Title: Geometry of quantum phases and emergent Newtonian dynamics

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Abstract: In the first part
of this talk I will discuss how one can characterize geometry of quantum phases and phase transitions based on the Fubini-Study metric, which characterizes the distance between ground state wave-functions in the external parameter space. This metric is closely related to the Berry curvature. I will show that there are new geometric invariants based on the Euler characteristic.

I will also show how one can directly measure this metric
tensor in simple dynamical experiments. In the second part of the talk I will discuss emergent nature of macroscopic equations of motion (like Newton's equations) showing that they appear in the leading order of non-adiabatic expansion. I will show that the Berry curvature gives the Coriolis force and the Fubini-Study metric tensor is closely related to the inertia mass. Thus I will argue that any motion (not necessarily motion in space) is geometrical in nature.

Geometry of quantum phases and emergent Newtonian dynamics

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CMT seminar, Perimeter Institute, 10/15/2013

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Outline

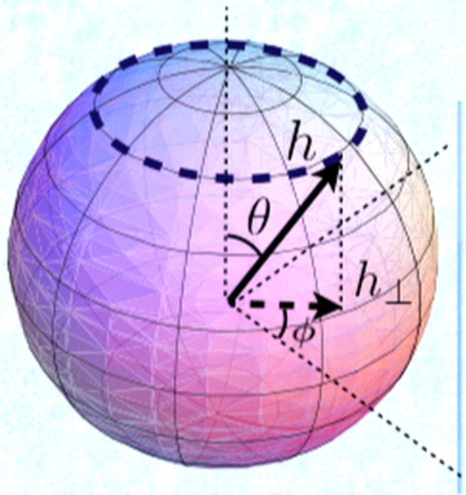
1. Geometric tensor, Berry curvature and the Fubini-Study metric tensor.

2. Geometry of the quantum XY chain. Phase diagram. Geometric invariants. Classification of singularities.

3. *Emergent Newtonian dynamics and the geometric origin of the mass.*

4. *Berry curvature and the Coriolis force. Dynamical (SU(2)) Quantum Hall effect.*

Geometric structure of the ground state manifold



Imagine a system at zero temperature described by some manifold of parameters $\vec{\lambda}$

E.g. $\vec{\lambda} = \{h_\theta, h_\phi\}$

Can we define the associated geometry of the ground state?

Geometric structure of the ground state manifold

Hamiltonian: $\mathcal{H} = \mathcal{H}(\vec{\lambda})$. Ground state wave-function: $\psi_0 = \psi_0(\vec{\lambda})$.

Consider the following change $\vec{\lambda} \to \vec{\lambda} + \delta\vec{\lambda}$

$$||\psi_0(\vec{\lambda}) - \psi_0(\vec{\lambda} + \delta\vec{\lambda})||^2 \approx 1 - |\langle\psi_0(\vec{\lambda})|\psi_0(\vec{\lambda} + \delta\vec{\lambda})\rangle|^2 = \chi_{\alpha\beta} d\lambda_\alpha d\lambda_\beta$$

$\chi_{\alpha\beta}$ - geometric tensor (Provost, Vallee, 1980)

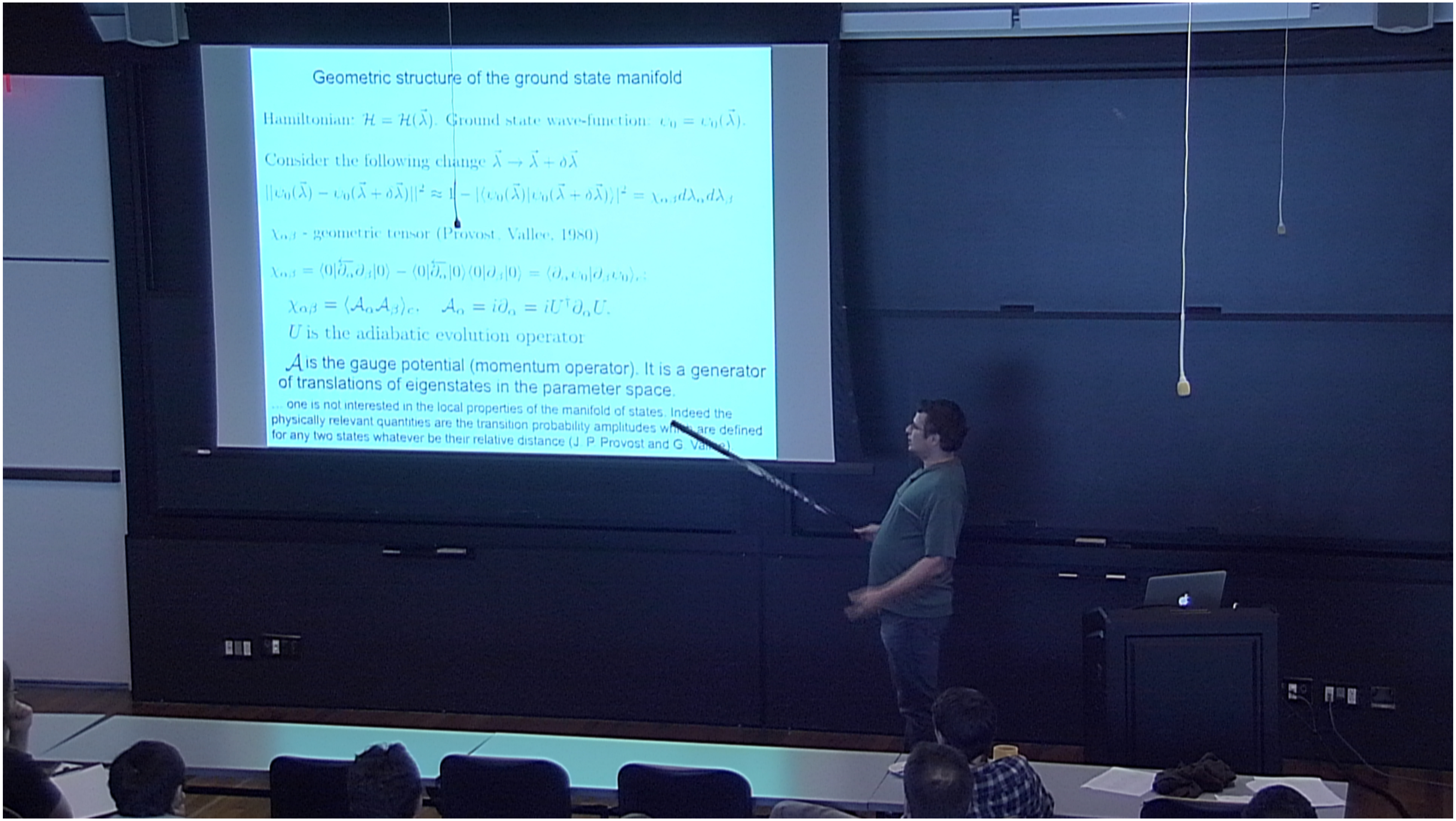
$$\chi_{\alpha\beta} = \langle 0|\overleftarrow{\partial_\alpha}\partial_\beta|0\rangle - \langle 0|\overleftarrow{\partial_\alpha}|0\rangle\langle 0|\partial_\beta|0\rangle = \langle\partial_\alpha\psi_0|\partial_\beta\psi_0\rangle_c;$$

$$\chi_{\alpha\beta} = \langle\mathcal{A}_\alpha\mathcal{A}_\beta\rangle_c, \quad \mathcal{A}_\alpha = i\partial_\alpha = iU^\dagger\partial_\alpha U,$$

U is the adiabatic evolution operator

$\mathcal{A}$ is the gauge potential (momentum operator). It is a generator of translations of eigenstates in the parameter space.

