Title: Minimal Distances and the RG Flow

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Abstract: We will discuss the renormalization group flow between different classes of CFTs in four dimensions and study possible lower bounds on the "distances" between these theories.

Minimal Distances and the RG Flow

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Based on Work to Appear

October 29, 2013

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Overview

- Theory space (and why we care)
- Special points in theory space: (S)CFTs, their data, and their constraints
- Probing theory space: the RG flow and some simple laws
- SUSY RG flows and SCFT repulsion
- $\mathcal{N} = 1$ SUSY and parametrically short RG flows
- $\mathcal{N} = 2$: constraints from symmetries and the bootstrap
- $\mathcal{N} = 4$ and the RG flow: minimal steps

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Theory Space

• It is a manifold, $\mathcal{M}_{\mathcal{T}}$. $\mathcal{P} \in \mathcal{M}_{\mathcal{T}}$ is a QFT (can consider different sub manifolds with different properties).

• We can study it by picking some reference point $\mathcal{P}$ (it is easier to study if $\mathcal{P}$ is a CFT). Deformations of $\mathcal{P}$

$$\delta\mathcal{L} = \lambda^I \mathcal{O}_I \ , \tag{1}$$

parameterize normal directions. Wilsonian analysis $\Rightarrow$ we can forget about highly irrelevant $\mathcal{O}_I$.

• Many subtleties: (presumably) infinite dimensional, dualities act on it in a non-trivial way, changes of coordinates $\sim$ changes of scheme. Topology and geometry highly non-trivial.

Theory Space (cont...)

• Given that the Standard Model is some effective IR description, can we constrain the UV (there are already some constraints, e.g., from the a theorem). Somehow looks like we need more than just EFT to under stand what LHC is telling us.

Special Points in Theory Space: (S)CFTs

• To get a foothold on some complicated system, it always pays to start with a more symmetric point. Pick some CFT, $\mathcal{T}$.

• CFT data is just a collection of simple numbers (OPE coefficients and spectrum) summarized nicely in the OPE:

$$\mathcal{O}(x)\mathcal{O}^\dagger(0) = \frac{1}{x^{2D_\mathcal{O}}} + f_{\mathcal{O}\mathcal{O}\mathcal{O}'} \cdot \frac{1}{x^{2D_\mathcal{O}-D_{\mathcal{O}'}}}\mathcal{O}'(0) + \cdots . \qquad (2)$$

• Some of the data describes collective behavior of the CFT. For example: τ_{ij}, a, c,.... We'll specialize mostly to four dimensions

$$T \sim a \cdot E + c \cdot W^2 . \qquad (3)$$

• This data is highly constrained

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Special Points in Theory Space: (S)CFTs (cont...)

• Unitarity is an important constraint. On 2-pt fns: $c, \tau_{ij} > 0$.

• Another set of constraints come from the associativity of the OPE (a.k.a., the "bootstrap" program) **[Rattazzi, Rychkov, Tonni, and Vichi, '08, ...]**.

• To understand this, consider four point functions in the CFT. We have

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2D}|x_{34}|^{2D}} , \qquad (4)$$

where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ are the conformally invariant cross ratios, and $g(u,v)$ is an "arbitrary" function.

Special Points in Theory Space: (S)CFTs (cont...)

- In fact, $g(u,v)$ has to obey some constraints. Invariance under $x_1 \leftrightarrow x_2$ and $x_1 \leftrightarrow x_3$ implies

$$g(u,v) = g(u/v, 1/v), \quad v^D g(u,v) = u^D g(v,u) \ . \tag{5}$$

- We also have

$$g(u,v) = 1 + \sum_{\phi_k \in \phi\times\phi - \{1\}} f^2_{\phi\phi\mathcal{O}} \cdot g_{\mathcal{O}}(u,v) \ , \tag{6}$$

with positive coefficients.

