Title: New lessons about hydrodynamics from gravity

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Abstract: Hydrodynamics of relativistic plasmas received, within the last 10 years, a lot of attention. The reason for it, on one hand, is the quest for theoretical understanding of the quark-gluon plasma created in heavy ion collisions and, on the other, advances in holographic duality and black hole physics in anti-de Sitter spacetimes. I will describe recent progress in answering foundational issues in hydrodynamics of strongly coupled systems, i.e. questions about its applicability and the character of hydrodynamic gradient expansion, that was achieved with the use of numerical techniques in anti-de Sitter spacetimes in the strong gravity regime.

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New lessons about hydrodynamics from gravity

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based on
1302.0697 [hep-th] MPH, R. A. Janik & P. Witaszczyk (**PRL 110 (2013) 211602**)

Why are we interested in hydrodynamics?

[gr-qc]: gravity under certain circumstances = hydrodynamics

membrane paradigm; fluid-gravity duality, blackfolds; gravity & turbulence.

[hep-ph] [nucl-th]: holography = ~ effective description of sQGP

viscosity and viscosity bound (?); new transport phenomena; cavitation (?).

[hep-th] gravity UV-completes certain hydrodynamic theories

thermalization & transition to hydrodynamics; relation to other DOFs.

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Modern relativistic (uncharged) hydrodynamics

hydrodynamics is an EFT of the slow evolution of conserved currents in collective media „close to equilibrium"

As any EFT it is based on the idea of the gradient expansion

DOFs: always local energy density ϵ and local flow velocity u^μ $(u_\nu u^\nu = -1)$

EOMs: conservation eqns $\nabla_\mu T^{\mu\nu} = 0$ for $T^{\mu\nu}$ *systematically* expanded in gradients

terms carrying 2 and more gradients

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\} - \eta(\epsilon)\, \sigma^{\mu\nu} - \zeta(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\}(\nabla \cdot u) + \ldots$$

perfect fluid stress tensor

microscopic input: **EoS** (famous) **shear viscosity** **bulk viscosity** (vanishes for CFTs)

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Applicability of hydrodynamics

terms carrying 2 and more gradients

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\} - \eta(\epsilon)\, \sigma^{\mu\nu} - \zeta(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\}(\nabla \cdot u) + \ldots$$

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Naively one might be inclined to associate hydrodynamic regime with small gradients.

But this is not how we should think about effective field theories! The correct way is to understand hydrodynamic modes as low energy DOFs.

Of course, there are also other DOFs in fluid.
The topic of my talk is to use holography to elucidate their imprint on hydro.

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Holographic plasmas and their degrees of freedom

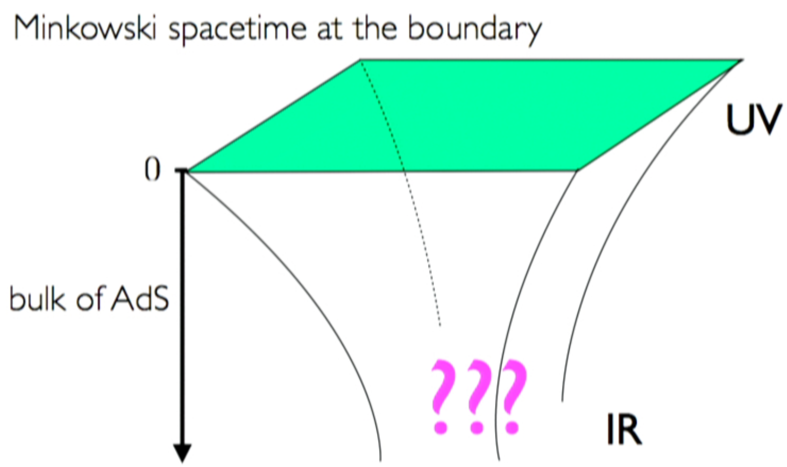
Holography

From applicational perspective AdS/CFT is a tool for computing correlation functions in certain strongly coupled gauge theories, such as $\mathcal{N} = 4$ SYM at large N_c and λ.

For simplicity I will consider AdS$_{1+4}$ / CFT$_{1+3}$ and focus on pure gravity sector.

$$R_{ab} - \frac{1}{2} R g_{ab} - \frac{6}{L^2} g_{ab} = 0$$

Different solutions correspond to states in a dual CFT with different $\langle T_{\mu\nu} \rangle$.