… one is not interested in the local properties of the manifold of states. Indeed the physically relevant quantities are the transition probability amplitudes which are defined for any two states whatever be their relative distance (J. P. Provost and G. Vallee).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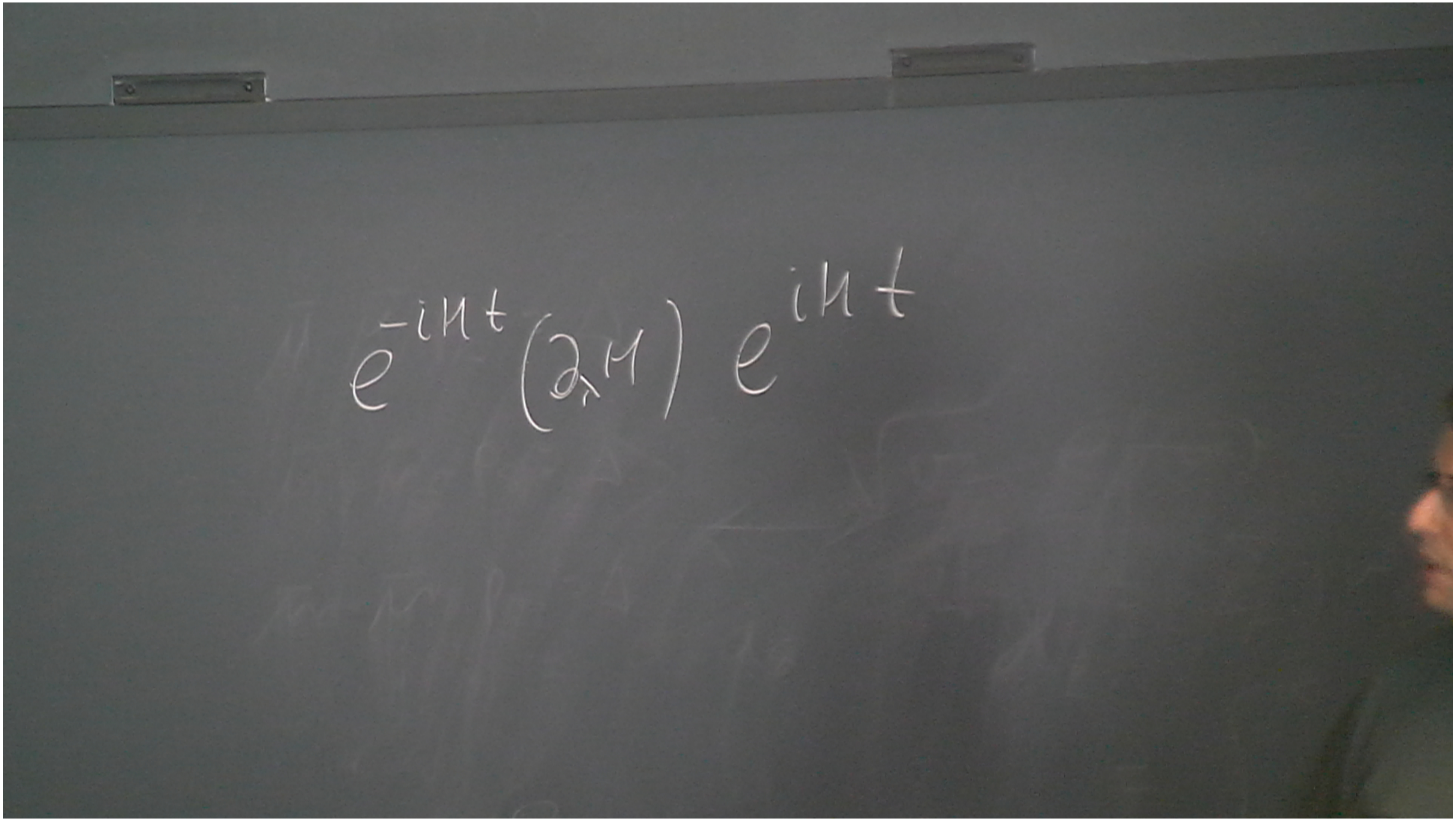
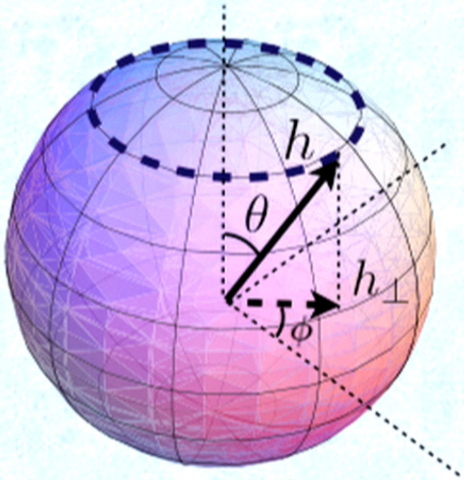

$e^{-iHt}(\partial_t H)\,e^{iHt}$

Example: spin ½ in a magnetic field

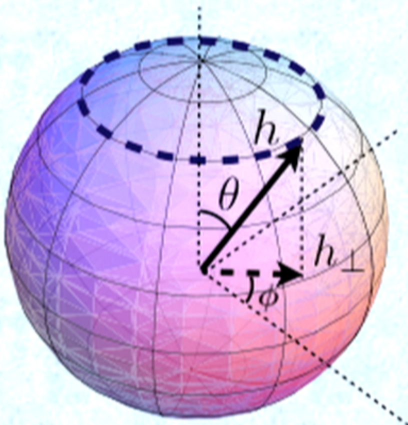


$$H = -\vec{h}\vec{\sigma}$$

$$|\psi_0\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathrm{e}^{-i\phi} \end{pmatrix}.$$

$$\chi_{\theta\theta} = \frac{1}{4}, \quad \chi_{\phi\phi} = \frac{1}{4}\sin^2(\theta), \quad \chi_{\theta\phi} = -\frac{i}{4}\sin(\theta).$$

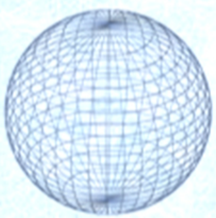
Geometric invariants.



Chern number: measures the effective ``magnetic field' flux

$$\oint_{\mathcal{A}} F_{\lambda\mu} d\lambda d\mu = 2\pi n$$

$$\int d\theta d\phi F_{\theta\phi} = \frac{1}{2}\int d\theta d\phi \sin(\theta) = 2\pi$$



$\chi = 2$

$\chi = 0$

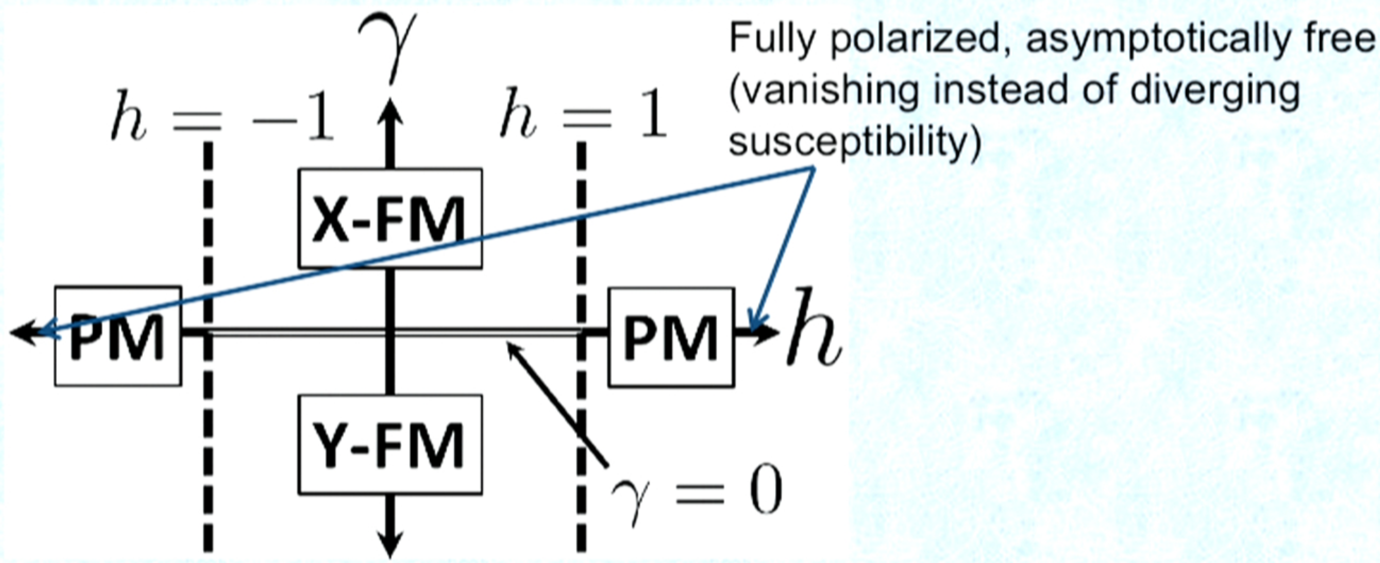
Euler characteristic: measures number of holes

$$\chi(\mathcal{M}) = \frac{1}{2\pi}\left[\int_{\mathcal{M}} K dS + \int_{\partial M} k_g dl\right]$$

