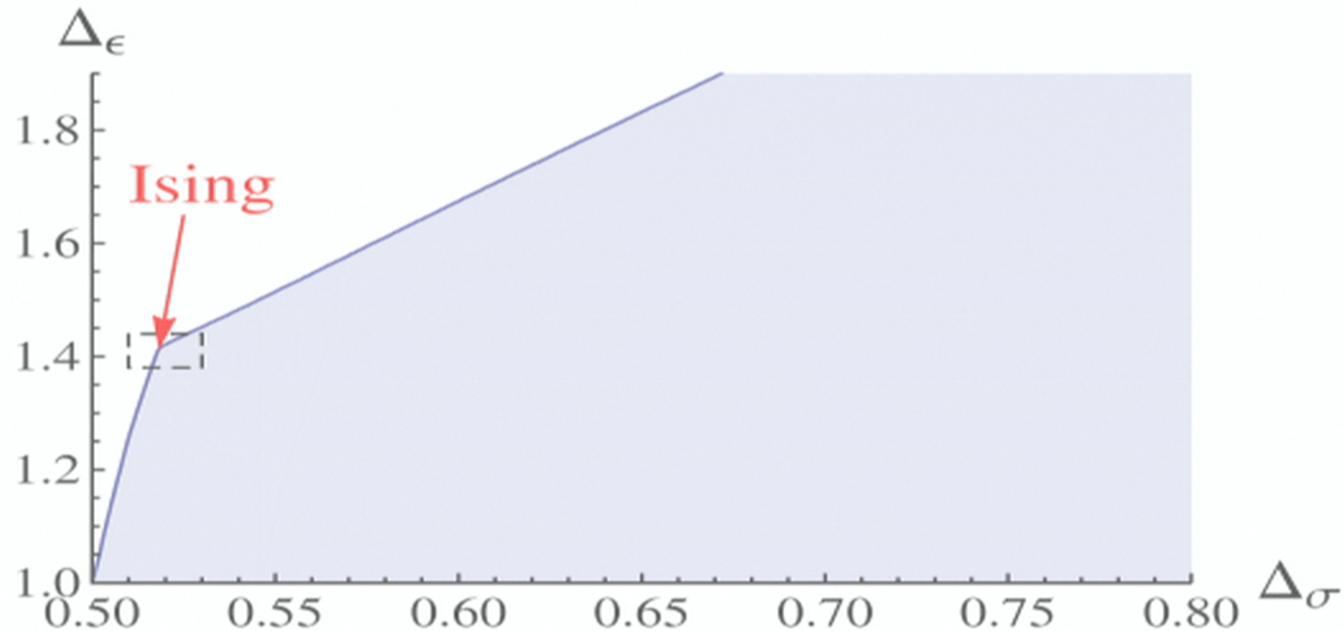
Title: Bootstrapping the O(N) Vector Models

Date: Oct 15, 2013 02:00 PM

URL: http://pirsa.org/13100074

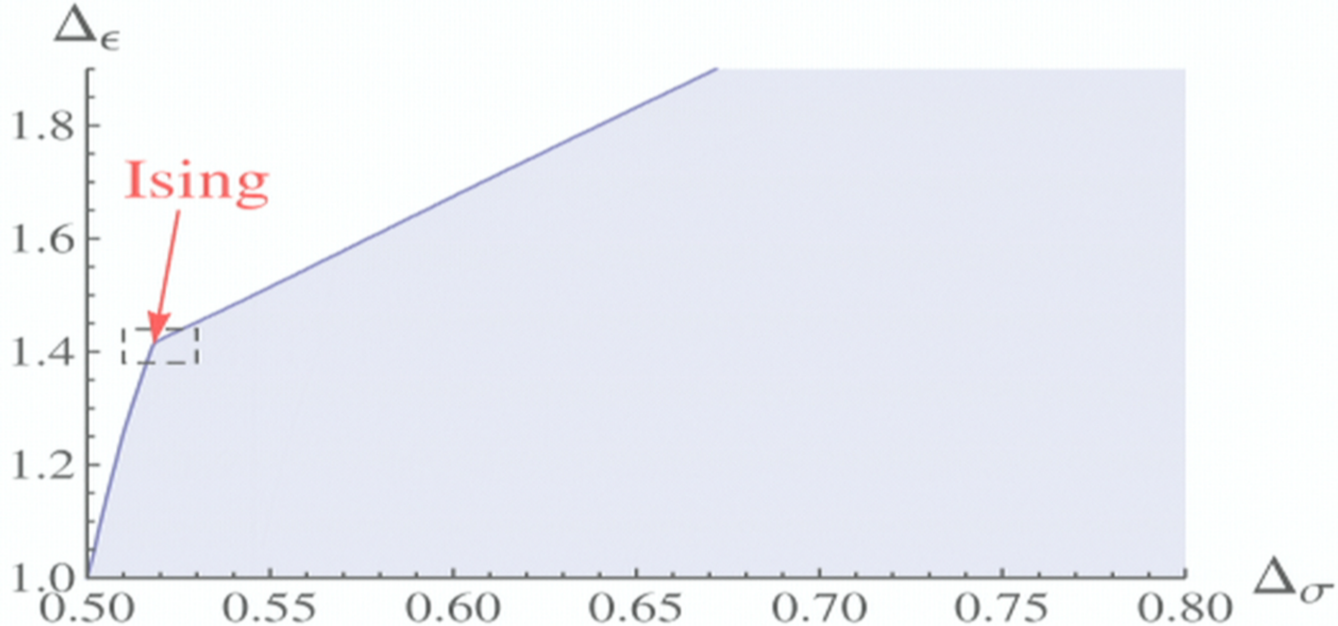
Abstract: We study the conformal bootstrap for 3D CFTs with O(N) global symmetry. We obtain rigorous upper bounds on the scaling dimensions of the first O(N) singlet and symmetric tensor operators appearing in the \phi_i x \phi_j OPE, where \phi_i is a fundamental of O(N). Comparing these bounds to previous determinations of critical exponents in the O(N) vector models, we find strong numerical evidence that the O(N) vector models saturate the bootstrap constraints at all values of N. We also compute general lower bounds on the central charge, giving numerical predictions for the values realized in the O(N) vector models. We compare our predictions to previous computations in the 1/N expansion, finding precise agreement at large values of N.

