Title: Vector Beta Function

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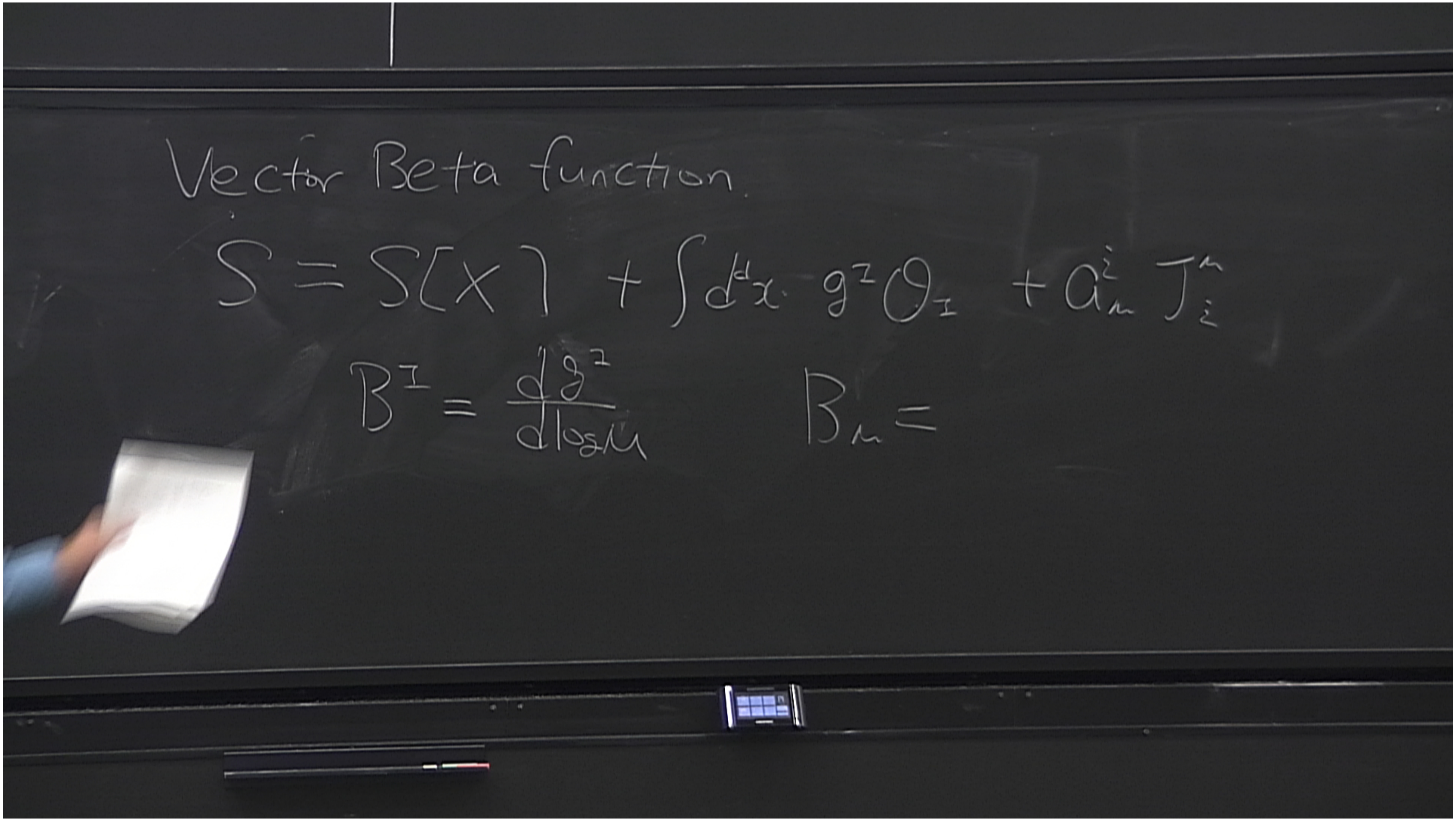
URL: http://pirsa.org/13100082

Abstract: We propose various properties of renormalization group beta functions for vector operators in relativistic quantum field theories. We argue that they must satisfy compensated gauge invariance, orthogonality with respect to scalar beta functions, Higgs-like relation among anomalous dimensions and a gradient property. We further conjecture that non-renormalization holds if and only if the vector operator is conserved. The local renormalization group analysis guarantees the first three within power counting renormalization. We verify all the conjectures in conformal perturbation theories and holography in the weakly coupled gravity regime.


Vector Beta function
$S = S[X] + \int d^d x \, g^I \mathcal{O}_I + a^i_\mu J^\mu_i$
$B^I = \frac{d g^I}{d \log \mu}$
$B_\mu =$

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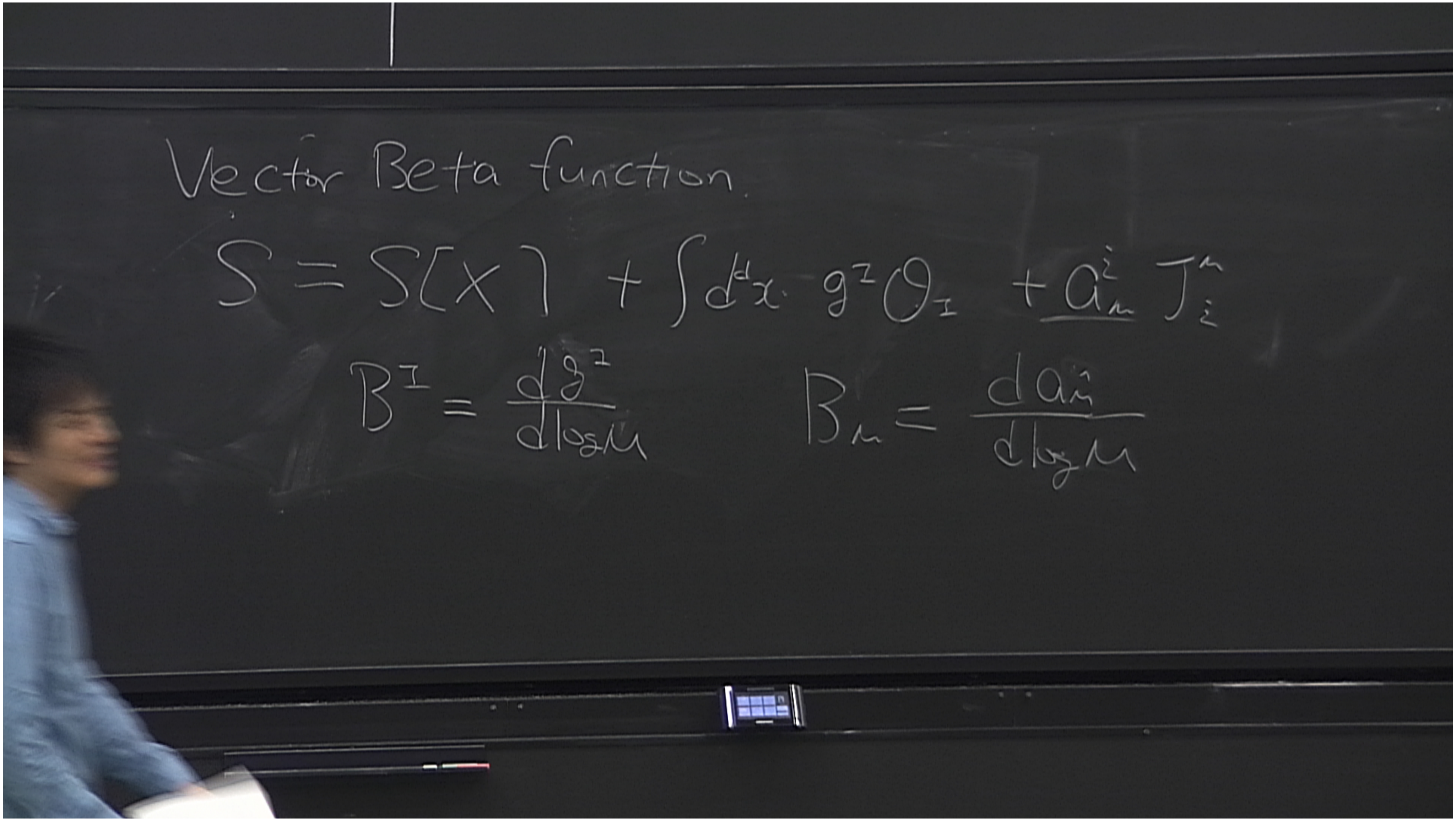
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2. Orthogonality.

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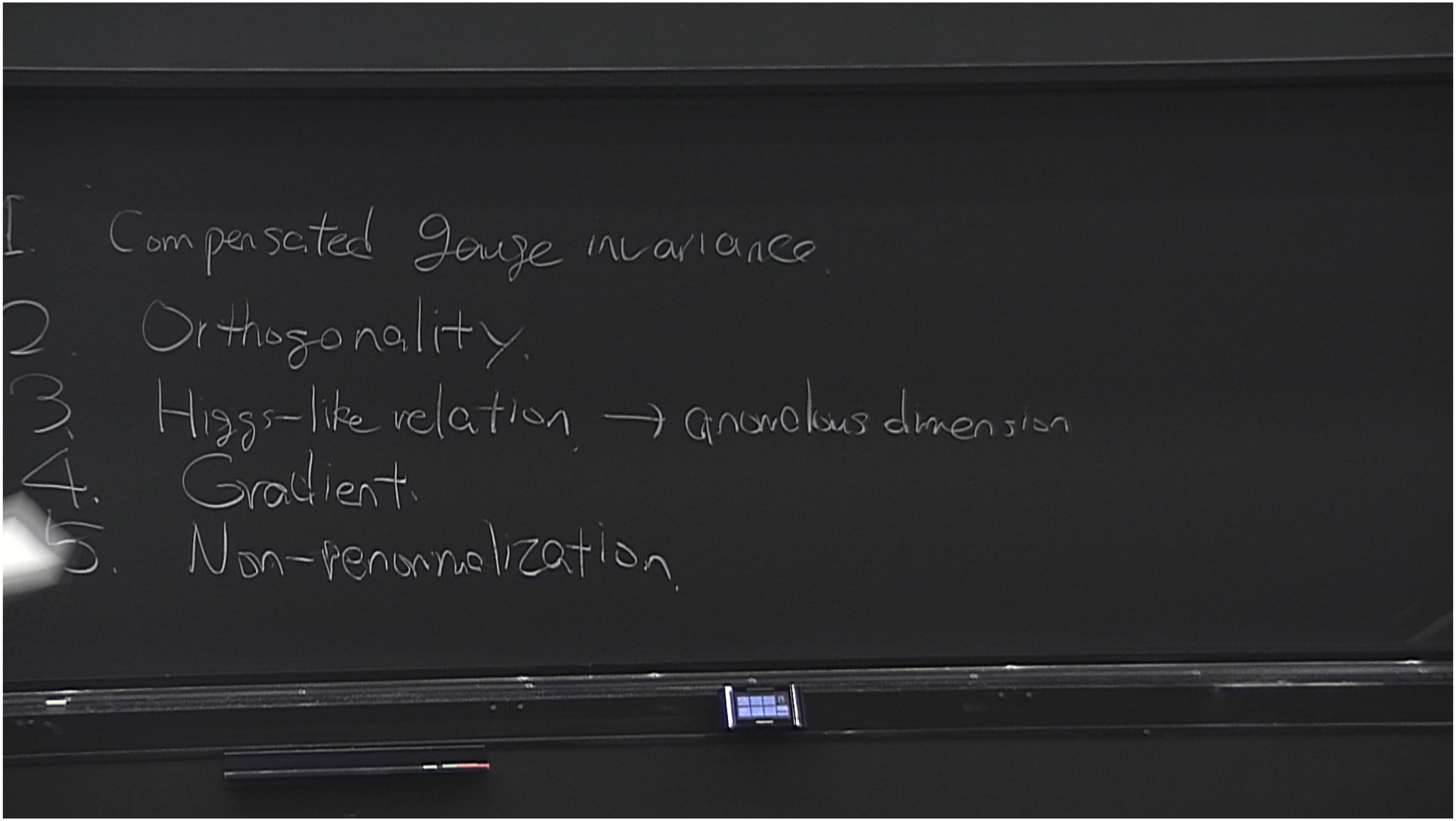
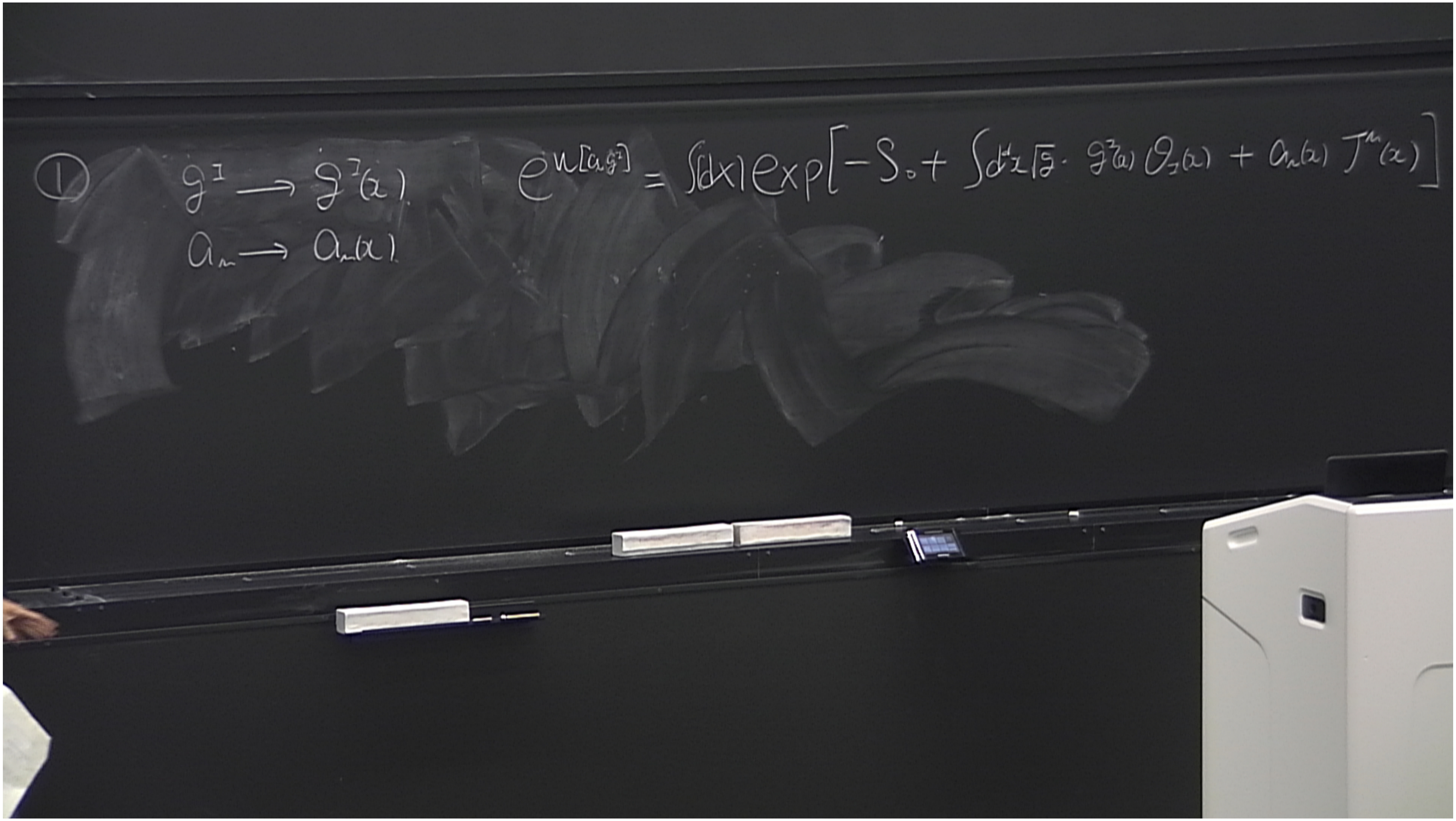
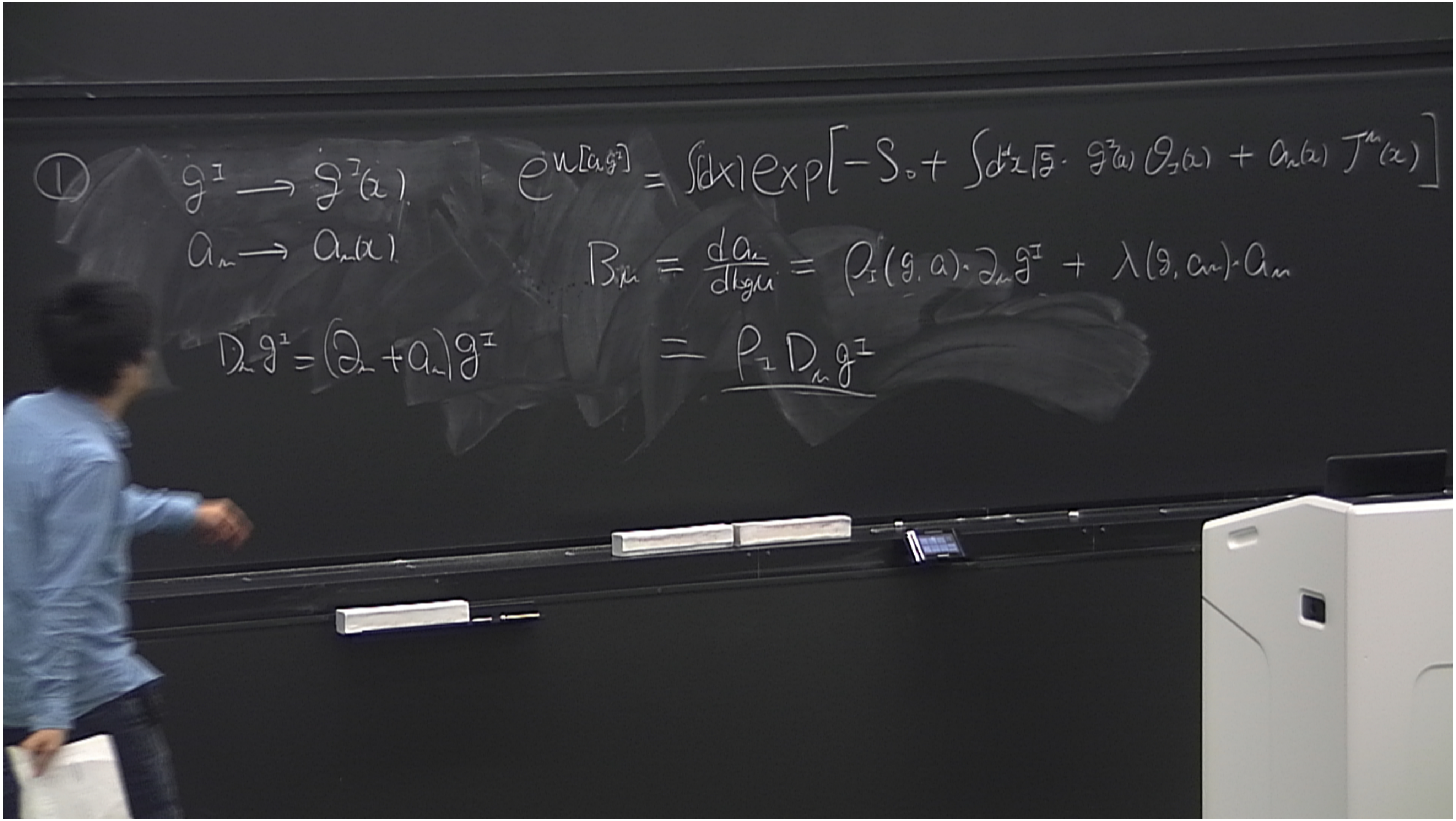
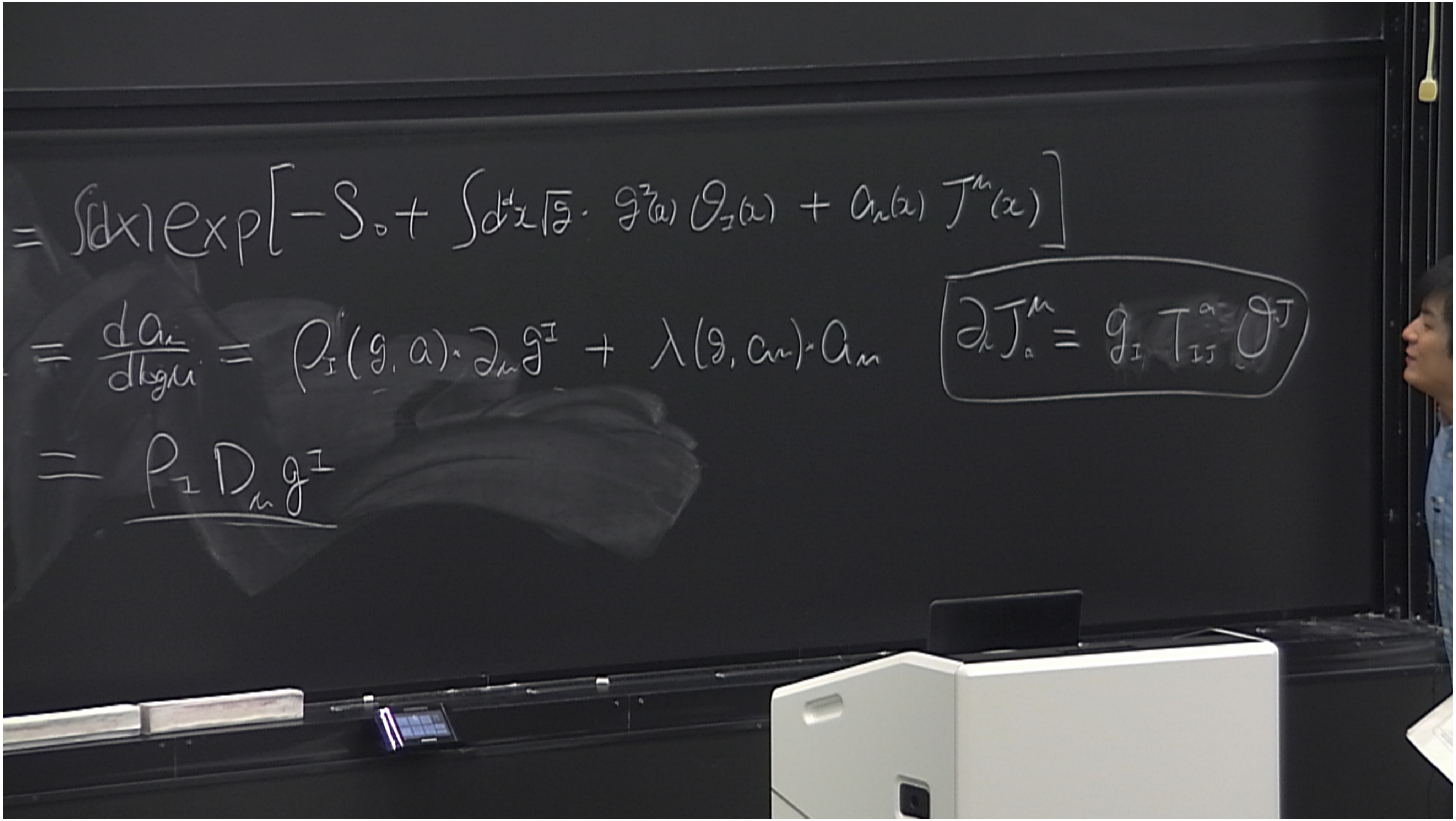
3. Higgs-like relation. → anomalous dimension
4. Gradient.

