Title: The Stokes groupoids

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Abstract: Ordinary differential equations become much less ordinary when they are allowed to have singularities.
Solving them naively in formal power series, one often obtains divergent series, just as in the perturbation series for physical observables in quantum field theory.

The asymptotic interpretation of this divergent series exhibits the famous Stokes phenomenon, an essential ingredient in any full description of the solutions to the system. I will explain a new viewpoint on singular ODE which illuminates the geometric meaning of the phenomena described above, and which can be applied to the problem of resummation of formal power series.

This viewpoint uses a very basic but underused tool in differential

geometry: Lie groupoids. A Lie groupoid is as natural and essential an object as a Lie group; I shall explain how to build examples of them and how to use them to solve singular differential equations.

The Stokes groupoids

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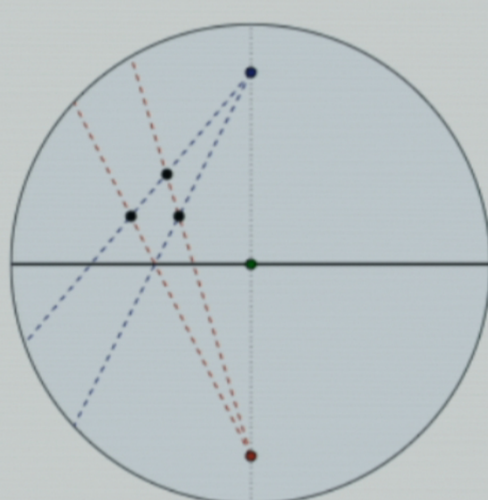
October 23, 2013



Based on arXiv:1305.7288 with Songhao Li and Brent Pym

Differential equations as connections

Any linear ODE, e.g.

$$\frac{d^2u}{dz^2} + \alpha\frac{du}{dz} + \beta u = 0,$$

can be viewed as a first order *system*: set $v = u'$ and then

$$\frac{d}{dz}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

This defines a flat connection

$$\nabla = d + \begin{pmatrix} 0 & -1 \\ \beta & \alpha \end{pmatrix} dz,$$

so that the system is

$$\nabla f = 0.$$

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Flat connections as representations

Recall that a connection ∇ on a vector bundle E gives, for each vector field $\mathcal{V}$, a differential operator on sections

$$\nabla_{\mathcal{V}} : \mathcal{E} \to \mathcal{E},$$

and that if the curvature vanishes,

$$\nabla_{[\mathcal{V}_1, \mathcal{V}_2]} = [\nabla_{\mathcal{V}_1}, \nabla_{\mathcal{V}_2}].$$

This means that we may view (E, ∇) as a *representation* of the Lie algebra of vector fields.

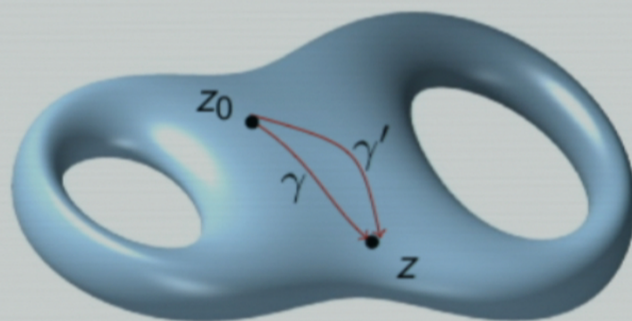
We say (E, ∇) is a representation of the "Lie algebroid" $\mathcal{T}_X$.

Solving ODE

Fix an initial point z_0. Solving the equation along a path γ from z_0 to z gives an invertible matrix

$$\psi(z)$$

mapping an initial condition at z_0 to the value of the solution at z.



This is called a *fundamental solution* and its columns form a basis of solutions.

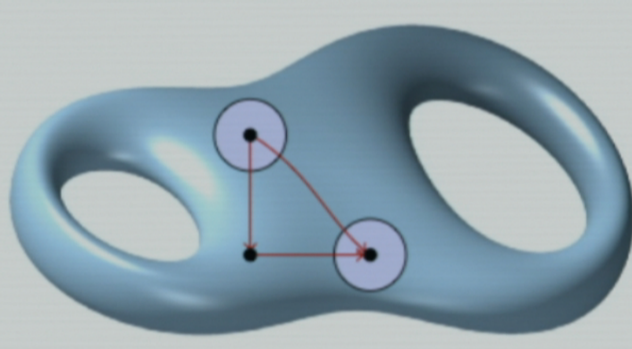
Also called *Parallel transport operator*, and depends only on the homotopy class of γ.

The fundamental groupoid

Define the **fundamental groupoid of** X:

$$\Pi_1(X) = \{\text{paths in } X\}/(\text{homotopies fixing endpoints})$$

- Product: concatenation of paths
- Identities: constant paths
- Inverses: reverse directions
- Manifold of dimension $2(\dim X)$



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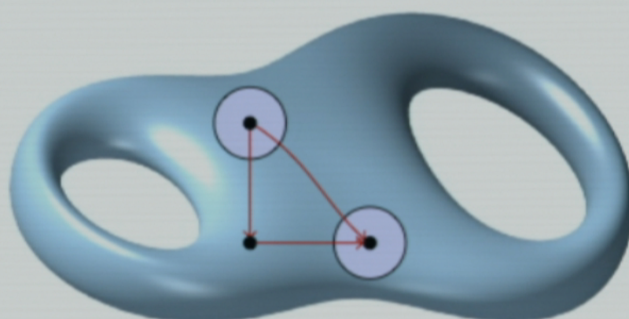

Parallel transport as a representation

The parallel transport gives a map

$$\Psi : \Pi_1(X) \to GL(n, \mathbb{C})$$

which is a **representation of** $\Pi_1(X)$:

$$\Psi(\gamma_1\gamma_2) = \Psi(\gamma_1)\Psi(\gamma_2)$$
$$\Psi(\gamma^{-1}) = \Psi(\gamma)^{-1}$$
$$\Psi(1_x) = 1$$

We call Ψ the **universal solution** of the system.

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Riemann–Hilbert correspondence

Correspondence between differential equations, i.e. flat connections

$$\nabla : \Omega^0_X(\mathcal{E}) \to \Omega^1_X(\mathcal{E}),$$

and their solutions, i.e. parallel transport operators

$$\Psi(\gamma) : \mathcal{E}_{\gamma(0)} \to \mathcal{E}_{\gamma(1)}.$$

$$\{\text{representations of } \mathcal{T}_X\} \underset{\text{Differentiation}}{\overset{\text{Integration}}{\rightleftarrows}} \{\text{representations of } \Pi_1(X)\}$$

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$$\boxed{\begin{tikzcd} \{\text{representations of } \mathcal{T}_{\mathsf{X}}\} \arrow[r, bend left, "\text{Integration}"] & \{\text{representations of } \Pi_1(\mathsf{X})\} \arrow[l, bend left, "\text{Differentiation}"] \end{tikzcd}}$$

Main problem: singular ODE

A singular ODE leads to a singular (meromorphic) connection

$$\nabla = d + A(z)z^{-k}dz.$$

For example, the Airy equation $f'' = xf$ has connection

$$\nabla = d + \begin{pmatrix} 0 & -1 \\ -x & 0 \end{pmatrix} dx,$$

and in the coordinate $z = x^{-1}$ near infinity,

$$\nabla = d + \begin{pmatrix} 0 & -1 \\ -z & -z^2 \end{pmatrix} z^{-3}dz.$$

Singular ODE

Singular ODE have singular solutions:

$$f' = z^{-2} f \qquad f = Ce^{-1/z}$$

Formal power series solutions often have zero radius of convergence:

$$\nabla = d + \begin{pmatrix} -1 & z \\ 0 & 0 \end{pmatrix} z^{-2} dz$$

has solutions given by columns in the matrix

$$\psi = \begin{pmatrix} e^{-1/z} & \hat{f} \\ 0 & 1 \end{pmatrix},$$

where formally $\hat{f} = \sum_{n=0}^{\infty} n! z^{n+1}$.