Recent Bootstrap Results [El-Showk, Paulos, Poland, Rychkov, DSD, Vichi '12]



- Using: conformal symmetry, unitarity, existence of a scalar σ.
- Something special about 3d Ising: CFT_3 that maximizes $\Delta_{\phi^2} - 2\Delta_\phi$

Recent Bootstrap Results [El-Showk, Paulos, Poland, Rychkov, DSD, Vichi '12]

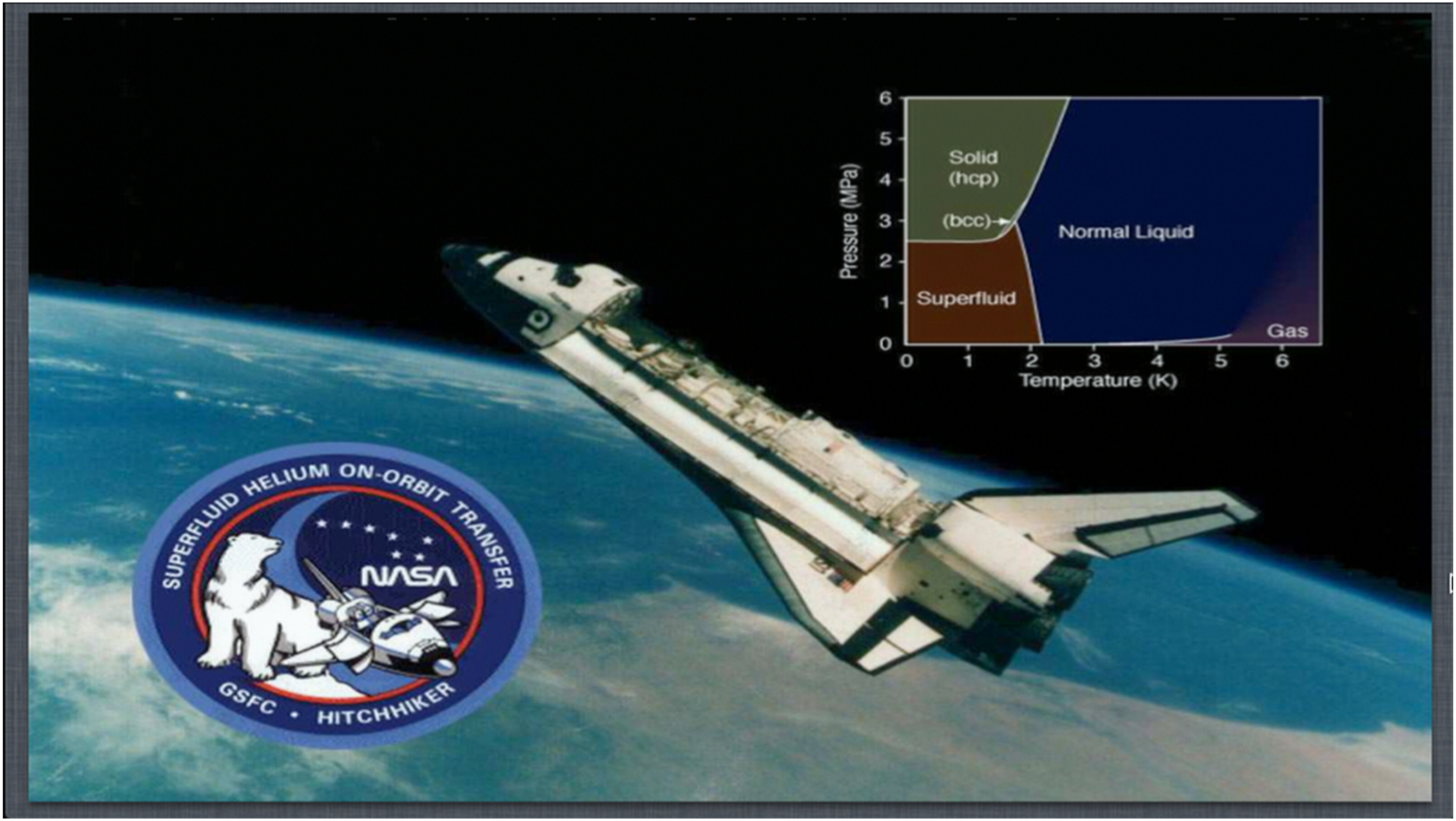


- Using: conformal symmetry, unitarity, existence of a scalar σ.
- Something special about 3d Ising: CFT_3 that maximizes $\Delta_{\phi^2} - 2\Delta_\phi$

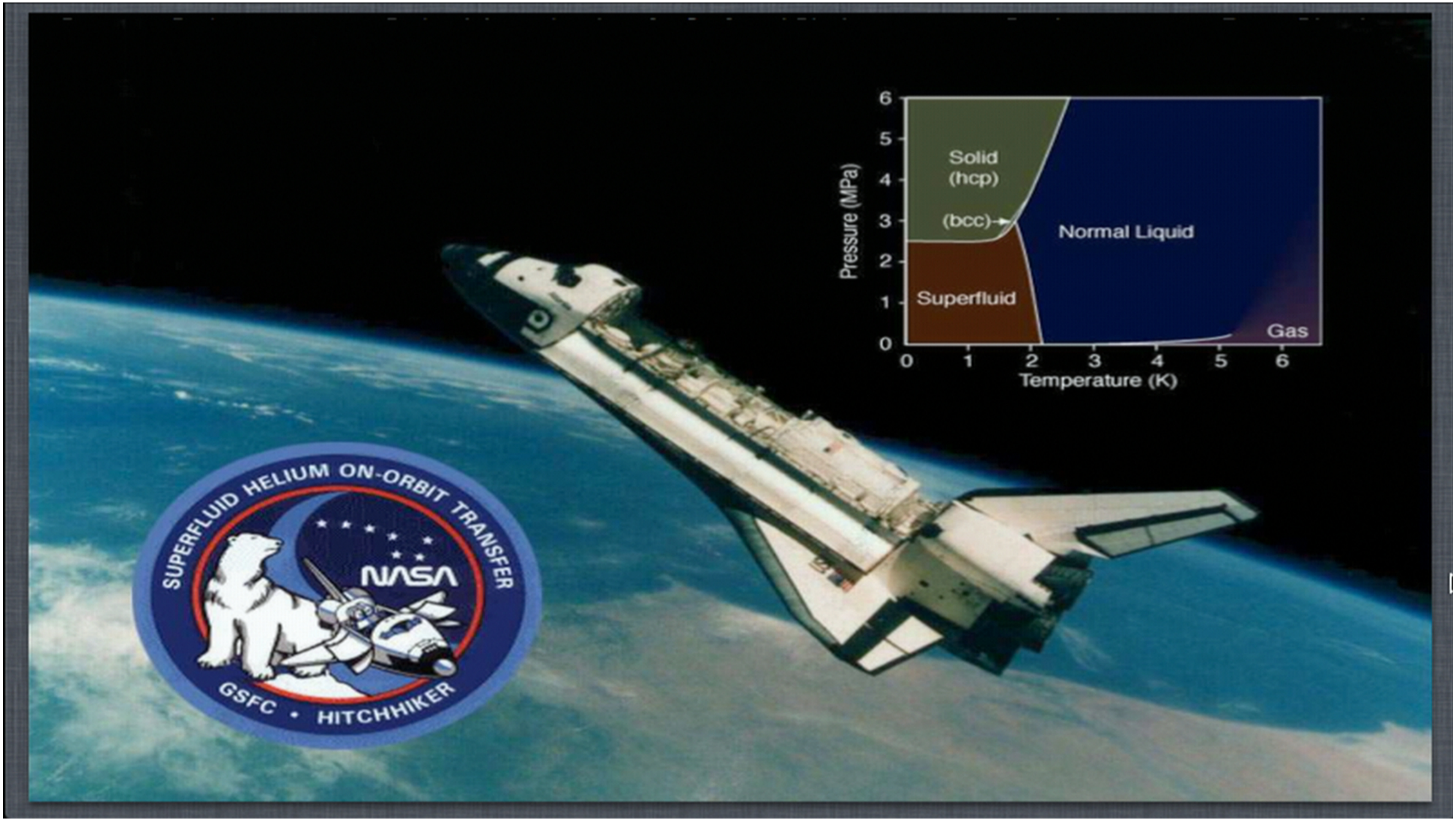
What about other CFT's?