XY – chain in the transverse field

$$H = -\sum_j \frac{1+\gamma}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{2}\sigma_j^y\sigma_{j+1}^y - h\sum_j \sigma_j^z$$

Maps to free fermions using Jordan-Wigner transformation.
(Lieb-Schultz-Mattis, 1961, S. Sachdev: Quantum phase transitions)



$$H(h,\gamma,\phi) = -\sum_j (\sigma_j^+ \sigma_{j+1}^- + h.c.) - \gamma \sum_j (\mathrm{e}^{i\phi} \sigma_j^+ \sigma_{j+1}^+ + h.c.) - h \sum_j \sigma_j^z$$

$$|\psi\rangle = \prod_k |\phi_k\rangle, \ |\phi_k\rangle = \begin{pmatrix} \cos(\theta_k/2)\mathrm{e}^{i\phi/2} \\ \sin(\theta_k/2)\mathrm{e}^{-i\phi/2} \end{pmatrix} \qquad \tan(\theta_k) = \frac{\gamma \sin(k)}{h - \cos(k)}$$

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Gauge potentials.

$$\mathcal{A}_{h,\gamma} = i\partial_{h,\gamma} = \frac{1}{2} \sum_k \partial_{h,\gamma} \theta_k \, \tau_k^y \qquad \mathcal{A}_\phi = -\frac{1}{2} \sum_k [\cos(\theta_k)\tau_k^z + \sin(\theta_k)\tau_k^x]$$

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Metric components

$$g_{hh} = \langle \mathcal{A}_h^2 \rangle_c = \frac{1}{4} \sum_k \left(\frac{\partial \theta_k}{\partial h} \right)^2 = \frac{1}{16} \begin{cases} \frac{1}{|\gamma|(1-h^2)} & |h| < 1 \\ \frac{|h|\gamma^2}{(h^2-1)(h^2-1+\gamma^2)^{3/2}} & |h| > 1 \end{cases}$$

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What next?

Try to visualize phase diagram (imbed into 3D space)

$$H(h,\gamma,\phi) = -\sum_j (\sigma_j^+ \sigma_{j+1}^- + h.c.) - \gamma \sum_j (\mathrm{e}^{i\phi} \sigma_j^+ \sigma_{j+1}^+ + h.c.) - h \sum_j \sigma_j^z$$

$h - \phi$ plane - cylindrical symmetry

$$dz^2 + dr^2 + r^2 d\phi^2 = g_{hh} dh^2 + g_{\phi\phi} d\phi^2$$

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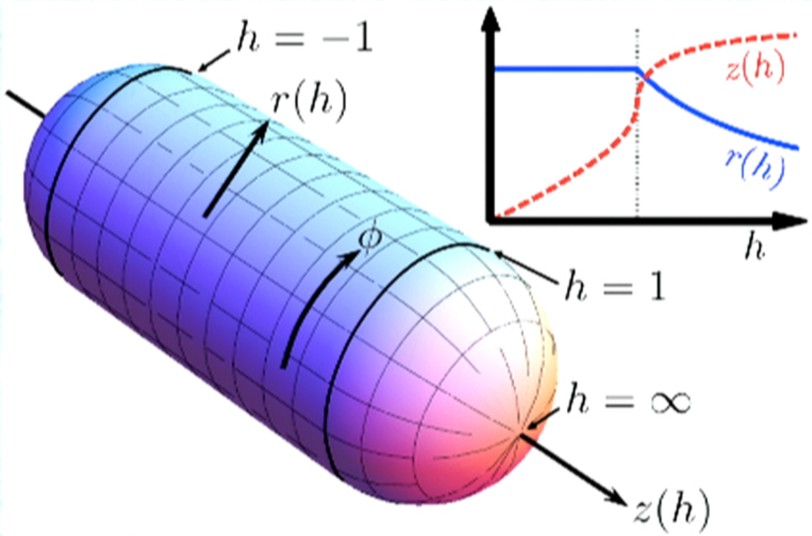

Integrable singularity: shape is regular

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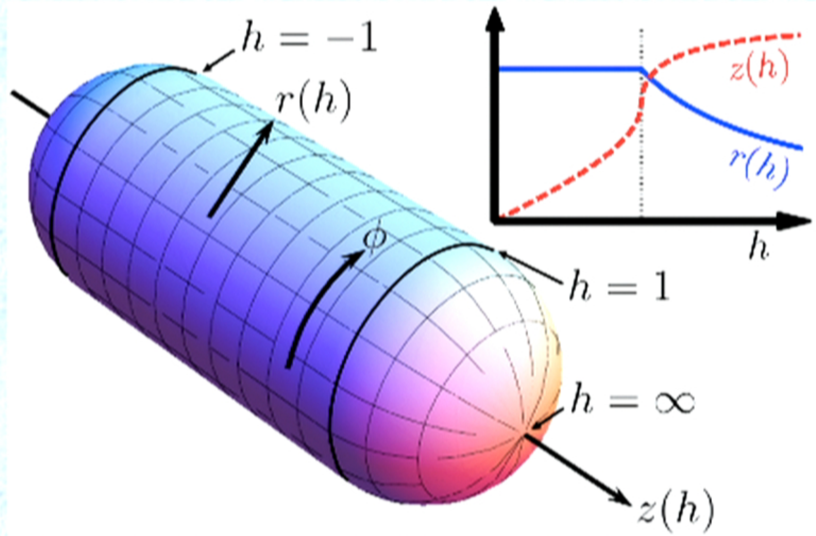

Euler characteristic

$\chi = \int K dS = 2$ overall

$$\chi = \chi_{pm} + \chi_{fm} + \chi_{pm}$$

$$\chi_{fm} = 0 = \int_{|h|<1} K dS,$$

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Protected (by geodesic curvature) geometric invariants

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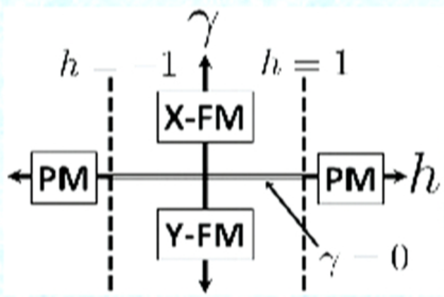
$\gamma - \phi$ plane - critical line piercing the plane



$\lambda_1 = \gamma \cos(\phi), \ \lambda_2 = \gamma(\sin\phi)$

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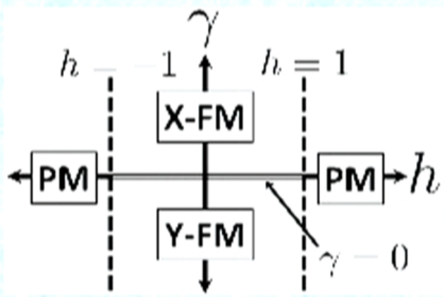