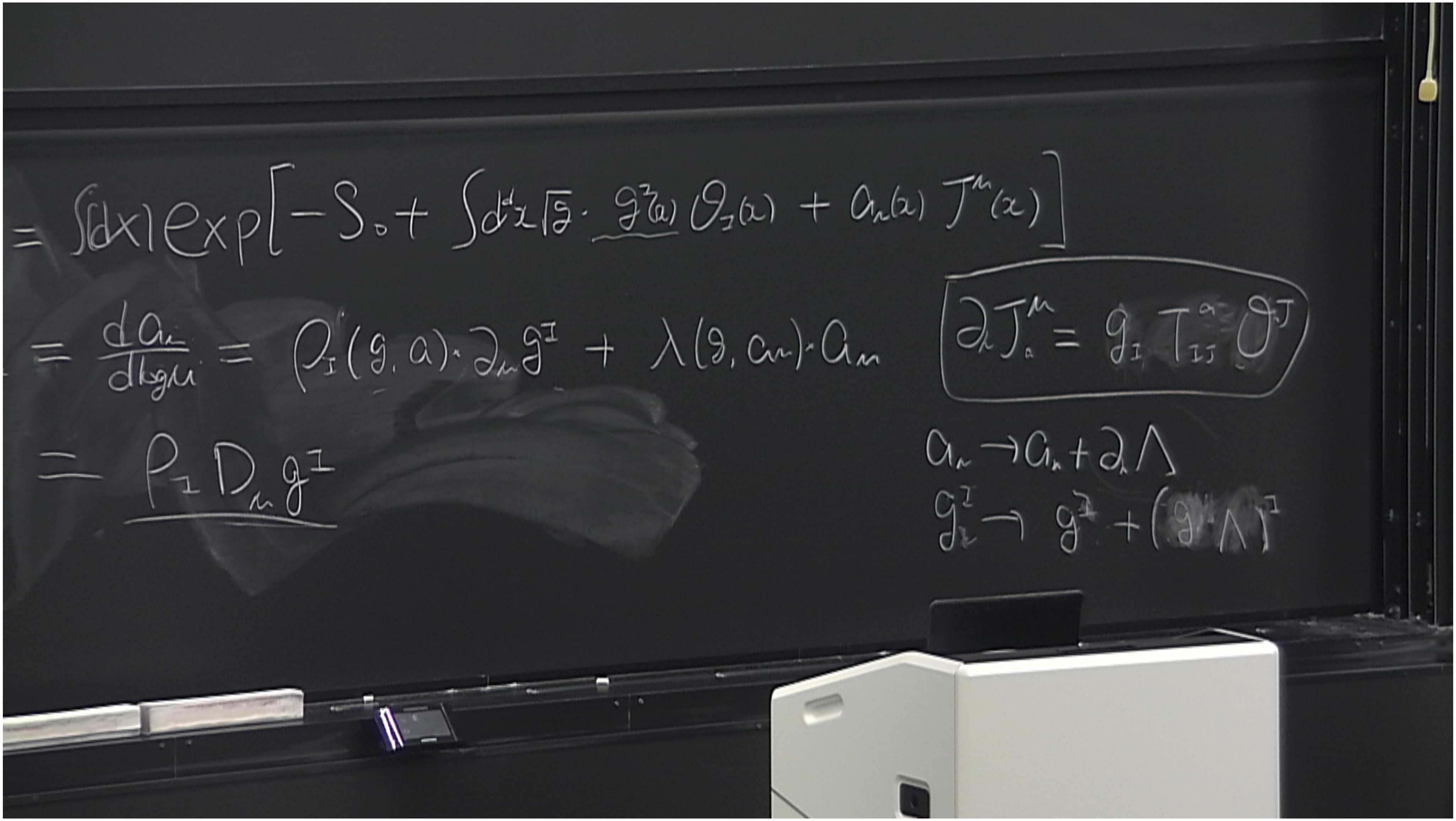

1. Compensated gauge invariance
2. Orthogonality
3. Higgs-like relation → anomalous dimension
4. Gradient
5. Non-renormalization


① $g^I \longrightarrow g^I(x)$
$a_m \longrightarrow a_m(x)$
$e^{W[a,g^I]} = \int [dx] \exp\left[-S_0 + \int d^dx \sqrt{g} \cdot g^I(x)\, \mathcal{O}_I(x) + a_m(x)\, J^m(x)\right]$


① $g^I \longrightarrow g^I(x)$
$a_\mu \longrightarrow a_\mu(x)$
$D_\mu g^I = (\partial_\mu + a_\mu) g^I$
$e^{W[a,g^I]} = \int [dx] \exp\left[-S_0 + \int d^4x \sqrt{g}\, g^I(x)\, O_I(x) + a_\mu(x) J^\mu(x)\right]$
$B_\mu = \frac{d a_\mu}{d \log \mu} = \rho_I(g, a) \cdot \partial_\mu g^I + \lambda(g, a_\mu) \cdot a_\mu$
$= \rho_I D_\mu g^I$



$= \int [dx] \exp\left[-S_0 + \int d^d x \sqrt{g} \cdot g^I(x)\, \mathcal{O}_I(x) + a_\mu(x)\, J^\mu(x) \right]$
$= \frac{d a_\mu}{d \log \mu} = \rho_I(g, a) \cdot \partial_\mu g^I + \lambda(g, a_\mu) \cdot a_\mu$
$\boxed{\partial_\mu J^\mu_a = g_I\, T^a_{IJ}\, \mathcal{O}^J}$
$= \rho_I D_\mu g^I$


$= \int [dx] \exp\left[-S_0 + \int d^d x \sqrt{g} \cdot g^I(x)\, \mathcal{O}_I(x) + a_\mu(x)\, J^\mu(x) \right]$
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$= \rho_I D_\mu g^I$
$\partial_\mu J^\mu_a = g_I T^a_{IJ} \mathcal{O}^J$
$a_\mu \to a_\mu + \partial_\mu \Lambda$
$g^I \to g^I + (\;\; \Lambda)^I$

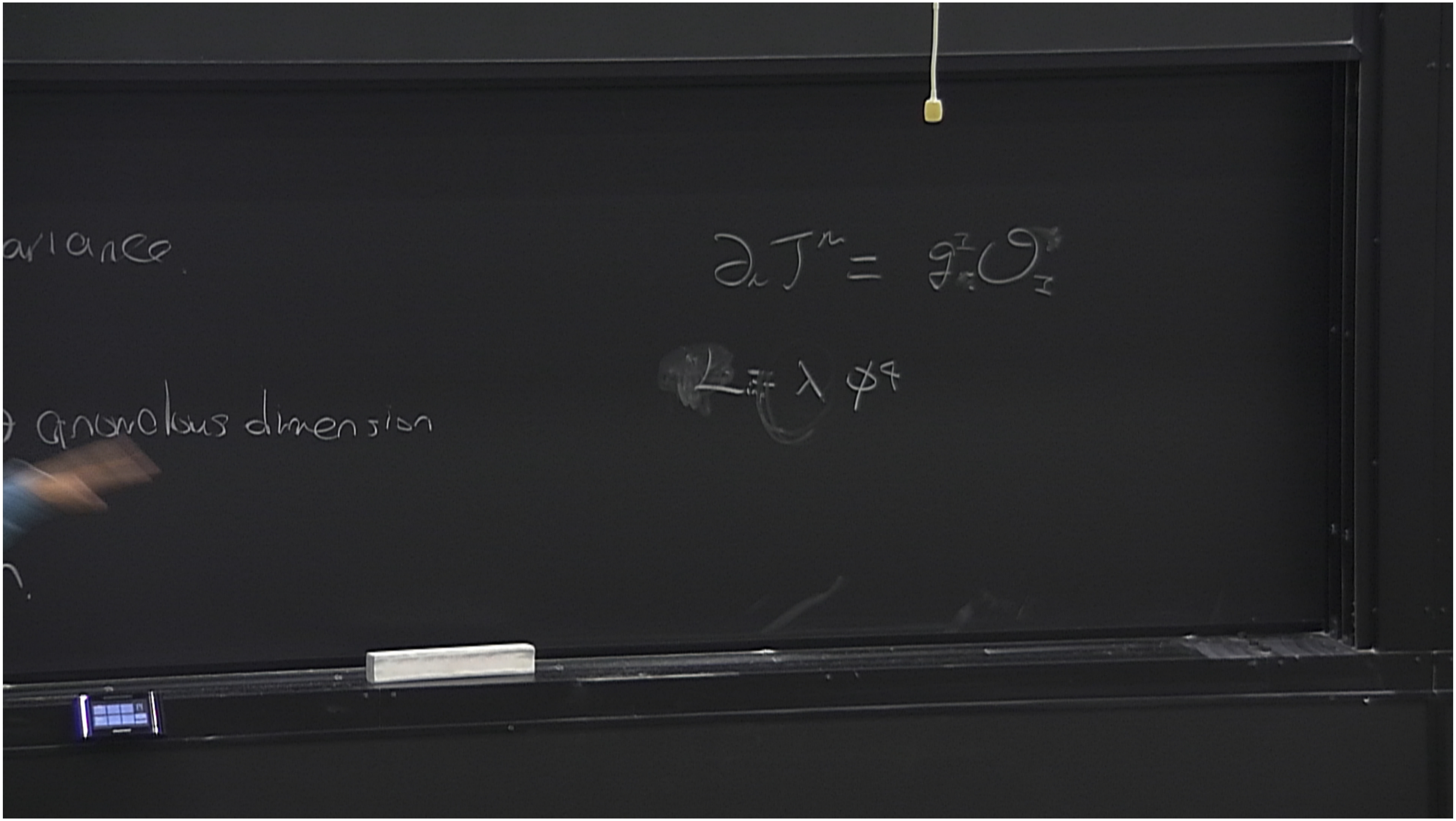
(2)