Simplest generalization of 3d Ising is the $O(N)$ model

- Critical point of N bosons $\vec{\phi}$ with $(|\vec{\phi}|^2)^2$ potential
- Large N
  - can use perturbation theory
  - conjectured dual to higher-spin theory in AdS.
- Small N
  - superfluid transition in ^{4}He ($N = 2$)
  - Curie transition in isotropic magnets Ni, Fe, EuO ($N = 3$).
  - many other condensed matter systems



Pressure (MPa)
Temperature (K)
Solid
(hcp)
(bcc)
Superfluid
Normal Liquid
Gas
0
1
2
3
4
5
6
SUPERFLUID HELIUM ON-ORBIT TRANSFER
NASA
GSFC • HITCHHIKER


Solid
(hcp)
(bcc)
Normal Liquid
Superfluid
Gas
Pressure (MPa)
Temperature (K)
0
1
2
3
4
5
6
SUPERFLUID HELIUM ON-ORBIT TRANSFER
NASA
GSFC • HITCHHIKER

Bootstrapping 3d Theories with $O(N)$ Symmetry

- Bootstrap with global symmetries is hard
- From experience: need *semidefinite programming* (as opposed to *linear programming*) [Poland, DSD, Vichi '11]
- To apply SDP in 3d, we need some new results about 3d conformal blocks

Outline

1. Bootstrap Review
2. Rational Approximations for Conformal Blocks
3. Results
4. Future Directions

CFT Review: Conformal Block Decomposition

Consider scalar primary ϕ in a CFT.

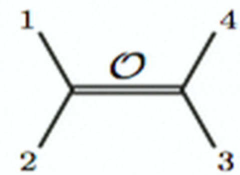
Use OPE to evaluate 4-point function

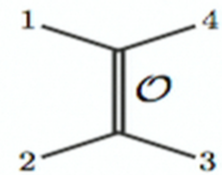
$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2\, g_{\Delta,l}(u,v)$$

- $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ conformally-invariant cross ratios.
- $g_{\Delta,l}(u,v)$ conformal block ($\Delta = \dim \mathcal{O}$ and $l =$ spin of $\mathcal{O}$)
- OPE coefficient $\lambda_{\mathcal{O}}$ is real, so $\lambda_{\mathcal{O}}^2$ is positive.

Crossing Symmetry

The answer should be independent of how we perform the OPE

$-$unit op. $=\sum$ 

$-\sum$ 

$$-(u^{-\Delta_\phi}-v^{-\Delta_\phi}) = \sum_{\mathcal{O}\in\phi\times\phi}\lambda_{\mathcal{O}}^2\left(u^{-\Delta_\phi}g_{\Delta,\ell}(u,v)-u\leftrightarrow v\right)$$

$$-F_{\text{unit}} = \sum_{\mathcal{O}\in\phi\times\phi}\lambda_{\mathcal{O}}^2 F_{\Delta,\ell}(u,v)$$

Bootstrap Bounds

$$-F_{\text{unit}} = \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 F_{\Delta,\ell}(u,v)$$

- Assume something about spectrum (e.g. all scalars $\mathcal{O}$ have $\Delta > \Delta_{\text{min}}$).
- Try to find a linear functional α such that

$$\begin{aligned} \alpha(-F_{\text{unit}}) &< 0, && \text{and} \\ \alpha(F_{\Delta,\ell}) &\geq 0, && \text{for all other } \mathcal{O} \in \phi\times\phi \end{aligned}$$

- If such α exists, we get a contradiction

$$0 > \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 \alpha(F_{\Delta,\ell}) \geq 0 \implies \text{spectrum ruled out}$$

Bootstrap Bounds

$$-F_{\text{unit}} = \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 F_{\Delta,\ell}(u,v)$$

- Assume something about spectrum (e.g. all scalars $\mathcal{O}$ have $\Delta > \Delta_{\text{min}}$).
- Try to find a linear functional α such that

$$\begin{aligned} \alpha(-F_{\text{unit}}) &< 0, && \text{and} \\ \alpha(F_{\Delta,\ell}) &\geq 0, && \text{for all other } \mathcal{O} \in \phi \times \phi \end{aligned}$$

- If such α exists, we get a contradiction

$$0 > \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 \alpha(F_{\Delta,\ell}) \geq 0 \quad \Longrightarrow \quad \text{spectrum ruled out}$$

Bounds from Semidefinite Programming

Our task:

- Find α such that $\alpha(-F_{\text{unit}}) < 0$ and $\alpha(F_{\Delta,\ell}) \geq 0$.

Semidefinite program:

- Find α such that $\alpha(\vec{v}) < 0$ and $\alpha(\vec{P}_i(x))$, where $\vec{P}_i(x)$ are polynomials in x.