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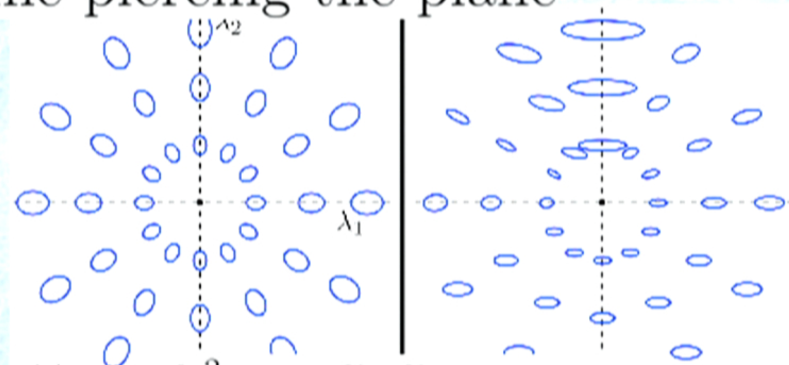
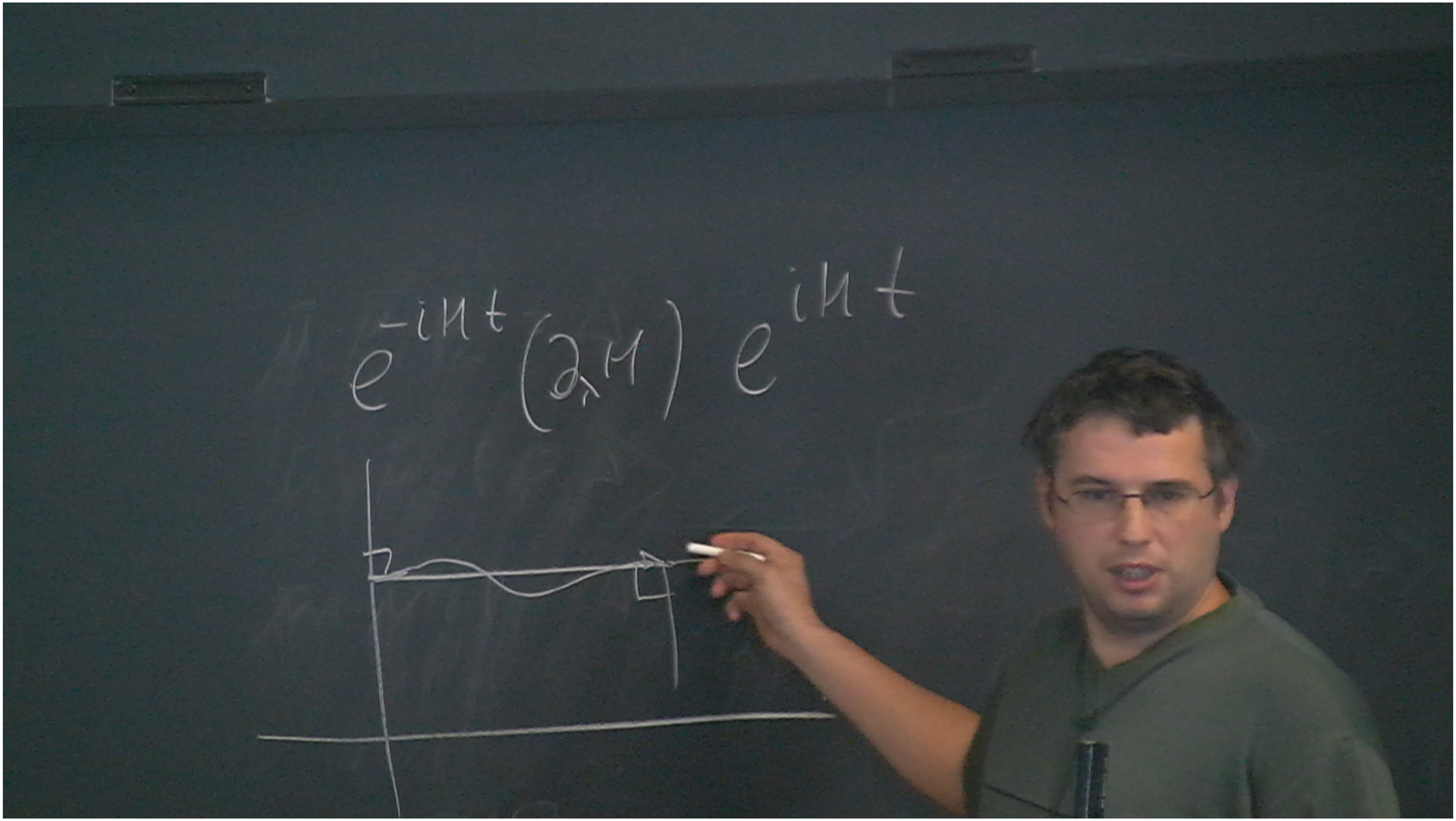


$ds^2 = g_{ij}d\lambda_i d\lambda_j = \text{const}$

$g_{\lambda_1\lambda_1} \sim A|\lambda|^\alpha,\ g_{\lambda_2\lambda_2} = B|\lambda|^\alpha$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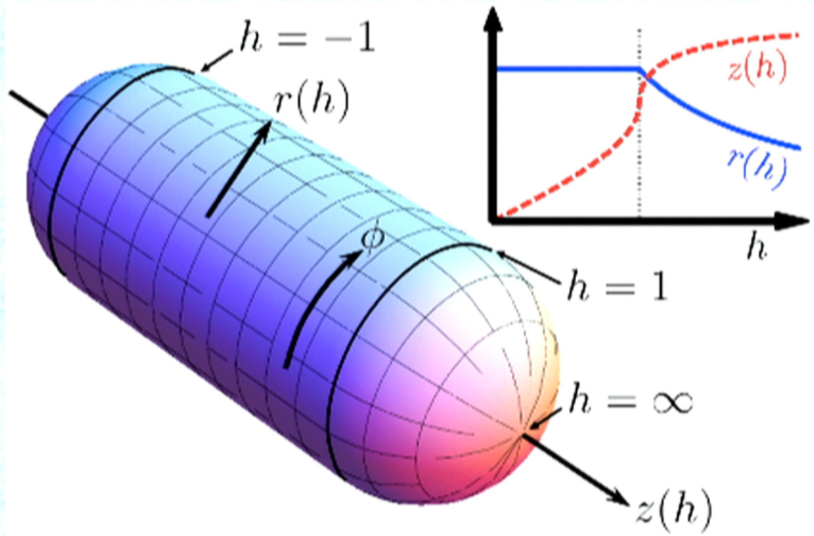

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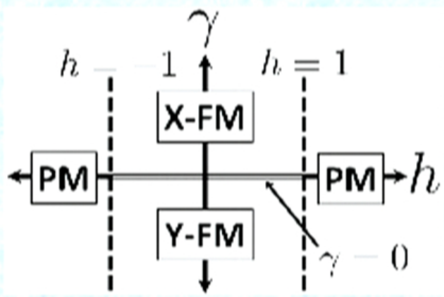

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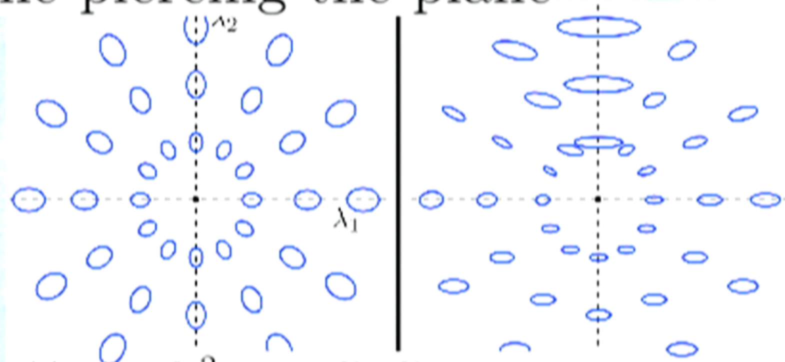

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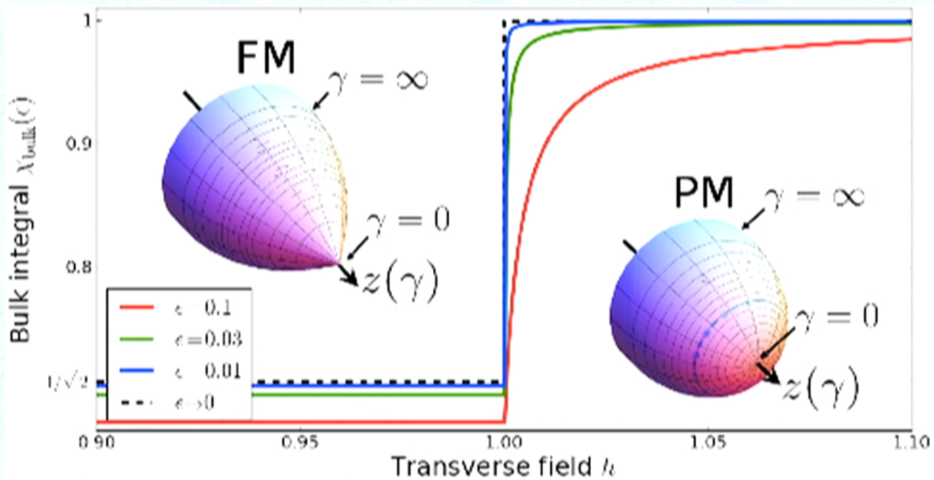
$$g_{\lambda_1\lambda_1} \sim A|\lambda|^\alpha,\ g_{\lambda_2\lambda_2} = B|\lambda|^\alpha$$