Resummation

Borel summation/multi-summation: recover actual solutions from divergent series:

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \left(\frac{1}{n! z} \int_0^{\infty} t^n e^{-t/z}\, dt \right)$$
$$= \frac{1}{z} \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_n t^n}{n!} \right) e^{-t/z}\, dt$$

The auxiliary series may now converge.

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Our point of view

The Stokes groupoids

Traditional solutions $\psi(z)$ may be/have

- multivalued
- not necessarily invertible
- essential singularities
- zero radius of convergence

because they are being written on the *wrong space*. The correct space must be 2-dimensional and must be adapted to the type of connections being considered: they are analogues of the fundamental groupoid.

The main idea

$\mathcal{T}_X(-D)$ as a Lie algebroid

View a meromorphic connection not as a representation of $\mathcal{T}_X$ with singularities on the divisor $D = k_1 \cdot p_1 + \cdots + k_n \cdot p_n$, but as a representation of the Lie algebroid

$$\mathcal{A} = \mathcal{T}_X(-D) = \text{sheaf of vector fields vanishing at D}$$
$$= \left\langle z^k \frac{\partial}{\partial z} \right\rangle$$

$\mathcal{A}$ defines a vector bundle over X which serves as a replacement for the tangent bundle $\mathcal{T}_X$.

tangent

$T(-D)$ = vector bundle / X
whose sections
are v. fields
vanishing on D.


tangent
T(−D) =
vector bundle/
whose sections
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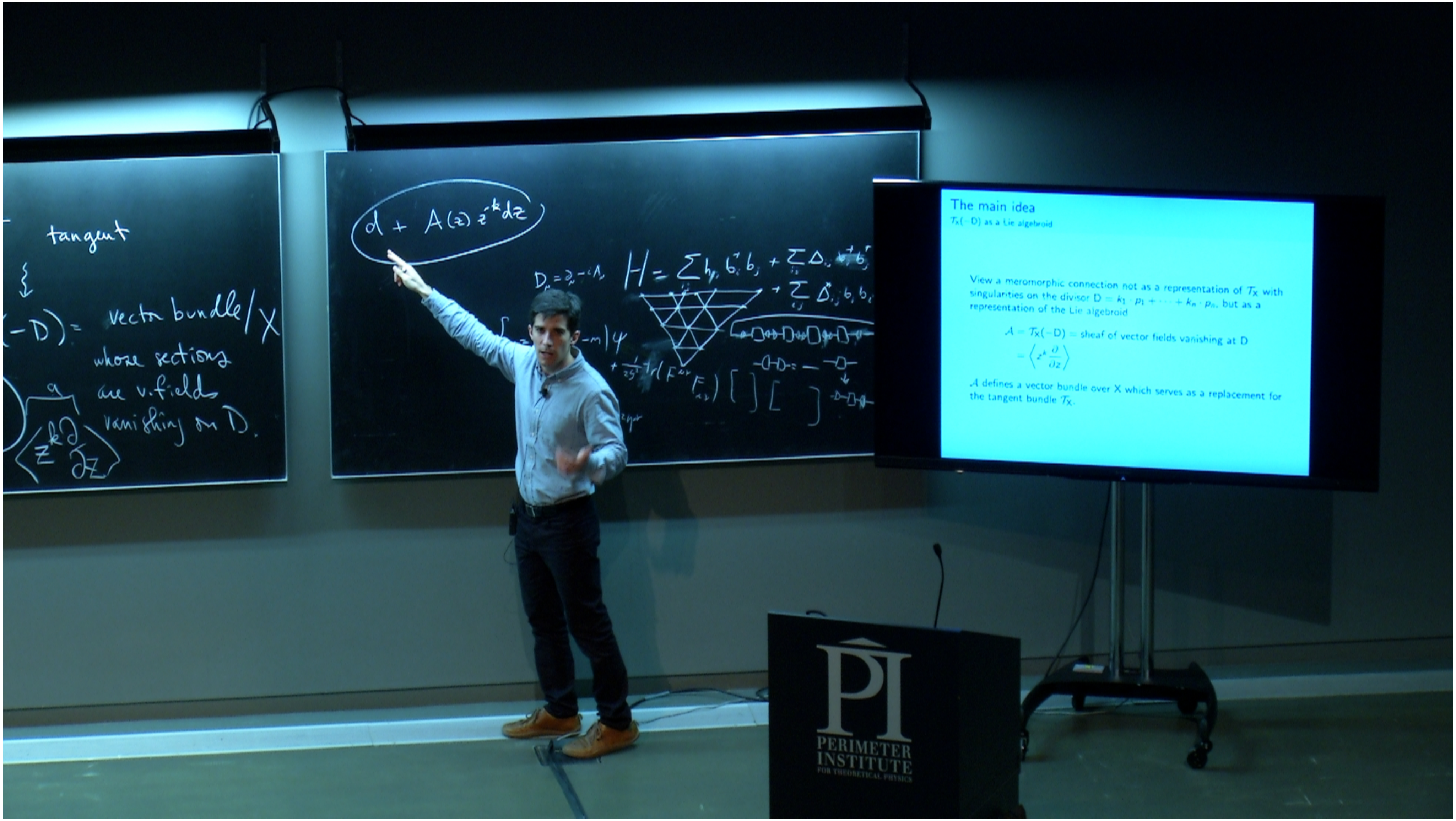
⟨z^k ∂/∂z⟩



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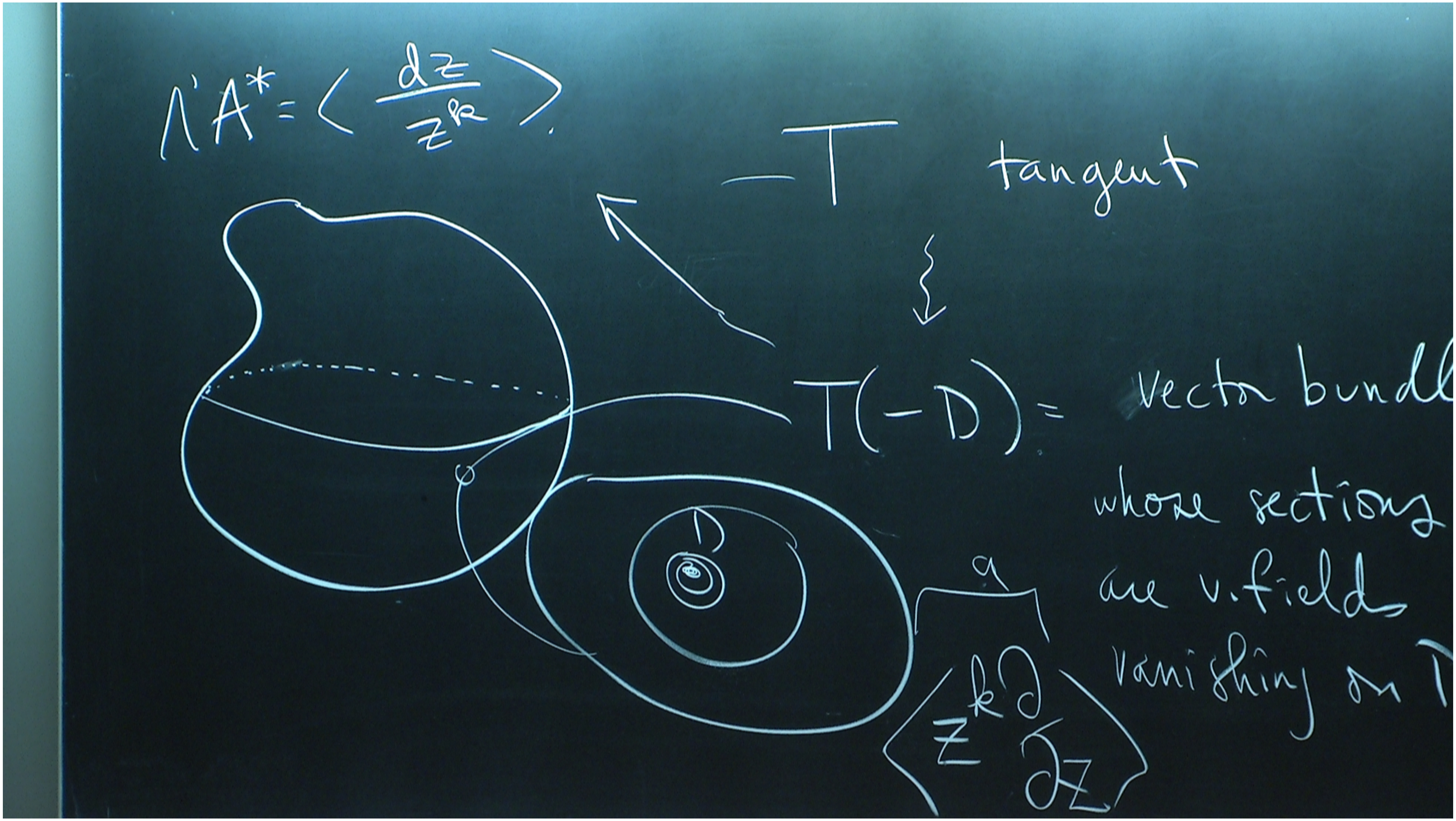
PERIMETER INSTITUTE FOR THEORETICAL PHYSICS



