

Title: Vector Beta Function

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Abstract: We propose various properties of renormalization group beta functions for vector operators in relativistic quantum field theories. We argue that they must satisfy compensated gauge invariance, orthogonality with respect to scalar beta functions, Higgs-like relation among anomalous dimensions and a gradient property. We further conjecture that non-renormalization holds if and only if the vector operator is conserved. The local renormalization group analysis guarantees the first three within power counting renormalization. We verify all the conjectures in conformal perturbation theories and holography in the weakly coupled gravity regime.

Vector Beta function

$$S = S[X] + \int d^d x \cdot g^2 \mathcal{O}_I + a_m^i J_i$$

$$B^I = \frac{dg^2}{d \log \mu} \quad B_m =$$

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Vector Beta function

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1. Compensated gauge invariance
2. Orthogonality.

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3. Higgs-like relation \rightarrow anomalous dimension
4. Gradient

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2. Orthogonality
3. Higgs-like relation \rightarrow anomalous dimension
4. Gradient
5. Non-renormalization

$$\textcircled{1} \quad g^I \rightarrow g^I(x), \quad e^{iW[g^I]} = \int dx \exp \left[-S_0 + \int dx \sqrt{-g} \cdot g^I(x) \mathcal{O}_I(x) + a_n(x) J^n(x) \right]$$

$$a_m \rightarrow a_m(x)$$

$$\textcircled{1} \quad g^I \rightarrow g^I(a), \quad e^{W[a; g^I]} = \int dx \exp \left[-S_0 + \int dx \sqrt{g} \cdot g^I(x) \mathcal{O}_I(x) + a_n(x) T^n(x) \right]$$

$$a_m \rightarrow a_m(a)$$

$$B_m = \frac{da_m}{dg^I} = P_I(g, a) \cdot \partial_m g^I + \lambda(g, a) \cdot a_m$$

$$D_m g^I = (\partial_m + a_m) g^I = \underline{P_I D_m g^I}$$

$$= \int dx^1 \exp \left[-S_0 + \int d^4x \sqrt{|g|} \cdot g^{\mu\nu} \mathcal{O}_2(x) + a_n(x) \mathcal{T}^\mu(x) \right]$$

$$= \frac{da_n}{dg^{\mu\nu}} = P_I(g, a) \cdot \partial_\mu g^I + \lambda(g, a) \cdot a_n$$

$$\partial_\mu \mathcal{T}^\mu = g_{IJ} T^{\alpha\beta} \partial^\alpha \mathcal{O}^\beta$$

$$= \underline{P_I D_\mu g^I}$$

$$= \int dx^4 \exp \left[-S_0 + \int d^4x \sqrt{|g|} \cdot \underline{g^{\mu\nu}} \mathcal{O}_2(x) + a_n(x) \mathcal{T}^\mu(x) \right]$$

$$= \frac{da_n}{d \ln \mu} = \rho_I(g, a) \cdot \partial_n g^I + \lambda(g, a) \cdot a_n$$

$$\partial_n \mathcal{T}^\mu = g_{IJ} T_{IJ}^{\mu\nu} \partial^J$$

$$= \underline{\rho_I D_n g^I}$$

$$a_n \rightarrow a_n + \partial_n \Lambda$$

$$g_{IJ}^I \rightarrow g_{IJ}^I + (g \cdot \Lambda)^I$$

②

$$\frac{da^a}{d \log \mu} = B^a = \rho^a_i D_m g^i$$

$$\frac{dg^i}{d \log \mu} = B^i$$

$$\rho^a_i B^i = 0$$

Stend

variance

$$\partial_n T^n = g^I \partial_I$$

general dimension

$$L = \lambda \phi^4$$

variance

$$\partial_n T^m = g^I \partial_I \mathcal{O}_I$$

→ general dimension

$$L = \lambda \phi^4$$

②

$$\frac{da^a}{d\log\mu} = B^a = \rho_i^a D_m g^i$$

$$\rho_I^a \times B^i = 0$$

$$\frac{dg^i}{d\log\mu} = B^i$$

④

$$B_n = \frac{\delta S[g^i, a_n]}{\delta a_n}, \quad B^i = \chi^i, \quad \frac{d\hat{\alpha}(g)}{d\log\mu}$$