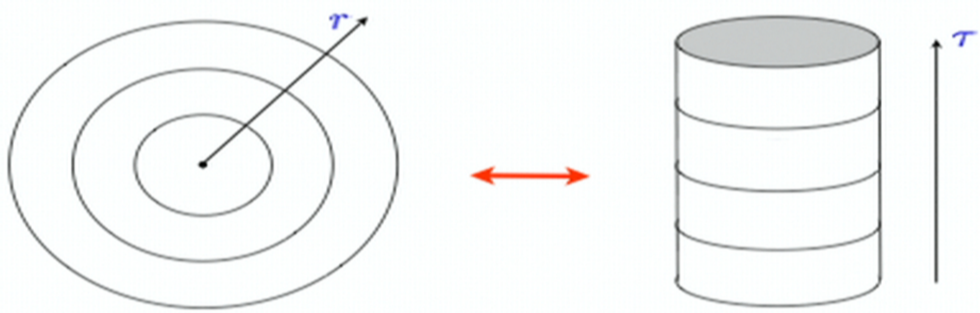
For $\alpha = \sum a_{mn}\partial_u^m \partial_v^n$, we must approximate

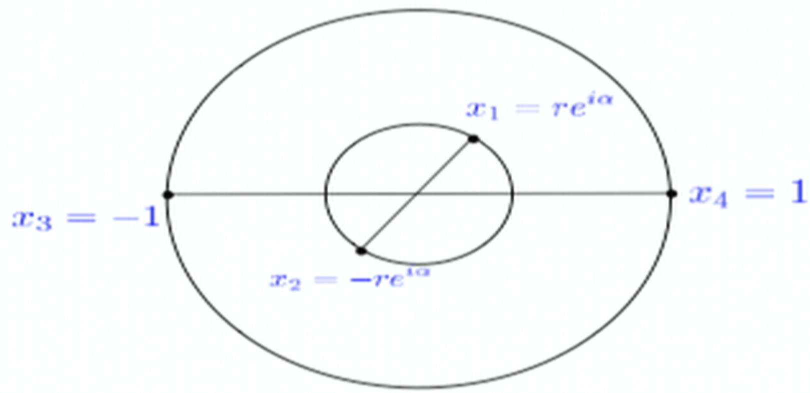
$$\partial_u^m \partial_v^n F_{\Delta,\ell}(u,v) \approx \chi_\ell(\Delta) P_{m,n,\ell}(\Delta)$$

where $\chi_\ell(\Delta)$ is positive and $P_{m,n,\ell}(\Delta)$ are polynomials.

CFT Review

- Radial quantization and the state-operator map $\mathcal{O}(0) \leftrightarrow |\mathcal{O}\rangle$





Conformal Block Decomposition

To compute a four-point function, insert a complete set of states in radial quantization

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$
$$= \sum_{\psi} \langle 0|\phi(x_1)\phi(x_2)|\psi\rangle \langle \psi|\phi(x_3)\phi(x_4)|0\rangle$$
$$= \sum_{\mathcal{O}} \left(\sum_{\psi=\mathcal{O},P\mathcal{O},PP\mathcal{O},\ldots} \frac{\langle 0|\phi(x_1)\phi(x_2)|\psi\rangle \langle \psi|\phi(x_3)\phi(x_4)|0\rangle}{\langle \psi|\psi\rangle} \right)$$
$$= \sum_{\mathcal{O}} \left(\lambda^2_{\phi\phi\mathcal{O}} \frac{g_{\Delta,\ell}(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \right)$$

where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$, and $\Delta = \dim \mathcal{O}$, $\ell = \text{spin}\, \mathcal{O}$.

Series Expansion for $g_{\Delta,\ell}$ [Hogervorst, Rychkov '13]

Classifying states $|\psi\rangle$ according to energy $\Delta + n$ and spin j, we get

$$g_{\Delta,\ell}(r,\alpha) = \sum A_{n,j} r^{\Delta+n} C_j^{(d/2-1)}(\cos\alpha)$$

For example $\ell = 0$ in d spacetime dimensions,

$$A_{n,0} = 1, \quad A_{2,0} = \frac{\Delta(d-2)^2}{2d(\Delta-(d/2-1))}, \quad A_{2,2} = \frac{2\Delta(d-1)}{d(\Delta+1)}, \ldots$$

Truncating at order $r^{\Delta+N}$ gives a rational function of Δ, times r^{Δ}. Poles all below unitarity bound.

Why Good Rational Approximations Exist

$$g_{\Delta,\ell} = \sum_{\psi=\mathcal{O},P\mathcal{O},P^2\mathcal{O}} \frac{\langle 0|\phi\phi|\psi\rangle\langle\psi|\phi\phi|0\rangle}{\langle\psi|\psi\rangle}$$

- Poles occur when $|\psi\rangle = P^n|\mathcal{O}\rangle$ is null for some n.
- This can occur at special Δ_i *below* the unitarity bound.
- Residue at a pole comes from summing up all descendants of the null state $|\psi\rangle$. This is a conformal block in itself.

$$g_{\Delta,\ell} = f_\infty(\Delta) + \sum \frac{c_i r^{\Delta-\Delta_i}}{\Delta-\Delta_i} g_{\Delta_i+n_i,\ell_i}$$

Similar to recursion relations for Virasoro blocks [Zamolodchikov '87].

