = \rho_0,\ V_1 = \rho_1 \oplus \rho_0, V_2 = \rho_2 \oplus \rho_1 \oplus \rho_0, \dots$$

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$$j(\tau) - 744 = \dim(V_{-1})\ q^{-1} + \sum_{i=1}^{\infty} \dim(V_i)\ q^i$$

But if the Monster M has a natural action on V, this also suggests that we can fruitfully study the McKay-Thompson series:

$$ch_\rho(g) = Tr(\rho(g)), \quad g \in M$$

$$T_g(\tau) = ch_{V_{-1}}(g)\ q^{-1} + \sum_{i=1}^{\infty} ch_{V_i}(g)\ q^i$$

For each conjugacy class in M (there are 194), we get such a series.

Now the point is the following. Let us imagine the modular function j is being generated by a partition function of a 2d CFT on a torus. If we twist the boundary conditions:

$$h\,\underset{g}{\boxed{}} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} h^d g^c\,\underset{g^a h^b}{\boxed{}}$$

we should still get a modular form for a subgroup of the modular group that preserves the form of the BC.

Thus, any conjecture for the decompositions is subjected to many checks, because the spaces of these modular forms are quite constrained.

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Technically, Conway and Norton found that the McKay-Thompson series give "Hauptmodules of the genus 0 subgroups of the modular group."

The Monster CFT

Frenkel, Lepowsky, Meurman; Dixon, Ginsparg, Harvey

In fact, there is a precise vertex operator algebra and an associated chiral CFT which "explains" Monstrous Moonshine. It is a Z2 quotient of string theory on the Leech lattice (the unique dim 24 even self-dual lattice with no points of length-squared 2).

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As the given compactification to 2d is rather peculiar and has not played a role in further developments in string theory, this has seemed to be a beautiful result, to be sure, but a curiosity.

II. Enter K3 and BPS States

A. Index computations

One can rarely solve a physical theory to complete satisfaction. In supersymmetric theories, one can often compute indices - data independent of certain deformations of the theory.

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For instance, consider a SUSY QFT with 4 supercharges (such as 4d N=1 or 2d (2,2)).

Take a Hermitian combination of supercharges with

$$Q^2 = H$$

Divide the states in Hilbert space into "bosons" and "fermions":

$$e^{2\pi i J_z}|b\rangle = |b\rangle$$
$$e^{2\pi i J_z}|f\rangle = -|f\rangle$$

Its conventional to define:

$$(-1)^F = e^{2\pi i J_z}$$

Now, all states at non-zero energy are paired:

$$|b\rangle \leftrightarrow f \equiv \frac{1}{\sqrt{E}} Q|b\rangle$$

On the other hand, the E=0 states (which are the lowest possible energy states, due to SUSY) could a priori exist in arbitrary "unpaired" numbers:

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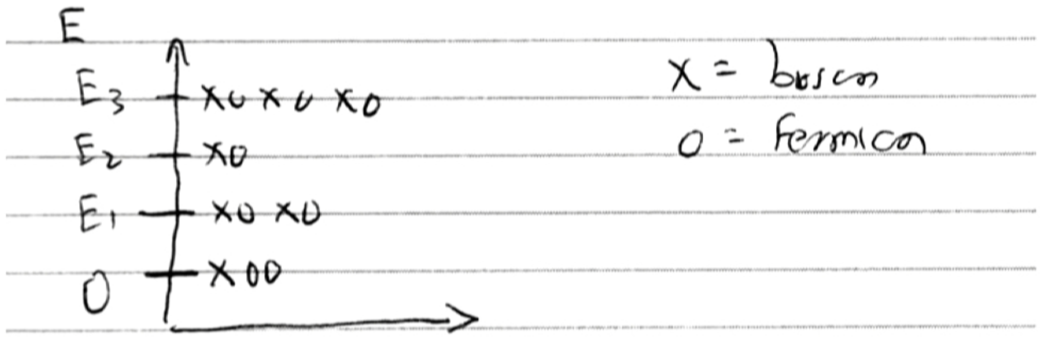

As you vary couplings, the finite energy states can move around at will. But notice that states can join or leave E=0 only in pairs! So:

$$\mathrm{Tr}(-1)^F = n_B - n_F$$

is an invariant, the "Witten index."

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The hero of this part of our story is a more refined invariant that can be defined in 2d theories with at least (0,2) supersymmetry: the elliptic genus.