②

$$\frac{da^a}{d\log\mu} = B_n^a = \rho_{I^a}^a D_m g^I$$

$$\rho_{I^a}^a B^I = 0$$

③

$$\frac{dg^I}{d\log\mu} = B^I$$

④

$$B_n = \frac{\delta S[g^I, a_n]}{\delta a_n}, \quad B^I = \chi^I, \quad \frac{d\hat{\alpha}(g)}{d\log g^I}$$

Conformal Perturbation Theory

$$S = S_{\text{CFT}} + \int d^d x \cdot g^I \mathcal{O}_I(x)$$

$$\langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle = \frac{\delta_{IJ}}{(x-y)^{2d}}$$

Conformal Perturbation Theory

$$S = S_{\text{CFT}} + \int d^d x \cdot g^I \mathcal{O}_I(x)$$

$$\langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle = \frac{\delta_{IJ}}{(x-y)^{2d}} + \frac{C_{IJK}}{(x-y)^d} \mathcal{O}_K(y) + \dots$$

Conformal Perturbation Theory

$$S = S_{\text{SCFT}} + \int d^d x \cdot g^I \mathcal{O}_I(x)$$

$$\langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle = \frac{\delta_{IJ}}{(x-y)^{2d}} + \frac{C_{IJK}}{(x-y)^d} \mathcal{O}_K(y) + \dots$$

$$\delta S = \log \mu \int d^d x g^J g^K C_{IJK} \mathcal{O}_I \quad \rightarrow \quad \beta^I = C_{IJK} g^J g^K$$

Conformal Perturbation Theory

$$S = S_{\text{CFT}} + \int d^d x \cdot g^I \mathcal{O}_I(x)$$

$$\mathcal{O}_I(x|y) = \frac{\delta_{IJ}}{(x-y)^{2d}} + \frac{C_{IJK}}{(x-y)^d} \mathcal{O}_K(y) + \frac{C_{IJ}(x-y)^{\mu}}{(x-y)^{d+2}}$$

$$\delta S = \log \mu \int d^d x g^J g^K C_{IJK} \mathcal{O}_I \quad \rightarrow \quad B^I = C_{IJK} g^J g^K$$

Conformal Perturbation Theory

$$S = S_{\text{CFT}} + \int d^d x \cdot g^I \mathcal{O}_I(x)$$

$$\langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle = \frac{\delta_{IJ}}{(x-y)^{2d}} + \frac{C_{IJK}}{(x-y)^d} \mathcal{O}_K(y) + \frac{C_{IJ}(x-y)^{\Delta}}{(x-y)^{d+2}}$$

$$\delta \log Z = \int d^d x g^J g^K C_{IJK} \mathcal{O}_I \rightarrow B^I = C_{IJK} g^J g^K$$

$$\delta \log Z = \int d^d x (g^I \partial_m g^J) C_{IJ} T^m \rightarrow B_m = C_{IJ} g^I \partial_m g^J$$

$$\pi + d/g = 0$$

has Rms

$$D_n J^m = C_{IJ} g^J \cdot \theta^I$$

$$g^I \theta_I(x)$$

$$\frac{C_{JK}}{(x-y)^d} \theta_K(y) + \frac{C_{IJ} (x-y)^m}{(x-y)^{d+2}} \cdot J_n(y)$$