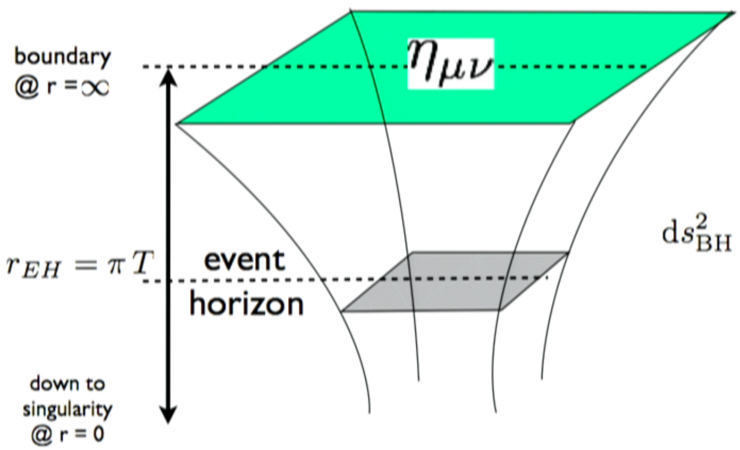


$$ds^2 = \frac{L^2}{z^2} \Big\{ dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu + \frac{2\pi^2}{N_c^2} \langle T_{\mu\nu} \rangle \, z^4 + \ldots \Big\}$$

Global equilibrium

Thermal deconfined hQFT = **Bulk black hole**

AdS-Schwarzschild black hole is described by the metric



$$ds^2_{\rm BH} = 2\mathrm{d}t\mathrm{d}r - r^2\left(1 - \frac{\pi^4 T^4}{r^4}\right)\mathrm{d}t^2 + r^2 \mathrm{d}\vec{x}^2$$

The plasma/black hole thermodynamics is given by

$$T_{\mu\nu} = \frac{1}{8}\pi^2 N_c^2 T^4 \,\mathrm{diag}\,(3,1,1,1)_{\mu\nu},\; s\Big|_{\lambda\to\infty} = \mathrm{Area}_{\rm EH}/4G_N = \frac{1}{2}N_c^2 \pi^2 V T^3 = \frac{3}{4}s\Big|_{\lambda\to 0}$$

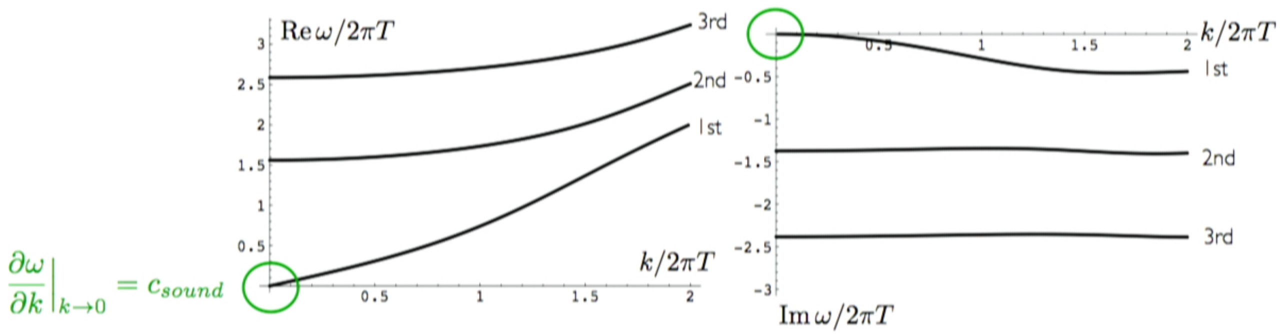
Excitations of strongly coupled plasmas

Kovtun & Starinets [hep-th/0506184]

Consider small amplitude perturbations ($\delta T_{\mu\nu}/N_c{}^2 \ll T^4$) on top of a holographic plasma

$$T_{\mu\nu} = \frac{1}{8}\pi^2 N_c^2 T^4 \,\mathrm{diag}\,(3,1,1,1)_{\mu\nu} + \delta T_{\mu\nu} \quad (\sim e^{-i\,\omega(k)\,t + i\,\vec{k}\cdot\vec{x}})$$

Dissipation leads to modes with complex $\omega(k)$, which in the sound channel look like



There are two different kinds of modes:

$\omega(k) \to 0$ **as** $k \to 0$: slowly evolving and dissipating modes (hydrodynamic sound waves)

all the rest: far from equilibrium (QNM) modes dampened over $t_{therm} = \mathcal{O}(1)/T$