Conical singularity



Universal (protected) bulk Euler characteristic

$$\chi_{bulk} = \int_{\gamma > \epsilon} K\,dS = \begin{cases} \frac{1}{\sqrt{2}} & |h| < 1 \\ 1 & |h| > 1 \end{cases}$$

$$\chi_{bulk} = \left(1 - \frac{\alpha}{2}\right)\sqrt{\frac{B}{A}}$$

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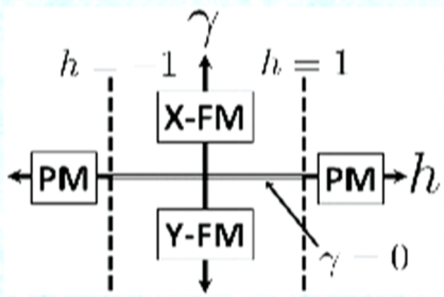

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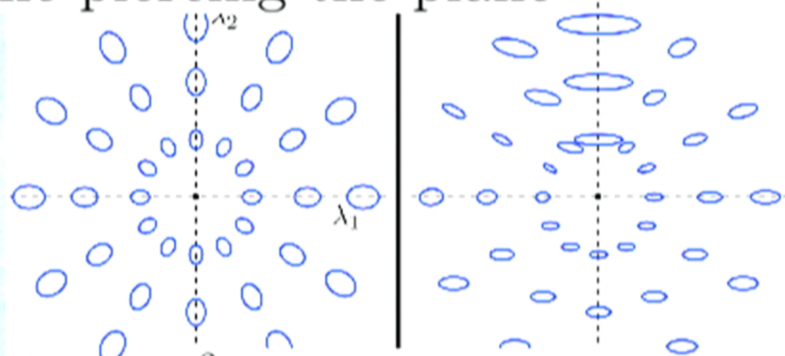

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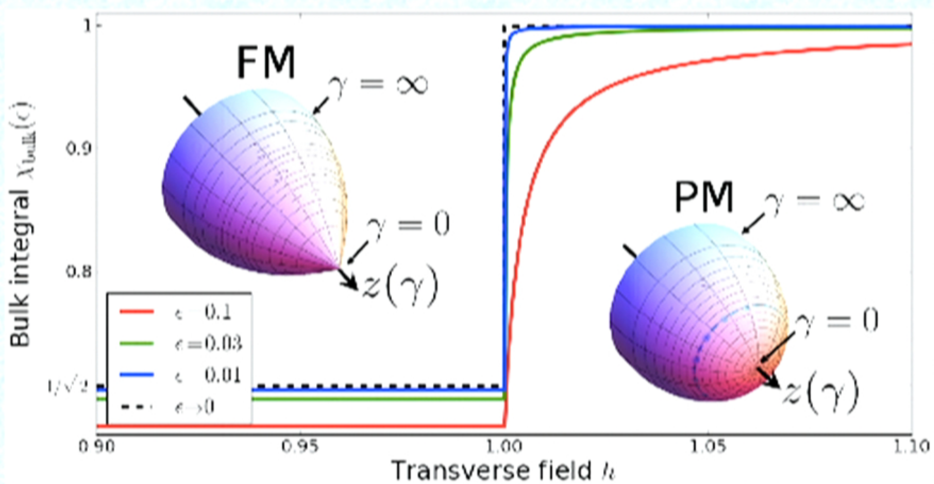

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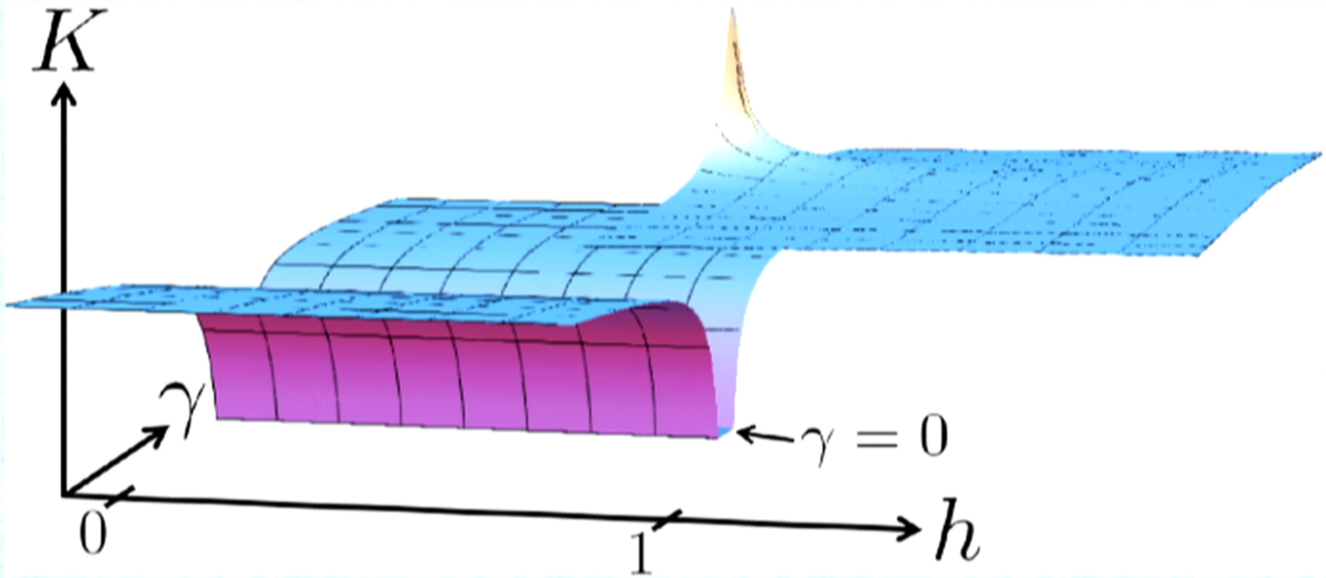
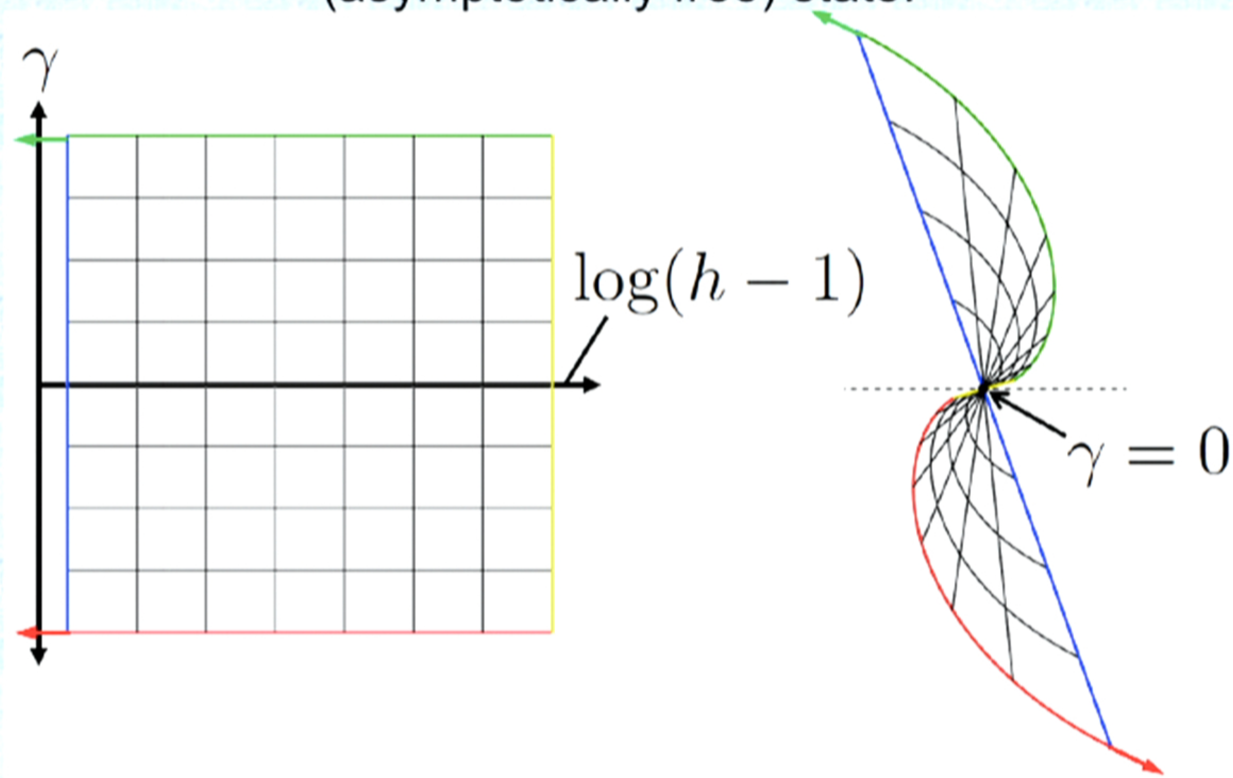
$\gamma - h$ plane - singular surface, hard to visualize



Nonintegrable curvature singularity near anisotropic transition and near the multicritical point.

