

Title: General Relativity for Cosmology - Lecture 4

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Abstract:

GR for Cosmology, Achim Kempf, Fall 2013, Lecture 4

Note Title

Differential forms (also called "exterior differential forms")

Preparation:

an arbitrary point

Consider the cotangent space $T_p(M)^*$ at p :

□ Each $\omega \in T_p(M)^*$ is a lin. map:

$$\omega: T_p(M) \rightarrow \mathbb{R}$$

$$\omega: \xi \rightarrow \omega(\xi)$$

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Each such ω is a covariant tensor, of rank $(0, 1)$

More generally, consider the covariant tensors of rank $(0, r)$:

Recall: $T_p(\mathcal{M})_r := T_p(\mathcal{M})^* \otimes \dots \otimes T_p(\mathcal{M})^*$
 r factors

Each $v \in T_p(\mathcal{M})_r$ is a multi-linear map:

$$v: T_p(\mathcal{M})^* \rightarrow \mathbb{R}$$

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r factors

□ Each $v \in T_p(\mathcal{M})_r$ is a multi-linear map:

$$v: T_p(\mathcal{M})^* \rightarrow \mathbb{R}$$

□ In particular, if $\xi_1, \dots, \xi_r \in T_p(\mathcal{M})$ then:

$$v: \xi_1 \times \dots \times \xi_r \rightarrow v(\xi_1, \dots, \xi_r)$$

□ For $r=0$ we define the set of differential 0-forms at $p \in M$ as:

$$\Lambda_0(p) := \mathbb{R} \quad \left(\begin{array}{l} \text{for 0 forms on the entire} \\ \text{manifold we will have } \Lambda_0 := \mathcal{F}(M) \end{array} \right)$$

□ For $r=1$ we define the set of

differential 1-forms

(or "Pfaffian forms")

at $p \in M$ through:

$$\Lambda_1(p) := T_p^*(M),$$

Definition: If $r > 1$ and $v \in T_p(M)_r$, then we define the "anti-symmetric part of v " as the image $\tilde{v} = A(v)$ of v under the linear antisymmetrization map A :

$$\tilde{v}(\xi_1, \dots, \xi_r) = A(v)(\xi_1, \dots, \xi_r) := \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) v(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)})$$

the sign (± 1) of the permutation σ

↖ group of all $r!$ permutations of $(1, 2, \dots, r)$

Why consider these?

They will be key for integration! Only antisym. cov. tensors transform under chart change so as

Concretely:

□ Consider $v := df \otimes dg$, for $f, g \in \mathcal{F}_p(\mathcal{M})$

□ Then $v(\xi_1, \xi_2) = df(\xi_1) dg(\xi_2)$ (which is $= \xi_1(f) \xi_2(g)$)

□ Apply A :

$$\tilde{v}(\xi_1, \xi_2) = Av(\xi_1, \xi_2) = \frac{1}{2} (df(\xi_1) dg(\xi_2) - df(\xi_2) dg(\xi_1))$$

□ \Rightarrow We can also write:

$$A(df \otimes dg) = \frac{1}{2} (df \otimes dg - dg \otimes df)$$

Proposition: A is a projector, i.e., it obeys

$$A \circ A = A$$

(i.e. it is a self-adjoint
lin. map with the only
eigenvalues being 0 and 1)

Check in above example:

$$\begin{aligned} A \circ A(d_f \otimes d_g) &= A\left(\frac{1}{2}d_f \otimes d_g - \frac{1}{2}d_g \otimes d_f\right) \\ &= \frac{1}{2}\left(\frac{1}{2}d_f \otimes d_g - \frac{1}{2}d_g \otimes d_f\right) - \frac{1}{2}\left(\frac{1}{2}d_g \otimes d_f - \frac{1}{2}d_f \otimes d_g\right) \\ &= \frac{1}{2}(d_f \otimes d_g - d_g \otimes d_f) \\ &= A(d_f \otimes d_g) \end{aligned}$$

Definition:

For $r > 1$ we define the space of differential r -forms (or 'exterior' r -forms) $\Lambda_r(p)$ at $p \in M$

as the subspace of totally anti-symmetric tensors

of rank $(0, r)$:

a vector space \rightarrow $\Lambda_r(p) := A T_p(M)_r$

a projector on a vector space \rightarrow

a vector space \rightarrow

\leadsto So if $v \in \Lambda_r(p)$ then $\tilde{v} = A(v) = v$

The wedge product :

Def: If $\omega \in \Lambda_s(p)$, $\nu \in \Lambda_r(p)$, and $r, s \neq 0$ then the wedge product \wedge yields a new differential form:

$$\wedge : \Lambda_r(p) \times \Lambda_s(p) \rightarrow \Lambda_{r+s}(p)$$

$$\wedge : (\omega, \nu) \rightarrow \omega \wedge \nu = \underbrace{\frac{(s+r)!}{s!r!}}_{\text{a normalization factor}} A(\omega \otimes \nu)$$

Def: For $c \in \Lambda_0$, $\omega \in \Lambda_s$ we have $c \wedge \omega = c\omega$

Example: For dx^i, dx^j we obtain:

Note: $dx^i \wedge dx^i = 0 \forall i$
 \Uparrow

Def: For $c \in \Lambda_0, w \in \Lambda_s$ we have $c \wedge w = cw$

Note: $dx^i \wedge dx^i = 0 \forall i$

Example: For dx^i, dx^j we obtain:

\nearrow

$$dx^i \wedge dx^j = (dx^i \otimes dx^j - dx^j \otimes dx^i)$$

Properties of \wedge :

□ bi-linear:

$$(w + v) \wedge \eta = w \wedge \eta + v \wedge \eta$$

$$(aw) \wedge v = w \wedge (av) = a(w \wedge v) \quad \text{if } a \in \mathbb{R}$$

□ associative:

Properties of \wedge :

□ bi-linear:

$$(\omega + \nu) \wedge \eta = \omega \wedge \eta + \nu \wedge \eta$$

$$(a\omega) \wedge \nu = \omega \wedge (a\nu) = a(\omega \wedge \nu) \text{ if } a \in \mathbb{R}$$

□ associative:

$$(\omega \wedge \nu) \wedge \eta = \omega \wedge (\nu \wedge \eta)$$

□ "graded" commutative:

$$\omega \wedge \nu = (-1)^{rs} \nu \wedge \omega \text{ if } \omega \in \Lambda_r, \nu \in \Lambda_s$$

(E.g. for $dx^i \in \Lambda_1(p)$:
 $dx^i \wedge dx^i = -dx^i \wedge dx^i$)



□ "graded" commutative:

(E.g. for $dx^i \in \Lambda_1(p)$:
 $dx^i \wedge dx^j = -dx^j \wedge dx^i$
 since $r=s=1$.)

$$\omega \wedge v = (-1)^{rs} v \wedge \omega \text{ if } \omega \in \Lambda_r, v \in \Lambda_s$$

□ We can use \wedge to build bases of $\Lambda_r(p)$:

Assume: $\{\theta^i\}_{i=1}^n$ is a basis of $\Lambda_1 = T_p(M)^*$.

(for example $\theta^i = dx^i$)

Then: $\{\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_r}\}_{1 \leq i_1 < i_2 < \dots < i_r \leq n}$
 is a basis of $\Lambda_r(p)$ for $r > 1$.

show this \rightarrow

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show this \nearrow

□ Therefore:

$$\dim(\Lambda_r(p)) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

\Rightarrow no diff. forms of degree $r > n$!

Example: For $p \in M = \mathbb{R}^3$ we have bases:

$$\Lambda_1 = \text{span}(dx^1, dx^2, dx^3)$$

$$\Lambda_2 = \text{span}(dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3)$$

$$\Lambda_3 = \text{span}(dx^1 \wedge dx^2 \wedge dx^3)$$

Definition:

$(\dim(\Lambda) = 2^n)$ $\rightarrow \Lambda(p) := \bigoplus_{i=0}^n \Lambda_i(p)$ equipped with the multiplication \wedge , is an associative algebra, called the exterior algebra or the Grassmann algebra over $T_p(M)$.