Lets assume the left movers and right movers both enjoy N=2 SUSY. The N=2 algebra requires:

$$\{Q_R^+, Q_R^-\} = L_0 = H_R$$

The one-loop partition function is:

$$Z(q, \gamma_L, \gamma_R) = Tr_{\mathcal{H}}(-1)^F q^{H_L} \bar{q}^{H_R} e^{i\gamma_L J_L + i\gamma_R J_R}$$

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Now, the N=2 algebra has irreducible highest weight representations (think of Verma modules...). Let us say that these are denoted by

$$R_\alpha, \bar{R}_\alpha$$

for the left/right N=2. Then we can write:

$$Z = \sum_\alpha Tr_{R_\alpha} (-1)^{F_L} \; q^{H_L} \; e^{i\gamma_L J_L} \; Tr_{\bar{R}_\alpha} (-1)^{F_R} \bar{q}^{H_R} e^{i\gamma_R J_R}$$

However, except in very special CFTs, we cannot solve for the full partition function. (It contains e.g. the spectrum of the space-time theory).

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So we can consider sacrificing some information to get an index, which will be computable.

* If we set

$$\gamma_L = \gamma_R = 0$$

we basically get the Witten index. Not a lot of info.

* However, we can consider setting only say

$$\gamma_R = 0$$

Then the right-moving bit of the sum becomes:

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So we only get contributions from right-moving ground states! (The elliptic genus can be defined for theories with only right-moving N=2 SUSY, since its all we use here).

It reduces then to the function:

$$Z(q, \gamma_L, 0) = \sum_{\alpha}' \mathrm{Tr}_{R_\alpha} \ (-1)^{F_L} \ q^{H_L} \ e^{i\gamma_L J_L} \ (-1)^{F_R}$$

This is clearly a more refined invariant than the Witten index - in fact, it is a modular form.

The elliptic genus of the worldsheet theory counts space-time BPS states.

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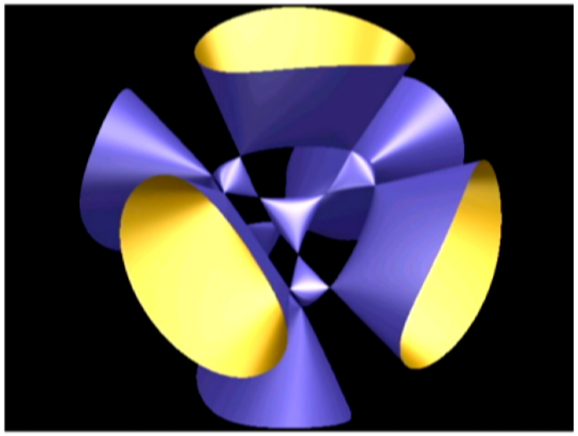
B. The elliptic genus of K3 and EOT

K3 is the simplest non-trivial Calabi-Yau space - i.e.,
it admits a family of Ricci flat Kahler metrics and preserves space-time supersymmetry.

Here is a picture:



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K3 compactifications of type II strings to 6d have played a central role in the theory. Many string dualities hinge on these compactifications. K3 also plays a role in AdS/CFT, in the canonical example of AdS3/CFT2 duality.

The elliptic genus was computed by Eguchi, Ooguri, Taormina and Yang in 1989. It is:

$$\phi_{K3}(\tau, z) = \mathrm{Tr}_{RR}\left(q^{L_0 - \frac{c}{24}}\ y^{J_L}\ (-1)^F\right)$$

$$q = e^{2\pi i \tau},\ y = e^{2\pi i z}$$

The result they found:

$$\phi_{K3}(\tau, z) = 8 \sum_{i=2}^{4} \frac{\theta_i(\tau, z)^2}{\theta_i(\tau, 0)^2}$$

The thetas are the classical Jacobi theta functions, whose q-expansions one can look up in Mathematica.

Representations of the N=4 worldsheet algebra are labelled by conformal spin h and an SU(2) quantum number l. Expanding the elliptic genus in terms of multiplets of the left-moving N=4 yields:

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$$\phi_{K3}(\tau,z) = 20\ ch^{\rm short}_{1/4,0}(\tau,z) - 2\ ch^{\rm short}_{1/4,1/2}(\tau,z) + \sum_{n=1}^{\infty} A_n\ ch^{\rm long}_{1/4+n,1/2}(\tau,z)$$

The values of the As are given by:

$$A_1 = 90 = \mathbf{45} + \overline{\mathbf{45}}$$

$$A_2 = 462 = \mathbf{231} + \overline{\mathbf{231}} \quad \longleftarrow \quad \text{dims of irreps of M24!}$$

$$A_3 = 1540 = \mathbf{770} + \overline{\mathbf{770}}$$

Eguchi, Ooguri, and Tachikawa noticed this in 2010, and conjectured a “Mathieu Moonshine” relating K3 compactification to the sporadic group M24.