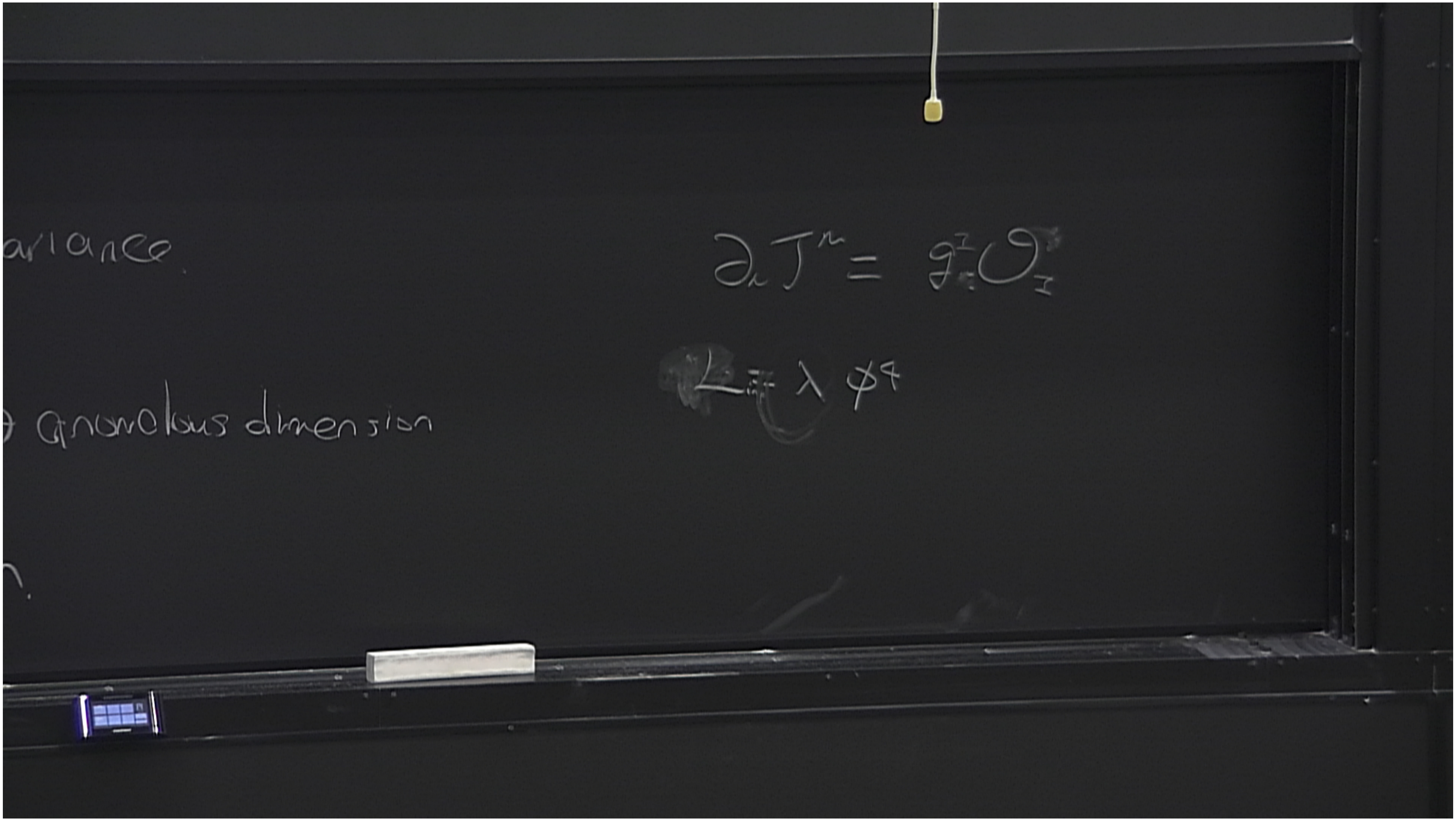
$$\frac{da^a_\mu}{d\log\mu} = B^a_\mu = \rho^a_I D_\mu g^I$$

$$\boxed{\rho_I^a \times B^I = 0}$$

Stand

$$\frac{dg^I}{d\log\mu} = B^I.$$


ariance.
∂_μ J^μ = g^I O_I
L = λ φ^4
anomalous dimension

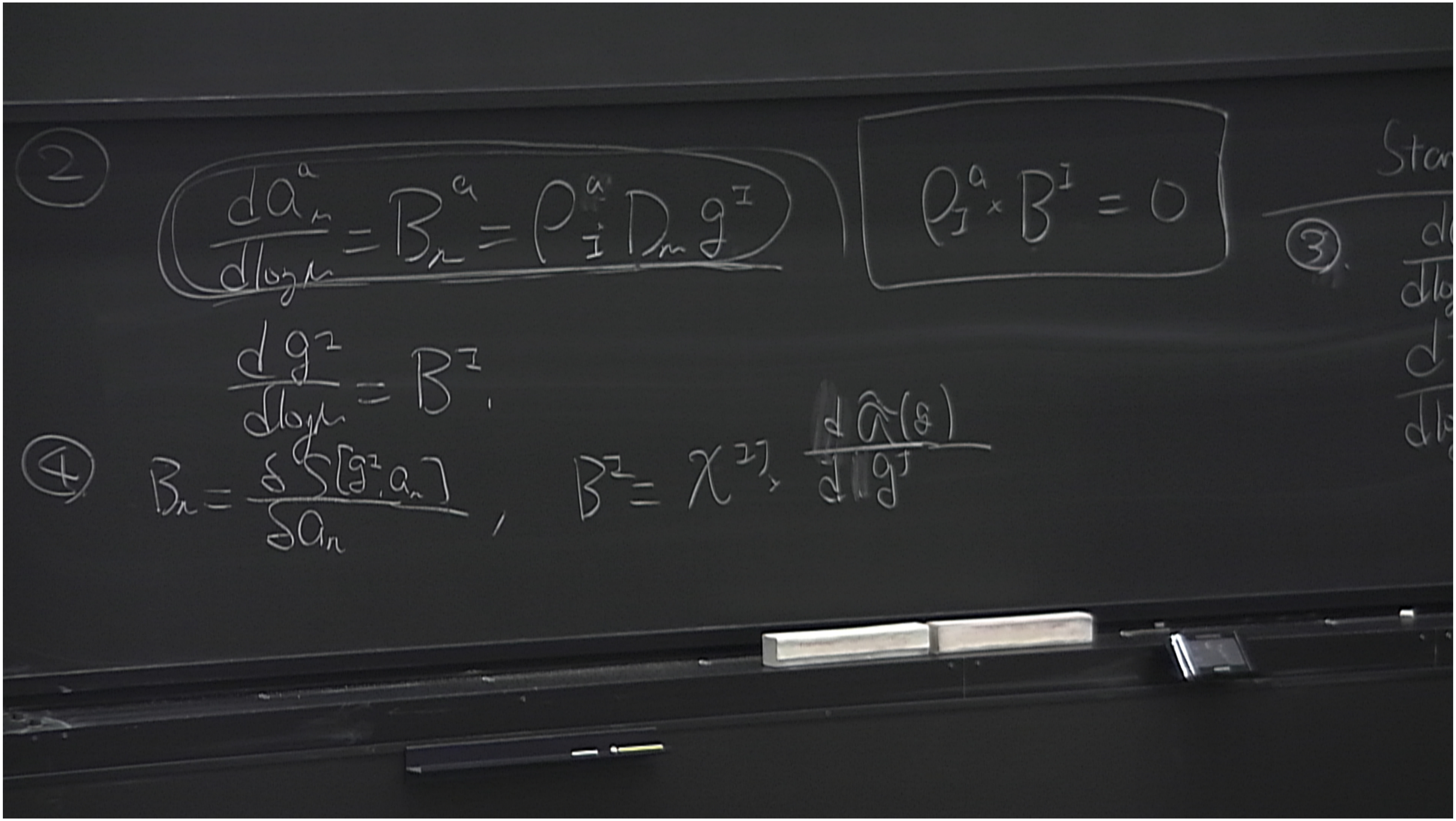

ariance
anomalous dimension
$\partial_\mu J^\mu = g^I \mathcal{O}_I$
$\mathcal{L} = \lambda \phi^4$

② $$\frac{da^a_\mu}{d\log\mu} = B^a_\mu = \rho^a_I D_\mu g^I$$

$$\boxed{\rho_I^a \times B^I = 0}$$

$$\frac{dg^I}{d\log\mu} = B^I.$$

④ $$B_\mu = \frac{\delta S[g^I, a_\mu]}{\delta a_\mu}, \quad B^I = \chi^{IJ} \frac{d\hat{a}(g)}{dg^J}$$

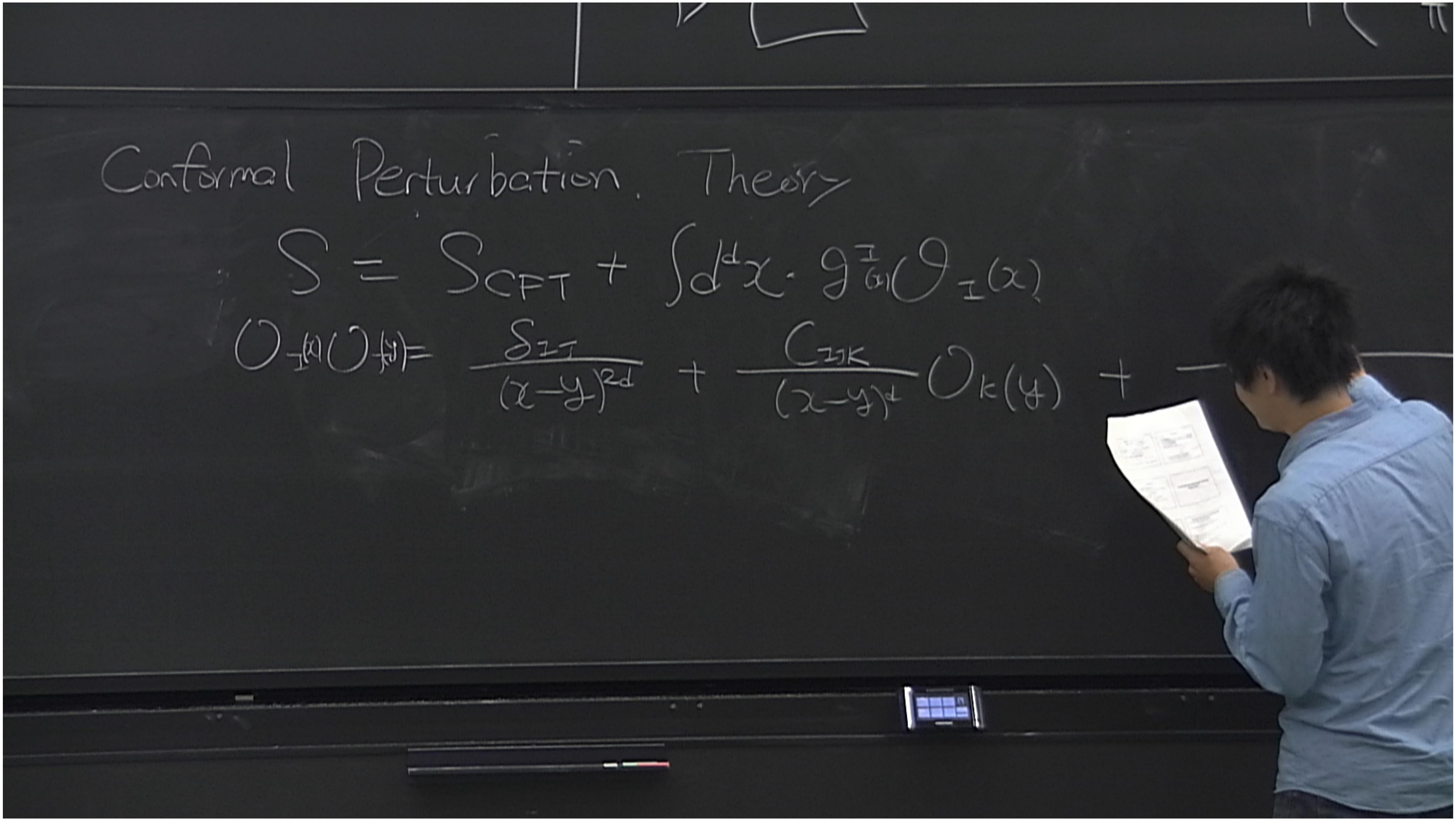


2
$\frac{da^a_\mu}{d\log\mu} = B^a_\mu = \rho^a_I D_\mu g^I$
$\rho^a_I B^I = 0$
$\frac{dg^I}{d\log\mu} = B^I$
4
$B_\mu = \frac{\delta S[g^I, a_\mu]}{\delta a_\mu}$, $B^I = \chi^{IJ} \frac{d\hat{a}(g)}{dg^J}$

Conformal Perturbation Theory

$$S = S_{CFT} + \int d^d x \cdot g^I(x) \, \mathcal{O}_I(x)$$

$$\mathcal{O}_I(x)\,\mathcal{O}_J(y) = \frac{\delta_{IJ}}{(x-y)^{2d}}$$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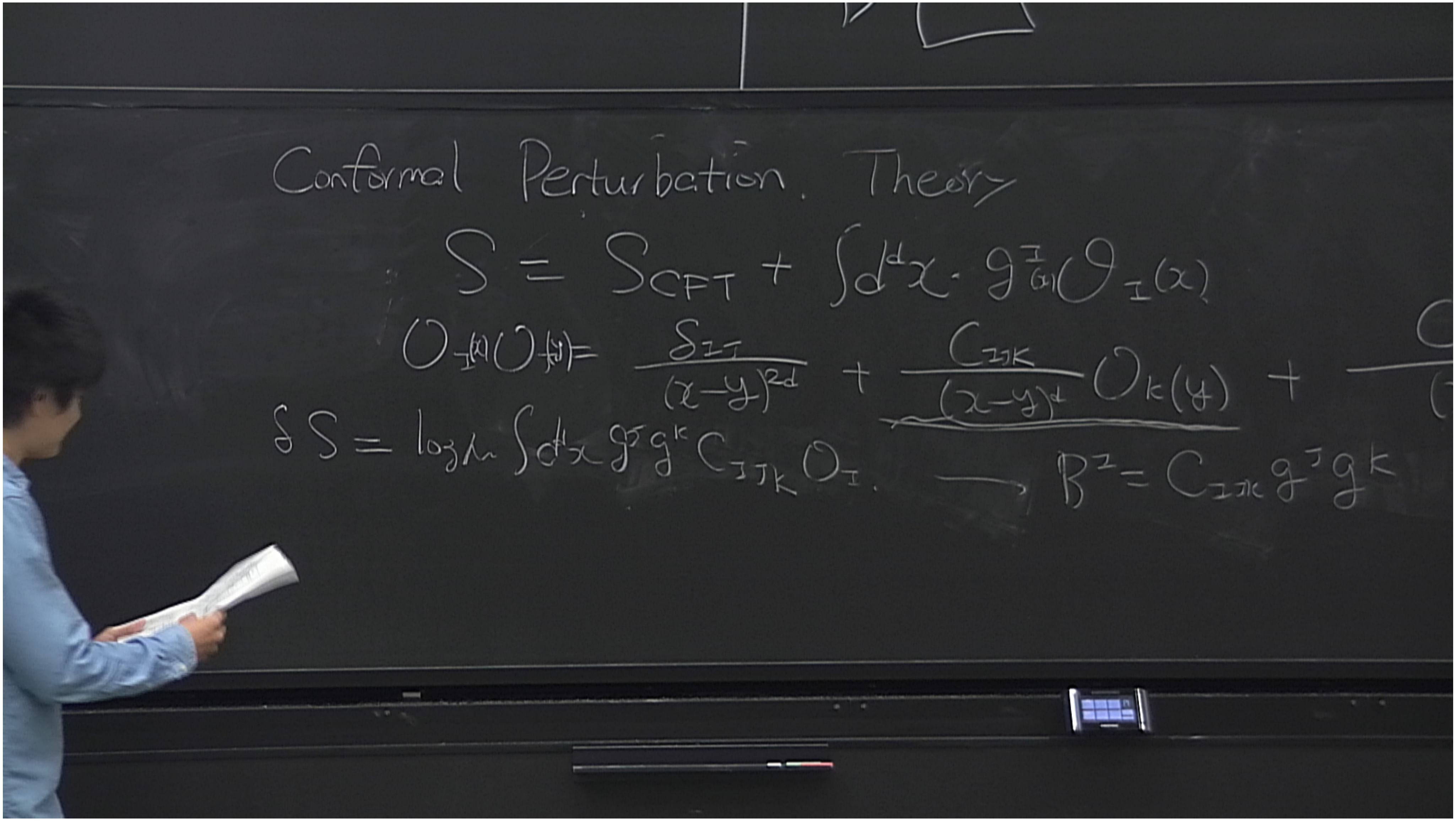
$S = S_{CFT} + \int d^dx \, g^I(x) \mathcal{O}_I(x)$
$\mathcal{O}_I(x)\mathcal{O}_J(y) = \frac{\delta_{IJ}}{(x-y)^{2d}} + \frac{C_{IJK}}{(x-y)^d}\mathcal{O}_K(y) + \cdots$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$\delta S = \log\lambda \int d^d x\, g^J g^K C_{IJK} \mathcal{O}_I \longrightarrow \beta^I = C_{IJK} g^J g^K$