Λ¹A* = < dz / z^k >
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tangent
T(−D) = vector bundl
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are v.fields
vanishing on D
D
< z^k ∂/∂z >

Lie algebroids

Introduction

Definition: A Lie algebroid $(\mathcal{A}, [,], a)$ is a vector bundle $\mathcal{A}$ with a Lie bracket on its sections and a bracket-preserving bundle map

$$a : \mathcal{A} \to \mathcal{T}_{\mathsf{X}},$$

such that $[u, fv] = f[u, v] + (L_{a(u)}f)v$.

A Lie algebroid behaves as $\mathcal{T}_{\mathsf{X}}$ does: de Rham complex $(\wedge^{\bullet}\mathcal{A}^{*}, d_{\mathcal{A}})$.

Examples:

- $\mathcal{A} = \mathcal{T}_{\mathsf{X}}(-\mathrm{D})$, with de Rham complex $d : \mathcal{O}_{\mathsf{X}} \to \Omega^1_{\mathsf{X}}(\mathrm{D})$.
- Fix $\lambda \in \mathbb{C}$ and let $\mathcal{A} = \mathcal{T}_{\mathsf{X}}$, with $[,] = \lambda[,]_{Lie}$ and $a = \lambda \cdot \mathbf{id}$, with de Rham complex $(\Omega^{\bullet}_{\mathsf{X}}, \lambda d)$
- Foliations $\mathcal{F} \subset \mathcal{T}_{\mathsf{X}}$, with foliated de Rham complex.

Lie algebroids

Representations

Definition: A representation of the Lie algebroid $\mathcal{A}$ is a vector bundle $\mathcal{E}$ with a flat $\mathcal{A}$-connection

$$\nabla : \mathcal{E} \to \mathcal{A}^* \otimes \mathcal{E}, \qquad \nabla(fs) = f\nabla s + (d_{\mathcal{A}} f)s.$$

Examples:

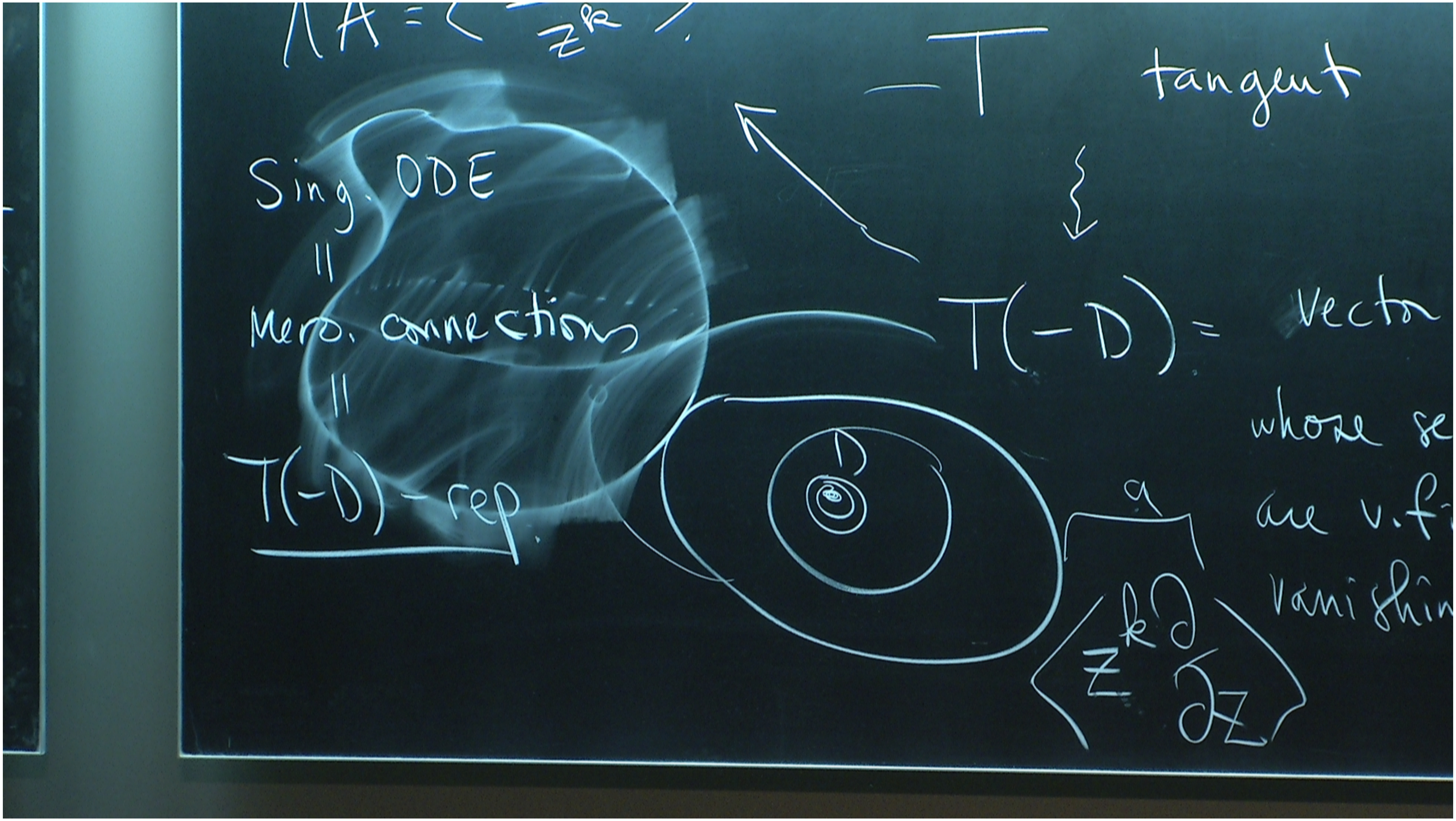
– For $\mathcal{A} = \mathcal{T}_X(-D) = \langle z^k \partial_z \rangle$, we have $\mathcal{A}^* = \langle z^{-k} dz \rangle$, and so

$$\begin{aligned} \nabla &= d + A(z)(z^{-k} dz) \\ &= (z^k \partial_z + A(z))\, z^{-k} dz, \end{aligned}$$

i.e. a meromorphic connection.

– For $\mathcal{A} = (\mathcal{T}_X, \lambda[,], \lambda \cdot \mathbf{id})$, we get a λ-connection:

$$\nabla(fs) = f\nabla s + \lambda df \otimes s.$$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Lie Groupoids

Introduction

Definition: A Lie groupoid G over X is a manifold of arrows g between points of X.