$$B^I = C_{IJK} g^J g^K$$

$$\rightarrow B_m = C_{IJ} g^I D_n g^J$$

$$P_2 B^I = C_{IJ} g^J C_{IKL} g^K g^L$$

$$\pi + d/g = 0$$

has R_{ab}

$$D_n J^m = C_{IJ} g^J \cdot \theta^I$$

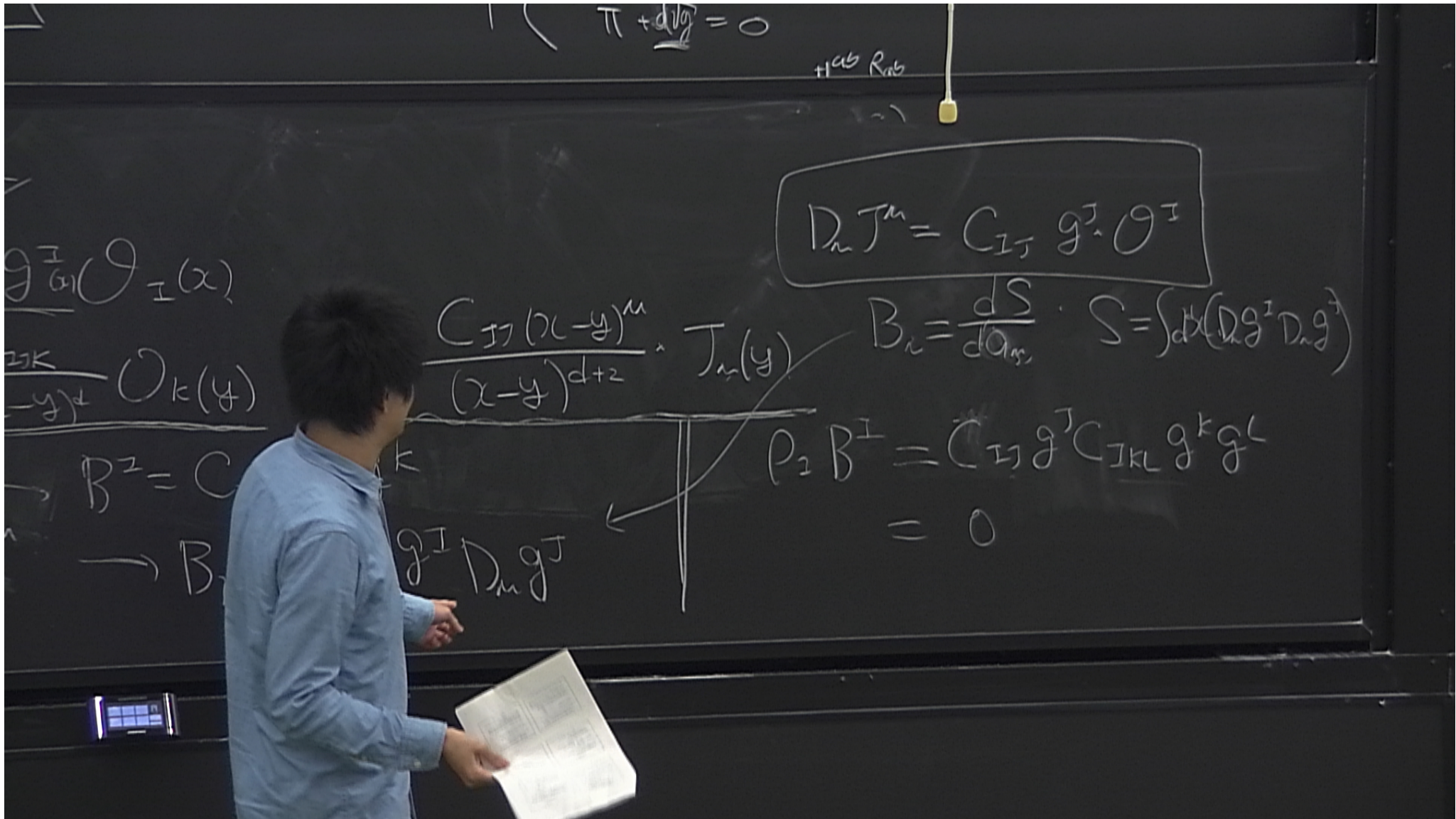
$$g^I \theta_I(x)$$

$$\frac{C_{JK}}{(x-y)^d} \theta_K(y) + \frac{C_{IJ} (x-y)^m}{(x-y)^{d+2}} \cdot J_n(y)$$