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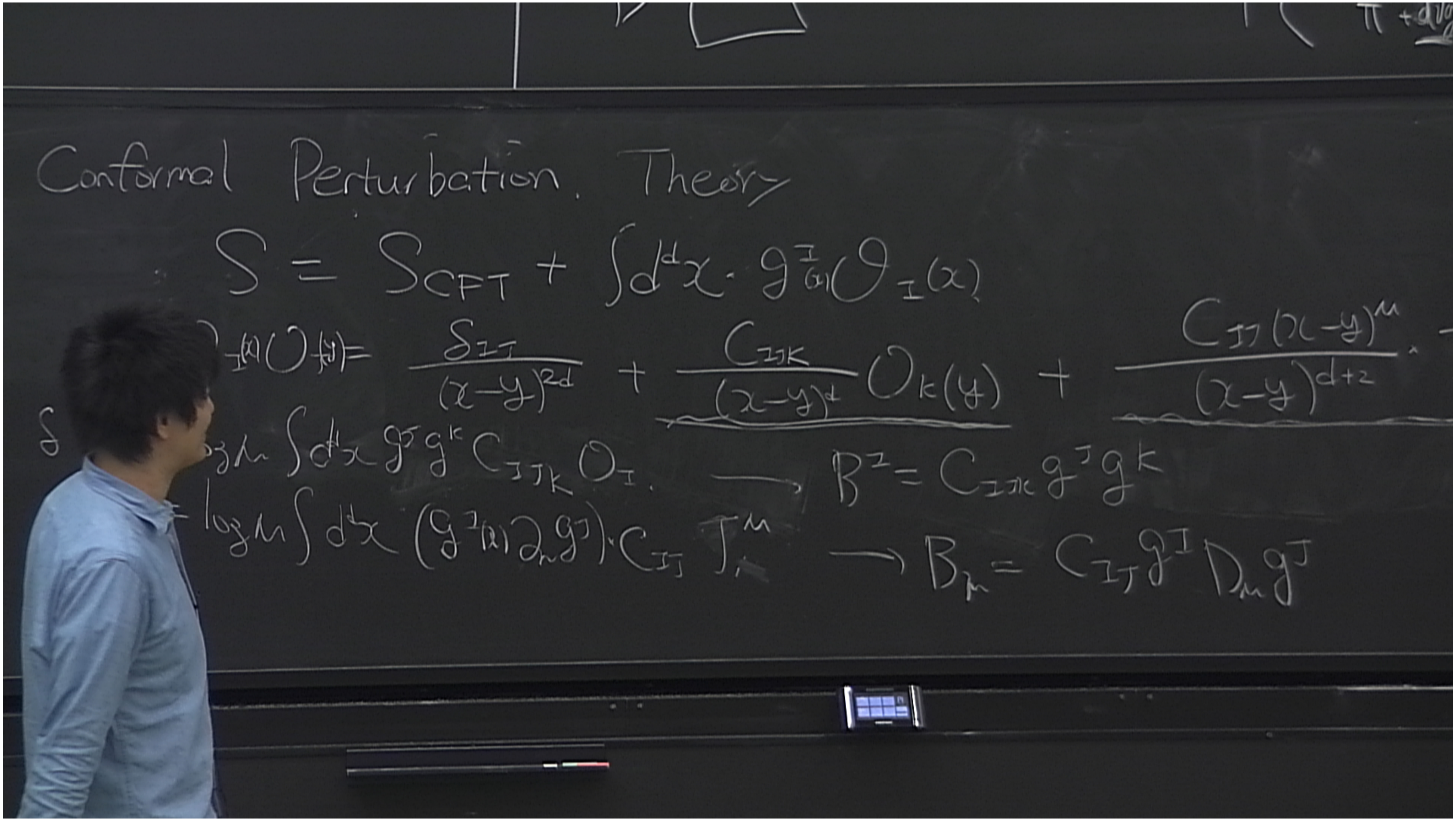
$$\delta S = \log\mu \int d^d x\, g^J g^K C_{IJK}\, \mathcal{O}_I \longrightarrow \beta^I = C_{IJK}\, g^J g^K$$

$$=$$


Conformal Perturbation Theory
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$\log\mu \int d^dx\, g^J g^K C_{IJK} \mathcal{O}_I \longrightarrow \beta^I = C_{IJK} g^J g^K$
$\log\mu \int d^dx\, (g^I(x) \partial_\mu g^J) \cdot C_{IJ} J^\mu \longrightarrow B_\mu = C_{IJ} g^I D_\mu g^J$

$\pi + d\sqrt{g} = 0$

$H^{ab} R_{ab}$

$$\boxed{D_\mu J^\mu = C_{IJ}\, g^J \cdot \mathcal{O}^I}$$

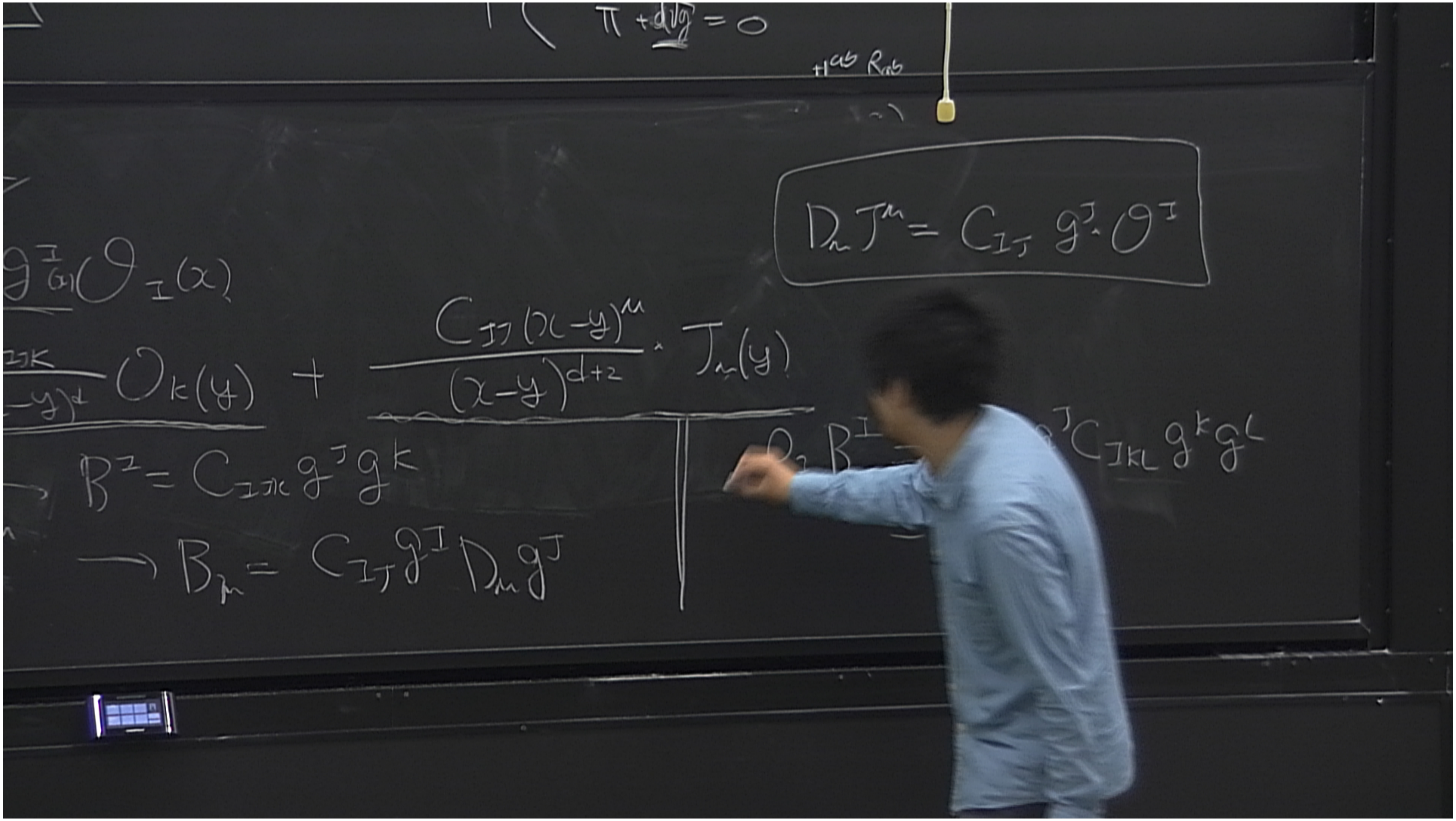
$g^I(x)\, \mathcal{O}_I(x)$

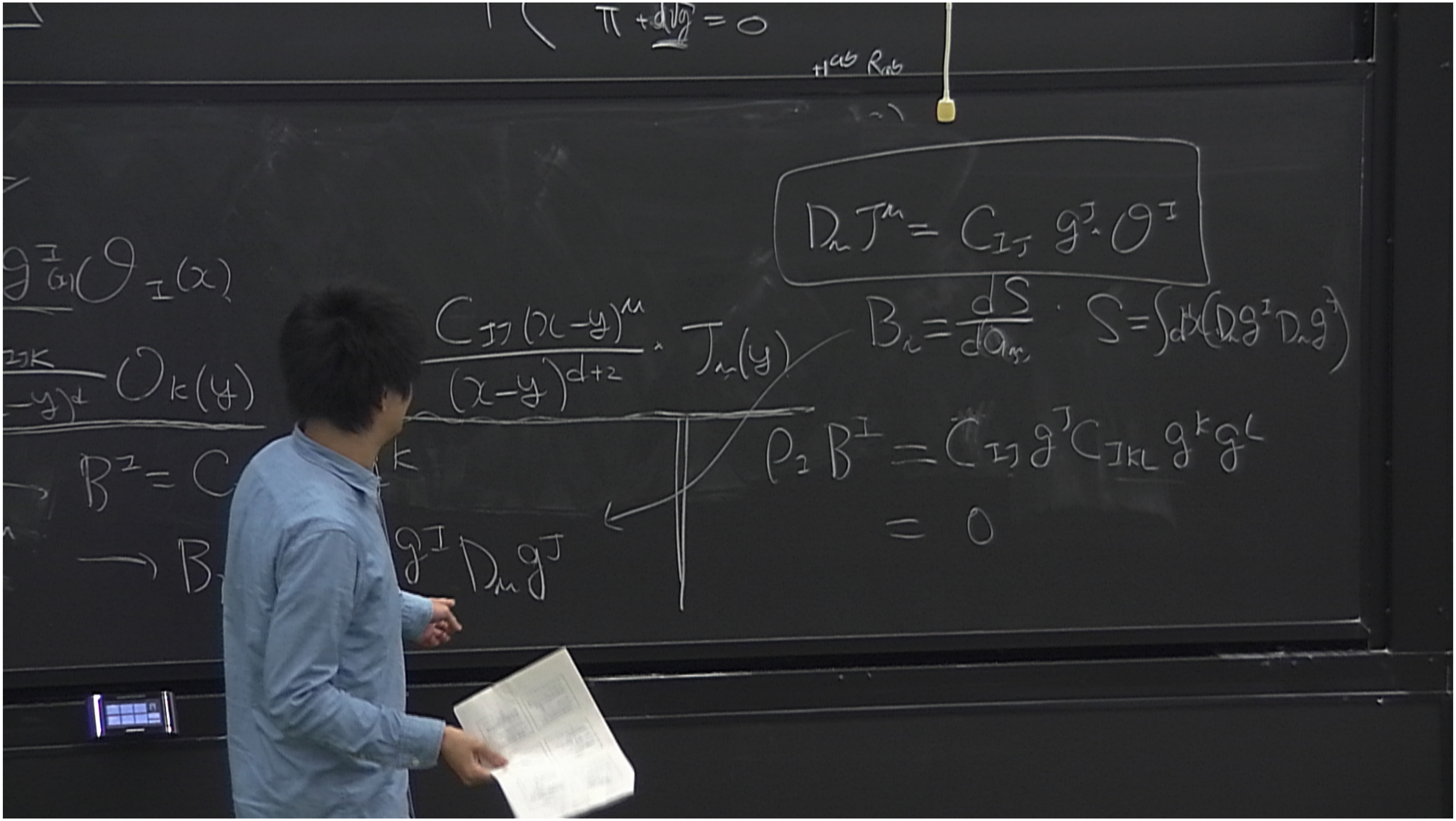
$$\frac{C_{IJK}}{(x-y)^d}\, \mathcal{O}_K(y) + \frac{C_{IJ}(x-y)^\mu}{(x-y)^{d+2}} \cdot J_\mu(y)$$

$\to B^I = C_{IJK}\, g^J g^K$

$\to B_\mu = C_{IJ}\, g^I D_\mu g^J$

$\rho_I B^I = C_{IJ}\, g^J C_{JKL}\, g^K g^L$


D_μ J^μ = C_IJ g^J O^I
B^I = C_IJK g^J g^K
B_μ = C_IJ g^I D_μ g^J


$\boxed{D_\mu J^\mu = C_{IJ}\, g^J \, \mathcal{O}^I}$
$B_\mu = \frac{dS}{d a_\mu}$
$\rho_I B^I = C_{IJ}\, g^J C_{JKL}\, g^K g^L$
$= 0$

$$\pi + d\sqrt{g} = 0$$

$$+ H^{ab} R_{ab}$$

$$\boxed{D_\mu J^\mu = C_{IJ}\, g^J \mathcal{O}^I}$$

$$g^I(x)\, \mathcal{O}_I(x)$$

$$\frac{C_{IJK}}{(x-y)^d}\, \mathcal{O}_K(y) + \frac{C_{IJ}(x-y)^\mu}{(x-y)^{d+2}}\, J_\mu(y)$$

$$B_\mu = \frac{dS}{da_\mu} \qquad S = \int d^dx \left(D_\mu g^I D_\mu g^J\right)$$

$$\rightarrow B^I = C_{IJK}\, g^J g^K$$

$$\rho_I B^I = C_{IJ}\, g^J C_{JKL}\, g^K g^L$$

$$= 0$$

$$\rightarrow B_\mu = C_{IJ}\, g^I D_\mu g^J$$

$\pi + d\sqrt{g} = 0$

$+ H^{ab} R_{ab}$

$V = C_{IJK}\, g^I g^J g^K$

$$\boxed{D_\mu J^\mu = C_{IJ}\, g^J \cdot \mathcal{O}^I}$$

$g^I_{(x)} \mathcal{O}_I(x)$

$\frac{C_{IJK}}{(x-y)^d} \mathcal{O}_K(y) + \frac{C_{IJ}(x-y)^\mu}{(x-y)^{d+2}} \cdot J_\mu(y)$

$B_i = \frac{dS}{d a_{\text{vec}}} \cdot \quad S_{\text{vector}} = \int d^d x \left(D_\mu g^I D_\mu g^J\right)$