Positive-times-polynomial

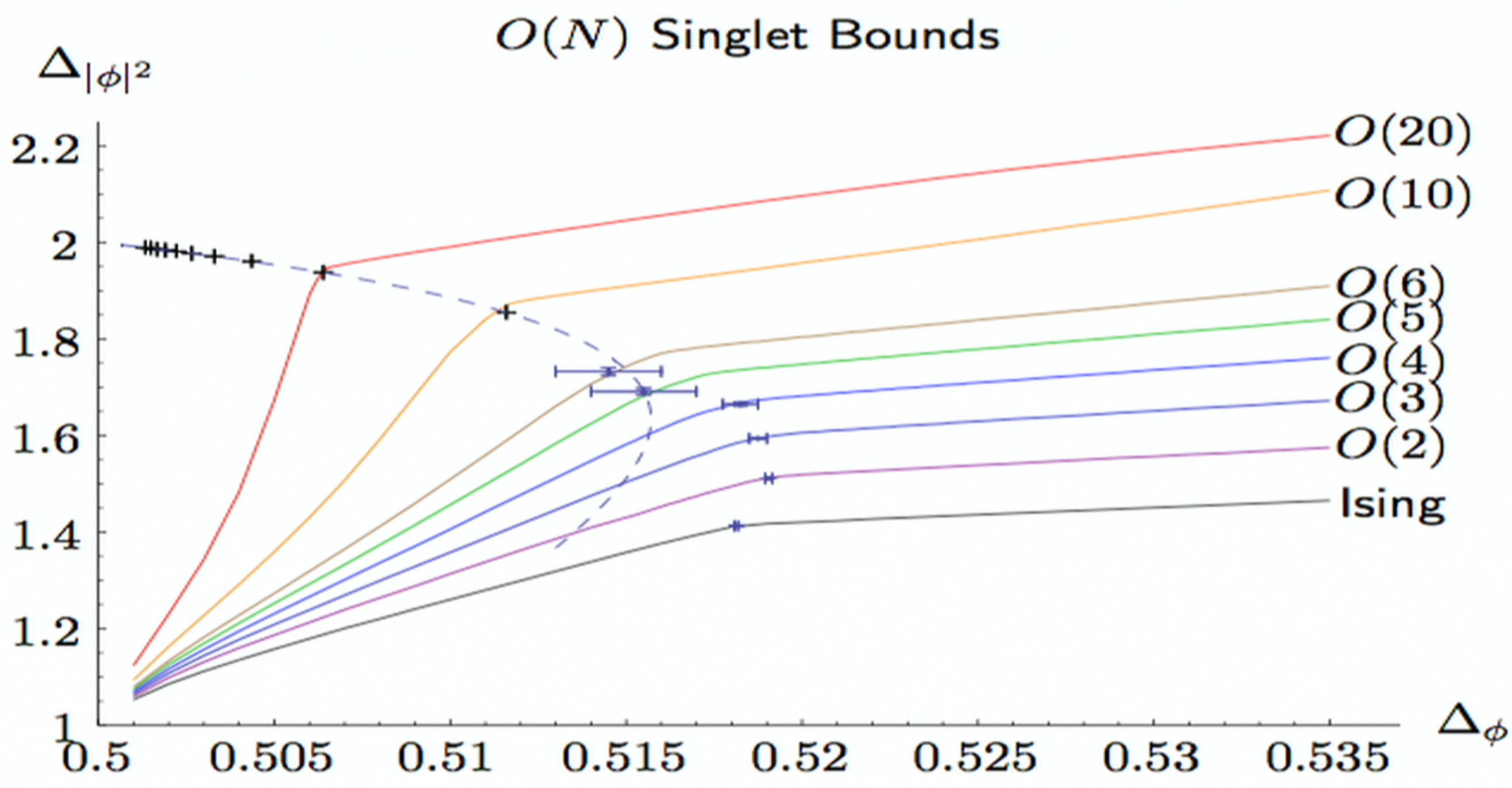
$$g_{\Delta,\ell} \;=\; f_\infty(\Delta) + \sum_i \frac{c_i r^{\Delta-\Delta_i}}{\Delta-\Delta_i} g_{\Delta_i+n_i,\ell_i}$$

The Δ_i get more and more negative. Thus, the residues $r^{\Delta-\Delta_i}$ decrease exponentially ($r \approx 0.17$ at crossing-symmetric point).

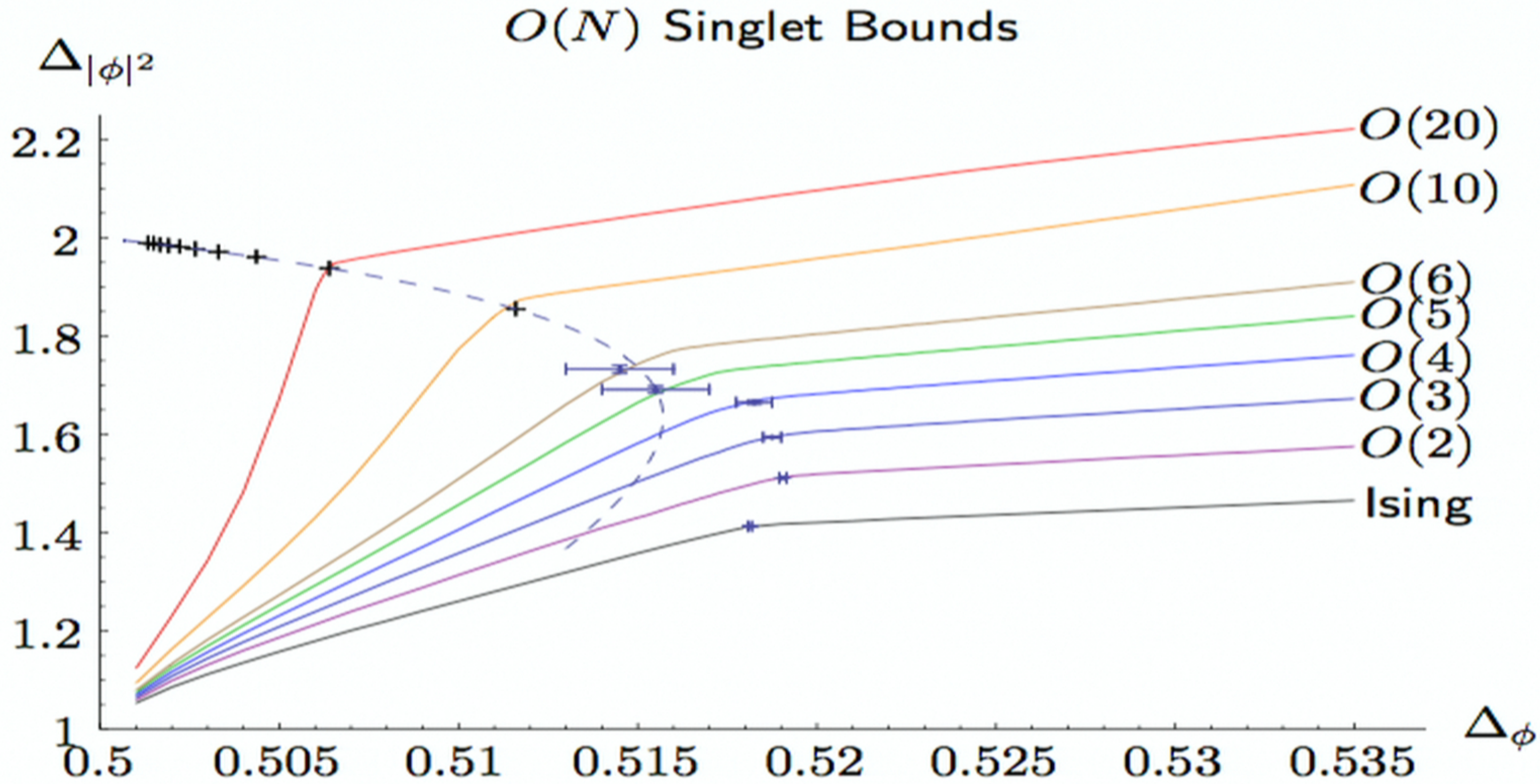
Truncating to a finite number of poles, we have