$$B^2 = C_{IJK} g^J g^K$$

$$\rightarrow B_m = C_{IJ} g^I D_n g^J$$

$$D_n B^I = C_{JKL} g^K g^L$$



$$\pi + d/g = 0$$

has R_{ab}

$$D_n J^m = C_{IJ} g^J \cdot \theta^I$$

$$g^I_{(n)} \theta_I(x)$$

$$C_{JK} \frac{\theta_K(y)}{(x-y)^d}$$

$$\frac{C_{IJ} (x-y)^m}{(x-y)^{d+2}} \cdot J_n(y)$$

$$B_n = \frac{dS}{d\alpha_n}, \quad S = \int dx (D_n g^I D_n g^J)$$

$$B^2 = C$$

$$\rightarrow B$$

$$g^I D_n g^J$$

$$P_2 B^I = C_{IJ} g^J C_{IKL} g^k g^L = 0$$

$$\pi + d \ln g^I = 0$$

has R_{ab}

$$D_n J^m = C_{IJ} g^J \cdot \theta^I$$

$$g^I g^J \theta_I(x)$$

$$\frac{C_{JK}}{(x-y)^d} \theta_K(y)$$

$$+ \frac{C_{IJ} (x-y)^m}{(x-y)^{d+2}} \cdot J_n(y)$$

$$B_n = \frac{dS}{d\alpha_n}, \quad S = \int d^d x (D_n g^I D_n g^J)$$

$$B^I = C_{IJK} g^J g^K$$

$$\rightarrow B_m = C_{IJ} g^I D_n g^J$$

$$P_1 B^I = C_{IJ} g^J C_{IKL} g^K g^L = 0$$

$$\pi + d \ln g = 0$$

has R_{ab}

$$V = C_{IJK} g^I g^J g^K$$

$$D_n J^m = C_{IJ} g^J \cdot \partial^I$$

$$g^I \partial_I \phi(x)$$

$$\frac{C_{IJK}}{(x-y)^d} \phi_K(y) + \frac{C_{IJ} (x-y)^m}{(x-y)^{d+2}} \cdot J_m(y)$$

$$B_n = \frac{dS}{d\alpha_m} \cdot S = \int_{\text{vector}} dx (D_n g^I D_n g^J)$$

$$P_2 B^I = C_{IJ} g^J C_{IKL} g^K g^L = 0$$

$$B^I = C_{IJK} g^J g^K$$

$$B_m = C_{IJ} g^I D_n g^J$$

$$\pi + d \ln g = 0$$

has R_{ab}

$$V = C_{IJK} g^I g^J g^K$$

$$D_n J^m = C_{IJ} g^J \cdot \partial^I$$

$$g^I \partial_I \phi_I(x)$$

$$\frac{C_{IJK}}{(x-y)^d} \phi_K(y)$$

$$+ \frac{C_{IJ} (x-y)^m}{(x-y)^{d+2}} \cdot J_n(y)$$

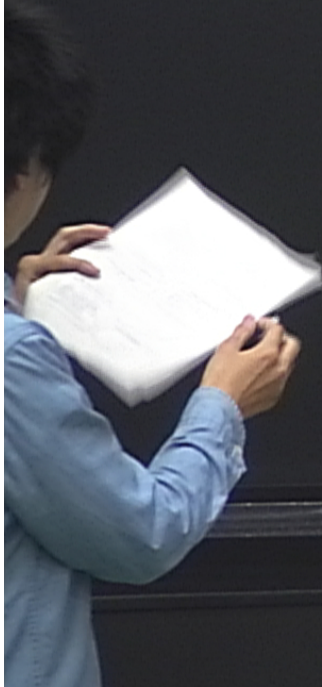
$$B_n = \frac{dS}{d\Omega_n}, \quad S = \int_{\text{vector}} dx (D_n g^I D_n g^J)$$

$$B^I = C_{IJK} g^J g^K + D g^I$$

$$P_1 B^I = C_{IJ} g^J C_{IKL} g^K g^L = 0$$

$$\rightarrow B_m = C_{IJ} g^I D_n g^J$$

$$\hat{P}_I B^T = 0$$



$$\boxed{\hat{P}_I B^I = 0}$$

$$e^{W[g, a_n]} = \int D\chi e^{-S + \left(\int d^4x \sqrt{g} g^{IJ} O_I + a_n J^n \right)}$$

$$\hat{P}_I B^I = 0$$

$$e^{W[g, a, \tau]} = \int D\chi e^{-S + \left(\int d^d x \sqrt{g} g^{\mu\nu} O_{\mu\nu} + a_n J^n \right)}$$

$$\Delta S = \int d^d x \sqrt{g} \cdot \left(26 g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} + 6 B \frac{\delta}{\delta g^{\mu\nu}} + 6 \hat{P}_I (D_\mu g^\mu{}^\nu) \frac{\delta}{\delta a_m} \right)$$