$\rightarrow B^I = C_{IJK}\, g^J g^K$

$\rightarrow B_\mu = C_{IJ}\, g^I D_\mu g^J$

$\rho_I B^I = C_{IJ}\, g^J C_{JKL}\, g^K g^L$

$= 0$

$$\pi + d\sqrt{g} = 0$$

$$H^{ab} R_{ab}$$

$$V = C_{IJK}\, g^I g^J g^K$$

$$\boxed{D_\mu J^\mu = C_{IJ}\, g^J \cdot O^I}$$

$$g^I_{(x)} O_I(x)$$

$$\frac{C_{IJK}}{(x-y)^d} O_K(y) + \frac{C_{IJ}(x-y)^\mu}{(x-y)^{d+2}} \cdot J_\mu(y)$$

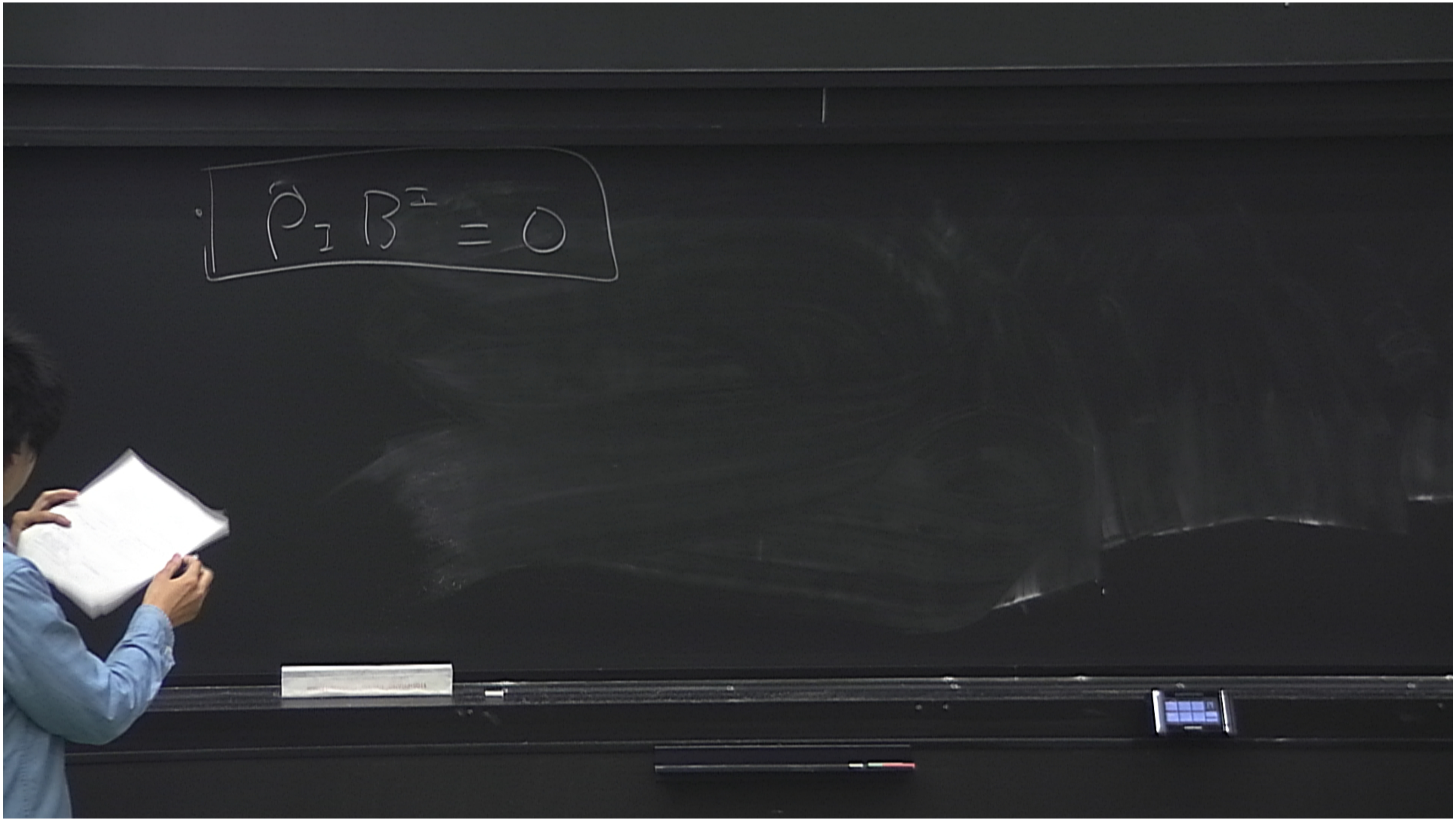
$$B_\mu = \frac{dS}{dg_{\mu}} \cdot \quad S_{\text{vector}} = \int d^dx\, (D_\mu g^I D_\mu g^J)$$

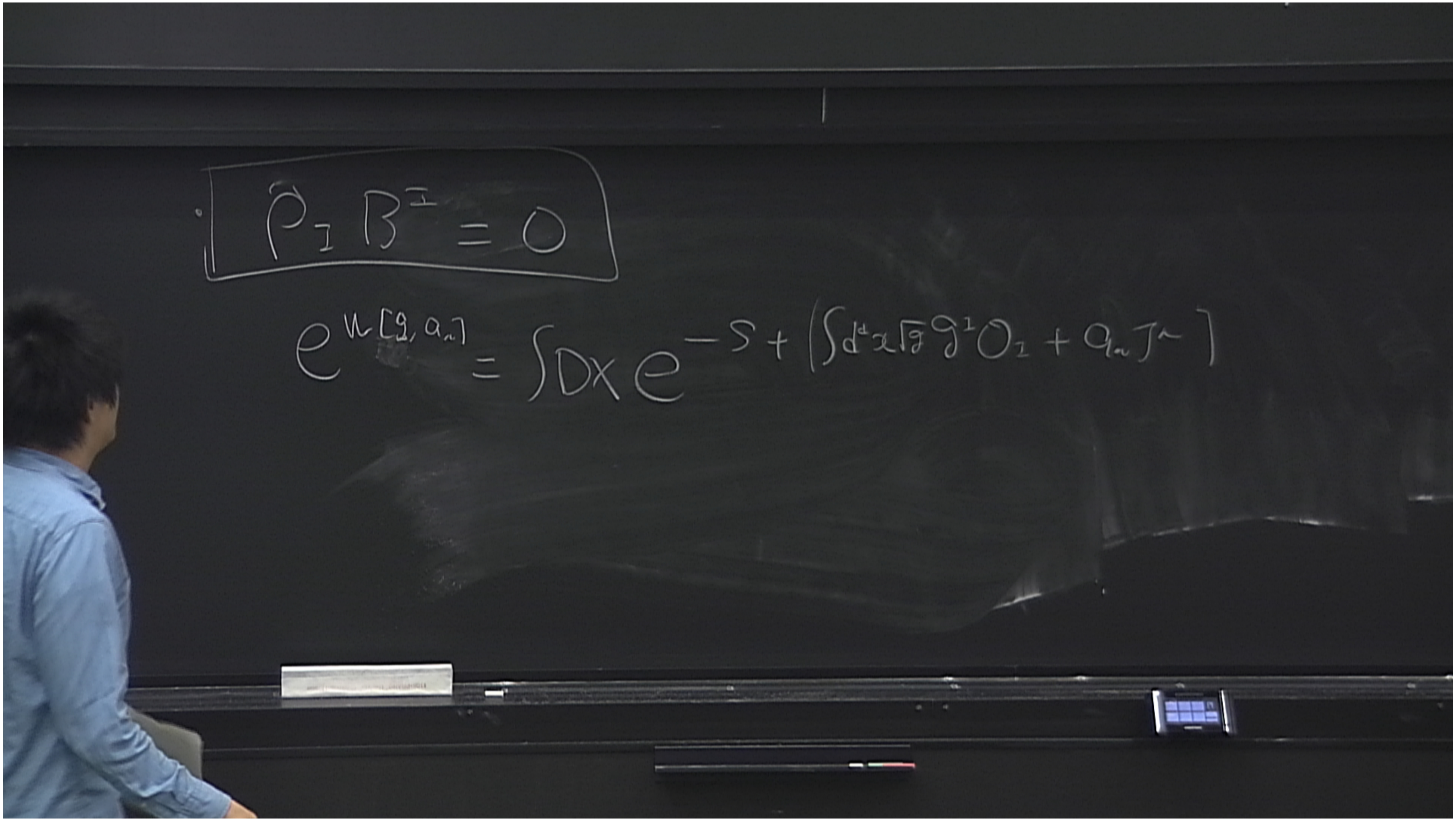
$$B^I = C_{IJK}\, g^J g^K \overset{?}{+} D g^J$$

$$\rho_I B^I = C_{IJ}\, g^J C_{JKL}\, g^K g^L$$

$$= 0$$

$$\longrightarrow B_\mu = C_{IJ}\, g^I D_\mu g^J$$


$\tilde{P}_I B^I = 0$

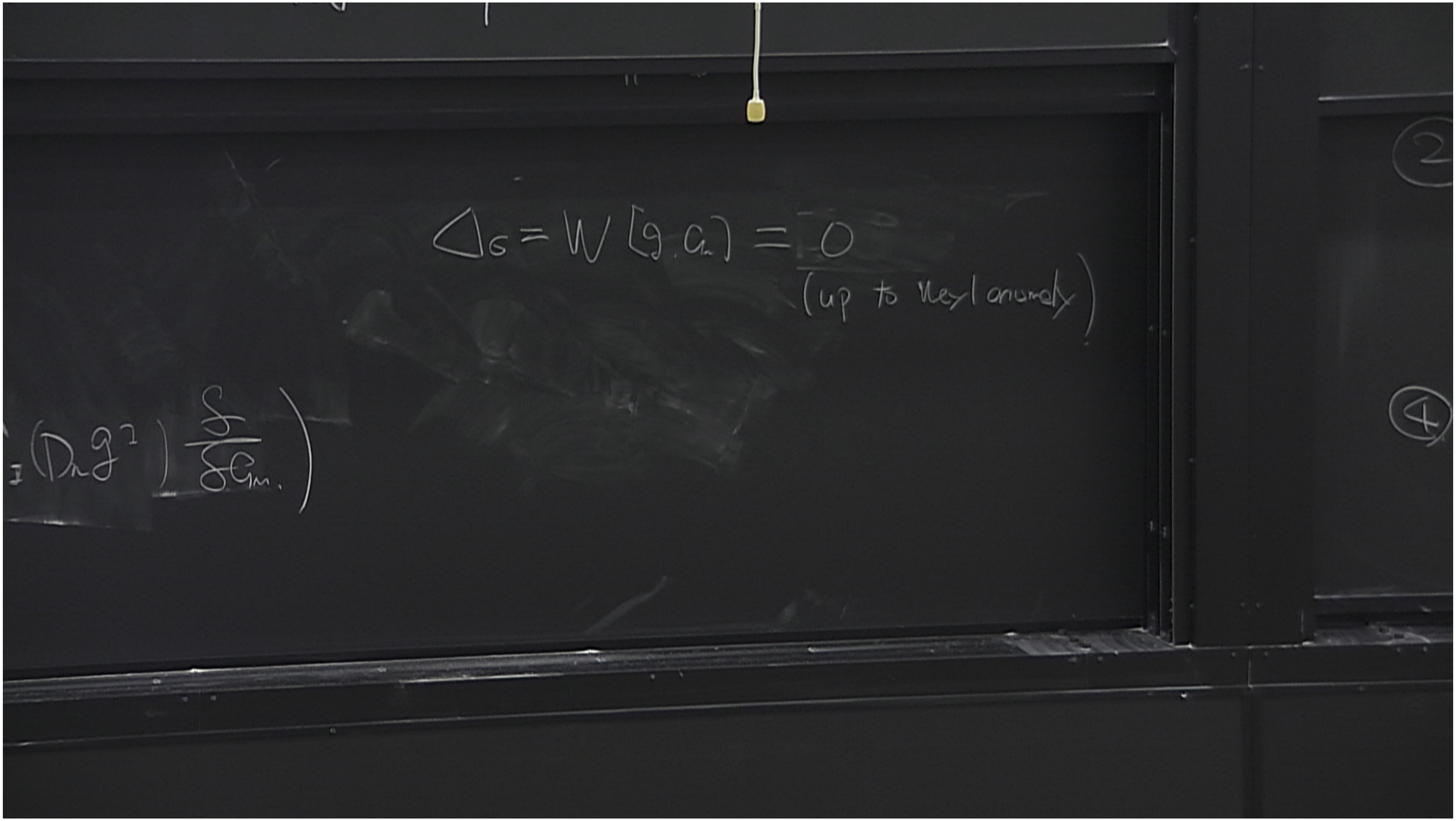


$\boxed{\hat{P}_I B^I = 0}$
$e^{W[g, a_\mu]} = \int Dx\, e^{-S + \left(\int d^d x \sqrt{g}\, g^I O_I + a_\mu J^\mu\right)}$

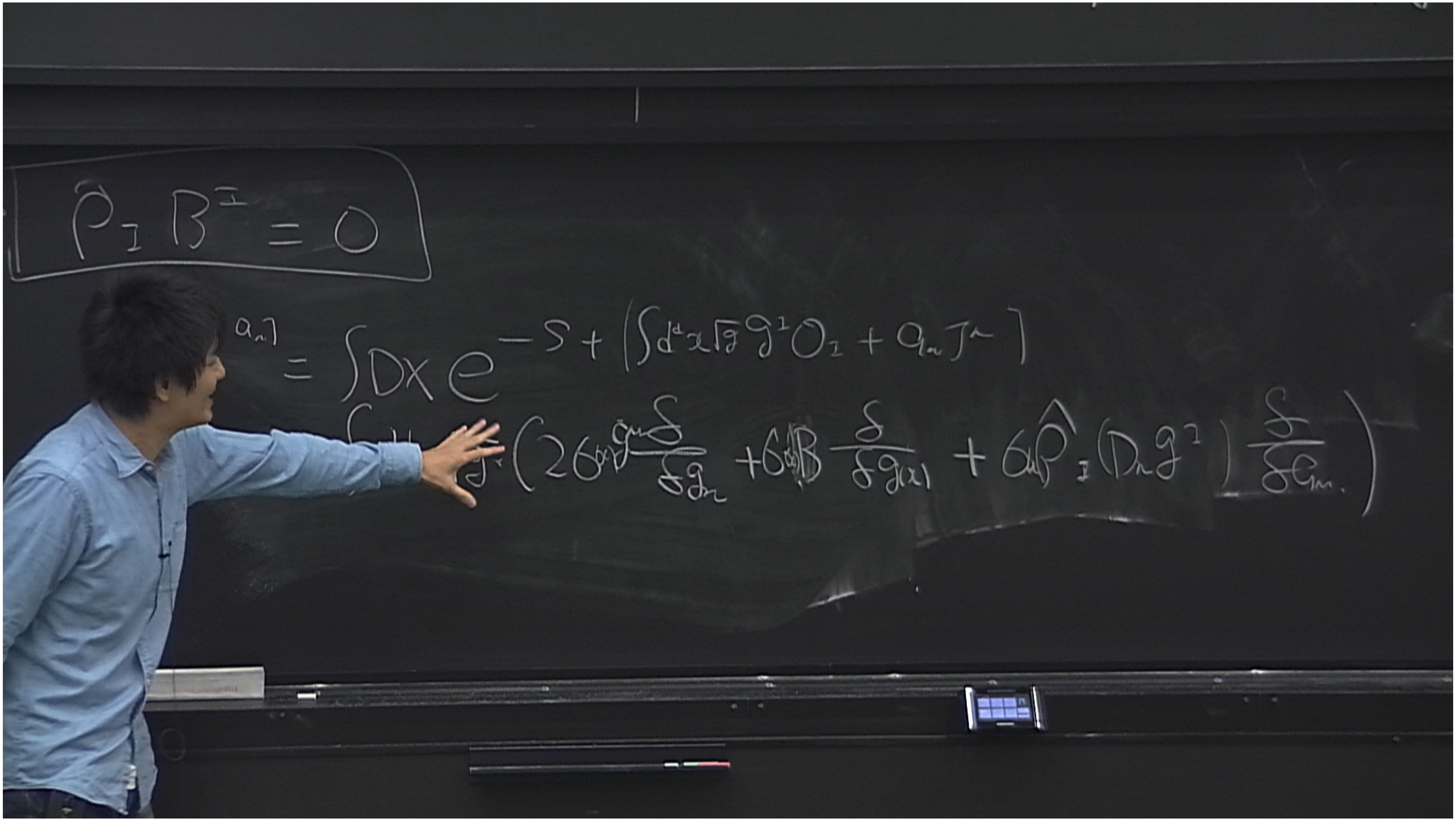
$$\boxed{\hat{\rho}_I B^I = 0}$$

$$e^{W[g, a_\mu]} = \int Dx\, e^{-S + \left(\int d^d x \sqrt{g}\, g^I O_I + a_\mu J^\mu\right)}$$