No obvious geometric invariants.

Universal metric near the fully polarized (asymptotically free) state.



Locally flat (non-singular curvature) but singular mapping.
Divergent Ricci curvature.

Why shall we care about the metric?

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Emergent Newtonian dynamics and the geometric origin of mass

(L. D'Alessio, A.P. arXiv:1309.6354)

Consider a dynamical macroscopic degree of freedom coupled to an arbitrary system.

$$H_{tot}(\vec{\lambda}) = H_0(\vec{\lambda}) + H(\vec{\lambda})$$

The Hamiltonian H_0 is just to build intuition, e.g.

$$H_0(\vec{\lambda}) = \frac{\vec{P}_{\vec{\lambda}}^2}{2M} + U(\vec{\lambda})$$

$M \to \infty$ means $\vec{\lambda}$ is an external field

Go the moving frame of the Hamiltonian $\mathcal{H}(\vec{\lambda})$

$$|\psi\rangle = U(\vec{\lambda})|\tilde{\psi}\rangle \;\Rightarrow\; i\partial_t|\tilde{\psi}\rangle = (U^\dagger \mathcal{H} U - \dot{\lambda}_\alpha \mathcal{A}_\alpha)|\tilde{\psi}\rangle, \quad \mathcal{A}_\alpha = iU^\dagger \partial_{\lambda_\alpha} U$$

$\tilde{\mathcal{H}} = U^\dagger \mathcal{H} U$ is diagonal, $\dot{\lambda}_\alpha \mathcal{A}_\alpha$ - Galilean term responsible for excitations

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Use the second order perturbation theory with respect to velocity

$$\frac{d\langle\mathcal{H}\rangle}{dt} = \dot{\lambda}_\alpha \langle \partial_{\lambda_\alpha} \mathcal{H}\rangle + \dot{Q}$$

$$\dot{Q}(t) = \dot{\lambda}_\alpha(t) \int_0^t dt' \int_0^\beta d\tau \, \dot{\lambda}_\beta(t-t') \langle \partial_\alpha \mathcal{H}(t) \partial_\beta \mathcal{H}(t-t'+i\tau)\rangle_c$$

Classical limit: D. A. Sivak and G. E. Crooks, PRL108,190602 (2012)

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$$\frac{d\langle \mathcal{H}\rangle}{dt} = \dot{\lambda}_\alpha \langle \partial_{\lambda_\alpha} \mathcal{H}\rangle + \dot{Q}$$

$$\dot{Q}(t) = \dot{\lambda}_\alpha(t) \int_0^t dt' \int_0^\beta d\tau \, \dot{\lambda}_\beta(t-t') \langle \partial_\alpha \mathcal{H}(t) \partial_\beta \mathcal{H}(t-t'+i\tau)\rangle_c$$

Classical limit: D. A. Sivak and G. E. Crooks, PRL108,190602 (2012)

Go the moving frame of the Hamiltonian $\mathcal{H}(\vec{\lambda})$

$$|\psi\rangle = U(\vec{\lambda})|\tilde{\psi}\rangle \;\Rightarrow\; i\partial_t|\tilde{\psi}\rangle = (U^\dagger \mathcal{H} U - \dot{\lambda}_\alpha \mathcal{A}_\alpha)|\tilde{\psi}\rangle, \quad \mathcal{A}_\alpha = iU^\dagger \partial_{\lambda_\alpha} U$$

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Perform the Taylor expansion and integrate over time

$$\dot{\lambda}_\beta(t-t') = \dot{\lambda}_\beta(t) - t'\ddot{\lambda}_\beta(t) + \dots$$

$$\dot{Q} = \dot{\lambda}_\alpha \eta_{\alpha\beta} \dot{\lambda}_\beta + \dot{\lambda}_\alpha \kappa_{\alpha\beta} \ddot{\lambda}_\beta$$

$$\eta_{\alpha\beta} = \pi\beta \sum_{n\neq m} \rho_m \langle m|\partial_\alpha \mathcal{H}|n\rangle\langle n|\partial_\beta \mathcal{H}|m\rangle \delta(E_n - E_m)$$

drag (on shell)

Berry and Robbins 1993.

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$$\kappa_{\alpha\beta} = \frac{1}{2} \int_0^\beta d\tau \langle \mathcal{A}_\alpha(-i\tau) \mathcal{A}_\beta(0) \rangle_c + \alpha \leftrightarrow \beta$$

mass (off shell)

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$$\frac{dE_{tot}}{dt} = 0 \;\Rightarrow\; \dot{\lambda}_\alpha (m_{\alpha\beta} + \kappa_{\alpha\beta})\ddot{\alpha}_\beta + \dot{\lambda}_\alpha \eta_{\alpha\beta}\dot{\lambda}_\beta + \dot{\lambda}_\alpha(\partial_\alpha U + \langle \partial_\alpha \mathcal{H}\rangle) = 0$$

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One-component parameter – find dissipative Newton's equation

$$(m + \kappa)\ddot{\lambda} + \eta\dot{\lambda} = f_\lambda$$

Can absorb H_0 into H, κ *is the total mass*

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Classical (high-temperature) limit

$$\kappa_{\alpha\beta} = \frac{\beta}{2}(\langle \mathcal{A}_\alpha \mathcal{A}_\beta \rangle + \alpha \leftrightarrow \beta) = \beta g_{\alpha\beta}$$

Mass is given by the Fubini-Study metric (times the inverse temperature)

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Free particles in space

$\mathcal{A}_x = p_x$, $\kappa = \beta\langle p_x^2\rangle_c$ is the equipartition theorem

Quantum limit: mass definition – quantum analogue of the equipartition theorem

Many degrees of freedom

$$(\kappa_{\alpha\beta} + F'_{\alpha\beta})\ddot{\lambda}_\alpha + (\eta_{\alpha\beta} - F_{\alpha\beta})\dot{\lambda}_\alpha = f_\alpha$$

η, F' - dissipative (on shell), κ, F - non-dissipative (off-shell)

Most general form of the Newton's equations with dissipation and the Coriolis force given by the Berry curvature

$$F_{\alpha\beta} = i\langle[\mathcal{A}_\alpha, \mathcal{A}_\beta]\rangle$$

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Low temperature (non-dissipative) limit – standard Newton's equations

Macroscopic (classical) Coriolis force can be used to extract the quantized Chern number.

Example: central spin model

$$\mathcal{H} = \frac{\vec{L}^2}{2I} - \vec{n} \sum_j \Delta_j \vec{\sigma}_j$$

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Gapped system, non-dissipative dynamics

$$I\dot{\vec{n}} = \vec{L} \times \vec{n}, \quad \dot{\vec{L}} = \vec{n} \times \langle -\frac{\partial H}{\partial \vec{n}} \rangle = \vec{n} \times \sum_i \Delta_i \langle \vec{\sigma}_i \rangle$$

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Use expansion in $\dot{\vec{n}}$

$$I\dot{\vec{n}} = \vec{L} \times \vec{n}, \; \dot{\vec{L}} = F_0\,\dot{\vec{n}} - \kappa_0\,(\vec{n} \times \ddot{\vec{n}})$$

$$\kappa_0 \;\equiv\; \sum_i \frac{\tanh(\beta\Delta_i)}{4\Delta_i}$$

$$F_0 \;\equiv\; \frac{1}{2}\sum_i \tanh(\beta\Delta_i)$$

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Equatorial plane $\theta=\pi/2$

$$(I+\kappa)\ddot{\theta} - F_0\dot{\phi} = 0$$

$$(I+\kappa)\ddot{\phi} + F_0\dot{\theta} = 0$$

Zero temperature – quantized Coriolis force even if gaps are non-uniform etc.