$$\Delta S = W [g, a_m] = 0$$

(up to next anomaly)

$$\left(D_{\mu} g^{\mu\nu} \right) \frac{\delta}{\delta g_{\mu\nu}}$$

$$\hat{P}_I B^I = 0$$

$$Z_{n+1} = \int D\chi e^{-S + \left(\int d^4x \sqrt{g} g^{\mu\nu} \partial_\mu \chi + q_n \int \chi \right)}$$

$$\left(26 \frac{\delta S}{\delta g_{\mu\nu}} + 6 \frac{\delta S}{\delta g_{\mu\nu}} + 6 \hat{P}_I (D_\mu g^\mu{}_\nu) \frac{\delta S}{\delta g_{\mu\nu}} \right)$$

$$\hat{P}_I B^I = 0$$

$$T^{\mu\nu} = B^2 O_I + B_m \times J^m$$

$$e^{W[g, a_m]} = \int D\chi e^{-S + \int d^d x \sqrt{g} g^{\mu\nu} O_I + a_m J^m}$$

$$\Delta S = \int d^d x \sqrt{g} \cdot \left(26 \frac{\delta S}{\delta g_{\mu\nu}} + 6 B \frac{\delta S}{\delta g_{\mu\nu}} + 6 \hat{P}_I (D_\mu g^\mu) \frac{\delta S}{\delta g_{\mu\nu}} \right)$$

$$z + B_m \times J^m$$

$$a_m J^m$$

$$(x) + \hat{G}_I (D_m g^I) \frac{\delta}{\delta g_m}$$

$$\Delta \sigma = W [g, a_m] = 0$$

(up to key anomaly)

$$[\Delta \sigma, \Delta \sigma] = 0$$

$$\mathbf{O}_I + \mathbf{B}_m \times \mathbf{J}^m$$

$$+ \mathbf{a}_m \mathbf{J}^m$$

$$\left(\frac{\delta}{\delta g_{\alpha\beta}} + \mathcal{L}_P \hat{\mathbf{P}}_I (D_m g^{\alpha\beta}) \frac{\delta}{\delta g_m} \right) \Rightarrow \left(\frac{d}{dt} \right)$$

$$\Delta = W [g, a_m] = 0$$

(up to next order)

$$\frac{\delta \Delta}{\delta g} = 0$$

$$O_I + B_m \times J^m$$

$$+ a_m J^m]$$

$$\Delta \sigma = W [g, a_m] = 0$$

(up to next order)

$$\left(\frac{\delta}{\delta g_{\alpha\beta}} + \hat{O}_I \hat{P}_I (D_m g^{\alpha\beta}) \frac{\delta}{\delta a_m} \right) \Rightarrow \int dt \sqrt{g} (\sigma_{\alpha\beta} \tilde{\sigma}^{\alpha\beta} - \tilde{\sigma}_{\alpha\beta} \sigma^{\alpha\beta}) \boxed{B^{-1} P_I} \frac{\delta}{\delta a_m} = 0$$

Holographic Computation



Holographic Computation

non-conserved vector = spontaneously broken gauge field

$$D^M F_{MN} = J_N$$

Holographic Computation

non-conserved \Rightarrow Spontaneously broken gauge field

$D^\mu F$

T

$$G^{IJ}(\Phi) T_{IJ} \Phi^k D_n \Phi^j$$

Holographic Computation

non-conserved vector = Spontaneously broken gauge field

$$D^M F_{MN} = \underline{J}_N = G^{IJ}(\Phi) \underline{T}_I^K \underline{\Phi}^L \underline{D}_L \underline{\Phi}^J$$

Holographic Computation

non-conserved vector = spontaneously broken gauge field

$$D^M F_{MN} = \underline{J_N} = \underline{G^{IJ}(\Phi) T^I_{J} \Phi^k D_n \Phi}$$
$$\rho_I(\Phi) = \underline{G_{IJ} T^{IJ} \Phi^k}$$