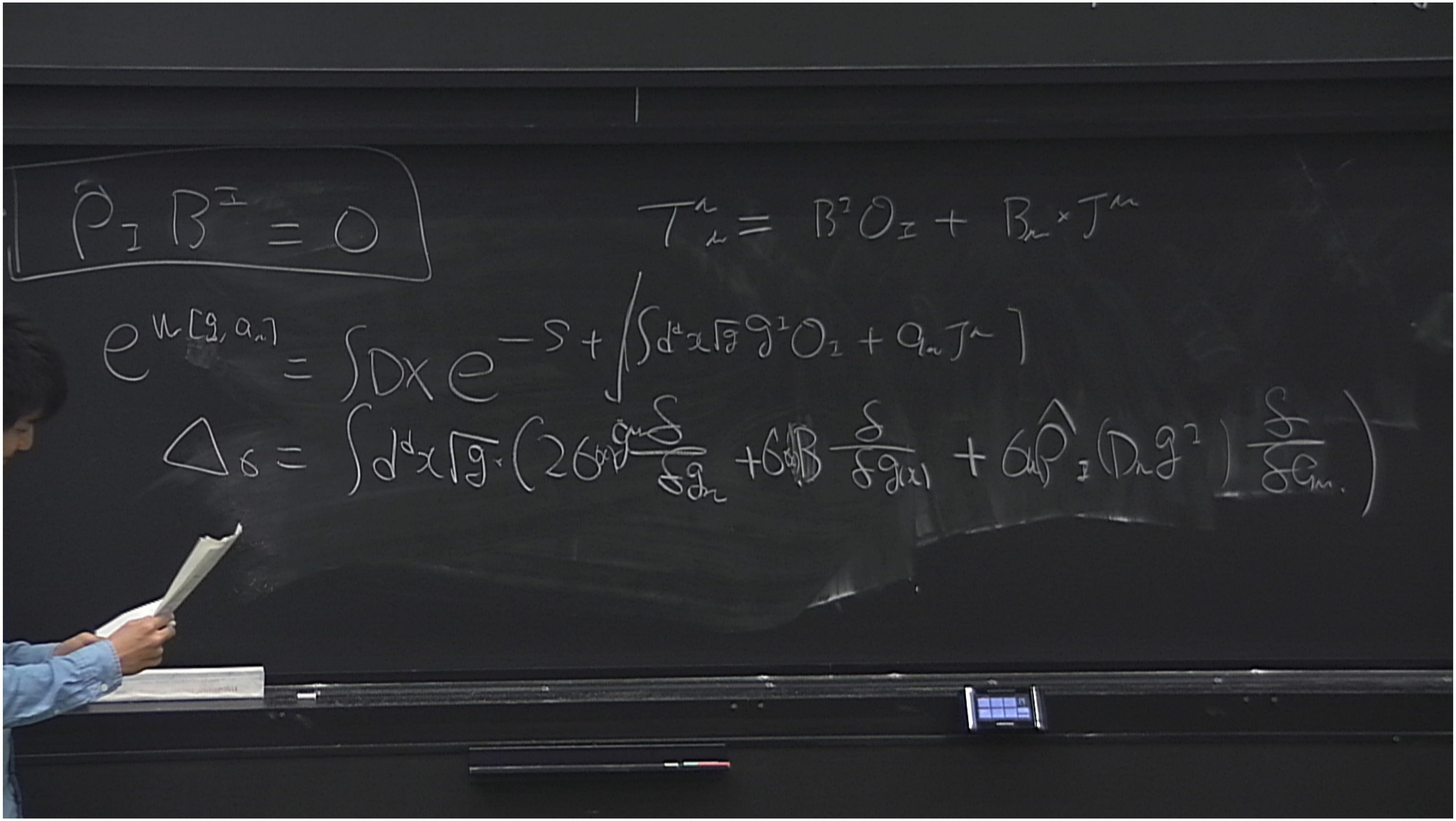
$$\Delta_\sigma = \int d^d x \sqrt{g}\cdot\left(26\, g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \sigma B \frac{\delta}{\delta g(x)} + \sigma \hat{\rho}_I (D_\mu g^I) \frac{\delta}{\delta a_\mu}\right)$$



$\Delta_G = W[g, a_m] = 0$
(up to Weyl anomaly)



$\hat{P}_I B^I = 0$
$a_\mu] = \int Dx\, e^{-S + (\int d^d x \sqrt{g}\, g^I O_I + a_\mu J^\mu)}$


$\boxed{\hat{\beta}_I B^I = 0}$
$T^\mu_\mu = B^I O_I + B_\mu J^\mu$
$e^{W[g, a_\mu]} = \int Dx\, e^{-S + \int d^dx \sqrt{g}\, g^I O_I + a_\mu J^\mu}$
$\Delta_\sigma = \int d^dx \sqrt{g} \left(2\sigma(x) g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \sigma(x) B \frac{\delta}{\delta g(x)} + \sigma(x) \hat{\rho}_I (D_\mu g^I) \frac{\delta}{\delta a_\mu} \right)$

$$+ \; B_\mu \times J^\mu$$

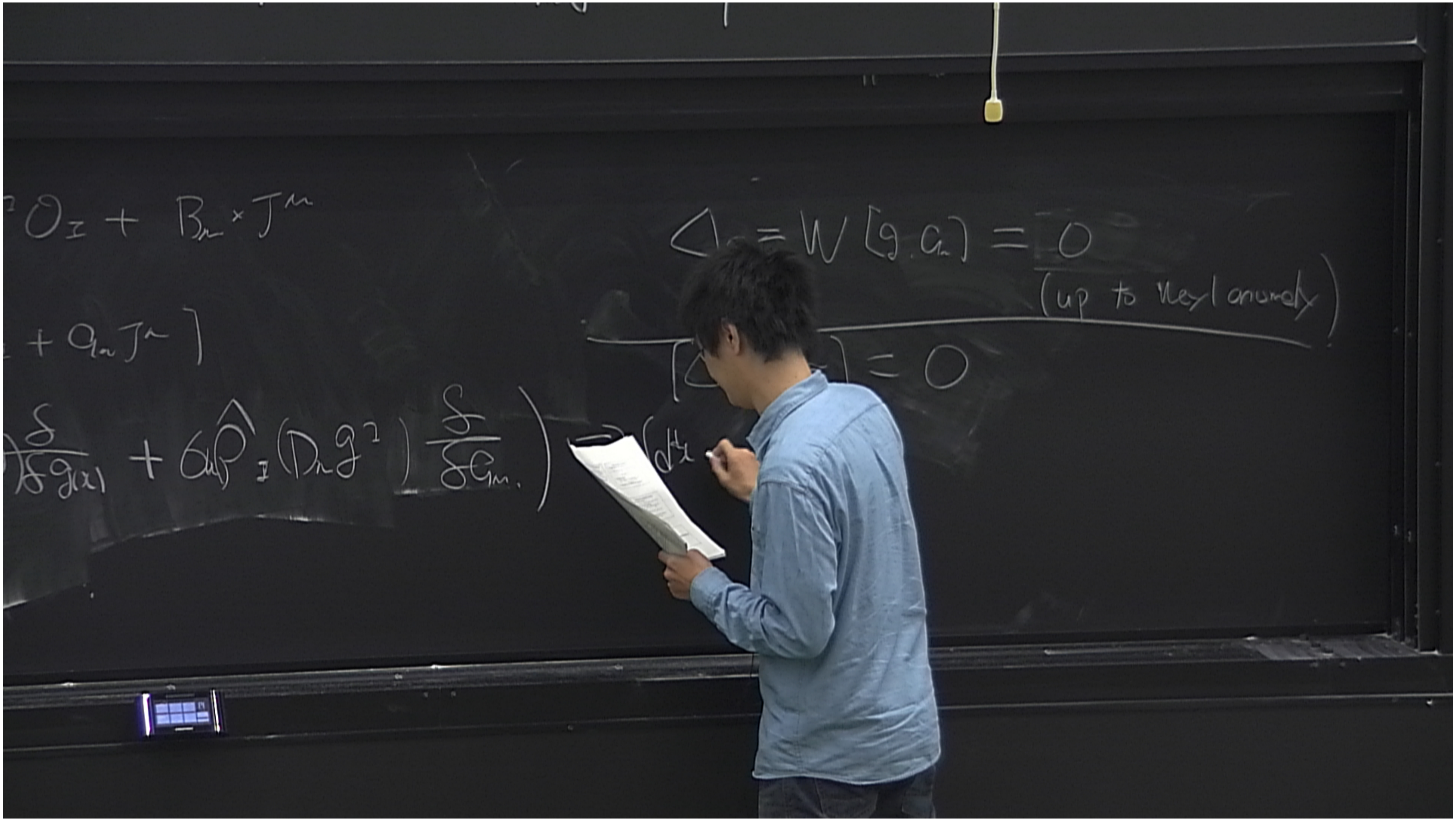
$$a_\mu J^\mu \,]$$

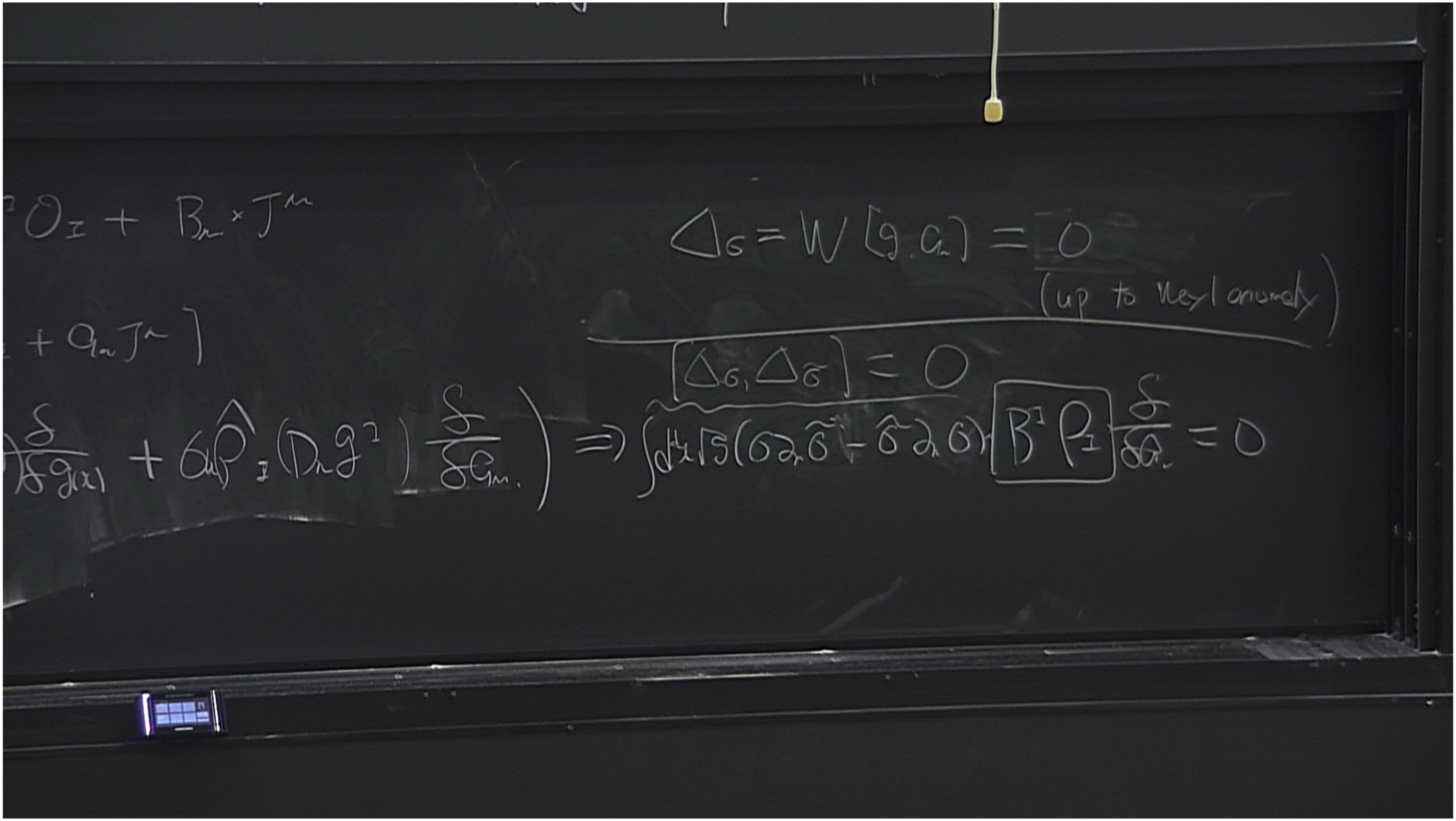
$$(x) + \sigma(x)\, \hat{\rho}_I \,(D_\mu g^I)\, \frac{\delta}{\delta a_\mu} \Big)$$

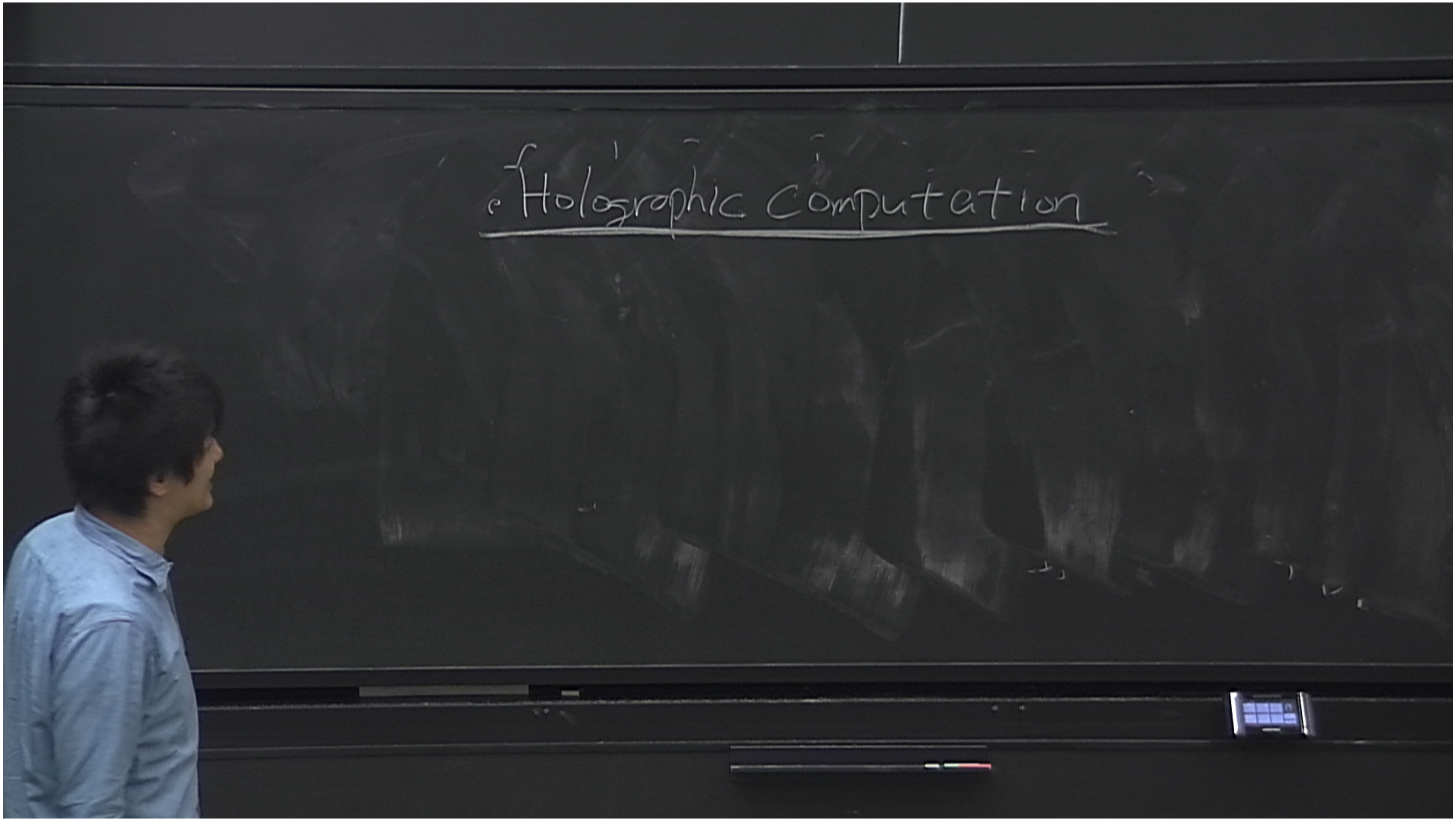
$$\Delta_\sigma = W[g, a_\mu] = 0$$

(up to Weyl anomaly)

$$[\Delta_\sigma, \Delta_{\tilde{\sigma}}] = 0$$


(up to Weyl anomaly)