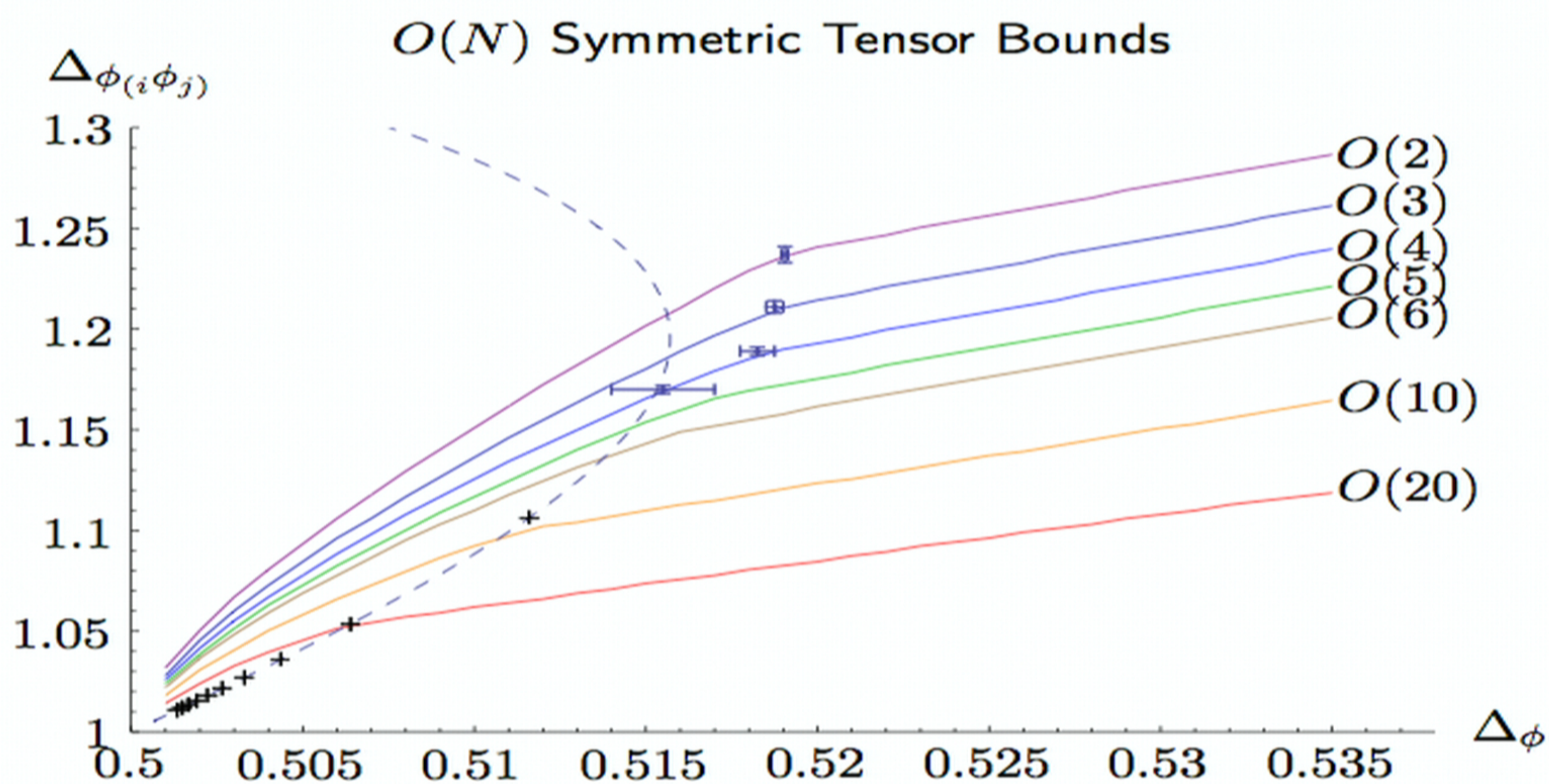
$$\partial_r^m \partial_\alpha^n g_{\Delta,\ell}(r,\alpha) \;\approx\; \underbrace{\frac{r^\Delta}{\prod_i(\Delta-\Delta_i)}}_{\text{positive}} \underbrace{P_{m,n,\ell}(\Delta)}_{\text{polynomial}}$$

This is enough to apply semidefinite programming.