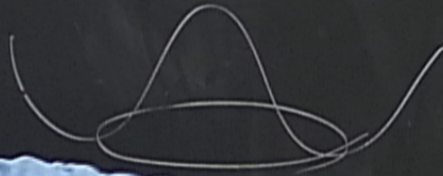
Holographic Computation

non-conserved vector = Spontaneously broken gauge field

$$D^M F_{MN} = \underline{J_N} = \left(G^{IJ}(\Phi) T^I_{J} \Phi^k D_n \Phi \right)$$
$$\rho_I(\Phi) = \underline{G_{IJ} T^{IJ} \Phi^k}$$

Holographic Computation

non-conserved vector = spontaneously broken



$$D^M F_{MN} = \underline{J}_N = \underline{G^{2J}(\Phi)}$$
$$\underline{P}_I(\Phi) = \underline{G_{IJ} T^{IJ} \frac{\delta \mathcal{L}}{\delta \Phi^I}}$$

H_{ab} K_{ab}

1- ...

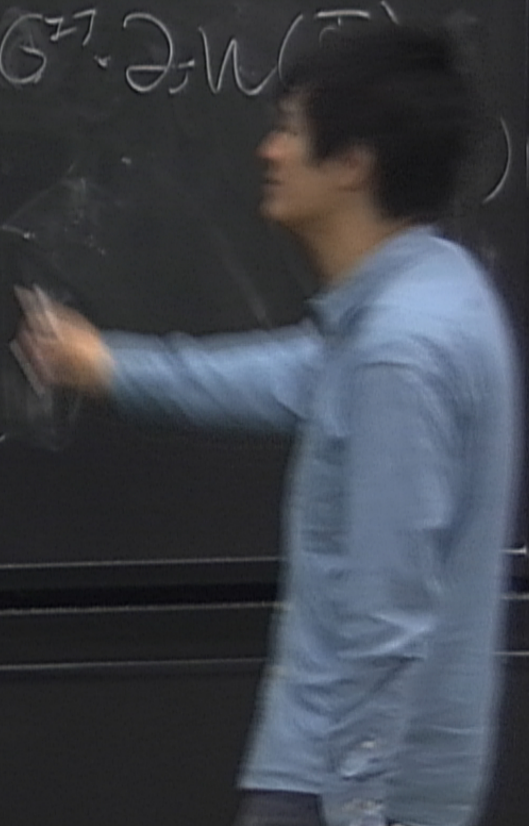
Orthogonality.

$$B^2(\Phi) = G^{2j} \partial_j W(\Phi)$$

Spontaneously broken gauge field

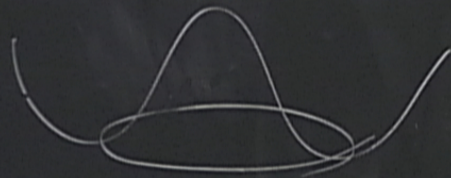
$$L = \frac{1}{2} G^{2j}(\Phi) T_{\mu\nu}^I \Phi^{\mu} D_{\nu} \Phi^{\nu}$$

$$T_{\mu\nu}^I = \frac{1}{i} (\Phi^{\mu} \partial_{\nu} \Phi^{\nu} - \Phi^{\nu} \partial_{\mu} \Phi^{\mu})$$



Holographic Computation

non-conserved vector = Spontaneously broken



$$D^M F_{MN} = \underline{J}_N = \frac{G^{2J}(\Phi) T^I}{G_{IJ} T^J \frac{\delta \Phi^I}{\delta \Phi^J}}$$