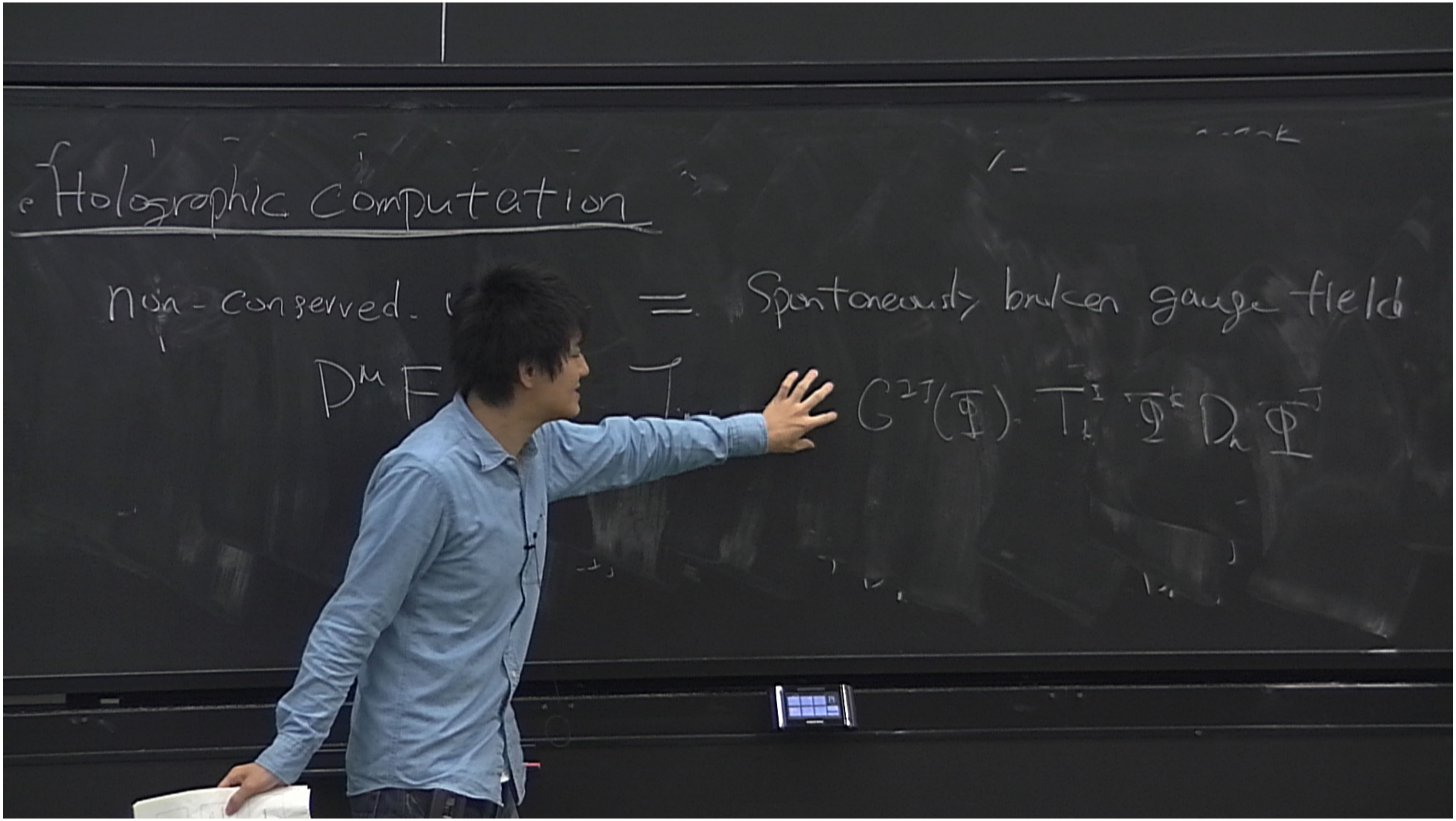

(up to Weyl anomaly)
[Δσ, Δσ̃] = 0


Holographic Computation

Holographic computation

non-conserved vector = Spontaneously broken gauge field

$$D^M F_{MN} = J_N$$


Holographic Computation
non-conserved.
= Spontaneously broken gauge field

Holographic computation

non-conserved vector = Spontaneously broken gauge field

$$D^M F_{MN} = \underline{J_N} = G^{IJ}(\Phi)\, T^I_k\, \underline{\Phi^k}\, \underline{D_N \Phi^J}$$

Holographic computation

non-conserved vector = Spontaneously broken gauge field

$$D^M F_{MN} = J_N = G^{IJ}(\Phi)\, T^I{}_k\, \Phi^k\, D_N \Phi^J$$

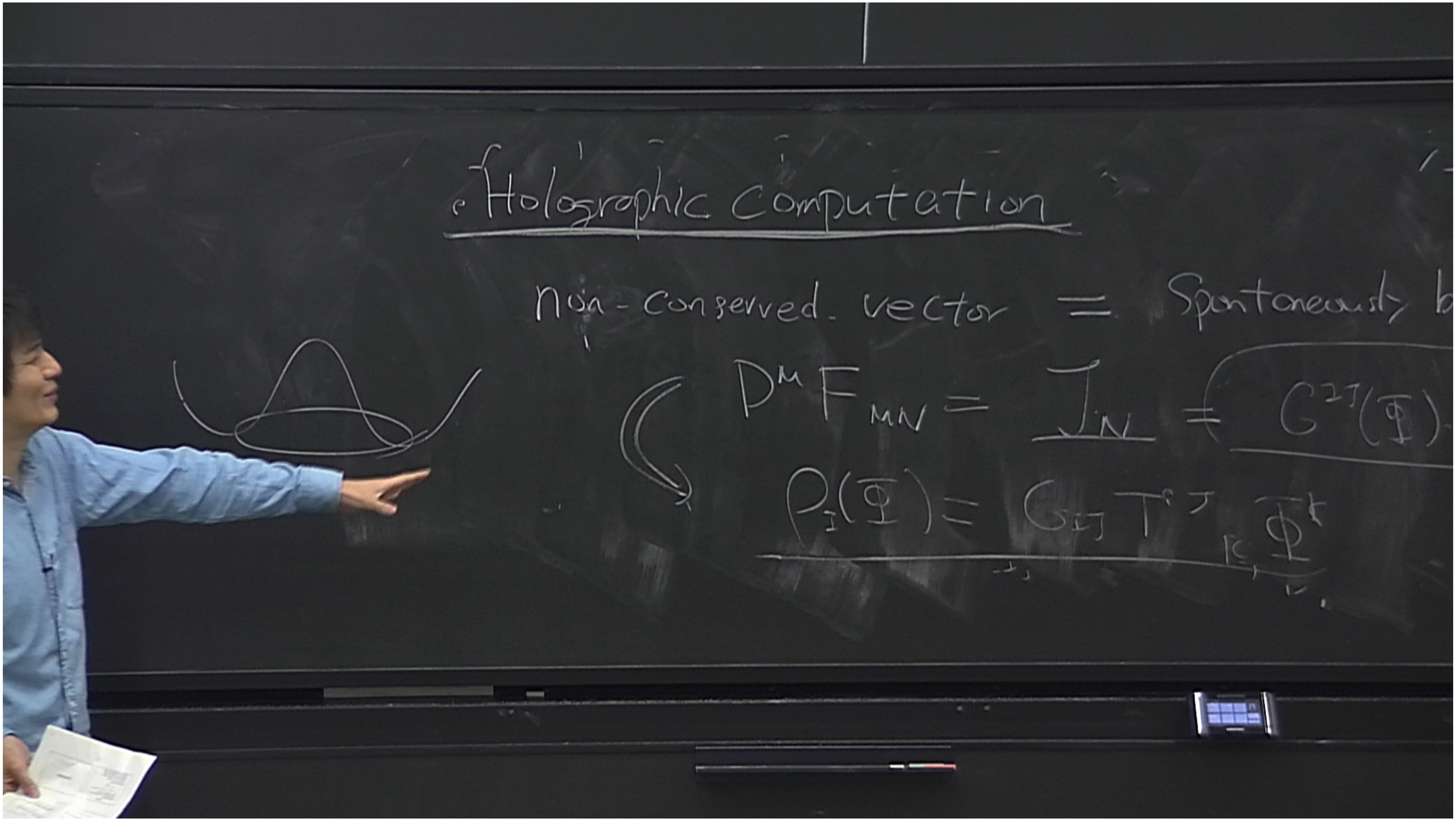
$$\rho_I(\Phi) = G_{IJ}\, T^J{}_k\, \Phi^k$$

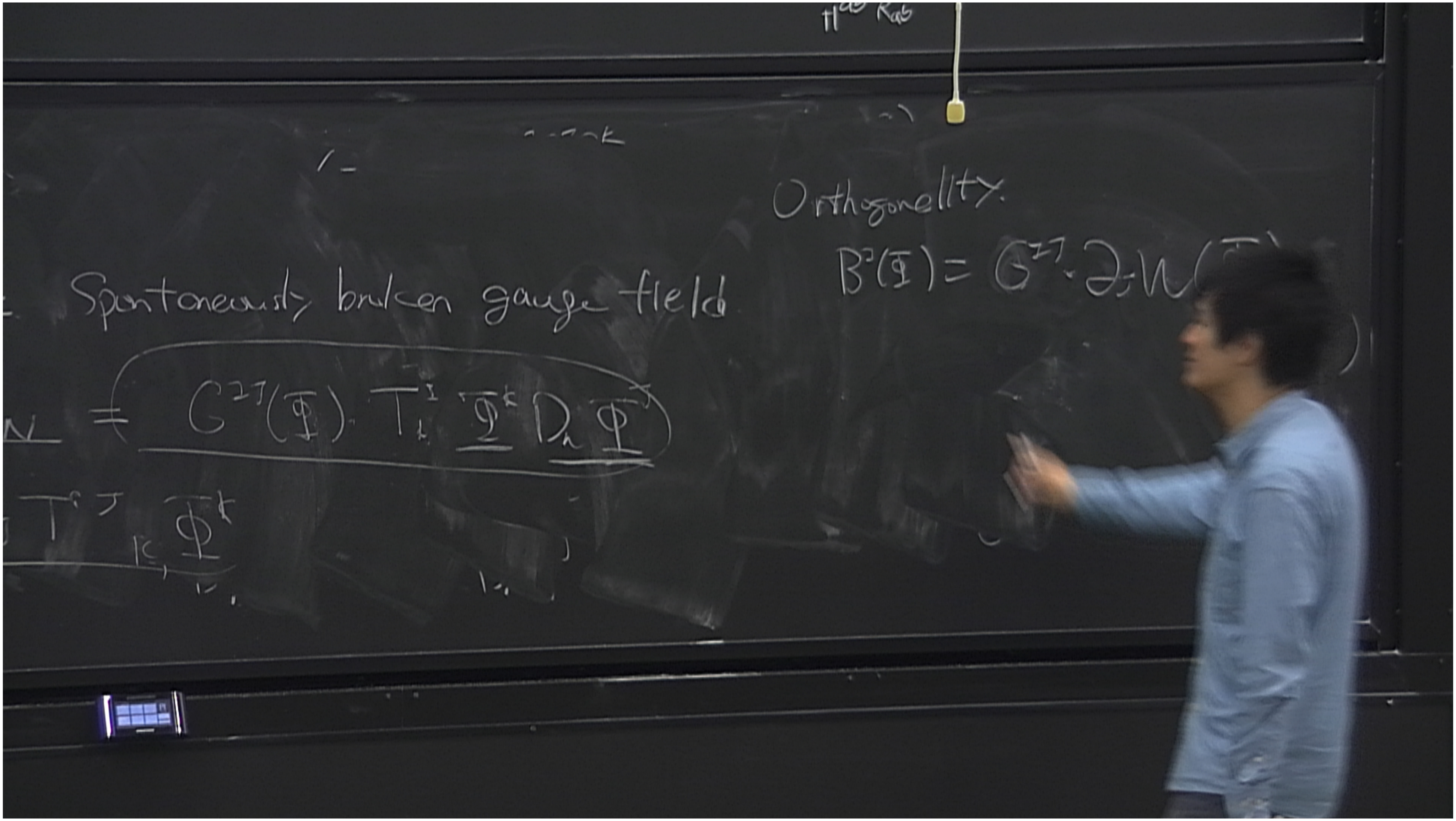
Holographic computation

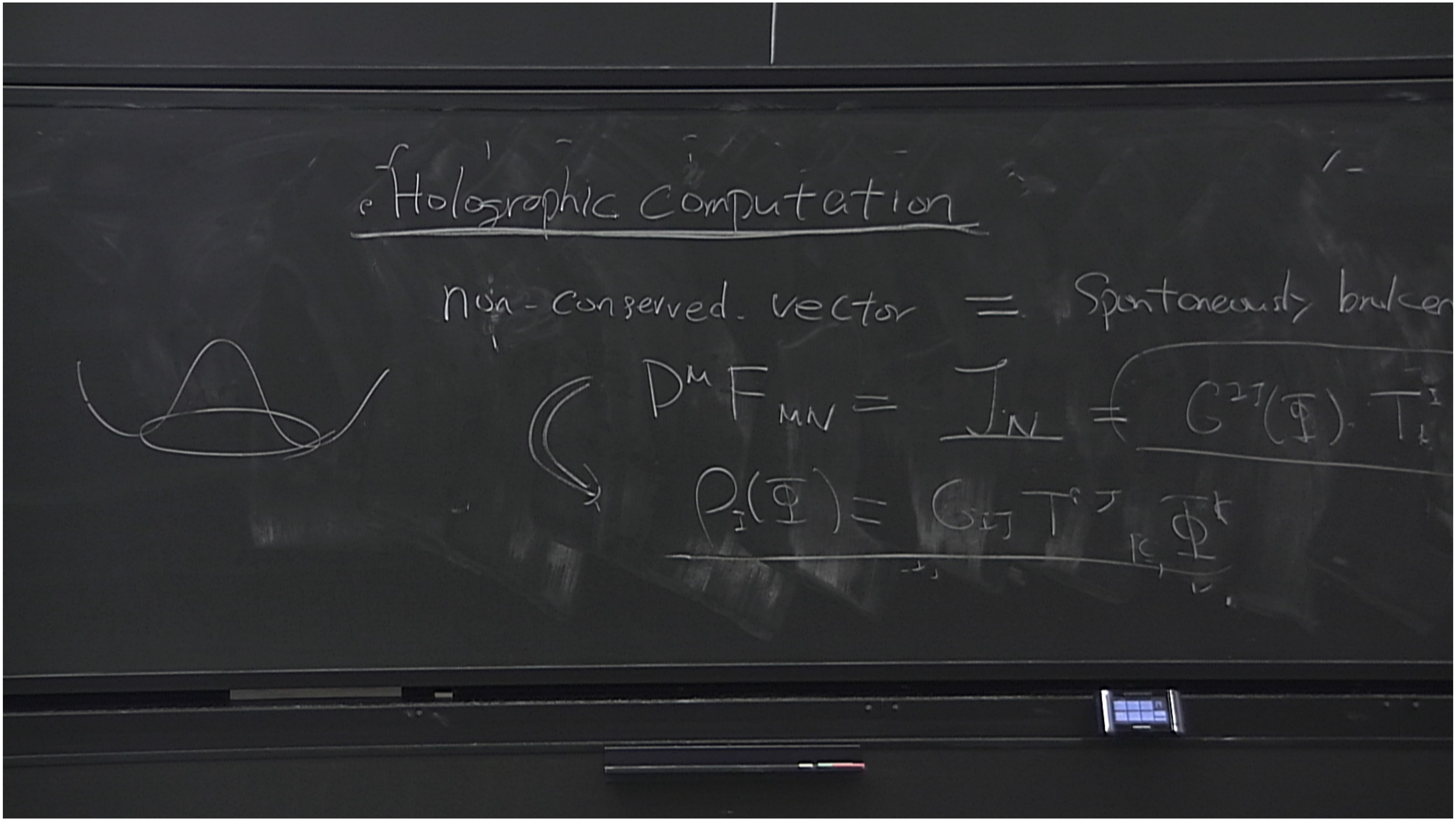
non-conserved vector = Spontaneously broken gauge field

