Title: Plasma without Plasma: Exploring Force-Free Magnetospheres

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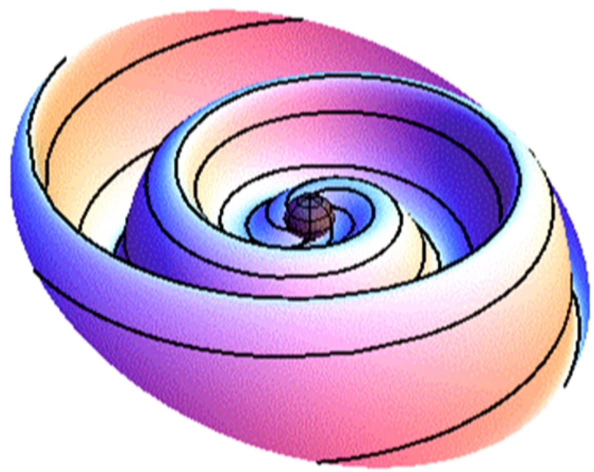
Abstract: Pulsars have enormous magnetic fields whose energy density dwarfs the rest mass density of their plasma magnetosphere. In this regime of a plasma, the particles drop out of the description, leaving a set of equations for the electromagnetic field alone. This non-linear, deterministic system is known as force-free electrodynamics, and turns out to have some beautiful and bizarre features. I will give a pedagogical introduction to these equations and their role in astrophysics and then discuss our recent contributions. We have taken a geometric viewpoint, using the null structure of spacetime to unify previous exact solutions and discover new ones. We have found non-stationary, non-axisymmetric solutions that describe the outer magnetosphere of pulsars, including those that are accelerated or torqued. We have derived the standard cartoon of the aligned pulsar magnetosphere from an explicit, minimal set of assumptions. Time permitting, I will also discuss some properties of black hole magnetospheres: Blandford-Znajek energy extraction, non-scattering force-free waves, and the no-ingrown-hair theorem.

Plasma without Plasma: Exploring Force-Free Magnetospheres



Sam Gralla
University of Maryland
Supported by a NASA Einstein Fellowship

arXiv:1305.6890 and *in prep*

collaborators: Daniel Brennan and Ted Jacobson

Magnetically Dominated Plasma

Consider a low-density plasma in a strong magnetic field. Assume there is

1) enough charge to affect the EM field, but…

2) not much mass: **field energy density >> rest mass density**

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The energy budget consists of a single term, the EM stress-energy.
Energy is Conserved.

$$\nabla^a T_{ab} = 0 \quad \Longrightarrow \quad \boxed{F_{ab} J^b = 0}$$

The Lorentz force density vanishes at every point in the plasma.

The 3+1 version is $\rho \vec{E} + \vec{J} \times \vec{B} = 0$

The plasma is **force-free.**

Force-Free Electrodynamics

$$\nabla_{[a}F_{bc]} = 0$$

$$\nabla_a F^{ab} = 4\pi J^b$$

Maxwell's Equations

$$F_{ab}J^b = 0$$

Vanishing Lorentz Force Density

($\rho\vec{E} + \vec{J} \times \vec{B} = 0$)

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Non-linear equations for the EM field of a magnetically dominated plasma

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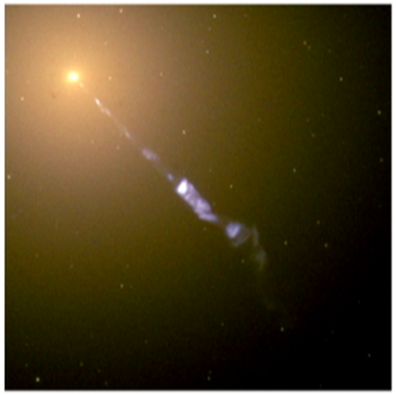
FFE in Nature

Pulsars (neutron stars)

Goldreich & Julian 1969

Very good arguments for a force-free magnetosphere.

Lots of interesting physics.