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Dynamical Hall effect: through the Coriolis force.

$$M_\mu \approx F_{\mu\lambda} v_\lambda$$

Velocity is like the current – couples to the vector potential

Interpretation: dynamical quantum Hall effect in the parameter space. Related results Thouless-Niu (1984), J.E. Avron et. al. (2011)

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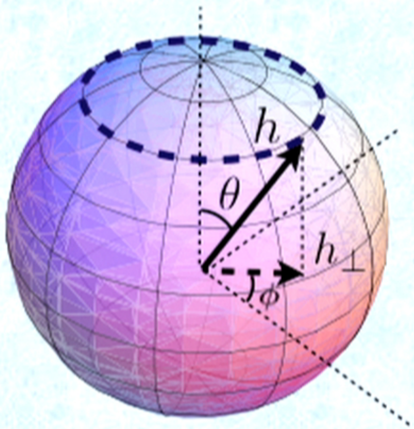
Standard (quantum) Hall effect is a particular example of the dynamical Hall effect:

$$\mathcal{E}_x = \frac{\partial A_x}{\partial t} = v_{A_x},\ j_y = -\frac{\partial \mathcal{H}}{\partial A_y} = M_y, \quad j_y = F_{yx} v_x,\ F_{yx} = \sigma_{yx}$$

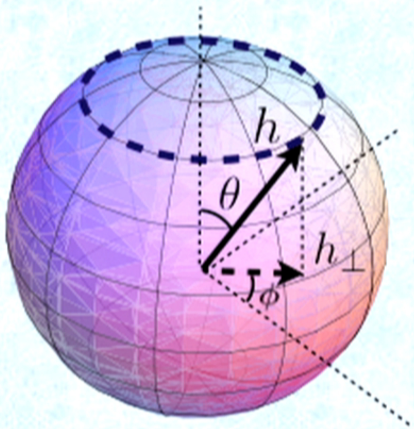
Example: spin in a magnetic field



Berry phase

$$\gamma = \oint_0^{2\pi} A_\phi d\phi = \pi(1-\cos(\theta)) = \pi\left(1-\frac{h_z}{h}\right)$$

Example: spin in a magnetic field



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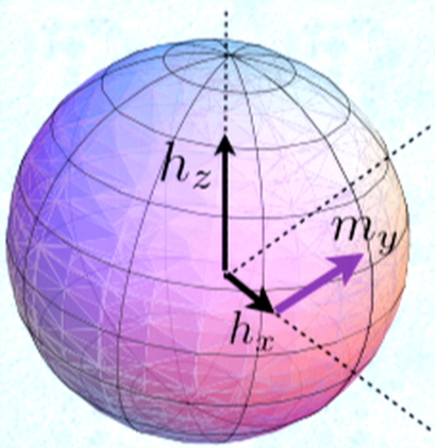
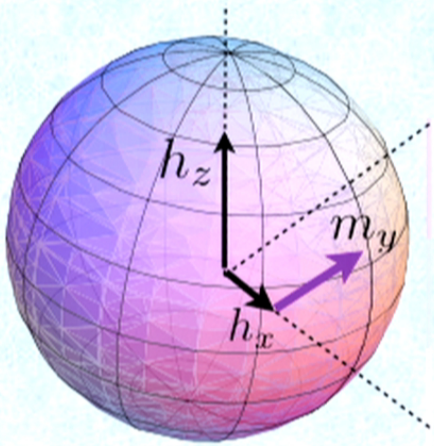


Dynamical Hall effect. Possible experimental sequence

1. Quench e.g. h_x linearly in time: $h_x(t) = h_x(0) + vt$, $t \in (-t_0, 0)$.
2. Measure $m_y(v)$. Check small v asymptotic. Linear slope is sufficient to prove breaking of the time-reversal symmetry
3. Extract the slope: $m_y(v) \approx \text{const} + \mathcal{F}_{xy}(h_x, h_y)v$
4. Repeat for other values of h_x, h_y.
5. Calculate the integral: $\gamma = \int_{\mathcal{A}} dh_x dh_y \mathcal{F}_{xy}(h_x, h_y)$

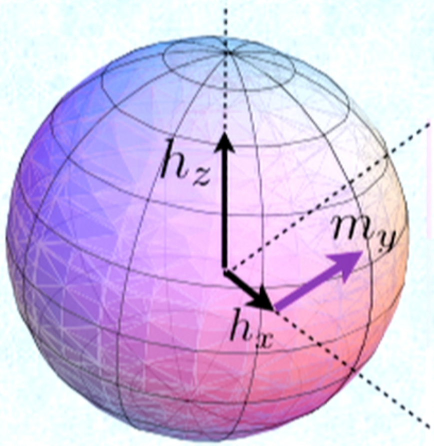
Numerical Simulation



Hamiltonian

$$\mathcal{H} = -h_z\sigma_z - (h_x + vt)\sigma_x$$

Numerical Simulation



Hamiltonian

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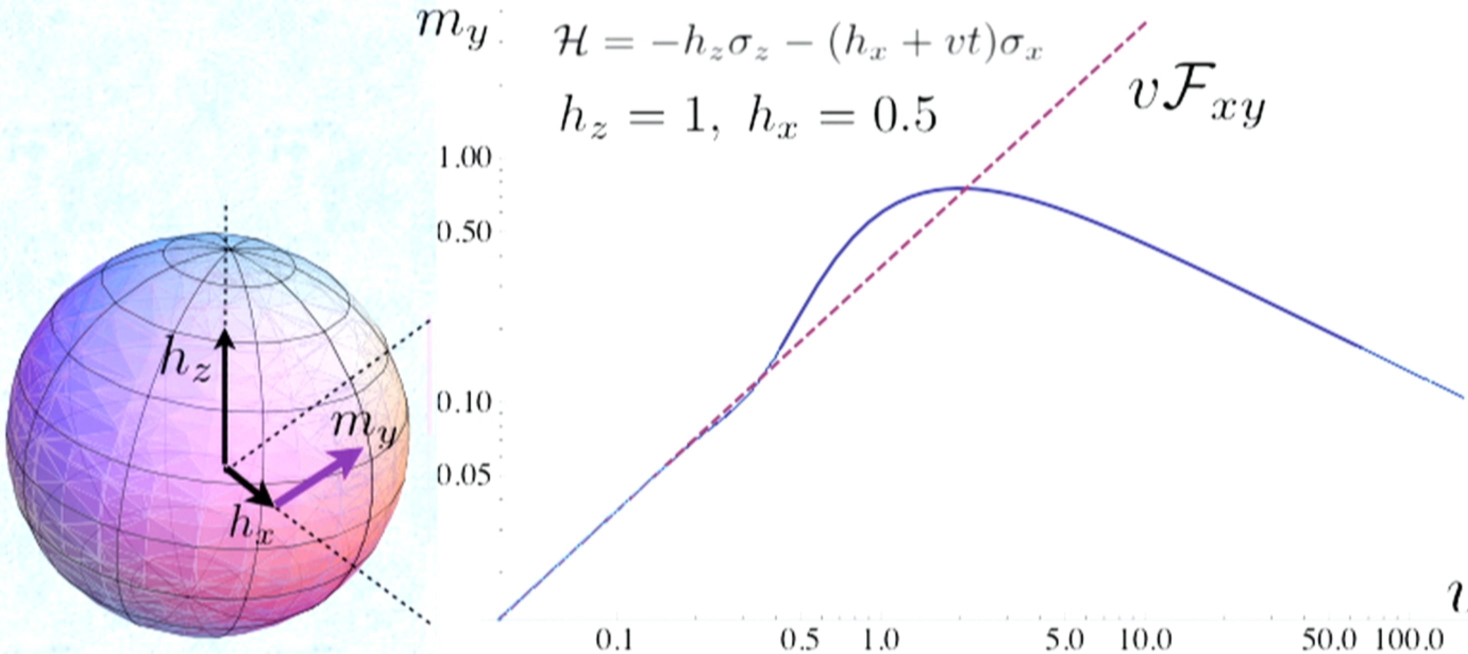
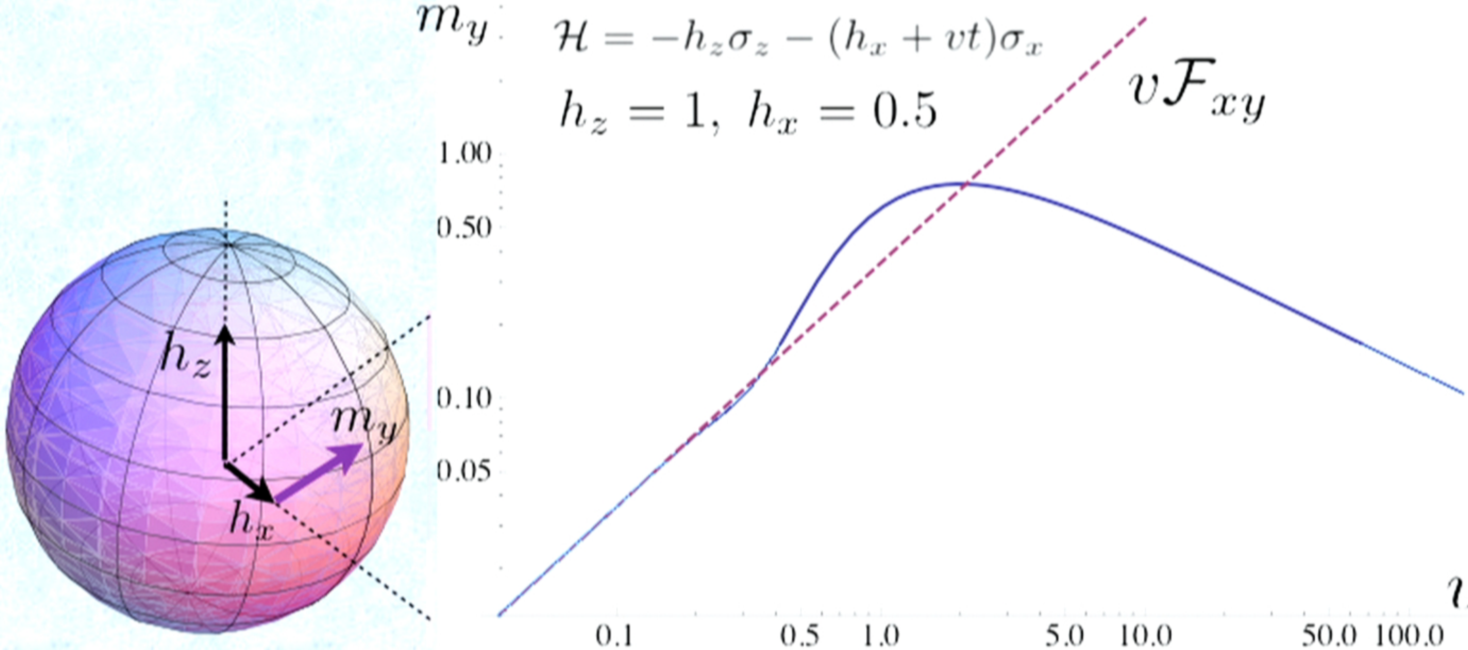
Wave function (in the eigenbasis)