Generalization to fields:

- ▢ A differential form field is a mapping that associates to each $p \in M$ an element:

$$\omega(p) \in \Lambda(p)$$

It is usually also called simply a differential form and denoted ω .

- ▢ They form the Grassmann algebra of differential forms, $\Lambda(M)$.

Recall:

Given an algebra, it is often useful to consider derivations of the algebra, i.e. to consider maps that obey the Leibniz rule. (we defined tangent vectors as derivations of $F_p(M)$)

Here: For the algebra $\Lambda(M)$, let us consider the exterior and the inner derivations:

Definition:

A linear map $\Phi: \Lambda(M) \rightarrow \Lambda(M)$ is called a derivation of degree k , if:

$$\Phi: \Lambda_s(M) \rightarrow \Lambda_{s+k}(M) \quad \text{for all } s$$

$$\Phi: \alpha \wedge \beta \rightarrow \Phi(\alpha) \wedge \beta + \alpha \wedge \Phi(\beta)$$

for all $\alpha, \beta \in \Lambda(M)$.

Leibniz rule

Also:

Definition:

A linear map $\Phi: \Lambda(M) \rightarrow \Lambda(M)$ is called an anti-derivation of degree k , if:

$$\Phi: \Lambda_s(M) \rightarrow \Lambda_{s+k}(M) \quad \text{for all } s$$

$$\Phi: \alpha \wedge \beta \rightarrow \Phi(\alpha) \wedge \beta + (-1)^i \alpha \wedge \Phi(\beta)$$

for all $\alpha \in \Lambda^i(M)$, $\beta \in \Lambda(M)$.

Anti-Leibniz rule

Proposition: (as we will show constructively)

Because of the Leibniz rule and linearity, any (anti-)derivation

$$\Phi: \Lambda(\mathcal{M}) \rightarrow \Lambda(\mathcal{M})$$

is already fully determined by its action only on $\Lambda_0(\mathcal{M})$ and on a basis of $\Lambda_1(\mathcal{M})$.

The exterior derivative

The exterior derivative,

$$d: \Lambda(M) \rightarrow \Lambda(M)$$

is the anti-derivation of degree $k=1$ which is defined through:

- a) $d: \Lambda_0(M) \rightarrow \Lambda_1(M)$
 $d: f \rightarrow df$ } action of d on $\Lambda_0(M)$
- b) $d: dx^i \rightarrow 0$ for all i . } action of d on a basis of $\Lambda_1(M)$

The exterior derivative,

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 $d: f \rightarrow df$ } action of d on $\Lambda_0(M)$
- b) $d: dx^i \rightarrow 0$ for all i . } action of d on a basis of $\Lambda_1(M)$!

In a chart:

We had: $d: f(x) \rightarrow df(x) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$

Now we have more generally, e.g., for

$$\beta = \sum_{i_1 < \dots < i_s} \beta_{i_1, \dots, i_s}(x) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s} \in \Lambda_s(M)$$

$\Lambda_0(M)$ $\in \Lambda_s(M)$
 \downarrow \wedge
 \Rightarrow like $f(x)$ above recall: $f \wedge w = fw$ when $f \in \Lambda_0$ and $w \in \Lambda$

$d: \beta \rightarrow d\beta$

namely, by applying the anti-Leibniz rule:

$$d\beta = \sum_{i_1 < \dots < i_s} \frac{\partial \beta_{i_1, \dots, i_s}(x)}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

Proposition: $d: \Lambda(M) \rightarrow \Lambda(M)$ obeys:

$$d \circ d = 0$$

Proof:

$$d \circ d(\beta) = \sum_{\substack{i_1 < \dots < i_s \\ j_1 k}} \frac{\partial^2 \beta_{i_1, \dots, i_s}(x)}{\partial x^{j_1} \partial x^k} dx^{j_1} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_s}$$