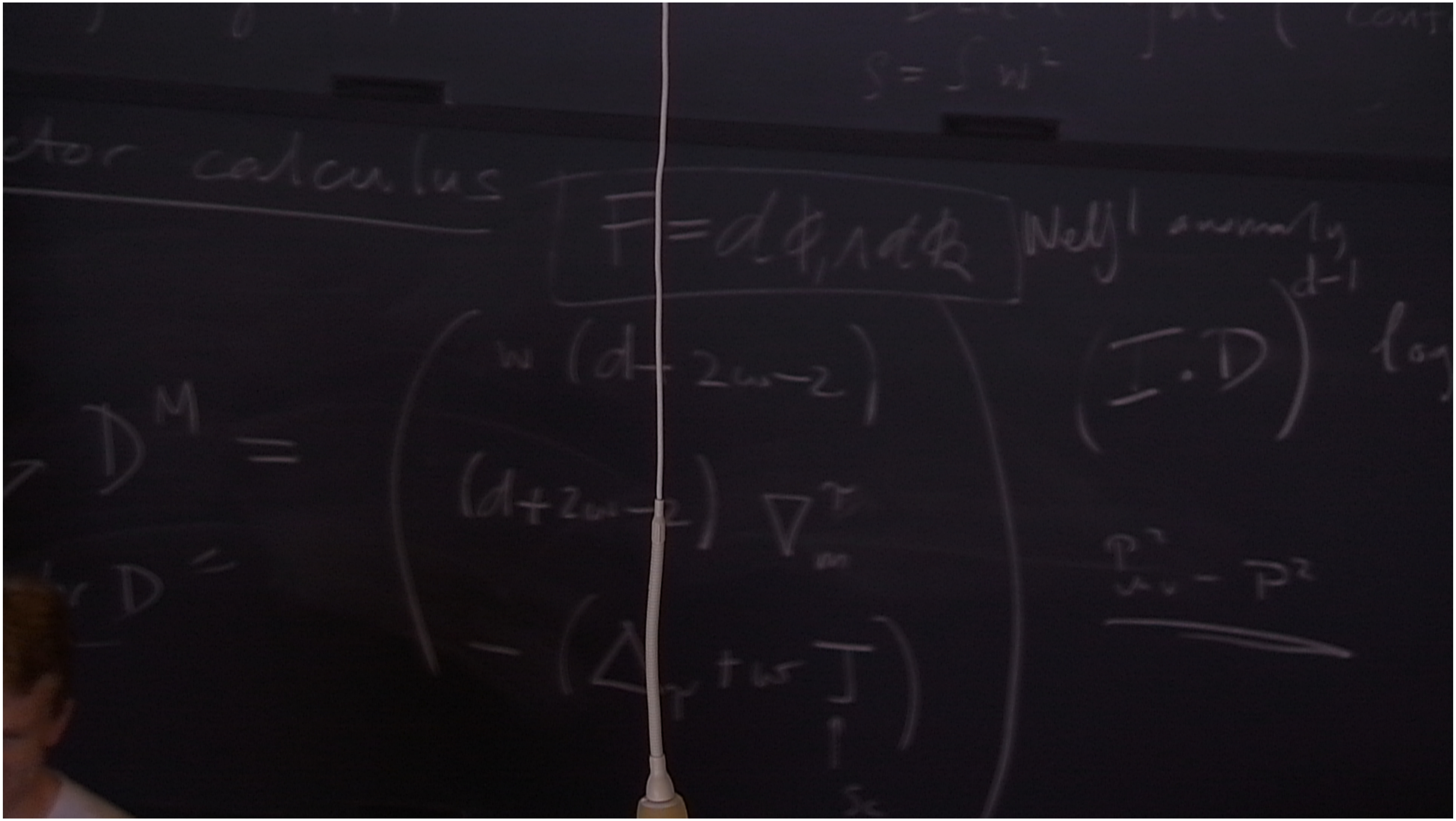


AGN (black holes)

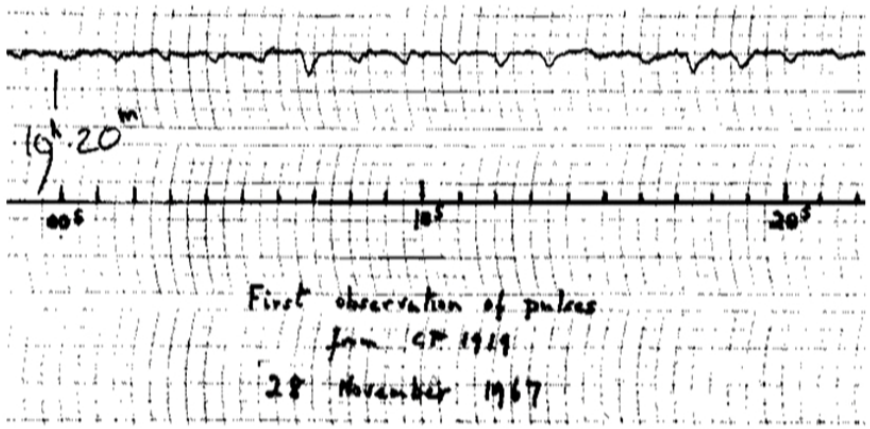
Blandford & Znajek 1976

Plausibility arguments for a force-free magnetosphere.

Lots of interesting physics.


ctor calculus
F = dφ₁ ∧ dφ₂
Weyl anomaly
(I · D)^{d-1} log
D^M =
(d + 2m − 2) ∇
P²_{μν} − P²

Pulsar



Bell and Hewish 1968

Little Green Men? No.

Pulsating Star? No.

Rotating, magnetized neutron star.

From P, P-dot: **B=10^{12} Gauss**

(compare to 10^4 for an MRI magnet)

Pulsar Magnetosphere

Goldreich & Julian 1969: If conducting and surrounded by vacuum we have

$$\boldsymbol{E}\cdot\boldsymbol{B} = -\left(\frac{\Omega R}{c}\right)\left(\frac{R}{r}\right)^{7} B_0{}^2 \cos^3\theta$$

Inside, E.B=0. At the surface layers there is a transition.

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Inside, E.B=0. At the surface layers there is a transition.

The electric fields are **huge**. The forces on an electron satisfy

$$\boxed{\frac{(\text{Electric Force})}{(\text{Gravitational Force})} \approx 10^{12}}$$

Charges would be ripped off. **A pulsar cannot be surrounded by vacuum.**
There must be charged particles outside... pulsars have a plasma magnetosphere.

Force-Free Pulsar Magnetosphere

How much Plasma? We need to neutralize the external E.B. For a first estimate suppose that charged particles of one species corotate with the star. Then we need (Goldreich and Julian '69)

$$\rho = \frac{\nabla \cdot E}{4\pi} = \frac{-\Omega \cdot B}{2\pi c} \frac{1}{[1 - (\Omega r/c)^2 \sin^2 \theta]}$$

For electrons and typical pulsar parameters, the number density is $n \approx 10^9$ /cm^3.

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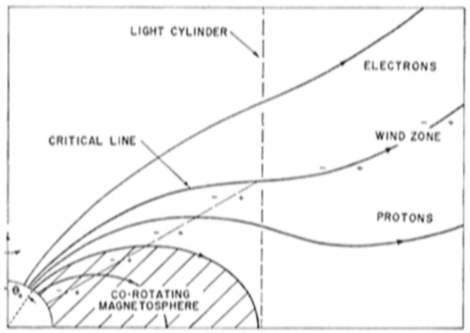
Energetically this is **nothing**. We have

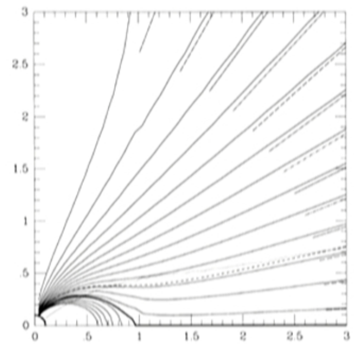
$$\boxed{\frac{\text{(rest mass density of particles)}}{\text{(energy density of fields)}} \approx 10^{-19}}$$

This plasma is magnetically dominated and hence force-free.