C. So what is M24 and why is it here?

The Mathieu group M24 is a sporadic group of order:

$$|\mathrm{M24}| = 2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23 \;=\; 244,823,040$$

"Automorphism group of the unique doubly even self-dual binary code of length 24 with no words of length 4 (extended binary Golay code)."

Or in plain English:

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* Consider a sequence of 0s and 1s

* Any length 24 word in G has even overlap with all codewords in G iff it is in G

* The number of 1s in each element is divisible by 4 but not equal to 4

* The subgroup of S24 that preserves G is M24

Why does M24 appear in relation to K3?

No one really knows! There are some hints from geometry.

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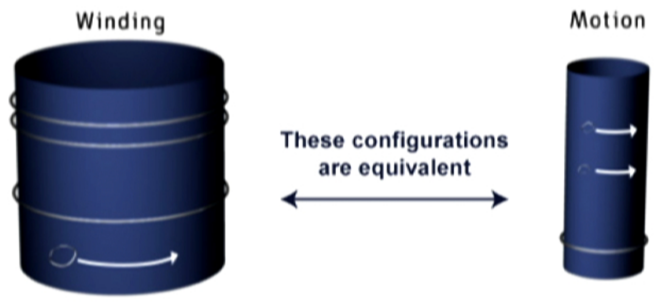
Theorem (Mukai, 1988): Any finite group of symplectic automorphisms of a K3 surface is isomorphic to a subgroup of the Mathieu group M23.

There is then a natural thought: could "stringy symmetries" (not visible in classical geometry) enhance the M23 of classical geometry to M24?



Theorem (Gaberdiel et al, 2011):

1. No K3 sigma model has M24 as a symmetry group.

2. Some K3 sigma models have symmetries that are not contained in M24.

3. All symmetry groups fit in Co1.

OK, so can we find more evidence of a real role of M24 in these theories?

Yes. The twining genera again give strong evidence that there really is an underlying module with M24 action.

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Cheng 2010 and many followers teach us:

Do the trace that yields the elliptic genus again, but with a g-insertion in the path integral:

$$\phi_g(\tau, z) = \mathrm{Tr}_{RR}\left(g\ q^{L_0 - c/24}\ e^{2\pi z J_L}(-1)^F\right)$$

If we identify g with a real symmetry of some concrete sigma-model, we can compute this directly, and see if it agrees with the "prediction" for the McKay-Thompson series of a suitable M24 conjugacy class.

Computable examples in orbifolds, Gepner models,...

Upshot: it works. So there is some M24 structure hiding here.

III. String duality, Calabi-Yau threefolds, and modular forms

The thing that interests me in the "new" Moonshine is that unlike the old one, it involves as a third leg Calabi-Yau geometries. I.e., the string vacua involved, have been more central to developments in string theory.

So far only K3 appeared. We remedy this now.

Cheng, Dong, Duncan, Harvey, SK, Wrase

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We move to the setting of 4d N=2 string vacua.

Such vacua arise in two simple avatars, related by duality:

SK, Vafa;
Ferrara, Harvey, Strominger, Vafa

Heterotic strings on K3xT2

$$\int_{K3} c_2(V_1) + c_2(V_2)$$
$$\equiv n_1 + n_2 = 24$$

dilaton S

Type IIA on Calabi-Yau threefolds

elliptic fibration over

$F_n, \; n_1 = 12 + n, n_2 = 12 - n$

size of base P^1

They can be lifted to 6d dualities in F-theory.

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There is a web of such vacua with various possible unHiggsed gauge groups and Coulomb branches, familiar also from studies of N=2 field theory. Dual descriptions of the heterotic gauge groups involve singular Calabi-Yau spaces.