$= 0$

$\Rightarrow \sum = 0$

Sym. in $j_1 k$

$dx^k \wedge dx^j = -dx^j \wedge dx^k$
i.e. antisym. in $j_1 k$

Example:

e.g.: electric potential

$$d\beta = \sum_{i_1 < \dots < i_s} \frac{\partial \beta_{i_1, \dots, i_s}(x)}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

+ terms of the form $d(dx^{i_1} \wedge \dots \wedge dx^{i_s})$

= 0 because when applying Leibniz rule to $d(dx^{i_1} \wedge \dots \wedge dx^{i_s})$ we eventually arrive at $d(dx^i) = 0 \dots$

Proposition:

$d: \Lambda(M) \rightarrow \Lambda(M)$ obeys:

$$d \circ d = 0$$

Example:

□ For $M = \mathbb{R}^3$ and $f \in \mathcal{F}(M)$ we have: e.g.: electric potential

$$df = \sum_{i=1}^3 \frac{\partial f}{\partial x^i} dx^i$$

□ Notice:

$(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3})$ is the
"Gradient field ∇f of f "

e.g., electric force on a test charge

□ Now assume $\gamma \in \Lambda_1(M)$ is an arbitrary (i.e. not necessarily gradient) covariant vector field:

$$\gamma = \sum_{i=1}^3 \gamma_i(x) dx^i \in \Lambda_1(\mathbb{R}^3)$$

□ Then:

$$\begin{aligned} d\gamma &= \sum_{i,j} \frac{\partial \gamma_i(x)}{\partial x^j} dx^j \wedge dx^i \\ &= \sum \left(\frac{\partial \gamma_i(x)}{\partial x^j} - \frac{\partial \gamma_j(x)}{\partial x^i} \right) dx^j \wedge dx^i \end{aligned}$$

from $dx^j \wedge dx^i = -dx^i \wedge dx^j$

$i=j$ does not occur because $dx^i \wedge dx^i = 0$

□ Then:

$$dy = \sum_{i,j} \frac{\partial \gamma^i(x)}{\partial x^j} dx^j \wedge dx^i$$

$i=j$ does not occur because $dx^i \wedge dx^i = 0$

from $dx^0 \wedge dx^1 = -dx^1 \wedge dx^0$

$$= \sum_{i < j} \left(\frac{\partial \gamma^i(x)}{\partial x^j} - \frac{\partial \gamma^j(x)}{\partial x^i} \right) dx^j \wedge dx^i$$

$$= - \overbrace{\left(\frac{\partial \gamma^1}{\partial x^2} - \frac{\partial \gamma^2}{\partial x^1} \right)}^{\beta_3 =} dx^1 \wedge dx^2$$

$$- \overbrace{\left(\frac{\partial \gamma^1}{\partial x^3} - \frac{\partial \gamma^3}{\partial x^1} \right)}^{\beta_2 =} dx^1 \wedge dx^3$$

$$- \underbrace{\left(\frac{\partial \gamma^2}{\partial x^3} - \frac{\partial \gamma^3}{\partial x^2} \right)}_{\beta_1 =} dx^2 \wedge dx^3$$

□ Notation:

$$- \underbrace{\left(\frac{\partial \gamma_2}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^2} \right)}_{\beta_1} dx^2 \wedge dx^3$$

□ Notice:

$(\beta_1(x), \beta_2(x), \beta_3(x))$ are the components of the curl

(We now know:
 β really is a 2-form!) \rightarrow

$\beta = \nabla \times \gamma$
 which is called a "pseudo vector field."

□ Recall:

□ Recall:

Gradient vector fields

are curl free: $\nabla \times (\nabla f) = 0$

It is special case of

$$d \circ d = 0$$

because if $\beta = d\gamma$ then:

$$d\beta = d^2\gamma = 0$$

Definition:

□ A differential form ω is called closed if:

$$d\omega = 0$$

□ A differential form ω is called exact if there exists a v so that

$$\omega = dv \quad (\text{v is like an anti-derivative!})$$

How are closedness and exactness related?