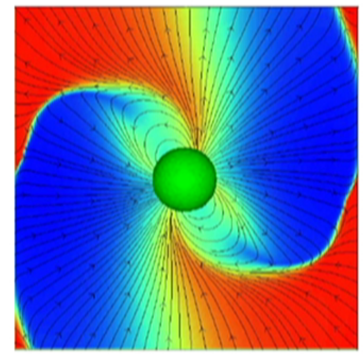
FFE Slideshow



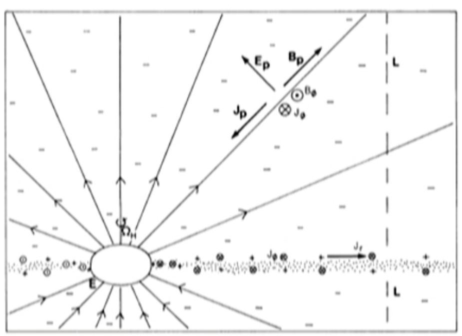
Goldreich and Julian 1969



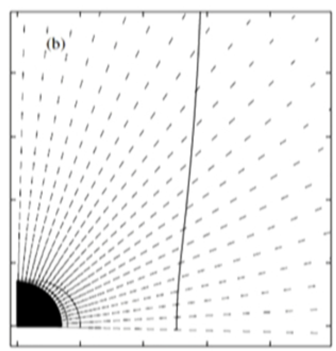
Contopoulos, Kazanas, Fendt 1999



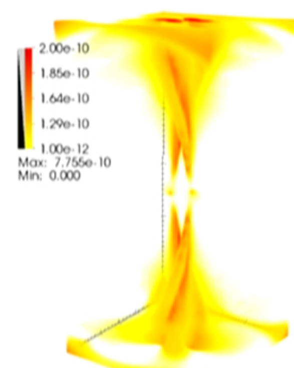
Spitkovsky 2006



Blandford & Znajek 1976



Komissarov 2002



Palenzuela, Garrett, Lehner, Liebling 2011

Force-Free Cheerleading

A beautiful set of equations.

A drop-dead astrophysical application (pulsars)

Lots of less certain astrophysical applications (GRBs, AGN, ...)

Basic physics: plasma dynamics in extreme EM fields

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Let's explore!!!

Outline

0. Introduction: Pulsars, Black Holes, Force-Free Electrodynamics
1. The Mathematics and Physics of FFE
2. Open Field Lines
3. Exact Solutions without Symmetry

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1. The Mathematics and Physics of FFE
2. Open Field Lines
3. Exact Solutions without Symmetry

References

1. "Exact Solutions to FFE in Black Hole Backgrounds" (arXiv:1305.6890)
2. "Accelerated Pulsar Magnetosphere" (to appear next week)
3. "Spacetime Approach to Force-Free Magnetospheres" (to appear soon)

...

Force-Free Electrodynamics

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FFE is Deterministic

To analyze FFE as an evolution problem requires a 3+1 split. In flat spacetime,

$$\partial_t \vec{E} = \nabla \times \vec{B} - \vec{j}, \qquad \nabla \cdot \vec{E} = \rho, \qquad \rho \vec{E} + \vec{j} \times \vec{B} = 0.$$
$$\partial_t \vec{B} = -\nabla \times \vec{E}, \qquad \nabla \cdot \vec{B} = 0,$$

(Maxwell evolution) (Maxwell constraint) (force-free)

After some algebra you can show that solutions to the above equations satisfy

$$\vec{j} = \frac{\vec{B}}{B^2}\left[\vec{B} \cdot (\nabla \times \vec{B}) - \vec{E} \cdot (\nabla \times \vec{E})\right] + \frac{\vec{E} \times \vec{B}}{B^2} \nabla \cdot \vec{E}.$$

The FF condition implies $\vec{E} \cdot \vec{B} = 0$. The following is an evolution problem,

$$\boxed{\begin{array}{ll} \partial_t \vec{E} = \nabla \times \vec{B} - \vec{j}, & \nabla \cdot \vec{B} = 0, \\ \partial_t \vec{B} = -\nabla \times \vec{E}, & \vec{E} \cdot \vec{B} = 0. \\ \text{(evolution eqns)} & \text{(constraints)} \end{array}}$$

Komissarov 2002, Pfeiffer & MacFayden 2013: system hyperbolic when $\boxed{B^2 - E^2 > 0.}$

Fieldsheets

Carter 1979, Uchida 1996

$F_{ab}J^b = 0$ and $F_{ab} = -F_{ba}$ → **2D plane of vectors annihilating F**

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Carter 1979, Uchida 1996

$F_{ab}J^b = 0$ and $F_{ab} = -F_{ba}$ → **2D plane of vectors annihilating F**

$B^2 - E^2 > 0$ → **plane is timelike**

timelike vectors in plane: frames where E=0

orthogonal vector: B-field in that frame

The Michel Monopole

Michel 1973

3+1

$$B_\phi = E_\theta = -B_0(\Omega R)\frac{R}{r}\sin\theta$$

$$B_r = B_0\left(\frac{R}{r}\right)^2$$

$$q = B_0 R^2$$

spacetime

$$F = q\, d(\cos\theta) \wedge [d\phi - \Omega d(t-r)]$$

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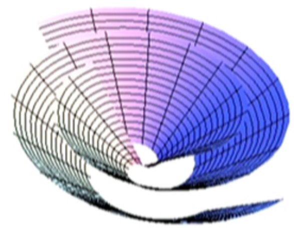
By inspection, the fieldsheets ($F_{ab}v^b = 0$) are spanned by



$$\ell^a = \left(\tfrac{\partial}{\partial t}\right)^a + \left(\tfrac{\partial}{\partial r}\right)^a$$

radial null vector

and

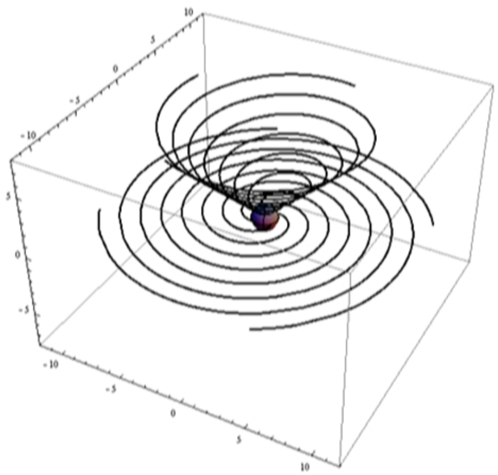
$$k^a = \left(\tfrac{\partial}{\partial t}\right)^a + \Omega\left(\frac{\partial}{\partial \phi}\right)^a$$