- Each arrow g has source $s(g) \in \mathsf{X}$ and target $t(g) \in \mathsf{X}$. The maps $s, t : \mathsf{G} \to \mathsf{X}$ are surjective submersions.
- There is an associative composition of arrows

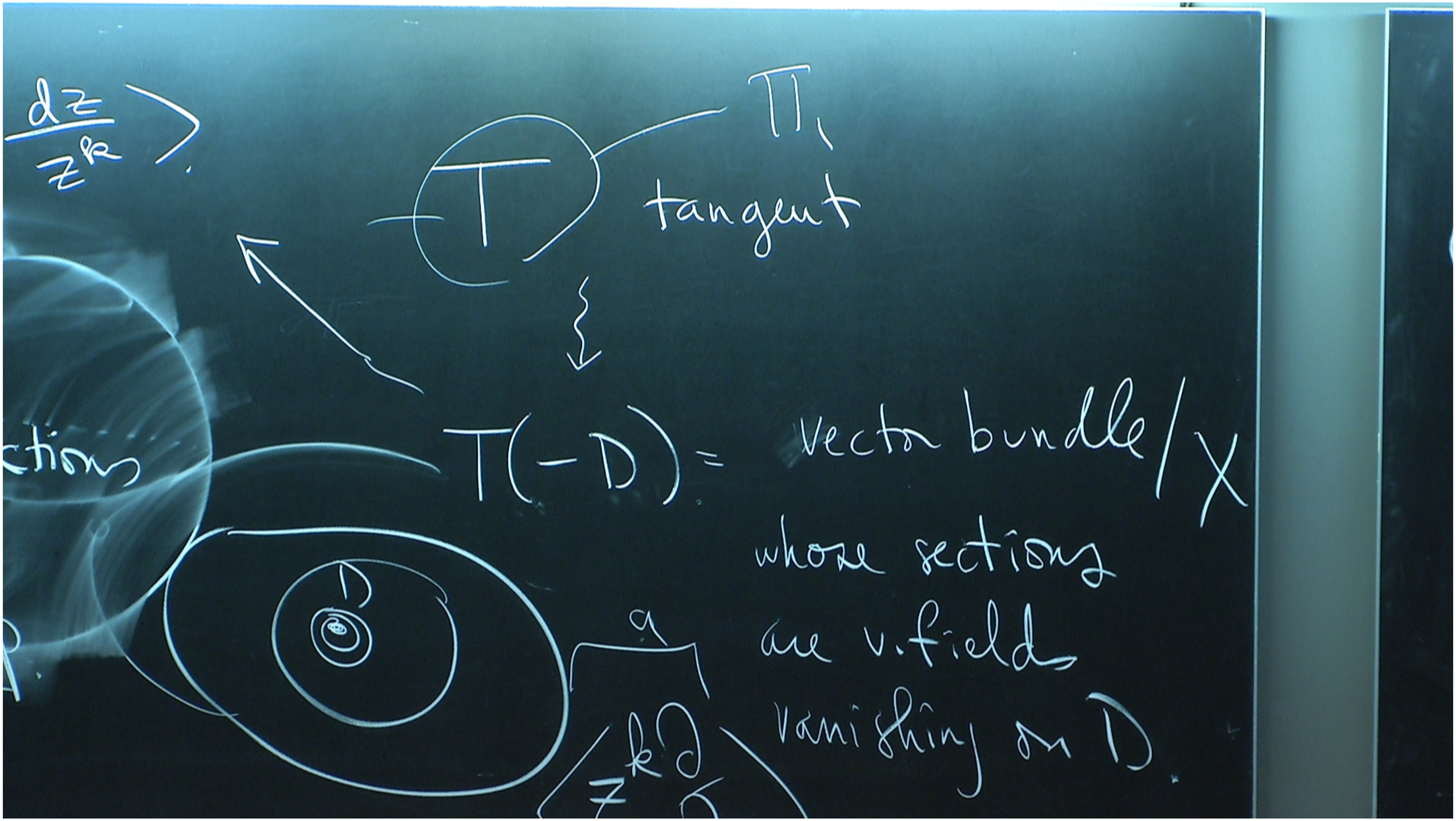
$$m : \mathsf{G}_s \times_t \mathsf{G} \to \mathsf{G}.$$

- Each $x \in \mathsf{X}$ has an identity $\mathrm{id}(x) \in \mathsf{G}$; this gives an embedding $\mathsf{X} \subset \mathsf{G}$.
- Each arrow has an inverse.

Examples:

- The fundamental groupoid $\Pi_1(\mathsf{X})$.
- The pair groupoid $\mathsf{X} \times \mathsf{X}$, in which

$$(x, y) \cdot (y, z) = (x, z).$$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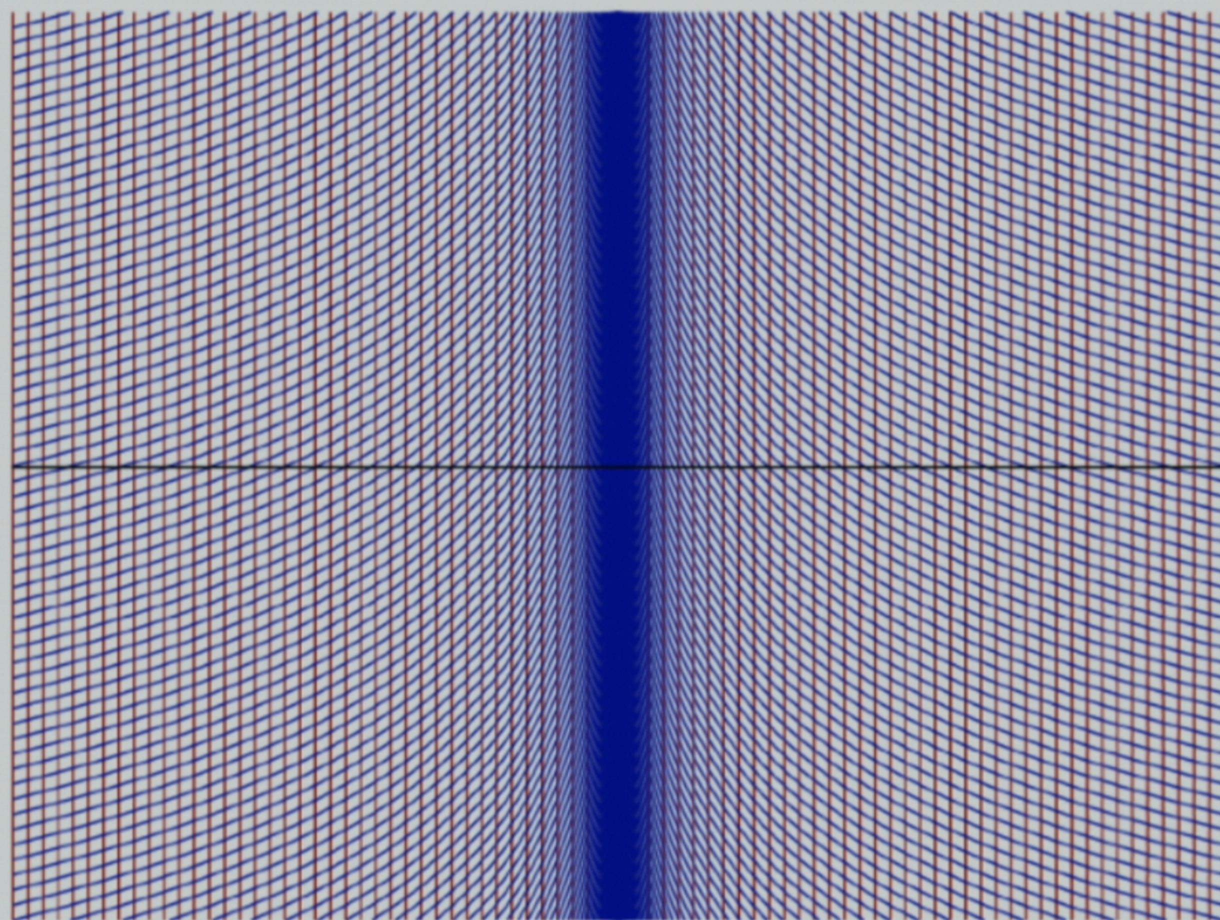


Action groupoid for $\mathbb{C}$ action on $\mathbb{C}$ given by $u \cdot z = e^u z$.
Vertical lines are s-fibres and blue curves are t-fibres.