Diagram 1: Higgs Tree

$$
\begin{array}{ccccccccc}
SU(3)_2 & \leftarrow & Sp(3) & \leftarrow & SU(6) & \leftarrow & SO(12) & & \\
 & & & & & & \downarrow & & \\
 & & & & \downarrow & & SO(11) & \leftarrow & E_7 \\
 & & & & & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & SU(5) & \leftarrow & SO(10) & \leftarrow & E_6 \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & SO(9) & \leftarrow & F_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & SO(8) & & \\
 & & & & & & \downarrow & & \\
\downarrow & & Sp(2) & \leftarrow & SU(4) & \leftarrow & SO(7) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \downarrow & & SU(3) & \leftarrow & G_2 & & \\
 & & & & \downarrow & & & & \\
SU(2)_2 & \leftarrow & SO(4) & \rightarrow & SU(2) & & & &
\end{array}
$$

Bershadsky, Intriligator, SK, Morrison, Sadov, Vafa

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A good quantity to look at is the "new supersymmetric index" of the heterotic (0,4) conformal field theory:

$$\mathcal{Z}_{new} = \frac{1}{\eta(q)^2} \mathrm{Tr}_R J_0 e^{i\pi J_0} q^{L_0 - c/24} \bar{q}^{\overline{L_0} - \bar{c}/24}$$

Cecotti, Fendley, Intriligator, Vafa

This quantity is particularly interesting because one can show that "morally":

$$\mathcal{Z}_{new} = -2i \left[\sum_{\text{BPS vectors}} q^{\Delta} \bar{q}^{\bar{\Delta}} - \sum_{\text{BPS hypers}} q^{\Delta} \bar{q}^{\bar{\Delta}} \right] .$$

Harvey, Moore

Beyond morals, it also shows up in the 1-loop threshold corrections to the space-time theory:

Antoniadis, Gava, Narain

$$\Delta_{\text{gauge/grav}} = \int \frac{d^2\tau}{\tau_2}\left[-\frac{i}{\eta(q)^2}\mathrm{Tr}_R\left(J_0 e^{i\pi J_0} q^{L_0-c/24}\bar{q}^{\overline{L_0}-\bar{c}/24} F_{\text{gauge/grav}}\right) - b_{\text{gauge/grav}}\right]$$

$$F_{\text{gauge}} = Q^2 - \frac{1}{8\pi\tau_2}, \qquad F_{\text{grav}} = E_2(q) - \frac{3}{\pi\tau_2} \equiv \hat{E}_2(q)$$

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

It follows that we expect

$$-\tau_2\frac{i}{\eta(q)^2}\mathrm{Tr}_R\left(J_0 e^{i\pi J_0} q^{L_0-c/24}\bar{q}^{\overline{L_0}-\bar{c}/24} F_{\text{gauge/grav}}\right)$$

to be modular invariant.

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to be modular invariant.

Focusing on the case of the gravitational threshold:

* Turn off all Wilson lines

* Then the index further factorizes as:

$$\mathcal{Z}_{new} = -2i \frac{1}{\eta(q)^2} \frac{\Theta_{\Gamma_{2,2}}}{\eta(q)^2} \mathcal{G}_{K3}$$

$$\begin{aligned} \Theta_{\Gamma_{2,2}}(q,\bar{q};T,U,\bar{T},\bar{U}) &= \sum_{p\in\Gamma_{2,2}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} = \sum_{p\in\Gamma_{2,2}} q^{\frac{1}{2}(p_L^2-p_R^2)} e^{-2\pi\tau_2 p_R^2} \\ &= \sum_{m_i,n_i\in\mathbb{Z}} e^{2\pi i\tau(m_1n_1+m_2n_2)-\frac{\pi\tau_2}{T_2U_2}|TUn_2+Tn_1-Um_1+m_2|^2} \end{aligned}$$

It follows that $\frac{\mathcal{G}_{K3}}{\eta(q)^4}$ has weight -2.

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Using the facts that :

See also:
Lopes Cardoso, Curio, Lust;
Stieberger;....

* $\mathcal{Z}_{new}$ has a 1/q pole

* This must come from $\frac{\mathcal{G}_{K3}}{\eta^4}$ because the torus sum has only positive powers

* Then $\eta^{20}\mathcal{G}_{K3}$ must be a holomorphic modular form of weight 10

we find that

$$\mathcal{G}_{K3} = \frac{E_4(q)E_6(q)}{\eta^{20}(q)}$$

independent of the choice of gauge bundles on K3.

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The normalization can be fixed by e.g. requiring cancellation of the 6d gravitational anomalies.