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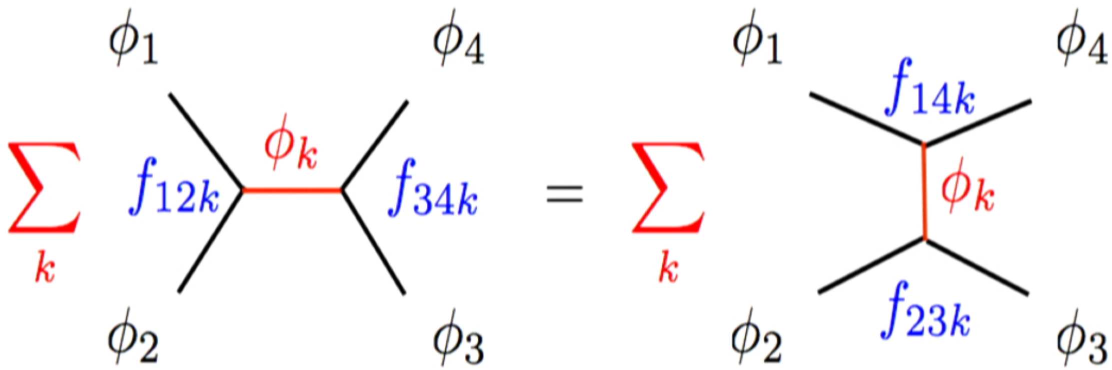


Special Points in Theory Space: (S)CFTs (cont...)

- Crossing symmetry implies the following constraint

$$1 = \sum_{\Delta,l} f^2_{\phi\phi\mathcal{O}} \left(\frac{v^D g_{\Delta,l}(u,v) - u^D g_{\Delta,l}(v,u)}{u^D - v^D} \right) \ . \qquad (7)$$

- Highly non-trivial to solve (non-trivial geometric statement in the space of functions of two variables).

Special Points in Theory Space: (S)CFTs (cont...)

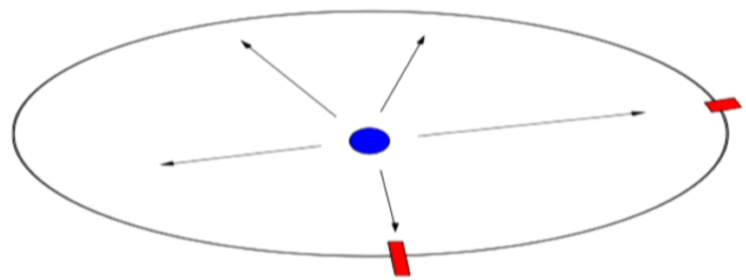
• Many beautiful results follow from this. For us, some significant ones include:

• $c_{\phi\phi\mathcal{O}} < M(D, \Delta)$, i.e., the CFT cannot have arbitrarily "strong" interactions **[Caracciolo and Rychkov, '09]**.

• $c > f(D)$, i.e., the CFT should contain some non-minimal amount of "stuff," **[Poland and Simmons-Duffin, '10; Rattazzi, Rychkov, and Vichi, '10]**

Special Points in Theory Space: (S)CFTs (cont...)

• Another interesting set of constraints come from positivity of the energy flux correlators **[Hofman and Maldacena, '08]**.



• Study

$$\mathcal{E}(\theta) \equiv \lim_{r\to\infty} \int_{-\infty}^{\infty} dt\ r^2\ n^i\ T_i^0(t, r\vec{n})\ . \qquad (8)$$

Special Points in Theory Space: (S)CFTs (cont...)

- And correlators

$$\langle \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\rangle \equiv \frac{\langle \mathcal{O}^\dagger \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\mathcal{O}\rangle}{\langle \mathcal{O}^\dagger\mathcal{O}\rangle} \qquad (9)$$

- By considering one point functions in states created by $T_{\mu\nu}$ and (SUSY) friends, can show from (conjectured) positivity

$$\begin{aligned} &\tfrac{31}{18} \geq \tfrac{a}{c} \geq \tfrac{1}{3} \quad (\mathcal{N}=0), \quad \tfrac{3}{2} \geq \tfrac{a}{c} \geq \tfrac{1}{2}, \quad (\mathcal{N}=1) \\ &\tfrac{5}{4} \geq \tfrac{a}{c} \geq \tfrac{1}{2}, \quad (\mathcal{N}=2)\ . \end{aligned} \qquad (10)$$

The RG Flow

• CFTs are also useful because they are UV/IR endpoints of the RG flow (at least in two dimensions; in four dimensions seems that interacting non-CFT endpoints would need very special properties).