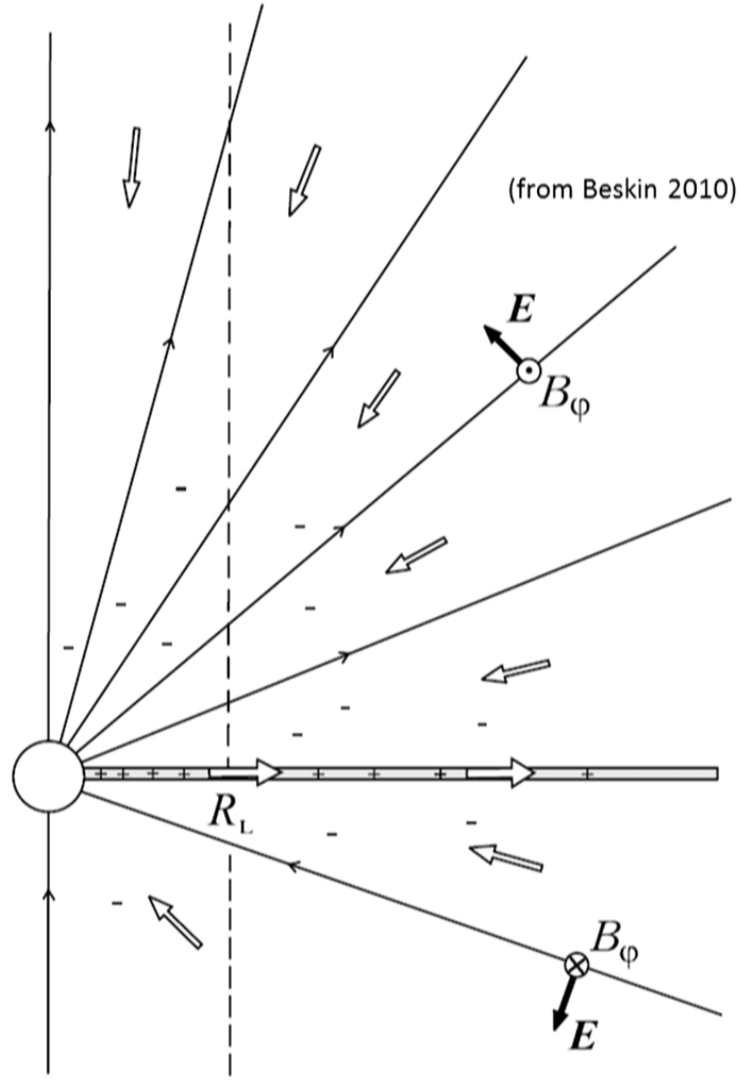
rotating frame vector

Split-Monopole Pulsar

Reverse the sign of the fields on each hemisphere, with a current sheet on the interface.

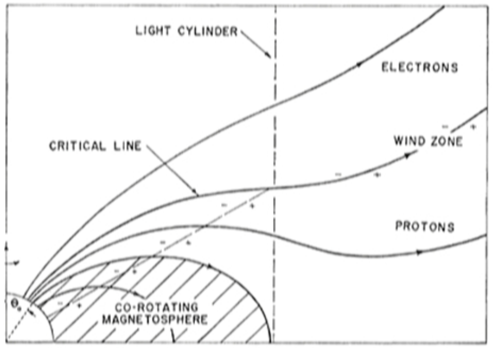
A “poor man’s dipole”.



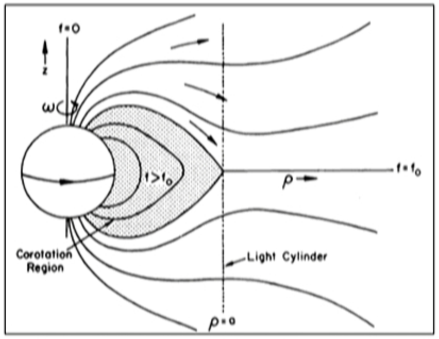


(from Beskin 2010)

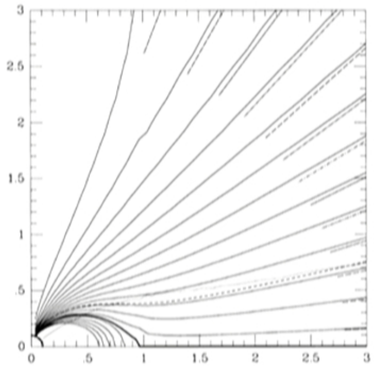
Dipole -> Monopole!



Goldreich and Julian 1969



Michel 1974



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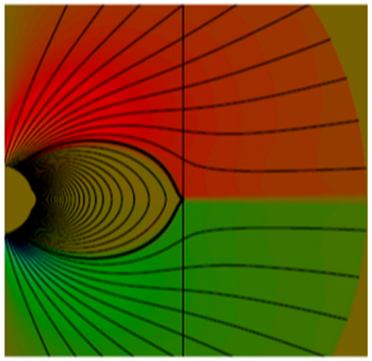

Spitkovsky 2006

Opening of Field Lines

The physical argument for the opening of field lines goes roughly as follows

1) Particles are stuck on field lines
2) Field lines corotate with the star
3) So, no closed line extends past light cylinder

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1) Set rotating dipole boundary conditions at the star and dipole initial conditions outside.
2) Evolve. When magnetic domination is violated, restore it somehow (e.g., reset E=B at grid level).
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We assume

i) The spacetime is vacuum, stationary, axisymmetric, asymptotically flat, and reflection symmetric about a plane (the "equatorial plane").

ii) The electromagnetic field outside the star is stationary, axisymmetric, and reflection symmetric, and force-free.

iii) The star is rigidly rotating and perfectly conducting.

iv) A "dipole-like" magnetization: field at stellar surface points in one direction in each hemisphere, and is never tangent to the surface.

We show

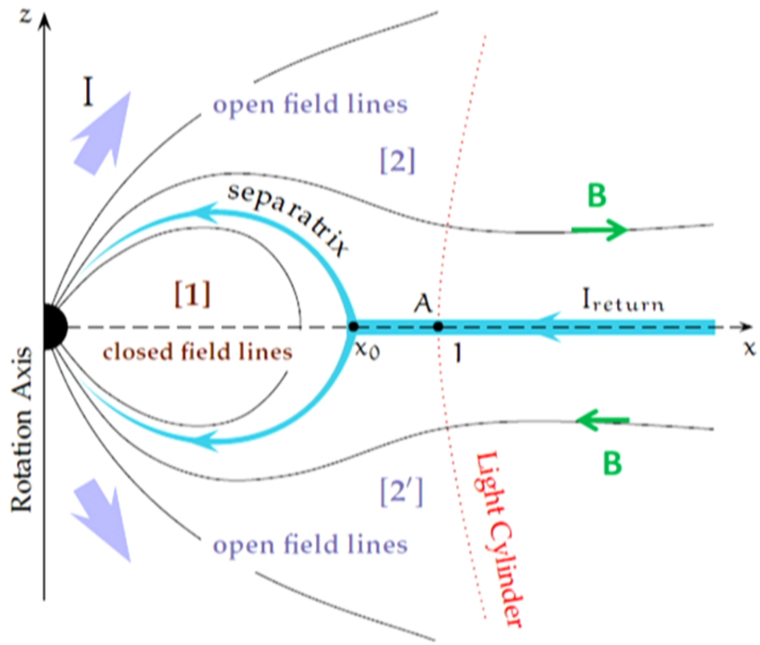
Closed field lines must remain within the light cylinder.