This universal structure contains a factor:

$$G(q) = -4E_6/\eta^{12}$$

$$G(q) = 24 g_{h=1/4,l=0}(q) + g_{h=1/4,l=1/2}(q) \sum_{n=0}^{\infty} A_n q^n$$

$$\begin{aligned} g_{h=1/4,l}(q) &= \left(\mathrm{ch}_s^{SO(12)} + \mathrm{ch}_c^{SO(12)}\right) \mathrm{ch}_{h=1/4,l}(q,-1) \\ &+ q^{1/4}\left(\mathrm{ch}_b^{SO(12)} + \mathrm{ch}_v^{SO(12)}\right) \mathrm{ch}_{h=1/4,l}(q,-q^{\frac{1}{2}}) \\ &- q^{1/4}\left(\mathrm{ch}_b^{SO(12)} - \mathrm{ch}_v^{SO(12)}\right) \mathrm{ch}_{h=1/4,l}(q,q^{\frac{1}{2}})\,. \end{aligned}$$

Importantly, as with the K3 elliptic genus,

$$A_n = -2, 90, 462, 1540, 4554, 11592, \ldots\,.$$

dims of irreps of M24!

How does this modular form manifest itself in the dual Calabi-Yau geometries?

* The 1-loop threshold corrections determine the 1-loop prepotential in the heterotic theory.

* In the dual Calabi-Yau compactification, one should then focus on the prepotential in the limit

$$\text{Perturbative heterotic limit} : t_{B_1} \to \infty,\ q_{B_1} \equiv e^{-2\pi t_{B_1}} \to 0\ .$$

The universal heterotic result showing the nice degeneracies should be (and is!) visible in curve counts in type IIA string theory.

Wednesday, October 2, 13

We make use of recent results in the mathematical physics literature. Expanding: Alim, Scheidegger; Klemm, Manschot, Woschke

$$F^{(g)}(q^A) = \sum f^{(g)}_{k,l}(q_F)\, q^k_{B_1} q^l_{B_2}$$

it has been shown that the coefficient "functions" are of the form

$$f^{(g)}_{k,l}(q_F) = \left(\frac{q_F^{\frac{1}{24}}}{\eta(q_F)}\right)^{2p(k,l)} P_{2g-2+p(k,l)}(E_2(q_F), E_4(q_F), E_6(q_F)),$$

$P_{2g-2+p(k,l)}$ is a quasi-modular form of weight $2g-2+p(k,l)$

$$p(k,l) = \frac{k}{2}\int_M c_2(M)\wedge J_2 + \frac{l}{2}\int_M c_2(M)\wedge J_1 \ .$$

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They furthermore satisfy interesting recursion relations. E.g. for n=0,1,2:

$$\frac{\partial f_{k,l}^{(g)}}{\partial E_2} = \frac{1}{24}\sum_{h=0}^{g}\sum_{s=0}^{k}\sum_{t=0}^{l}(ns(k-s)-s(l-t)-t(k-s))f_{s,t}^{(g-h)}f_{k-s,l-t}^{(h)} - \frac{1}{24}(2kl+(n-2)k-2l-nk^2)f_{k,l}^{(g-1)},$$

where the terms with k=0 can be compared to the perturbative heterotic string.

The simplest non-trivial term of interest satisfies:

$$\frac{\partial f_{0,1}^{(0)}}{\partial E_2} = 0\ .$$

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Using this together with the fact that

$$\int c_2(M) \wedge J_1 = 24$$

for these K3-fibered Calabi-Yau threefolds, we find:

$$f_{0,1}^{(0)}(q_F) = -\frac{1}{4\pi^3}\frac{q_F E_4(q_F) E_6(q_F)}{\eta(q_F)^{24}}$$

The same modular form appears!

It previously appeared in an integrand of a 1-loop amplitude. Here, it appears as a function of space-time moduli in the type II model.

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However, computing the one-loop heterotic prepotential in the same limit, using the result for the new supersymmetric index, one indeed finds: c.f. Kaplunovsky, Louis; de Wit, Kaplunovsky, Louis, Lust; Harvey, Moore

$$F^{het} = \ldots - \frac{1}{4\pi^3} q_F q_{B_2} \sum_{l \geq -1} c(l) q_F^l + \mathcal{O}(q_{B_1}, q_{B_2}^2) \ .$$

Here:

$$\frac{E_4 E_6}{\eta^{24}} = \sum_{m \geq -1} c(m) q^m = \frac{1}{q} - 240 + \cdots$$

Perfect agreement!

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Ending comments:

* One can check twining genera for certain simple M24 conjugacy classes in (0,4) LG models and orbifolds.

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* The building blocks of the new supersymmetric index show very intriguing structure related to larger sporadic groups (Conway??).

* Monster twining genera appear in GW invariants of some Calabi-Yau models of the sort I described...