• RG flow has very complicated phenomena: accidental symmetries, emergent phenomena, etc.

• Can identify universal CFT quantities and try to use them to constrain the RG flow. Examples include

$$c_{UV} > c_{IR} \quad (D=2), \quad F_{UV} > F_{IR} \quad (D=3), \quad a_{UV} > a_{IR} \quad (D=4) \tag{11}$$

[Zamolodchikov, '86], **[Casini and Huerta, '12; Myers, et. al., Jafferis et. al, Liu et. al.]**, and **[Komargodski and Schwimmer, '11]**

The RG Flow (cont...)

- More general picture with theory on S_D?

- Sometimes if have more symmetry (e.g., SUSY, R-symmetry) can potentially say more, $\tau_U^{UV} > \tau_U^{IR}$, **[MB]**.

- Morally, CFT constraints from OPE associativity, conformal collider physics, etc. should also constrain things away from criticality (at least if we are in a nearby patch of theory space). After all, CFT physics is smooth (bounded interaction strengths, etc.).

- We will argue that this is true, at least in certain special theories.

SUSY RG flows and SCFT repulsion

- We would like to study the distribution of SCFTs in theory space.

- **Question:** In particular, for SUSY RG flows with differing amounts of SUSY is their some minimal length to the RG flow?

- In some sense, we want to know how "repulsive" SCFTs are under the RG flow (i.e., how far away are $\mathcal{T}_{UV}$ and $\mathcal{T}_{IR}$—careful to be scheme-independent!).

- Note that non-trivial conformal manifolds are pretty common in $\mathcal{N} \geq 1$ SUSY, but we will study this question under a relevant perturbation of some UV SCFT, $\mathcal{T}_{UV}$.

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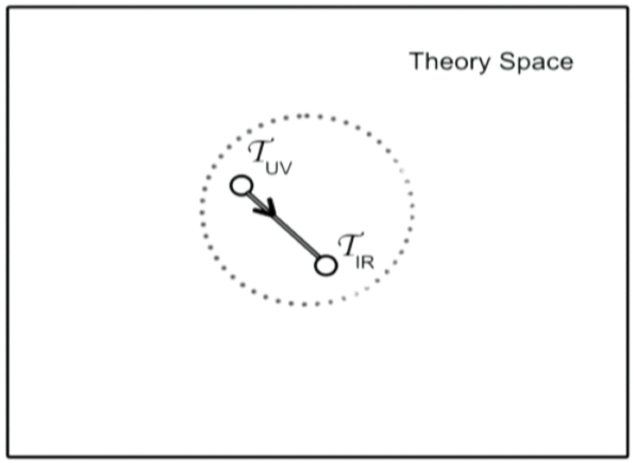
• Our intuition is that the less supersymmetric the theory, the richer the dynamics in the neighborhood of $\mathcal{T}_{UV}$, and the more "likely" it is that the RG flow can come to a stop in the same patch of theory space.

SUSY RG flows and SCFT repulsion (cont...)

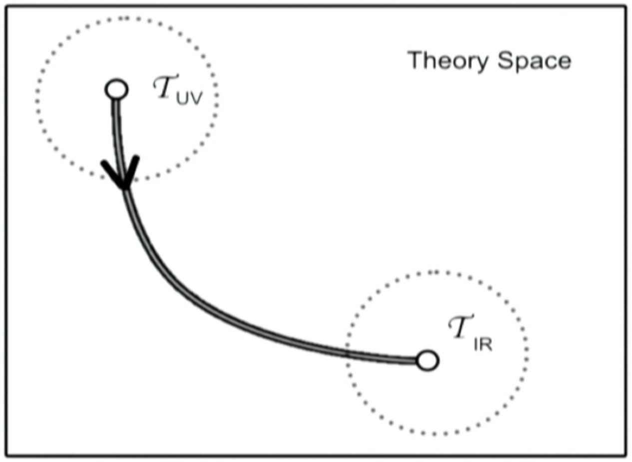
- In pictures:



is a common possibility for minimally SUSY theories.