$$\psi = a_1|0\rangle + a_2|1\rangle$$

$$a_2 \approx \frac{i}{2}v\frac{h_z}{(h_z^2+h_x^2)^{3/2}}$$



$$m_y = \langle\psi|\sigma_y|\psi\rangle \approx a_2\langle 0|\sigma_y|1\rangle + a_2^\star\langle 1|\sigma_y|0\rangle \approx \frac{v}{2}\frac{h_z}{(h_x^2+h_z^2)^{3/2}} = v\mathcal{F}_{xy}$$



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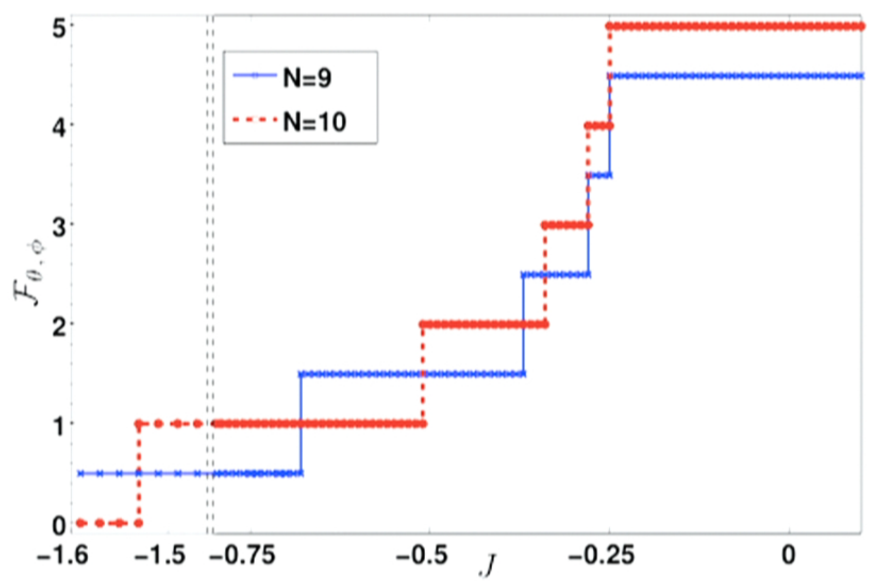
$$h^2\int d\theta d\phi \frac{M_\phi}{v_\theta} = 2\pi n$$ Use rotational symmetry $$\frac{M_\phi}{v_\theta} = \frac{n}{2h^2}\sin\theta$$

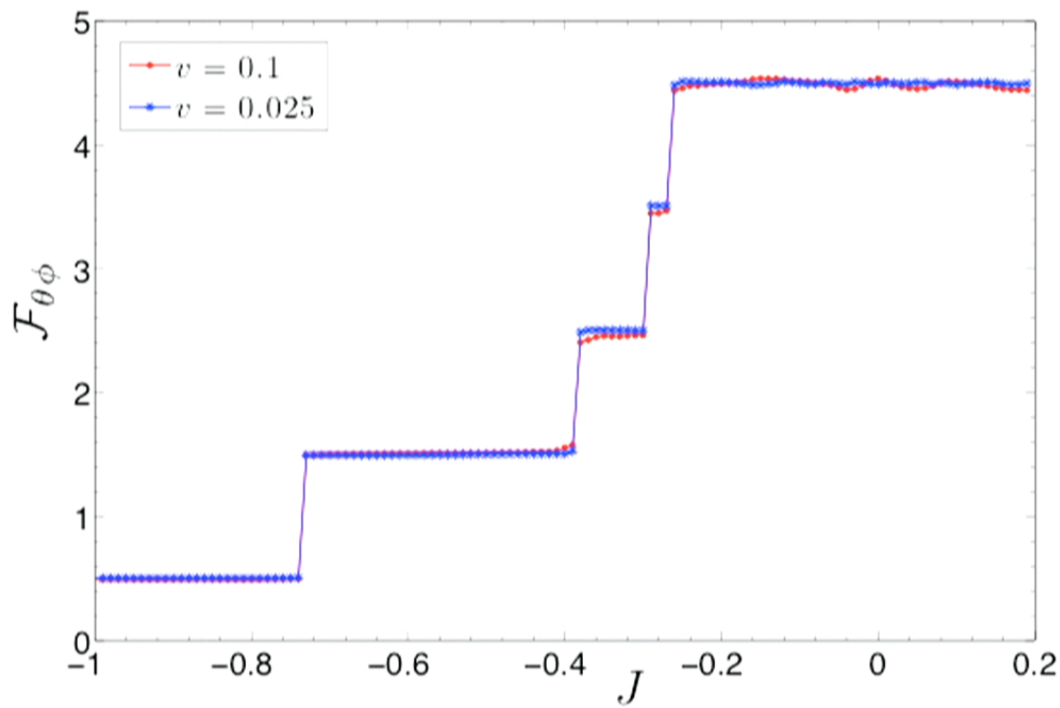
Example: interacting Heisenberg chain

$$\mathcal{H} = -\sum_{j=1}^{N} \vec{h}\vec{\sigma}_j - J\sum_{j=1}^{N-1} \vec{\sigma}_j\vec{\sigma}_{j+1}, \qquad h_x(t) = \sin\left(\frac{v^2t^2}{2\pi}\right),\ h_z(t) = \cos\left(\frac{v^2t^2}{2\pi}\right),\ h_y(t) = 0.$$



Disordered Heisenberg chain

$$\mathcal{H} = -\vec{h}\sum_{j=1}^{N}\xi_j\vec{\sigma}_j - J\sum_{j=1}^{N-1}\eta_j\vec{\sigma}_j\vec{\sigma}_{j+1}, \quad \text{Box distribution: } \xi_j, \eta_j \in [0.75, 1.25]$$



Dynamics near quantum critical point

(in progress with A. Katz and M. Kolodrubetz)

Conclusions!

- Can use Fubini-Study metric to characterize phases and phase transitions. Metric tensor can be measured through various dynamical responses

- Robust geometric invariants characterizing phases based on Gauss curvature. Unlike critical exponents they are independent of the parametrization.

- Emergent Newtonian dynamics for macroscopic degrees of freedom with mass directly related to metric (in the classical limit).

- Can extend quantum Hall effect to SU(2) and other situations. Only need to have a closed manifold of parameters.