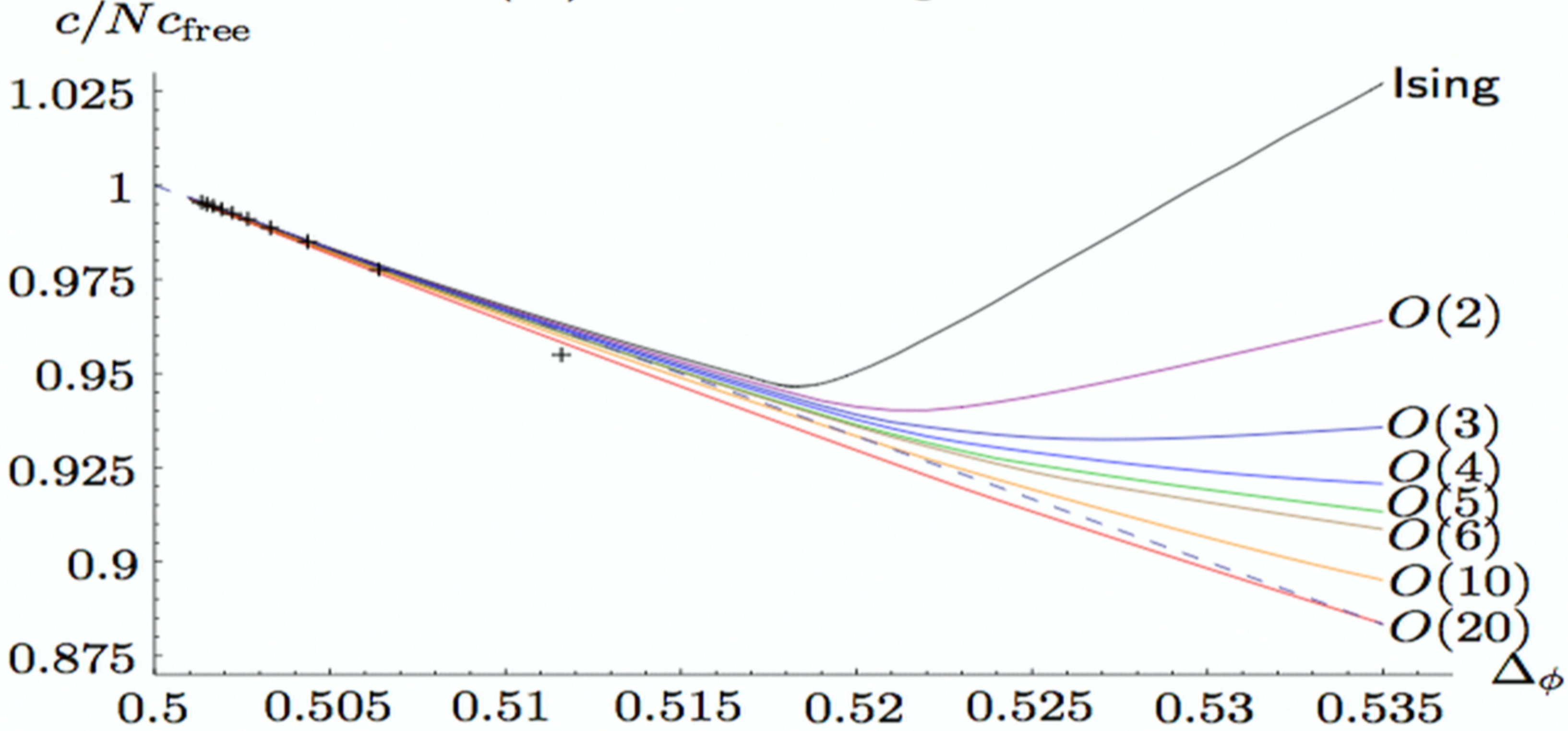


$O(N)$ Singlet Bounds
$\Delta_{|\phi|^2}$
2.2
2
1.8
1.6
1.4
1.2
1
0.5
0.505
0.51
0.515
0.52
0.525
0.53
0.535
Δ_ϕ
$O(20)$
$O(10)$
$O(6)$
$O(5)$
$O(4)$
$O(3)$
$O(2)$
Ising


$O(N)$ Singlet Bounds
$\Delta_{|\phi|^2}$
2.2
2
1.8
1.6
1.4
1.2
1
0.5
0.505
0.51
0.515
0.52
0.525
0.53
0.535
Δ_ϕ
$O(20)$
$O(10)$
$O(6)$
$O(5)$
$O(4)$
$O(3)$
$O(2)$
Ising


$O(N)$ Symmetric Tensor Bounds
$\Delta_{\phi_{(i}\phi_{j)}}$
1.3
1.25
1.2
1.15
1.1
1.05
1
0.5
0.505
0.51
0.515
0.52
0.525
0.53
0.535
Δ_ϕ
$O(2)$
$O(3)$
$O(4)$
$O(5)$
$O(6)$
$O(10)$
$O(20)$