Notice some things we do not assume:

- Flat spacetime (or any particular spacetime)
- A spherical star (or any particular shape)
- Precise Magnetic Dipole

Current Sheet

Open field lines + Reflection Symmetry → Current Sheet



(picture from Timokhin 2006)

Inclined Pulsar Magnetosphere

Spitkovsky 2006

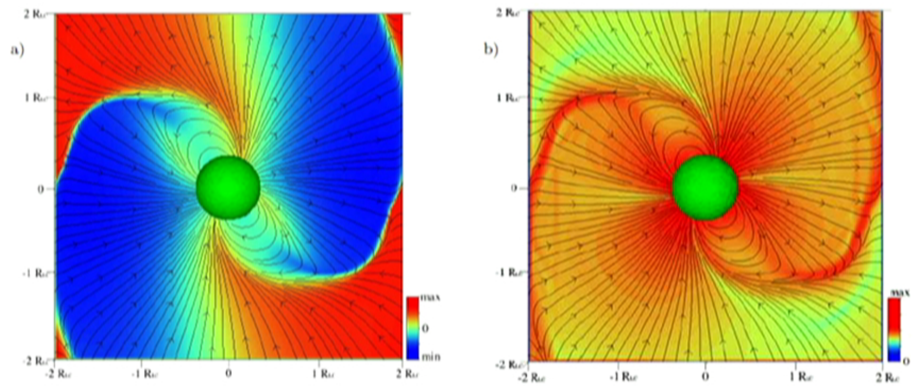


FIG. 2.— Oblique pulsar magnetosphere with magnetic inclination $\alpha = 60°$ in the corotating frame: a) Magnetic fieldlines in the μ-Ω_* plane. Color is the magnetic field perpendicular to the plane; b) same as a) but color represents absolute value of the total current $|\nabla \times \mathbf{B}|$.

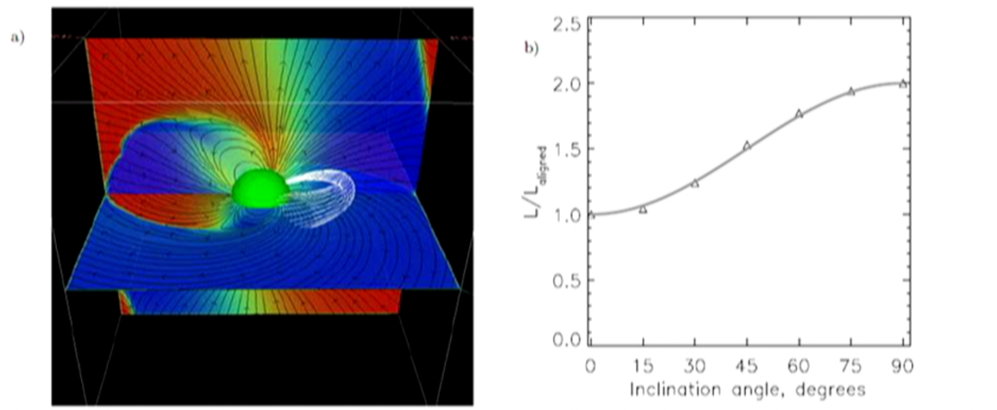


FIG. 3.— a) Slices through the 60° magnetosphere. Shown are fieldlines in the horizontal and vertical plane, color on the vertical plane is the perpendicular field, on the horizontal plane – toroidal field. Sample 3D flux tube is traced in white. b) Spindown luminosity in units of aligned force-free luminosity as a function of inclination. Triangles represent simulation data, and the line is a fit with eq. (3).

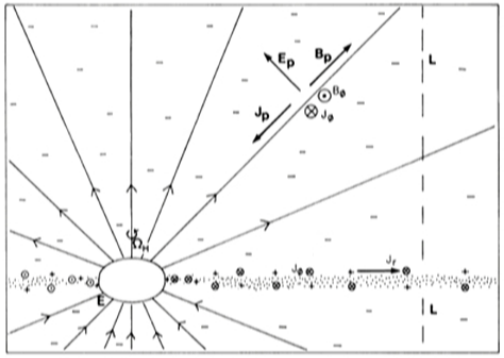
Black Hole Magnetosphere

Black Holes can't have a closed zone at all (MacDonald&Thorne 1982).
We call this the **no ingrown hair** theorem.

Blandford and Znajek 1977 promoted the Michel solution to linearized Kerr,

$$F = q\, d(\cos\theta) \wedge [d\phi - \Omega d(t - r)]$$

(Boyer-Lindquist coordinates)



They also corrected to $O(a^2)$.

Regularity at horizon → $\Omega = \frac{1}{2}\Omega_H$

This is mainly a toy model, but it could possibly form following the collapse of a pulsar (Lyutikov 2011). More realistically, should have an accretion disk.

Physics Recap

1) Stationary force-free fields can transport energy and angular momentum
2) A rotating, magnetized conductor does this
3) A spinning black hole can do it too
4) Closed Field lines tend to open: monopole exterior with current sheet
5) Current sheet wiggles in inclined case
6) No closed field lines at all for black holes

An Observation

The Michel Monopole has Charge and Current satisfying

$$|\vec{j}| = \rho$$

In spacetime language, the charge-current four-vector is lightlike or **null.**

$$\boxed{g_{ab} J^a J^b = 0}$$

The only previously known exact solution in Kerr (Menon&Dermer2006) has null current too!

$$I^\nu = -\frac{2}{a^2\sqrt{-g}}\frac{d}{d\theta}[\Lambda \frac{\cos\theta}{\sin^4\theta}]\, l^\nu \ .$$

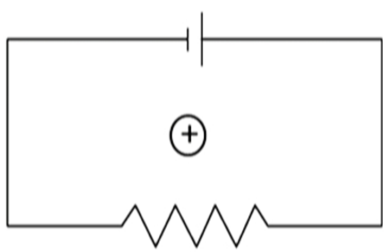
Ingoing principle null direction

To Emphasize…

Null current does *not* mean that charges move at the speed of light.