Holographic Computation

non-conserved vector = spontaneously broken

$$D^M F_{MN} = \underline{J}_N = \frac{G^{2J}(\Phi) T^J}{G^{IJ}(\Phi) T^I}$$

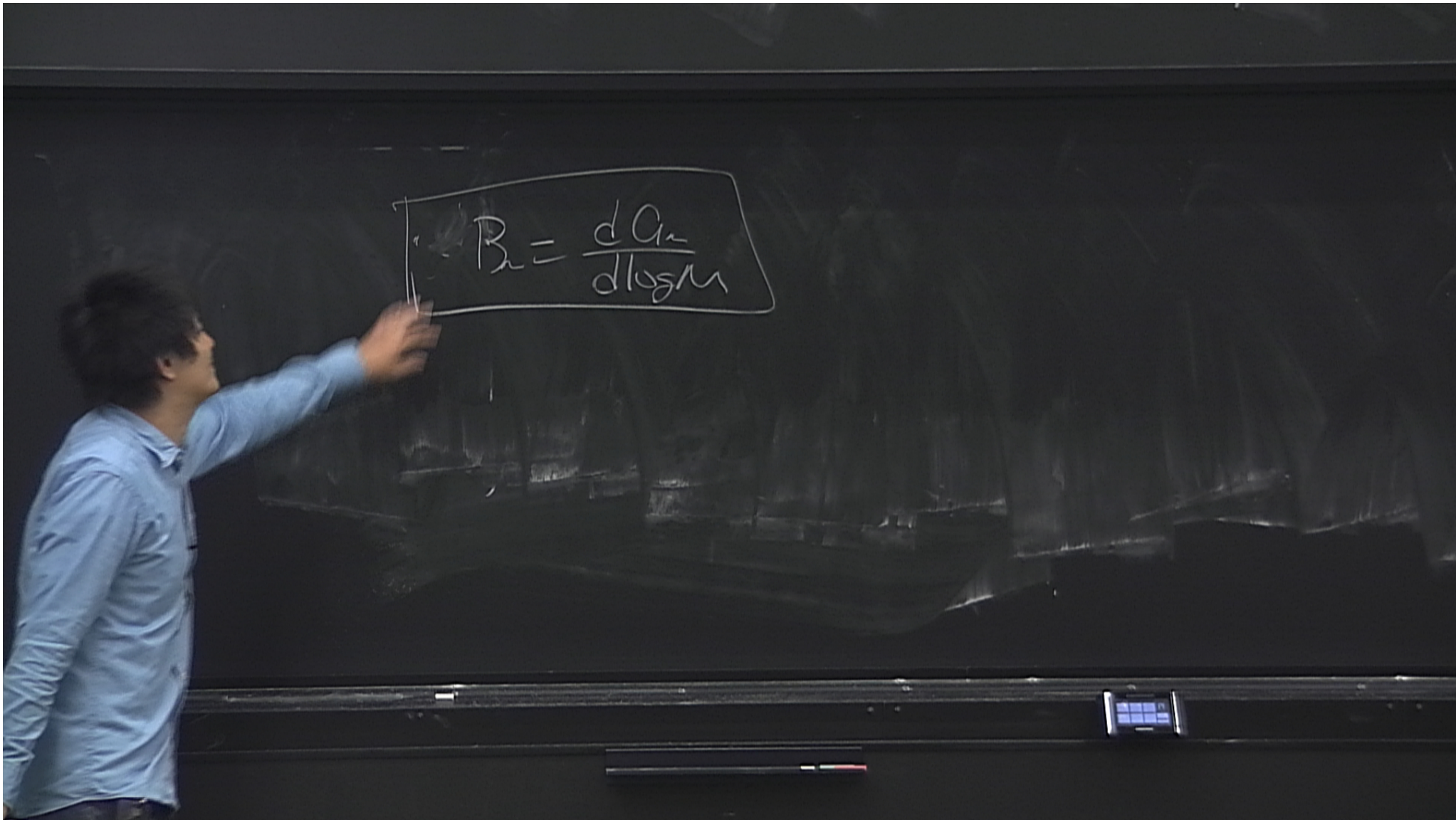
$$\rho_I(\Phi) = \frac{G_{IJ} T^J \Phi^I}{G_{IJ} T^J \Phi^I}$$

$$\hat{P}_I B^I = 0$$

$$T^{\mu}_{\nu} = B^2 \delta_{\mu\nu} + B_{\mu} \times J^{\mu}$$

$$\langle [g, a_{\mu}] \rangle = \int D\chi e^{-S + \int d^d x \sqrt{g} g^{\mu\nu} \partial_{\mu} \chi + a_{\mu} J^{\mu}}$$

$$\Delta S = \int d^d x \sqrt{g} \cdot \left(26 \frac{g_{\mu\nu} \delta g^{\mu\nu}}{\delta g^{\mu\nu}} + 6 \frac{B^2 \delta B^2}{\delta B^2} + 6 \hat{P}_I (D_{\mu} g^{\mu I}) \frac{\delta}{\delta \hat{P}_I} \right)$$



$$B_n = \frac{dA_n}{d \log M}$$