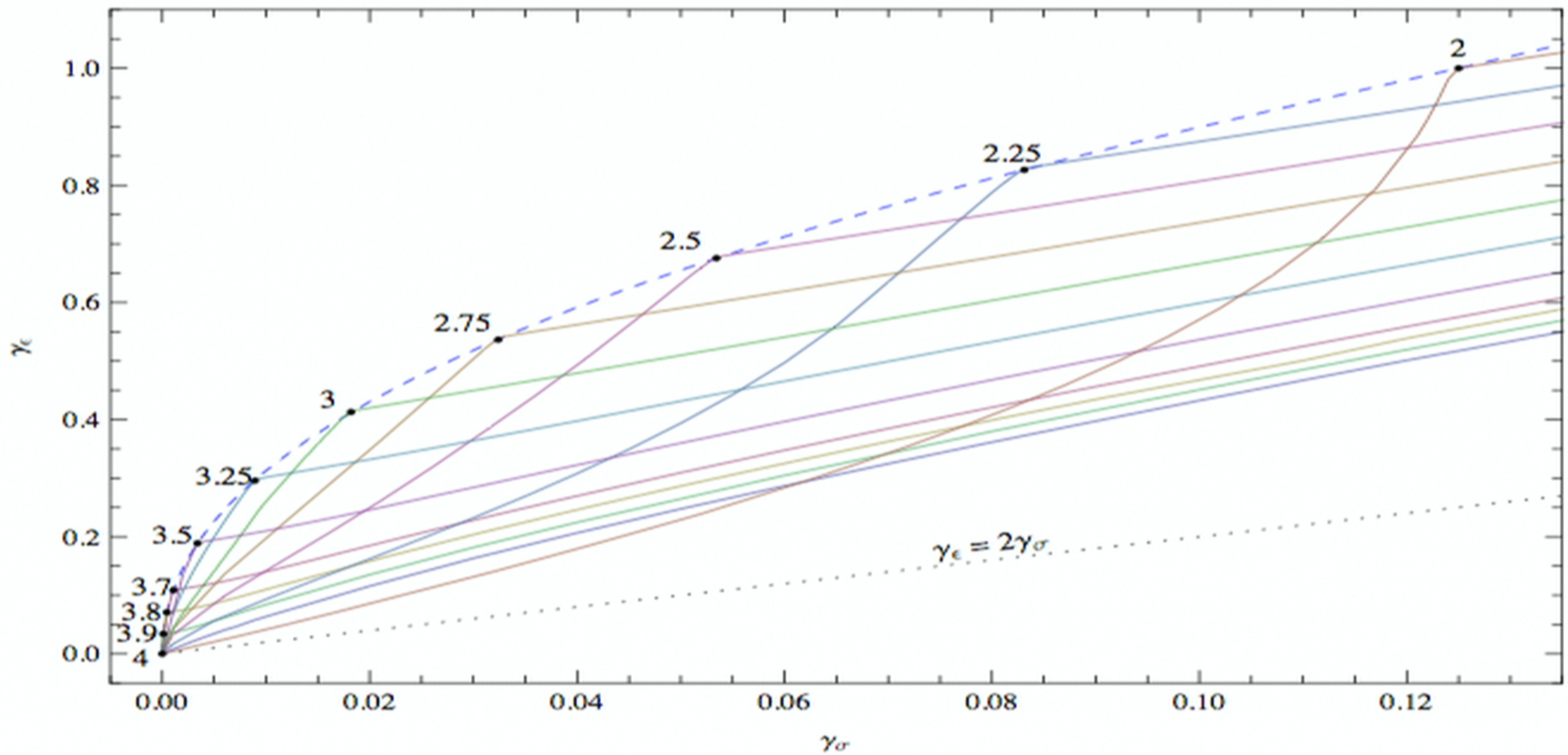
$O(N)$ Central Charge Bounds

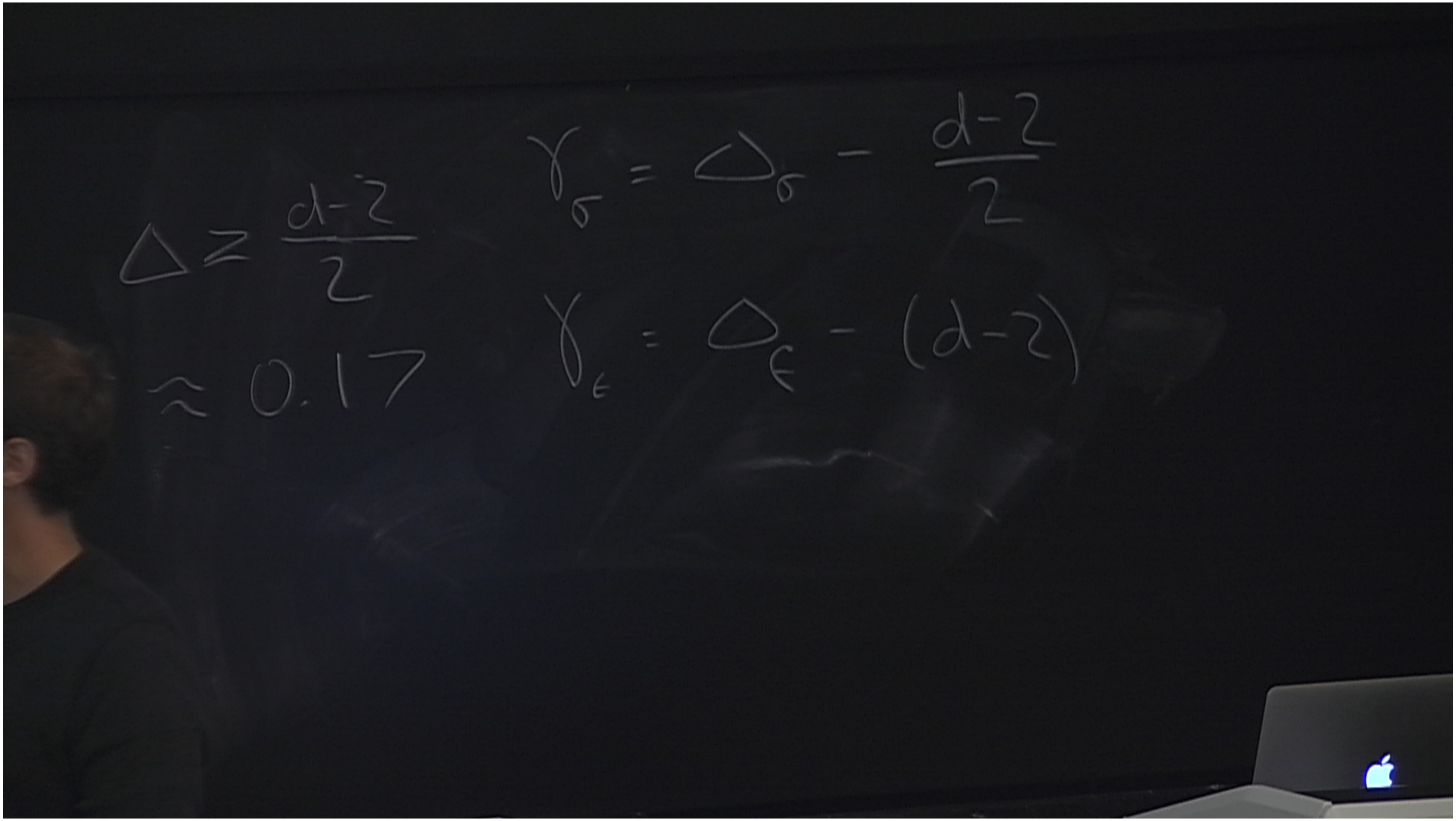


Predictions

N	Δ_ϕ	$\Delta_{\|\phi\|^2}$	$\Delta_{\phi_{(i}\phi_{j)}}$	$c/N c_{\text{free}}$
1	0.51813(5)	$1.4119^{+0.0005}_{-0.0015}$	–	$0.946600^{+0.000022}_{-0.000015}$
2	0.51905(10)	$1.5118^{+0.0012}_{-0.0022}$	$1.23613^{+0.00058}_{-0.00158}$	$0.94365^{+0.00013}_{-0.00010}$
3	0.51875(25)	$1.5942^{+0.0037}_{-0.0047}$	$1.2089^{+0.0013}_{-0.0023}$	$0.94418^{+0.00043}_{-0.00036}$
4	0.51825(50)	$1.6674^{+0.0077}_{-0.0087}$	$1.1864^{+0.0024}_{-0.0034}$	$0.94581^{+0.00071}_{-0.00039}$
5	0.5155(15)	$1.682^{+0.047}_{-0.048}$	$1.1568^{+0.009}_{-0.010}$	$0.9520^{+0.0040}_{-0.0030}$
6	0.5145(15)	$1.725^{+0.052}_{-0.053}$	$1.1401^{+0.0085}_{-0.0095}$	$0.9547^{+0.0041}_{-0.0027}$
10	0.51160	$1.8690^{+0.000}_{-0.001}$	$1.1003^{+0.000}_{-0.001}$	0.96394
20	0.50639	$1.9408^{+0.000}_{-0.001}$	$1.0687^{+0.000}_{-0.001}$	0.97936

Fractional Spacetime Dimensions [El-Showk, Paulos, Poland, Rychkov, DSD, Vichi]




$\Delta \geq \frac{d-2}{2}$
≈ 0.17
$\gamma_\sigma = \Delta_\sigma - \frac{d-2}{2}$
$\gamma_\epsilon = \Delta_\epsilon - (d-2)$

Future Bootstrap Directions

- Incorporate more CFT consistency constraints, e.g. $\langle\sigma\sigma\epsilon\epsilon\rangle$ and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$ in 3d Ising.
- Bounds on non-scalar operators, e.g. symmetry currents and stress tensors. (Perhaps make contact with [Hofman, Maldacena '08].) Higher spin blocks computable [DSD '12].
- Other dimensions (e.g., 6d, 8d) and/or more SUSY. (Recently [Beem, Rastelli, van Rees '13])
- Bulk AdS interpretation.
- Improve analytic understanding.