(and assuming Einstein, not Bateman, is right, they don't)

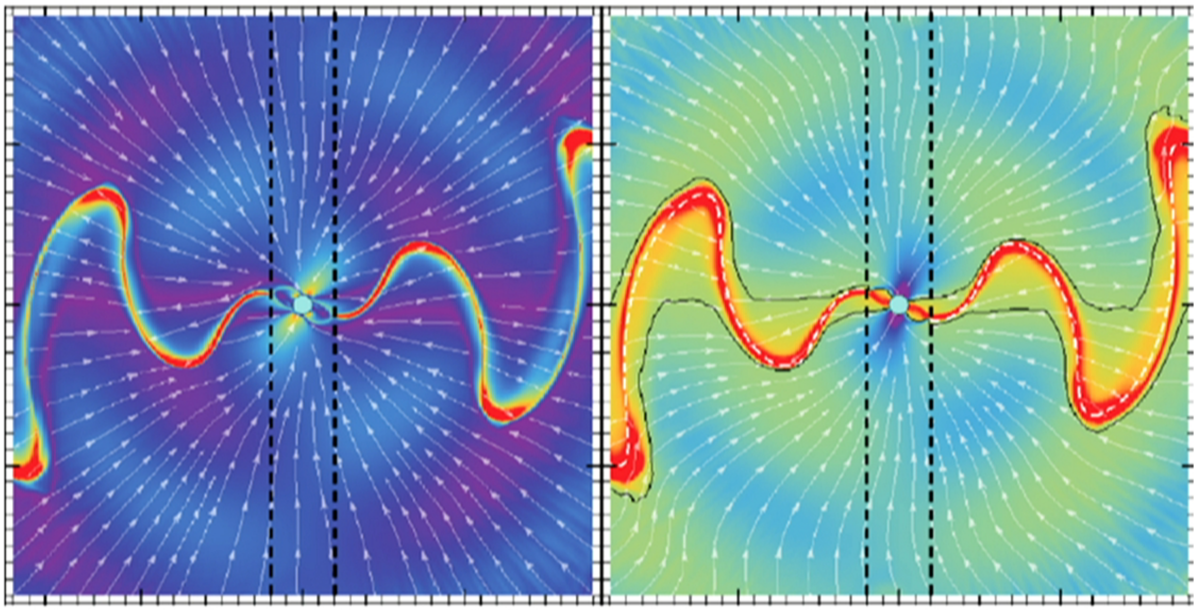
Null current just means that the *net* charge density is equal to the *net* current density.



$|\vec{j}| = \rho$, a null four-current.

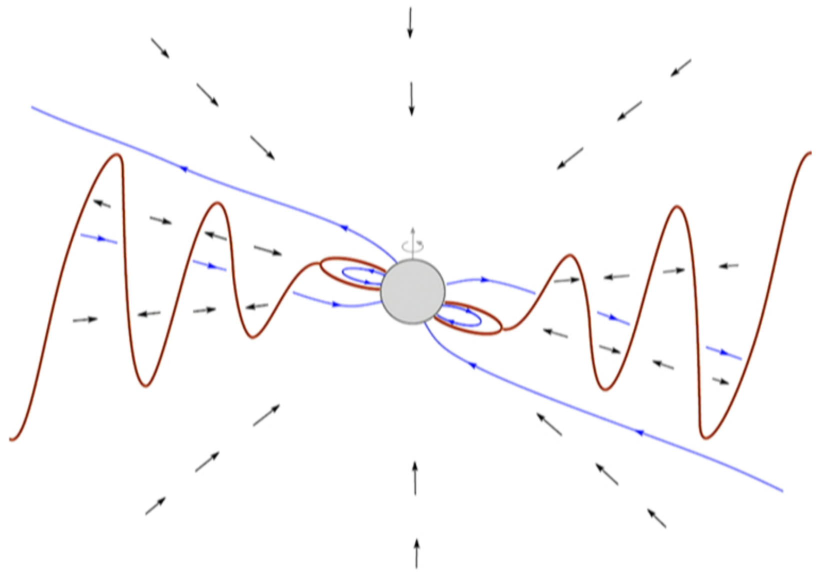
Inclined Pulsar Magnetosphere

Kalapotharakos, Contopoulos, Kazanas 2011



Three-Current Density　　Charge Density and Magnetic field

Cartoon Version



The outer magnetosphere has radial, null four-current.

Spacetime and Null Structure

There is a long history of leveraging the null structure of spacetime in GR research. The Kerr metric was found this way. Its perturbations were unlocked this way.

A key tool is the Newman-Penrose formalism, based on a *null tetrad*,

$$\{\ell^a, n^a, m^a, \bar{m}^a\} \qquad \ell^a, n^a \text{real}, \;\; m^a \text{complex}.$$

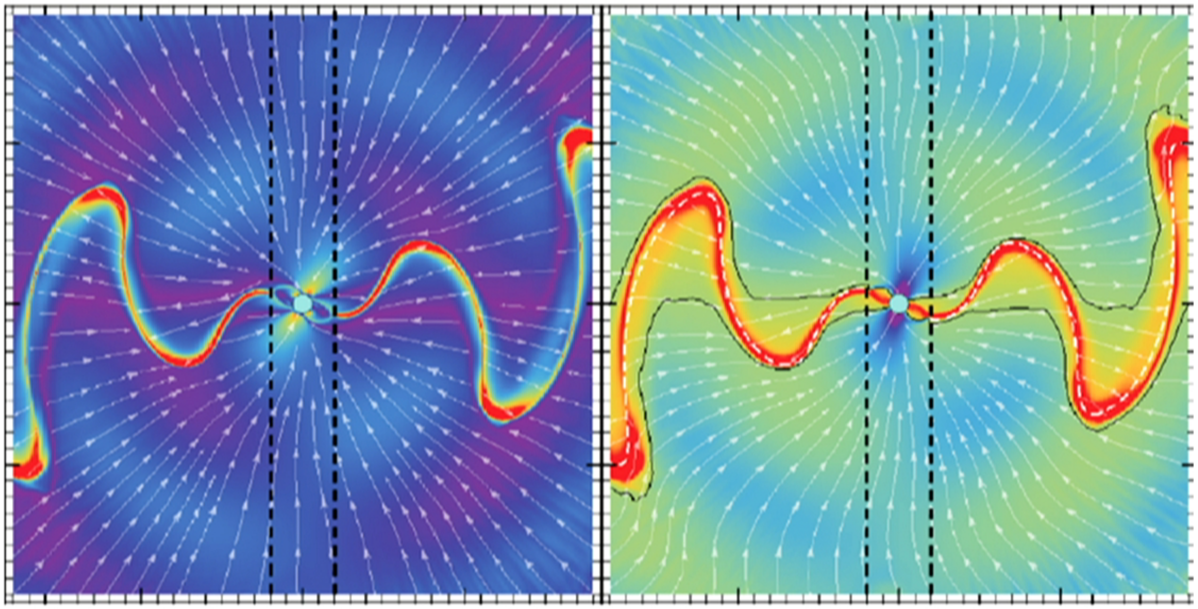
$$\begin{aligned} \ell \cdot n = 1, m \cdot \bar{m} = -1 \\ \ell \cdot m = n \cdot m = 0 \\ \ell \cdot \ell = n \cdot n = m \cdot m = 0 \end{aligned} \qquad (\text{signature } (+,-,-,-))$$