SUSY RG flows and SCFT repulsion (cont...)

- But we expect



is typical for non-minimal SUSY.

Distance Between Theories

- How do we define distance between theories?

- CFTs have a natural (Zamolodchikov) metric:

$$G^0_{IJ} = \langle \mathcal{O}_I(x)\mathcal{O}_J(0)\rangle \cdot x^{2D_I} \ . \tag{12}$$

If the $\mathcal{O}_I$ are Hermitian, unitarity guaranties $G^0_{IJ} > 0$.

- Non-trivial to take Zamolodchikov metric and extend away globally on theory space. However, in the vicinity of a CFT should still make sense (can compute radiative corrections).

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Distance Between Theories (cont...)

• One possibility (at least if $\mathcal{T}_{IR}$ can be written in terms of UV operators with some small changes in couplings)

$$d(\mathcal{T}_{UV},\mathcal{T}_{IR}) = \int_\gamma \left(G_{IJ}d\lambda^I d\lambda^J\right)^{\frac{1}{2}} \sim \left(G^0_{IJ}\delta\lambda^I\delta\lambda^J\right)^{\frac{1}{2}}$$
$$= \left(\delta_{IJ}\delta\lambda^I\delta\lambda^J\right)^{\frac{1}{2}} = d_0(\mathcal{T}_{UV},\mathcal{T}_{IR}) \ , \qquad (13)$$

γ is a minimal length geodesic in the space of theories. This is reparametrization invariant (i.e., scheme independent).

• Presumably above is well-defined if $\mathcal{T}_{IR}$ is within some $d < D_{\text{univ}}$ of $\mathcal{T}_{UV}$. Otherwise, would find that the above expression breaks down for arbitrarily small d. Impossible since OPE coefficients in UV are bounded and UV beta-functions cannot be arbitrarily large (as measured by the metric). Not clear how to extend globally (but this won't matter for our main question).

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Distance Between Theories (cont...)

- For large distance, $d > D_{\text{univ}}$, above expression generally breaks down (move to new patch of theory space, e.g., free magnetic phase of SQCD).

- Another measure of distance (normal to conformal manifold, so less general) is $\delta a = a_{UV} - a_{IR}$. Well-defined and can often be measured in different patches of theory space via dualities (e.g., SQCD free-magnetic range).

- Have to be careful with $\delta a = 0$ examples, but these are very special.

- **(Slightly) More Precise Question:** For differing amounts of SUSY along RG flow can we find theories with parametricaly small δa and d? Bounds are clearly related.

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The General SUSY Setup

- Start from some general SCFT; unitarity constrains the deformation to be of the form

$$\delta W = \lambda^i \mu^{3-D_i} \mathcal{O}_i \ , \quad D_i = 3 - \Delta_i \ , \tag{14}$$

Suppose that relevant deformation is dominant (turning on any vevs subdominant with exception of $\mathcal{N} = 4$ example we will study).

- UV physics controlled by OPEs:

$$\mathcal{O}_i(x)\mathcal{O}^\dagger_{\bar{j}}(0) = \frac{\delta_{i\bar{j}}}{|x|^{6-2\Delta_i}} + \frac{T^A_{i\bar{j}}}{|x|^{4-\Delta_i-\Delta_j}} J_A + \frac{c^I_{i\bar{j}}}{|x|^{6-D_I-\Delta_i-\Delta_j}} L_I + \cdots \ . \tag{15}$$

The General SUSY Setup (cont...)

- Smoothness of physics implies should look for RG flows initiated by almost marginal relevant deformation, $\delta W = \lambda \mu^{3-\epsilon}\mathcal{O}$, i.e., $\Delta \sim \epsilon$. Suppose single $U(1)$ (label it as U) with $U(\mathcal{O}) = \epsilon$ (gives R-symmetry and hope of finding perturbative IR solution). This actually holds for d. For small a need additional assumptions (non-fine-tuned decreasing a function).