This depends on the global topology of the manifold! (because anti-derivatives are in a sense global)

Simplest case: Assume M is contractible

i.e., $\exists \overset{\text{continuous}}{F}: [0, 1] \times M \rightarrow M$

so that $F(0, x) = x \quad \forall x$

$F(1, x) = x_0 \quad \forall x$

\uparrow some fixed pt e 25 / 30

Poincaré lemma:

On any contractible manifold:

$$\gamma \text{ exact} \iff \gamma \text{ closed}$$

E.g.

□ \mathbb{R}^m is contractible

□ $\mathbb{R}^m \setminus \{p\}$ is not contractible

↑ some arbitrary point $p \in M$

In general:

We only have

$$\gamma \text{ exact} \Rightarrow \gamma \text{ closed}$$

which is $d^2 = 0$.

Remark:

□ Study of closed forms / exact forms is one of few handles on global topology of diff. manifolds, e.g., to see if it has holes.

→ See cohomology theory

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→ See cohomology theory.

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□ Another tool for studying the global topology of M comes with the notion of **vector bundle**. There is only one for contractible mflds but others can have many non-isomorphic ones. The cohomology theory based on that idea is called **K-theory**.

□ Recall that, e.g., for a suitable vector bundle B :



$$\pi^{-1}(U_1) \cong U_1 \times \mathbb{R}^n$$

$$\pi^{-1}(U_2) \cong U_2 \times \mathbb{R}^n$$

But: $B \not\cong M \times \mathbb{R}^n$

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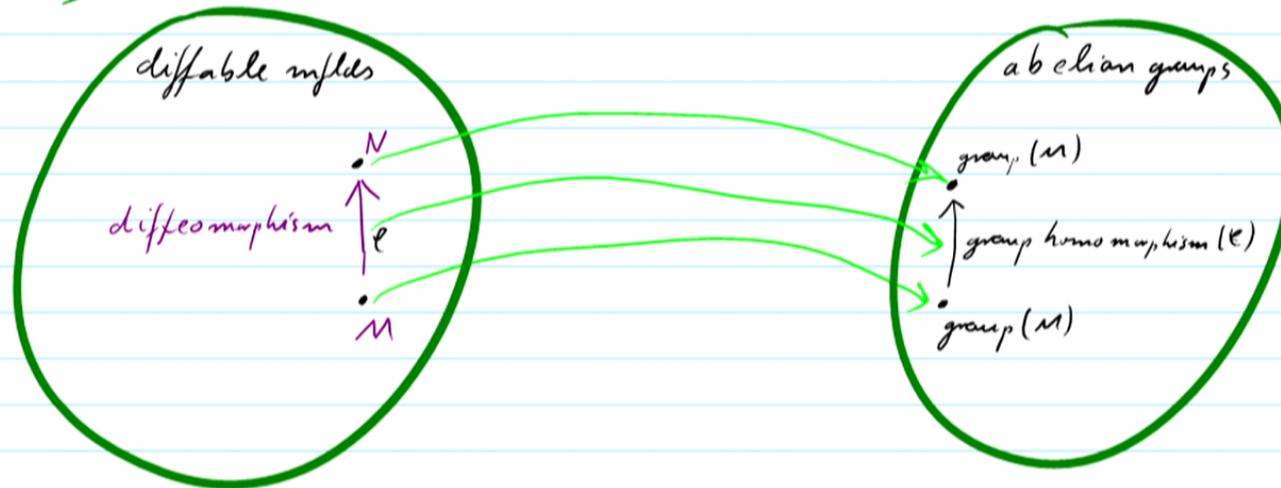
But: $B \neq M \times \mathbb{R}^m$

Roughly: All cohomology theories are maps ("natural transf.")

diffable mflds

abelian groups

Roughly: All cohomology theories are maps ("natural trans.")



so that, if $\text{group}(M) \neq \text{group}(N) \Rightarrow M$ not diffeomorph to N

Note: for no cohom. theory the reverse arrow holds, i.e. none is complete.

Note: This is what **category theory** was developed for.