Instead of working with tensors, work with complex scalars. For the EM field,

$$\begin{aligned} \phi_0 &= F_{ab} \ell^a m^b \\ \phi_1 &= \tfrac{1}{2} F_{ab} (\ell^a n^b + \bar{m}^a m^b) \\ \phi_2 &= F_{ab} \bar{m}^a n^b. \end{aligned}$$

Inclined Pulsar Magnetosphere

Kalapotharakos, Contopoulos, Kazanas 2011



Three-Current Density   Charge Density and Magnetic field

The current is null and radial in the outer magnetosphere!

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Radial, Null Current

We completely solve the equations for radial, null current in flat spacetime.

The **general** solution is

$$F_{ab} = F^{\mathrm{q}}_{ab} - 2\ell_{[a}\nabla_{b]}\psi,$$

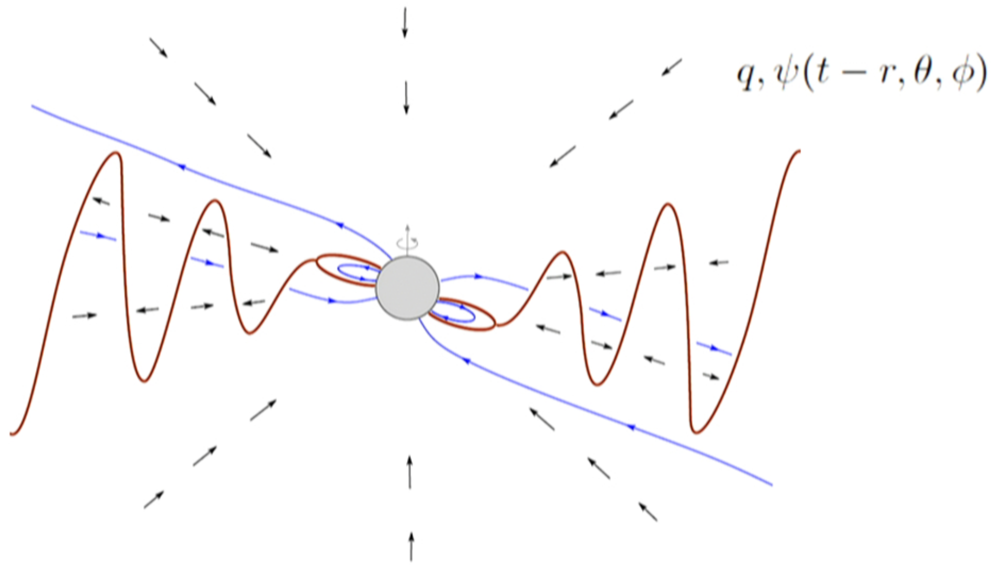
field of a magnetic monopole of charge q

an *arbitrary* function with $\ell^a \nabla_a \psi = 0$

In spherical coordinates,

$$F_{r\theta} = -F_{t\theta} = \partial_\theta \psi(t-r,\theta,\phi)$$
$$F_{r\phi} = -F_{t\phi} = \partial_\phi \psi(t-r,\theta,\phi)$$
$$F_{\theta\phi} = q\sin\theta,$$

Exact Outer Magnetosphere?



We have **all** solutions with null, radial current. To find the right one, match to numerical simulations. Then one has an exact analytic description of the outer pulsar magnetosphere, including the dynamics of the current sheet.

Torqued Split Monopole

$$\psi = B_0 R^2 \omega \cos\theta, \qquad q = B_0 R^2 \qquad \text{(rotating spherical conductor).}$$

(Michel 1973)

$$\psi = B_0 R^2 \omega(t-r) \cos\theta, \qquad q = B_0 R^2 \qquad \text{(variable rotation speed).}$$

(Lyutikov 2011)

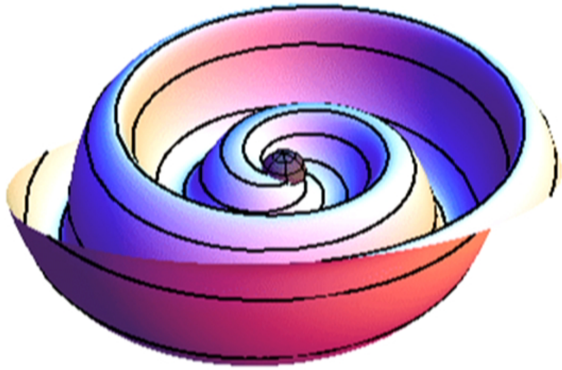
$$\psi = B_0 R^2 \omega(t-r)\{\cos\theta \cos[\theta_0(t-r)] + \sin\theta \cos[\phi - \phi_0(t-r)] \sin[\theta_0(t-r)]\},$$
$$q = B_0 R^2 \qquad \text{(variable rotation speed and axis).}$$

(new)

This last choice represents the most general rigid motion of a conducting sphere whose center is fixed!

Inclined Split Monopole

(Bogovalov 1999)



(a movie of the star and the current sheet)

Inclined, Torqued, Glitching Split Monopole, which does the Macarena at t=13sec.

[No movie available yet]

But our class does contain this solution!

$$\psi = B_0 R^2 \omega(t-r)\{\cos\theta\cos[\theta_0(t-r)] + \sin\theta\cos[\phi-\phi_0(t-r)]\sin[\theta_0(t-r)]\},$$
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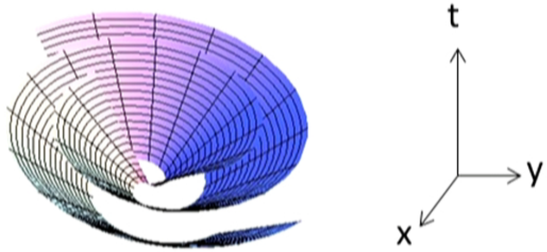
+ sheet

What determines the sheet behavior?