- Impose cut-offs on operator collisions in superspace and get

$$\beta = -\epsilon\lambda + 4\pi^4 \frac{\epsilon^2}{\tau_U}\lambda^3 + \cdots \,, \tag{16}$$

OPE coefficient bounds preclude 1-loop term from being large.

$$d \sim d_0(\mathcal{T}_{UV}, \mathcal{T}_{IR}) = \lambda_* = \left(\frac{\tau_U}{4\pi^4\epsilon}\right)^{\frac{1}{2}} \,. \tag{17}$$

The General SUSY Setup (cont...)

- Provided that $\tau_U \sim \epsilon^N$ with $1 < N \leq 2$, d_0 can be parametrically small.

- Also have

$$\delta a \sim -\frac{64\pi^4}{3}\int_0^{\lambda_*} d\lambda \cdot \beta = \frac{4}{3}\tau_U \ , \qquad (18)$$

- Should be case that U contains trace anomaly (since $\delta a \sim \beta^2$).

- In fact, this is true

$$\bar{D}^{\dot{\alpha}} R_{\alpha\dot{\alpha}} = \bar{D}^2 D_\alpha U \ . \qquad (19)$$

$\mathcal{N} = 1$ Theories

- For an $\mathcal{N} = 1$ theory, given any $r > 0$, can construct an RG flow with $\delta a, d < r$.

- Simplest examples are BZ flows $SU(N_c)$ with $N_f = N$ and $N_c = xN = \frac{1}{3}(1 + \epsilon)N$:

$$\beta_g = -\frac{g^3}{16\pi^2}\left(3Nx - N - \frac{1}{8\pi^2}N^2xg^2\right) + \cdots , \qquad (22)$$

- But δa not parametrically small

$$\delta a = \frac{16N^2x^2}{3}\int \frac{dg}{g^2}\beta_g = \frac{2N^2\epsilon^2}{9} + \cdots . \qquad (23)$$

Reason is Zamolodchikov metric not well-defined in UV ($d \sim d_0 = \infty$).

$\mathcal{N} = 1$ **Theories (cont...)**

- Should study theories with well-defined Zamolodchikov metric (no free gauge fields in UV or IR).

- Simple example aSQCD with $N_c, N_f \to \infty$ and $x = \frac{1}{2}(1+\epsilon)$. At interacting fix urned point, turn on

$$\delta W = \lambda \varphi \text{Tr } X^2 \ . \tag{24}$$

This has $D = 3 - \epsilon + \cdots$.

- Find

$$\delta a = \frac{4}{3}\epsilon^2 + \cdots \ll 1 \ . \tag{25}$$

Essentially only excite one d.o.f. Similarly, $d \sim d_0 \sim \epsilon$.

$\mathcal{N} = 2$ **Theories**

- Things are intuitively different here. For example, no BZ fixed points since β is one-loop exact (e.g., hypermultiplets have vanishing anomalous dimension).

- Alternative argument. When we turn on the gauge coupling, we have only non-chiral symmetries present and so any R-symmetry we can construct will not be a candidate IR superconformal R-symmetry since we would find $a_{IR} = c_{IR} = 0$.

- To understand how general this picture is, it is useful to introduce some basic concepts.

$\mathcal{N} = 2$ Theories (cont...)

• On the other hand, J is universal and couples to (partners of) gauge fields and matter

$$J \sim 2|\Phi|^2 - |Q|^2 - |\tilde{Q}|^2 \ . \tag{29}$$

• In more general language, the two-point function of J is just c and so gives a measure of the number of degrees of freedom at $\mathcal{T}_{UV}$

$$\tau_J \sim c \ . \tag{30}$$

Conclusions

• Argued that δa and d cannot be parametrically small for $\mathcal{N}=2$ theories. Can we find some numerical lower bounds?

• Also, we argue that approximately conserved currents cannot mix with the IR superconformal R-current. Can we give an explicit bound on how approximately conserved these currents must be?

• Can check that parametrically close $\mathcal{N}=1$ fixed points are allowed if UV is $\mathcal{N}=2$. Under assumption of a single stress tensor and UV flavor symmetry, one can show that UV central charges cannot be too large relative to deviations from marginality of the almost marginal relevant deformations.

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