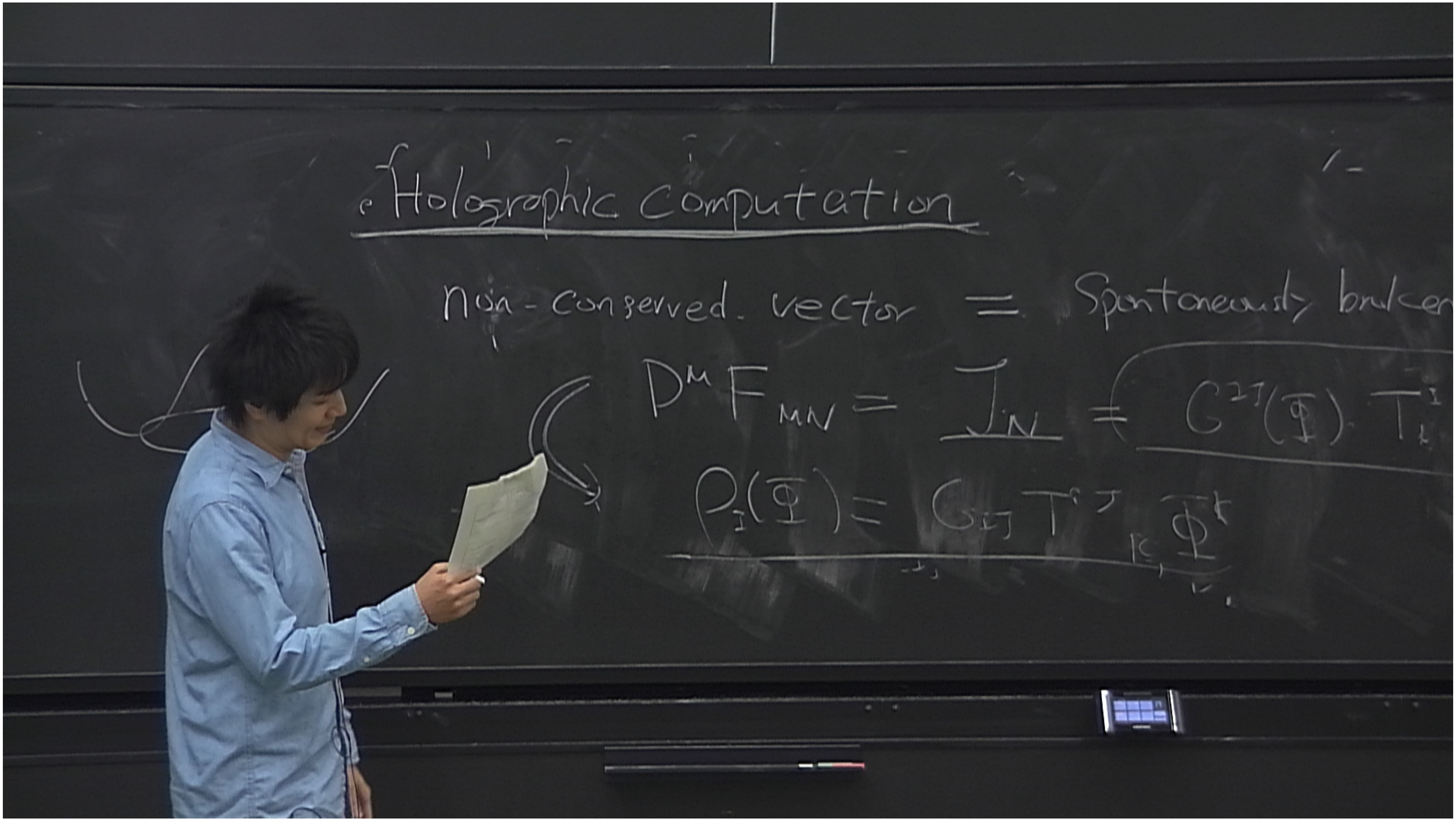
$$D^M F_{MN} = J_N = G^{IJ}(\Phi)\, T^I{}_k\, \bar{\Phi}^k D_N \Phi^J$$

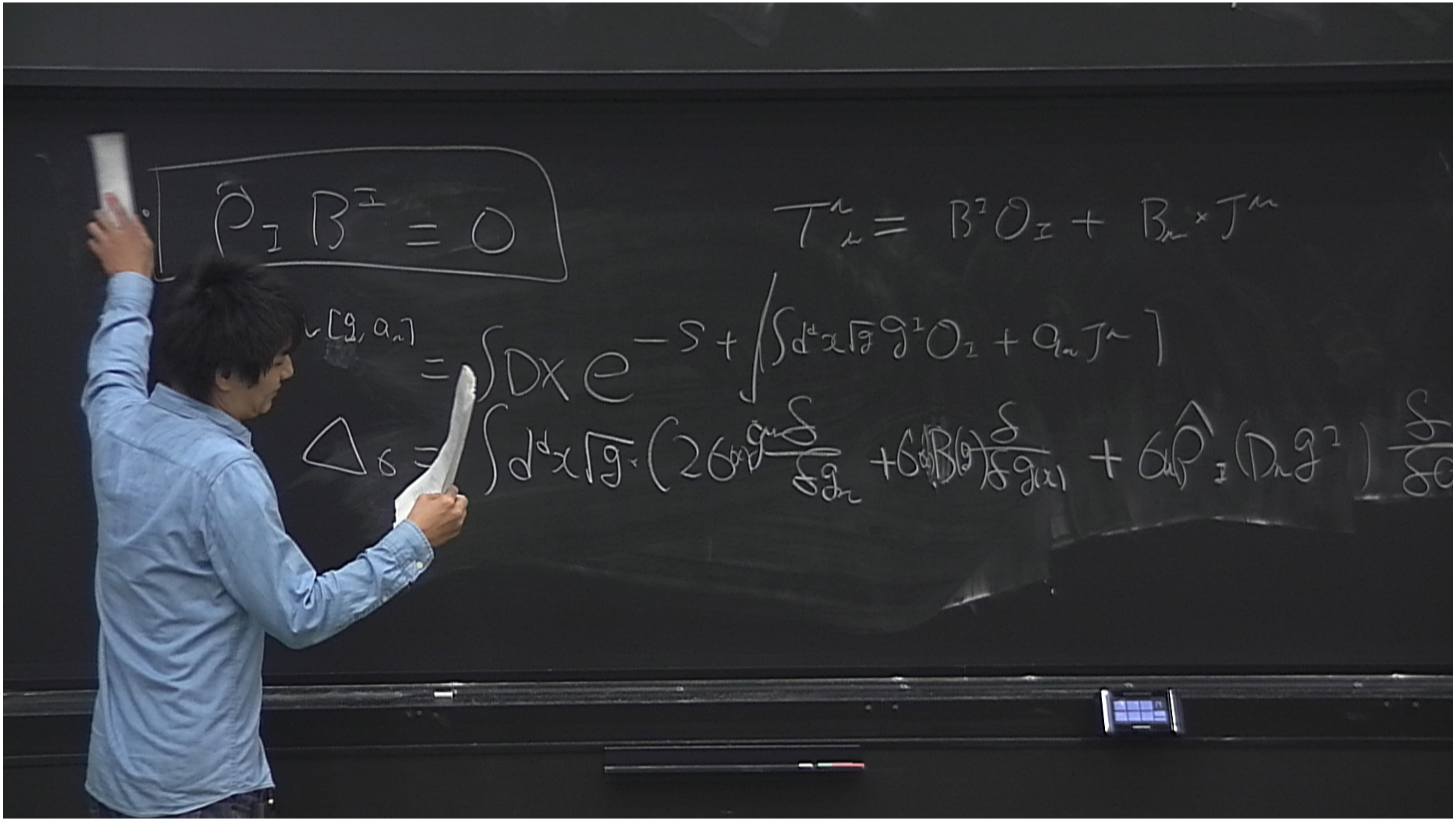
$$\rho_I(\Phi) = G_{IJ}\, T^J{}_k\, \Phi^k$$

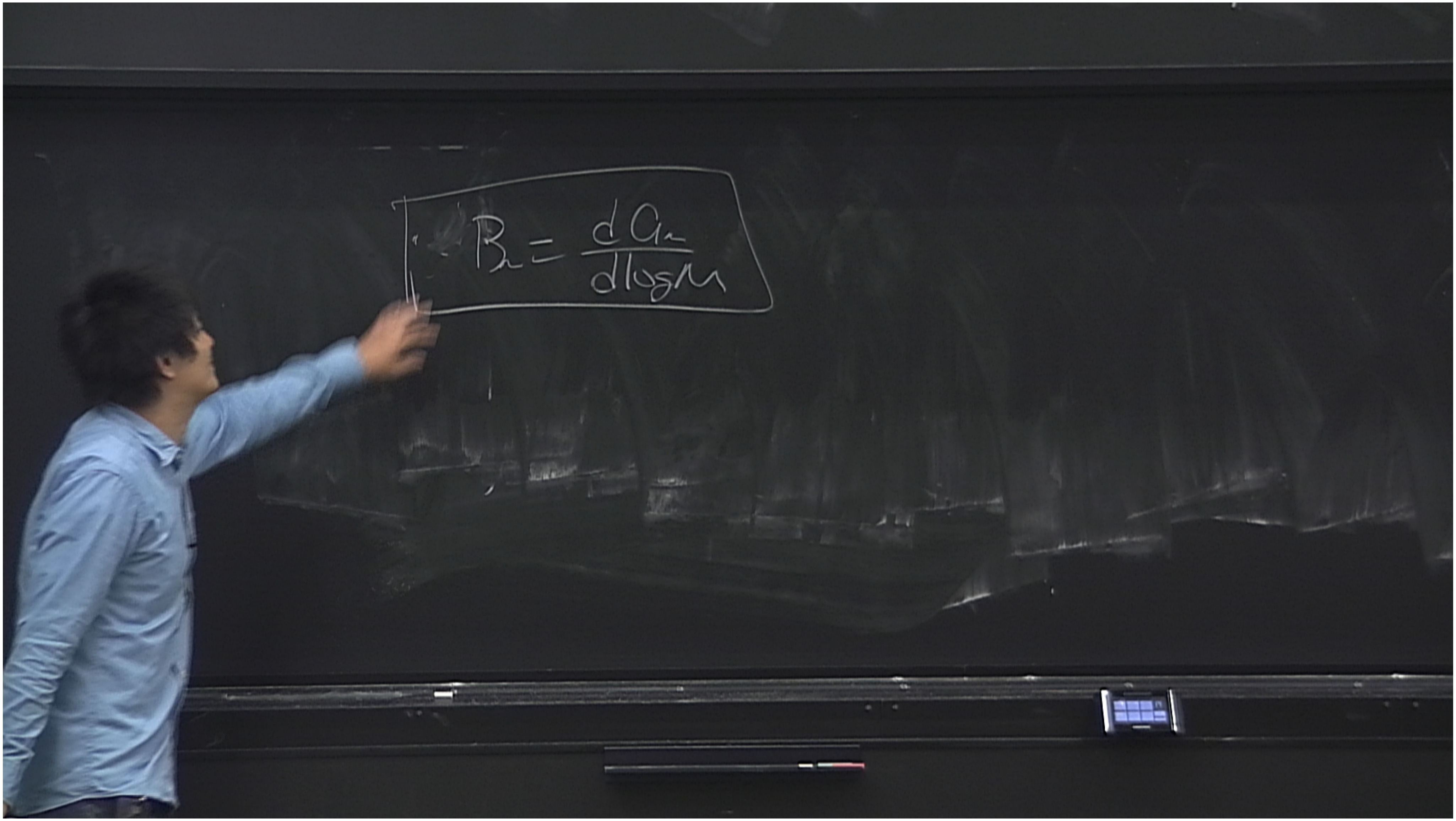

Holographic Computation
non-conserved vector = Spontaneously b
$D^{M} F_{MN} = J_{N} = G^{IJ}(\Phi)$
$\rho_I(\Phi) = G_{IJ} T^{J}{}_{K} \Phi^{K}$

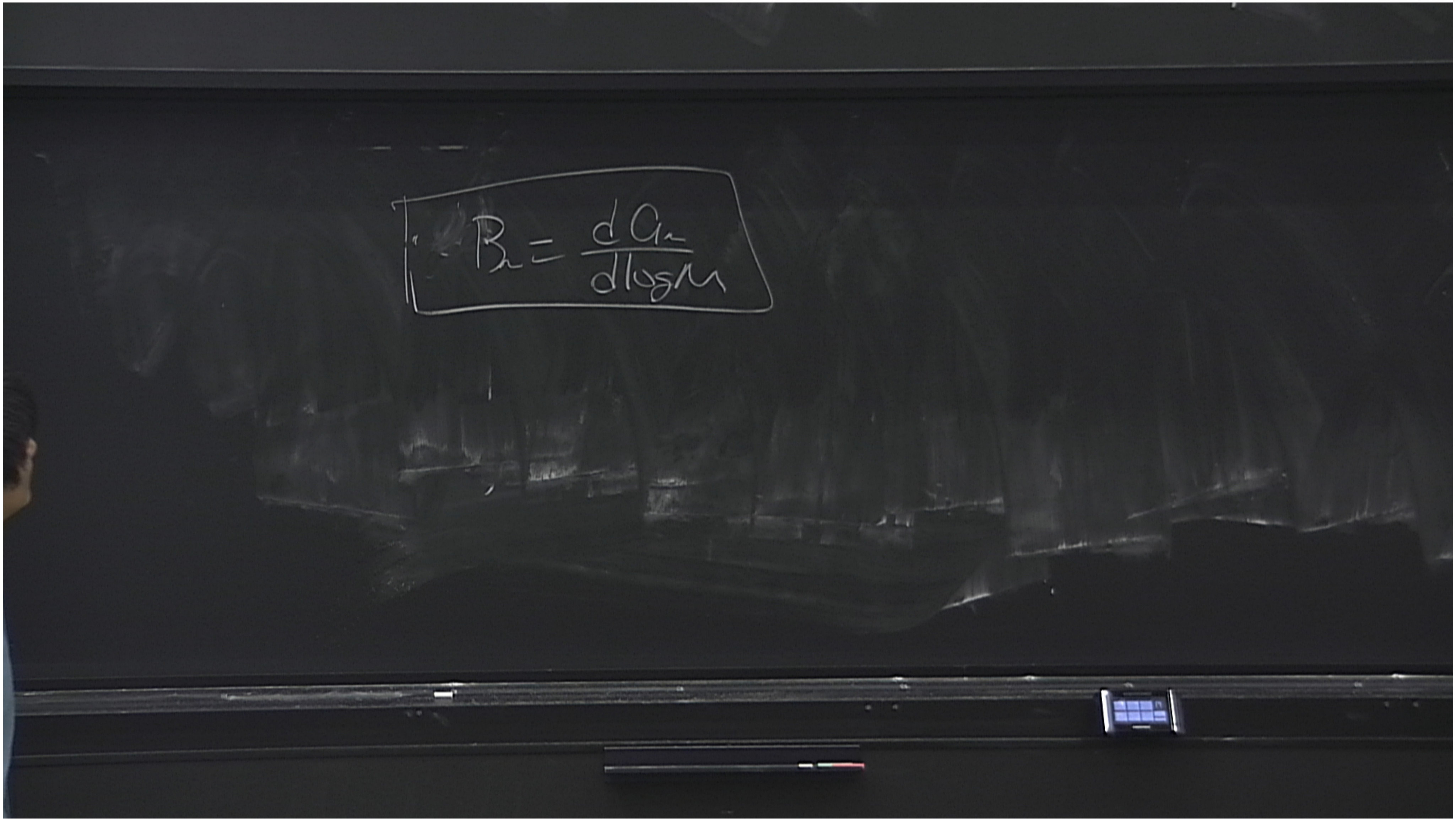

Orthogonality.
Spontaneously broken gauge field


Holographic computation
non-conserved vector = Spontaneously broken
$D^M F_{MN} = J_N =$
$\rho_I(\Phi) = G_{IJ} T^{J}{}_{K} \Phi^K$


Holographic computation
non-conserved vector = Spontaneously broken


β̂_I B^I = 0
T^μ_μ = B^I O_I + B_μ × J^μ


$B_n = \frac{d a_n}{d \log M}$


$\beta_n = \frac{d a_n}{d \log M}$