Thm: For matching a force-free solution to a rescaled copy of itself, the surface layer worldvolume must be ruled by a family of fieldsheets.

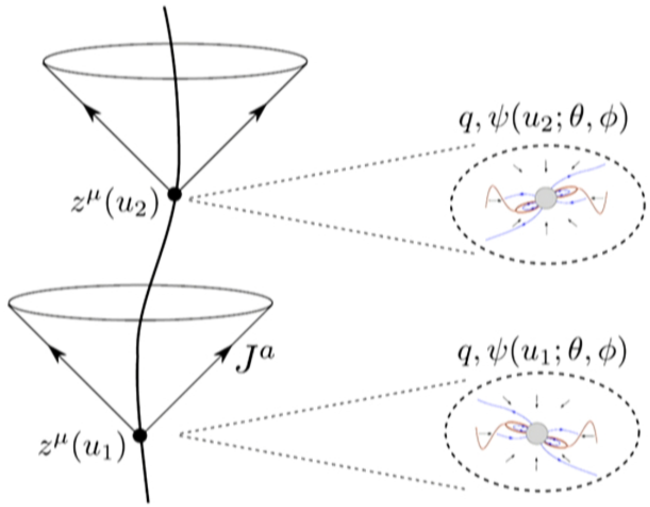


z=0

(Michel field surfaces in the equatorial plane)

Note also: sometimes a notion of “rotation of field lines” is discussed in the literature. This notion arises by using the field surfaces to identify lab frame (t=const) magnetic field lines at different times.

Accelerated Pulsar Magnetosphere



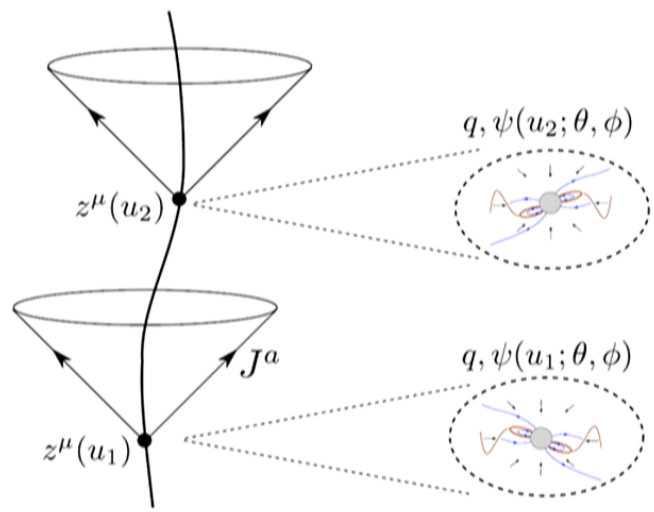
Remarkably, one can still solve the equations for null current along the light cones of an arbitrary timelike worldline. The answer is again

$$F_{ab} = F^{\mathrm{q}}_{ab} - 2\ell_{[a}\nabla_{b]}\psi, \qquad \ell^a \nabla_a \psi = 0$$

$$u^a \ell_a = 1$$

field of *moving* magnetic monopole

Accelerated Pulsar Magnetosphere



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Accelerated Pulsar Power

The energy flux is

$$\mathcal{P}(u) = \frac{2}{3} q^2 a^2 + \frac{1}{4\pi} \int_S g_S^{ab} \nabla_a \psi \nabla_b \psi \; d\Omega,$$

Larmor-type flux   Pulsar-type flux

For a dipole pulsar dimensional considerations imply $q \propto \mu\Omega$. Then

$$\Delta \mathcal{P}_{\text{accel.}} \sim \mu^2 \Omega^2 a^2$$

It should be possible to fix the numerical coefficient by matching to simulations of unaccelerated pulsars.

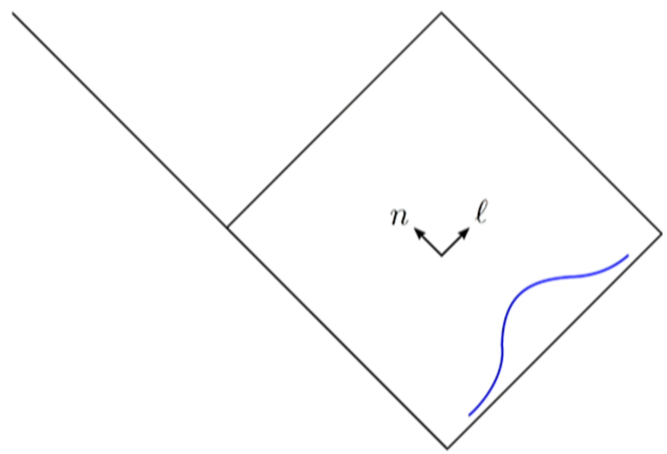
Compare to $\mu^2 \dot{a}^2$ for a vacuum accelerated dipole.

Non-Scattering Waves

In Schwarzschild in ingoing EF coordinates, our ingoing solutions are

$$F = dv \wedge dS + q \sin\theta d\theta \wedge d\phi \;, \quad S = S(v, \theta, \phi)$$

The free function can be thought of as giving initial data on past null infinity…



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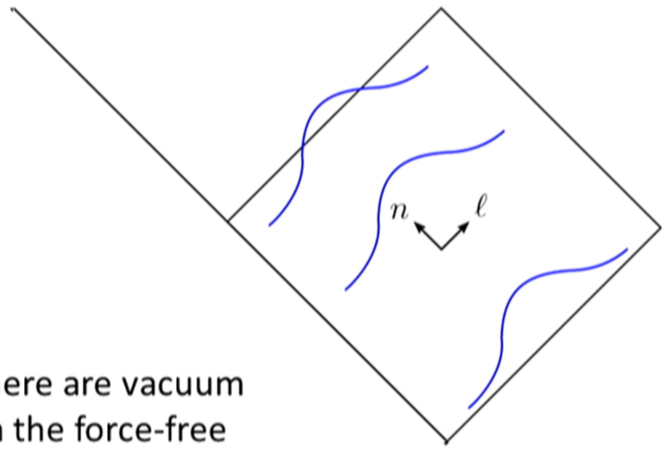

…and the wave just goes in.

This is surprising given that vacuum EM waves always scatter off black holes.

Robinson's theorem says that, *locally,* there are vacuum Maxwell solutions with this property. In the force-free case global solutions can exist, and we have bona fide non-scattering waves.