Lie Groupoids

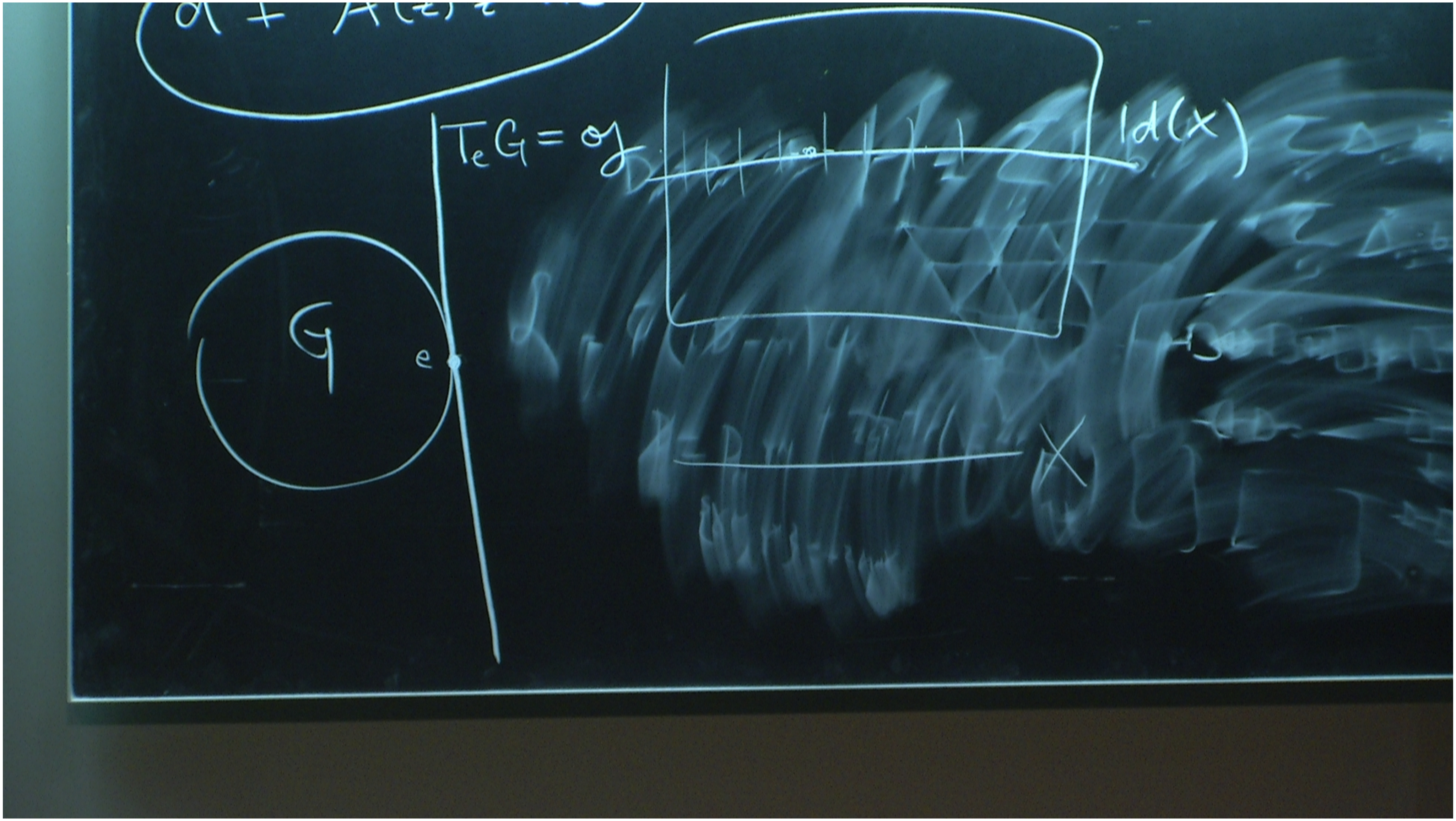
Relation to Lie algebroids

The Lie algebroid $\mathcal{A}$ of a Lie groupoid G over X is defined by:

$$\mathcal{A} = N(\mathrm{id}(\mathsf{X})) \cong \ker s_*|_{\mathrm{id}(\mathsf{X})}.$$

- Sections of $\mathcal{A}$ have unique extensions to right-invariant vector fields tangent to s-foliation $\mathcal{F}$. Thus $\mathcal{A}$ inherits a Lie bracket.
- t-projection defines the anchor a:

$$t_* : \mathcal{A} \to \mathcal{T}_{\mathsf{X}}.$$


$T_eG = \mathfrak{g}$
$Id(x)$
G
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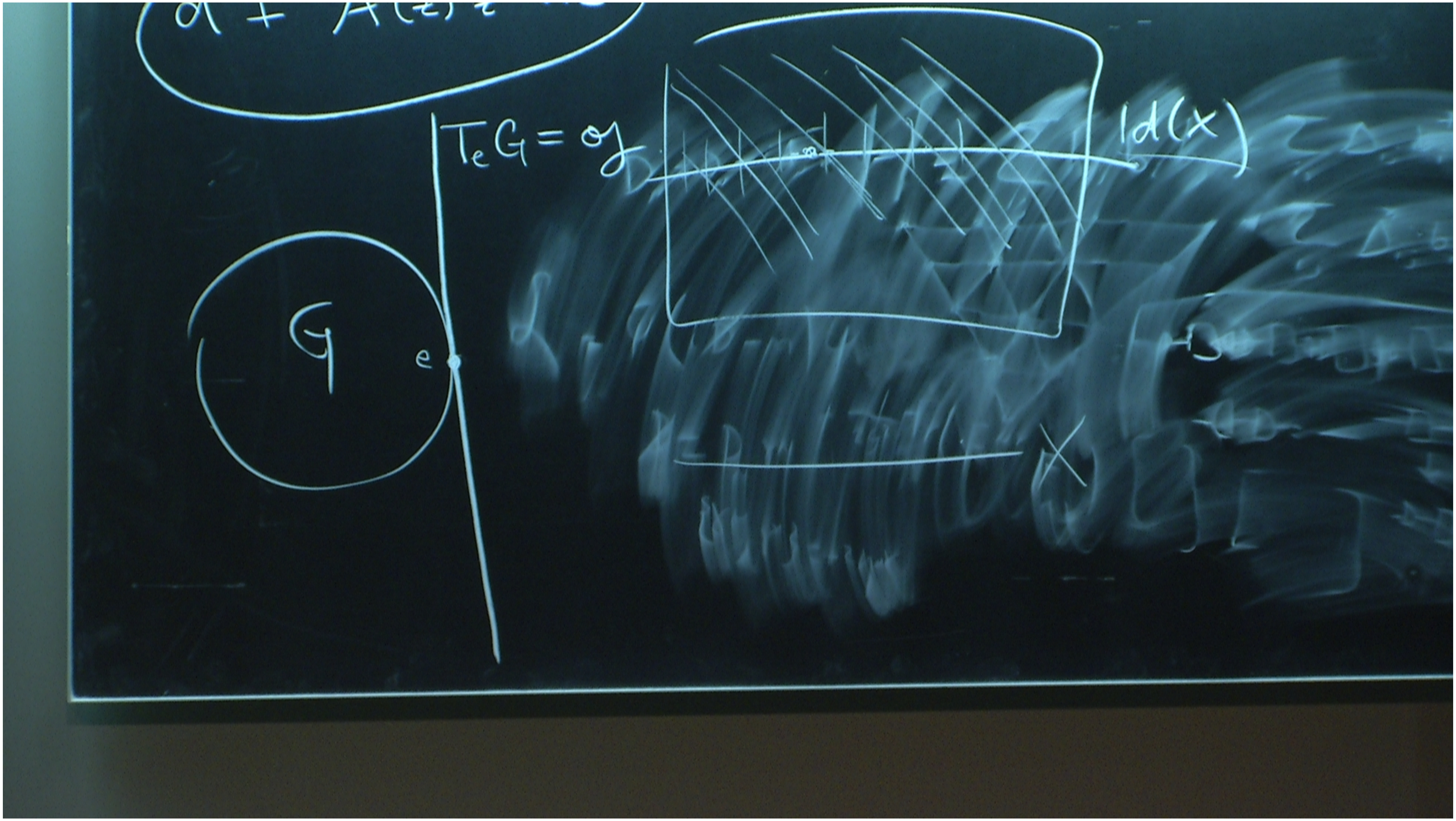
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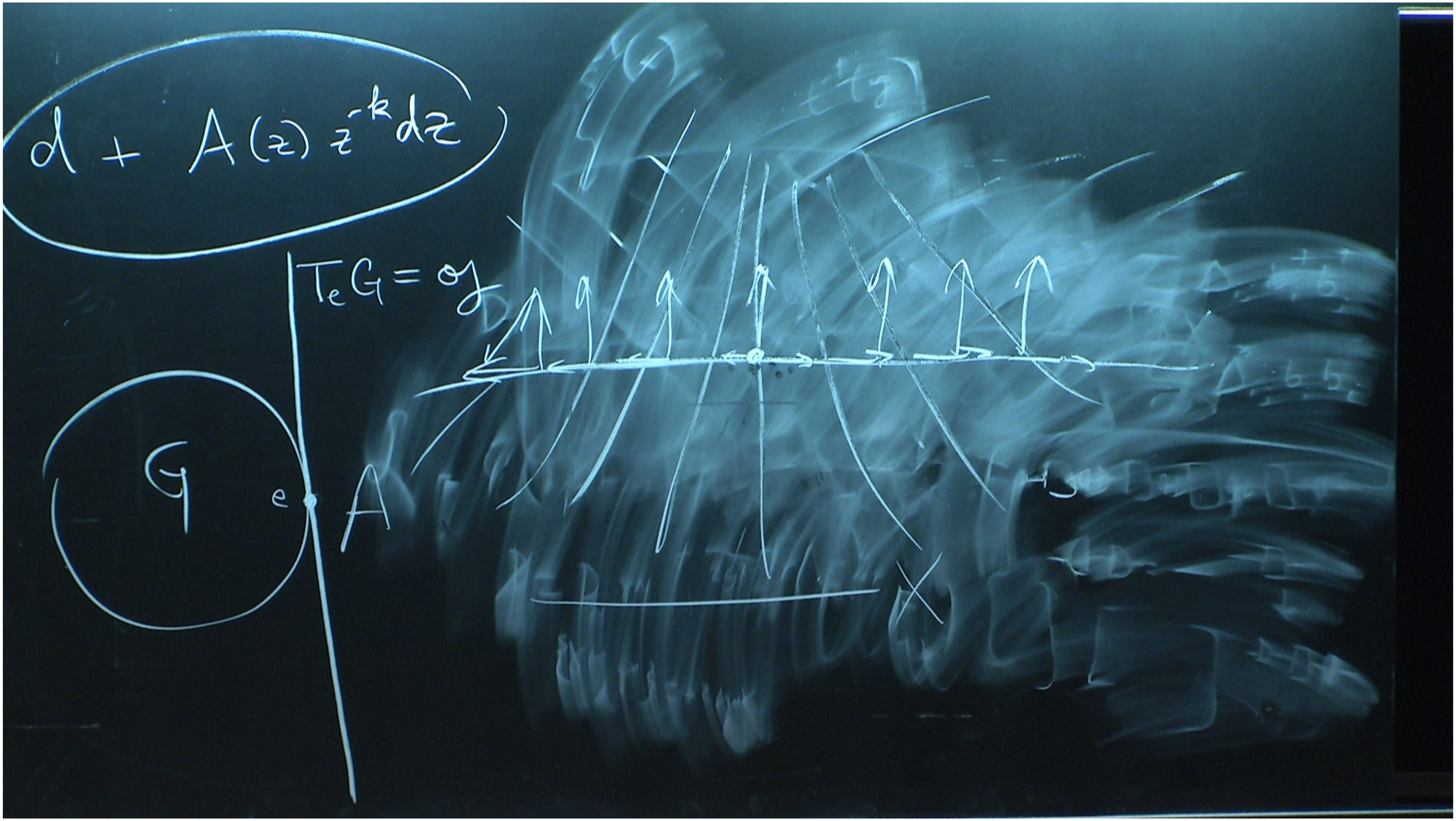


$d + A(z)\, z^{-k} dz$
$T_e G = \mathfrak{g}$
G
e
A

$\mathrm{Rep}(\text{Lie algebroid } A)$ (E, ∇)

$\Big\updownarrow$ $\leftarrow$ usual exist & uniqueness thm for smooth ODE

$\mathrm{Rep}(\text{Lie groupoid})$ $\Psi : \mathcal{G} \xrightarrow[\text{hol.}]{C^\infty} GL(n, \mathbb{C}$

Lie Groupoids

Lie III Theorem

In this way, we obtain an equivalence

$$\mathbf{Rep}(\mathcal{A}) \leftrightarrow \mathbf{Rep}(\mathsf{G}),$$

using nothing more than the usual existence and uniqueness theorem for nonsingular ODEs.

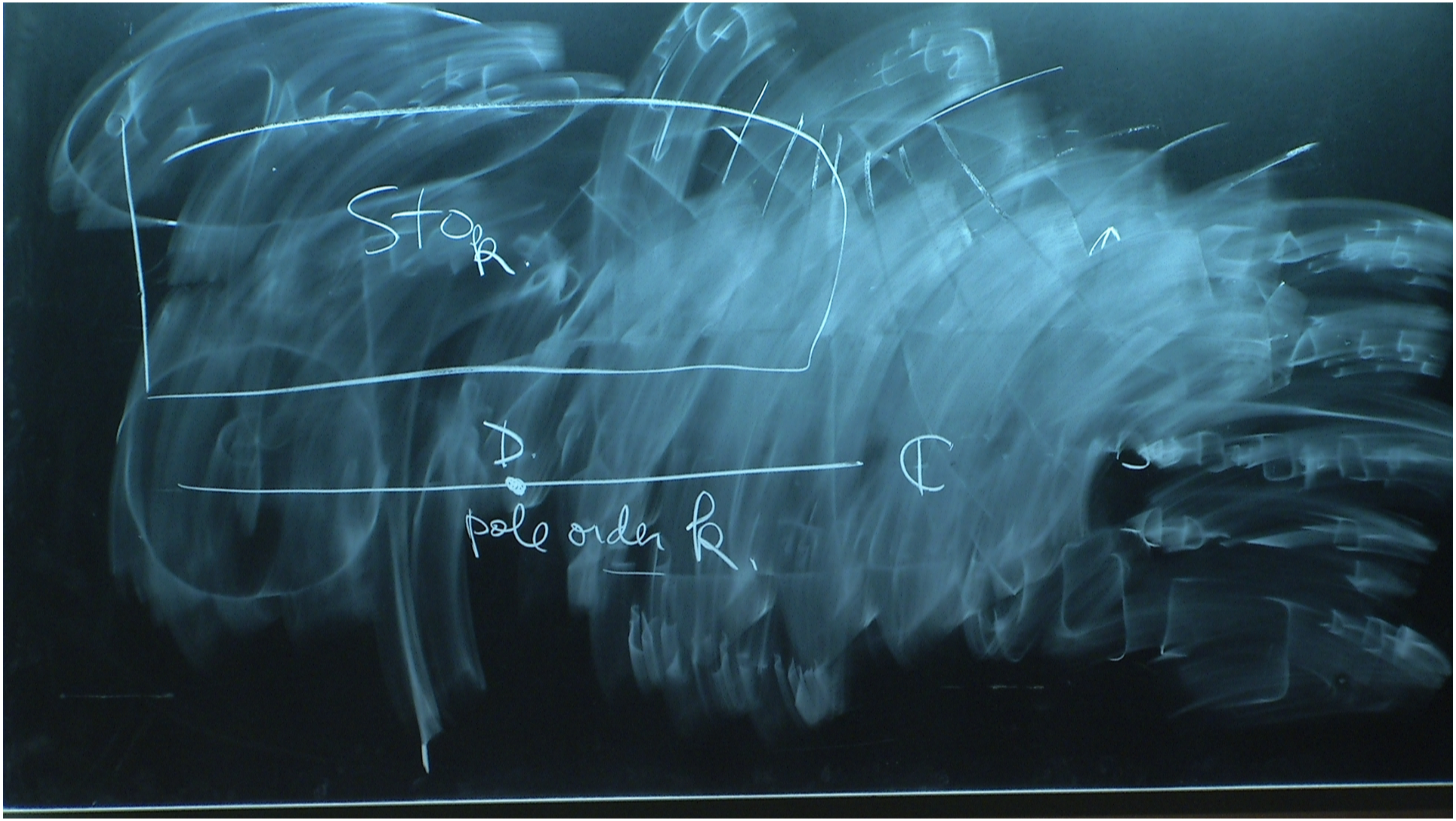
Concrete Examples

Stokes groupoids

Example: $\mathrm{Sto}_k = \Pi_1(\mathbb{C}, k \cdot 0) = \mathbb{C} \times \mathbb{C}$ with

$$\begin{aligned} s(z, u) &= z \\ t(z, u) &= \exp(uz^{k-1})z \\ (z_2, u_2) \cdot (z_1, u_1) &= (z_1, u_2 \exp((k-1)u_1 z_1^{k-1}) + u_1). \end{aligned}$$

For $k = 1$, coincides with action groupoid, but for $k > 1$ not an action groupoid.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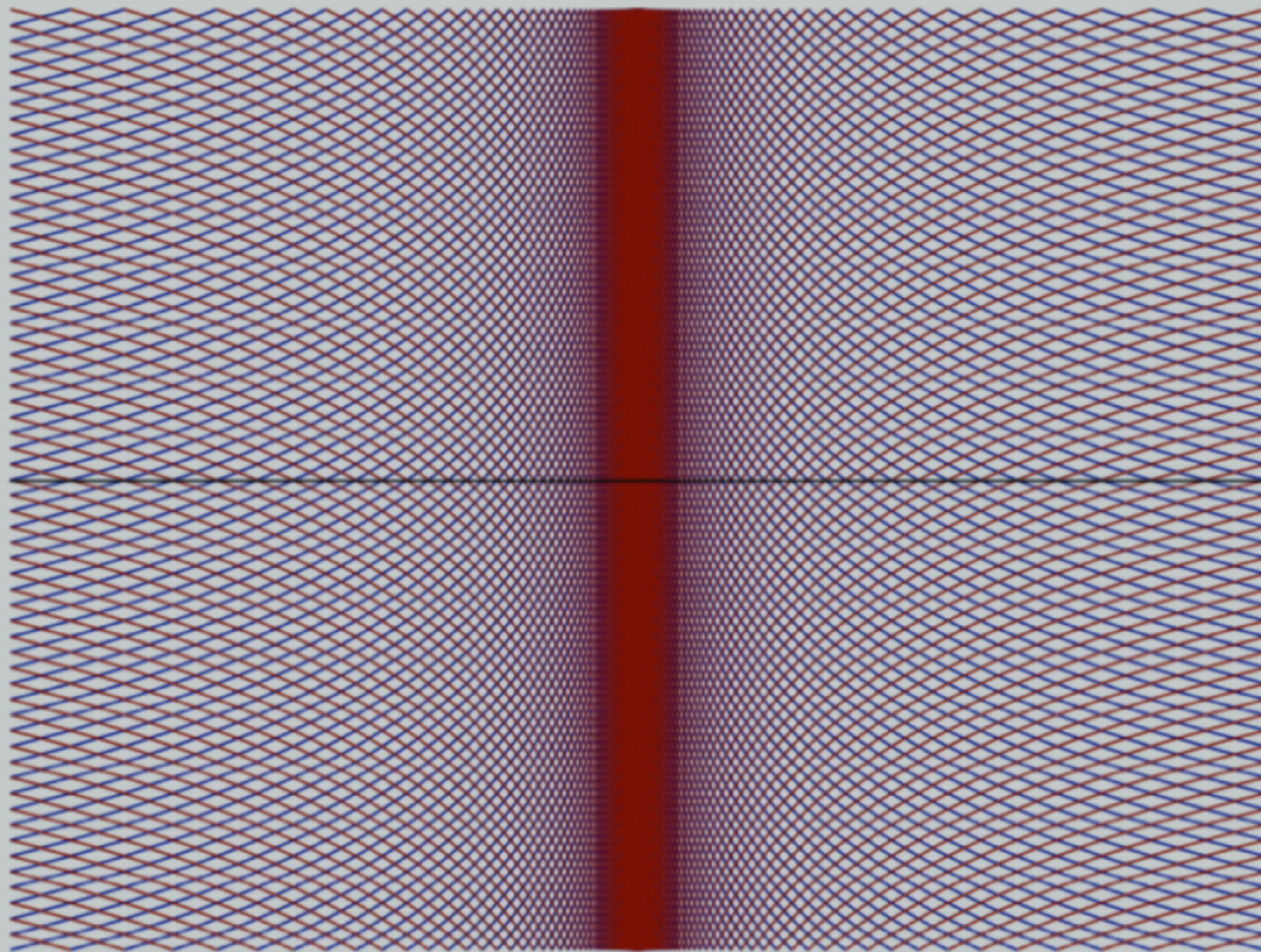
pole order k.

Concrete Examples

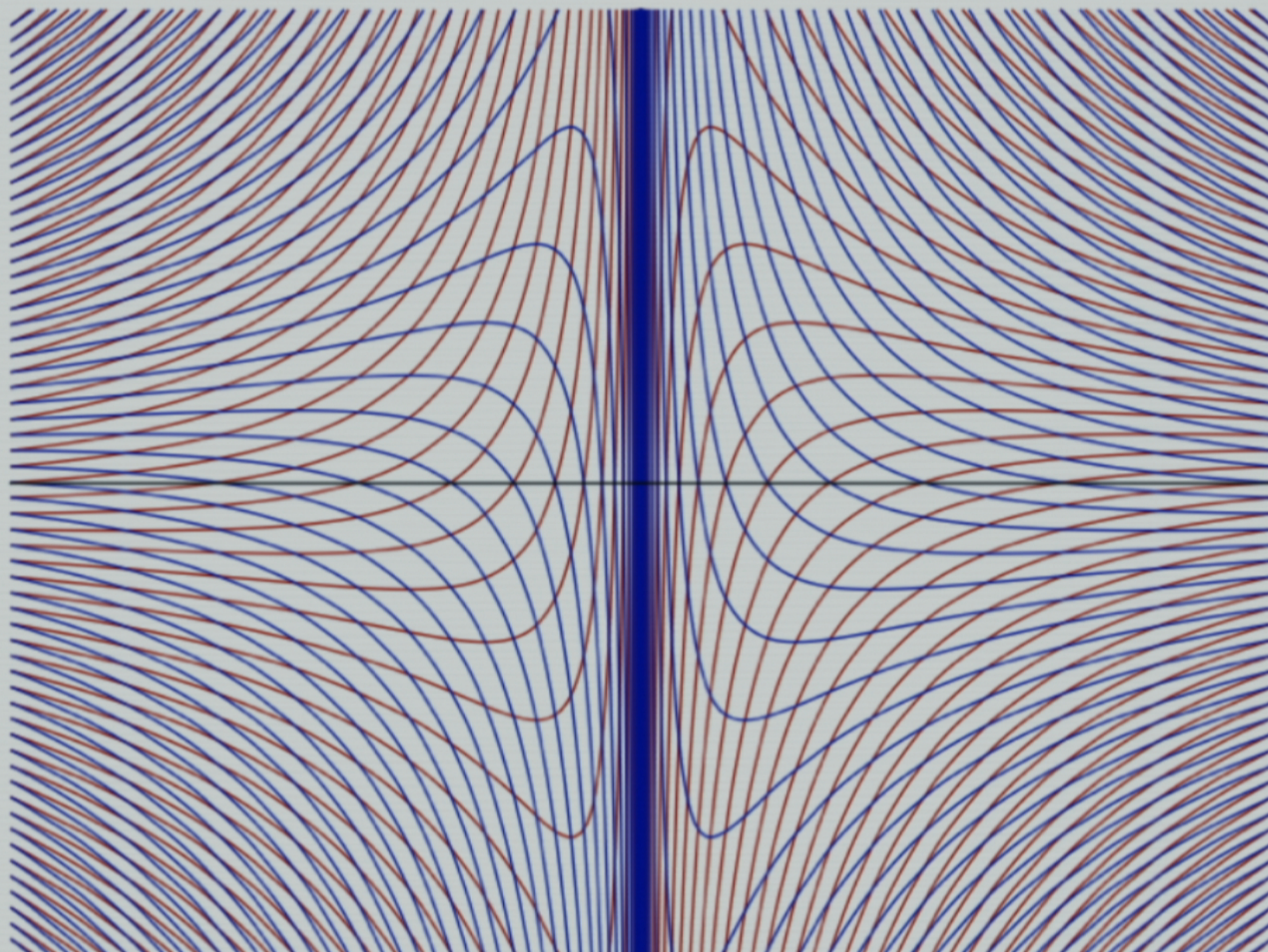
Stokes groupoids

We can write Sto_k more symmetrically:

$$s(z,u) = \exp(-\tfrac{1}{2}uz^{k-1})z$$
$$t(z,u) = \exp(\ \ \tfrac{1}{2}uz^{k-1})z$$



Sto_1 groupoid for 1st order poles on $\mathbb{C}$



Sto_2 groupoid for 2nd order poles on $\mathbb{C}$

Applications

Universal domain of definition for solutions to ODE

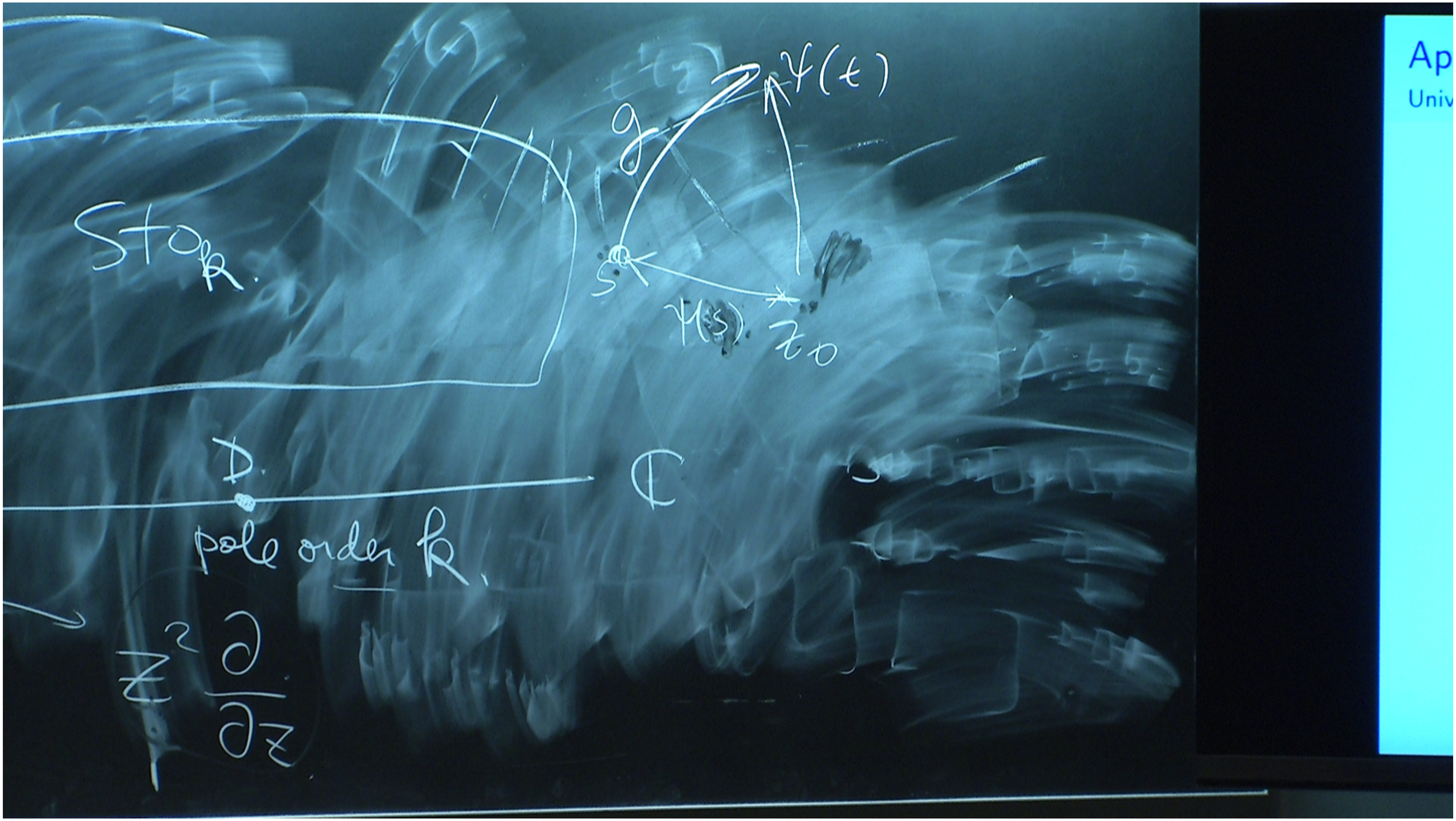
Theorem: If ψ is a fundamental solution of $\nabla\psi = 0$, i.e. a flat basis of solutions, and if ∇ is meromorphic with poles bounded by D, then ψ may be

- multivalued
- non-invertible
- singular,

however

$$\boxed{\Psi = t^*\psi \circ s^*\psi^{-1}}$$

is single-valued, smooth and invertible on the Stokes groupoid.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Applications

Summation of divergent series

Recall that the connection

$$\nabla = d + \begin{pmatrix} -1 & z \\ 0 & 0 \end{pmatrix} z^{-2} dz$$

has fundamental solution

$$\psi = \begin{pmatrix} e^{-1/z} & \widehat{f} \\ 0 & 1 \end{pmatrix},$$

$$\text{where formally } \widehat{f} = \sum_{n=0}^{\infty} n! z^{n+1}.$$

∇ is a representation of $\mathcal{T}_{\mathbb{C}}(-2 \cdot 0)$, and so the corresponding groupoid representation Ψ is defined on Sto_2. For convenience we use coordinates (z, μ) on the groupoid such that

$$s(z, \mu) = z, \quad t(z, \mu) = z(1 - z\mu)^{-1}.$$

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Applications

Summation of divergent series

$$\Psi = t^*\psi \circ s^*\psi^{-1} = t^* \begin{pmatrix} e^{-1/z} & \widehat{f} \\ & 1 \end{pmatrix} s^* \begin{pmatrix} e^{-1/z} & \widehat{f} \\ & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} e^{-(1-z\mu)/z} & t^*\widehat{f} \\ & 1 \end{pmatrix} \begin{pmatrix} e^{1/z} & -s^*\widehat{f} \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\mu} & t^*\widehat{f} - e^{\mu}s^*\widehat{f} \\ & 1 \end{pmatrix}$$

But we know a priori this converges on the groupoid:

Results

With this point of view, we obtain:

- An explicit family of groupoids depending on a curve with effective divisor (X, D), which serve as the *universal domains of definition* for systems with singularities bounded by D.
- A global and categorified treatment of the isomonodromy flat connection on the relative moduli space of meromorphic connections.
- A description of the Stokes data as a Čech representation of a groupoid 1-cocycle.
- A new approach to resummation of formal solutions to ODEs with irregular singularities.

"Divergent series are, in general, something terrible and it is a shame to base any proof on them. We can prove anything by using them and they have caused so much misery and created so many paradoxes . . . we hardly find, in mathematics, any infinite series whose sum may be determined in a rigorous fashion, which means the most essential part of mathematics has no foundation. For the most part, it is true that the results are correct, which is very strange. I am working to find out why, a very interesting problem."

–N. Abel