@ RHIC: $0.25\,\mathrm{fm} \times 500\,\mathrm{MeV} = 0.63$

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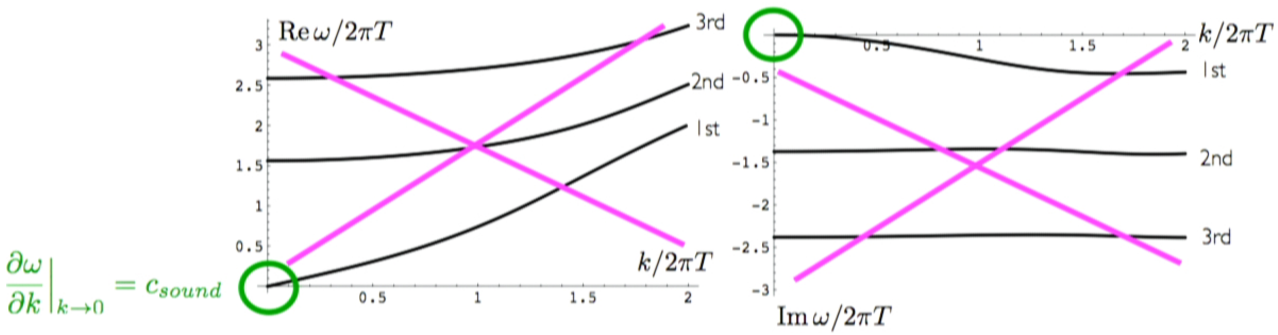
Lesson 1:

for hydrodynamics to work all the other DOFs need to relax.



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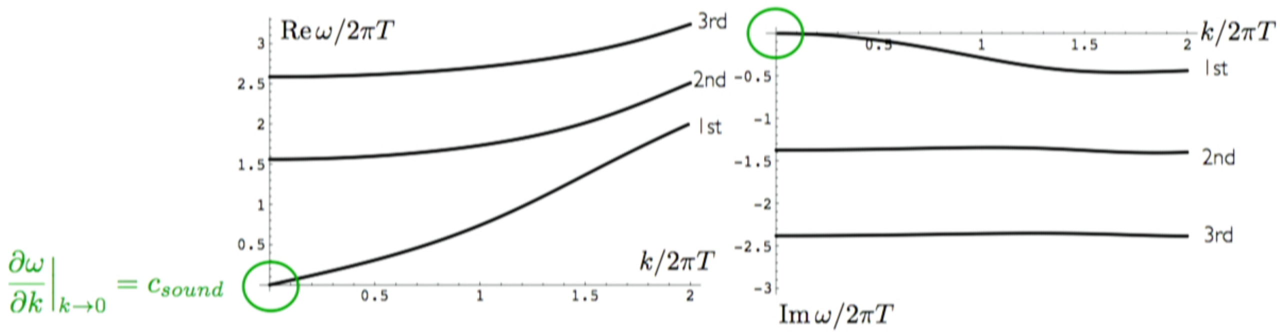
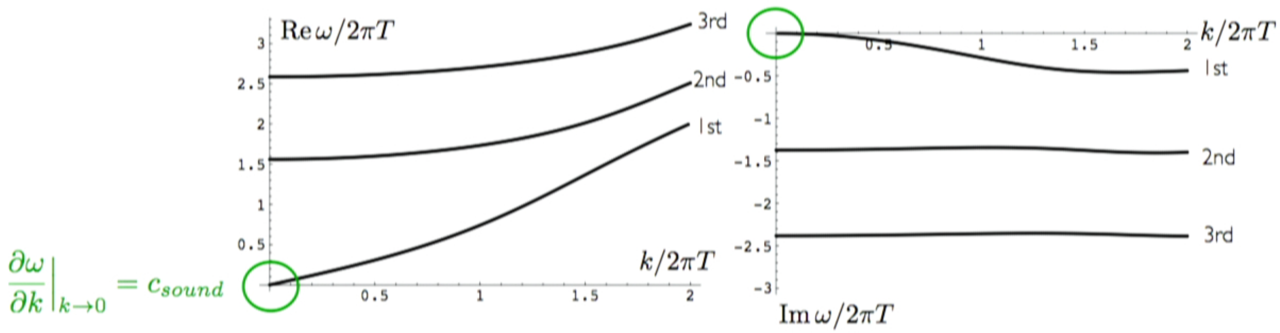


Observation:

No matter how long one waits, there will be always remnants of n-eq DOFs

Lesson 2:

Hydrodynamic gradient expansion cannot converge



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Lesson 2:

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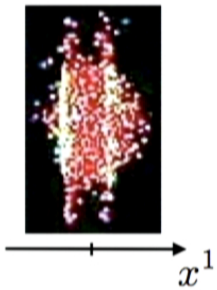
Fantastic toy-model [Bjorken 1982]



The simplest model in which one can test these ideas is the **boost-invariant flow** with **no transverse expansion.**

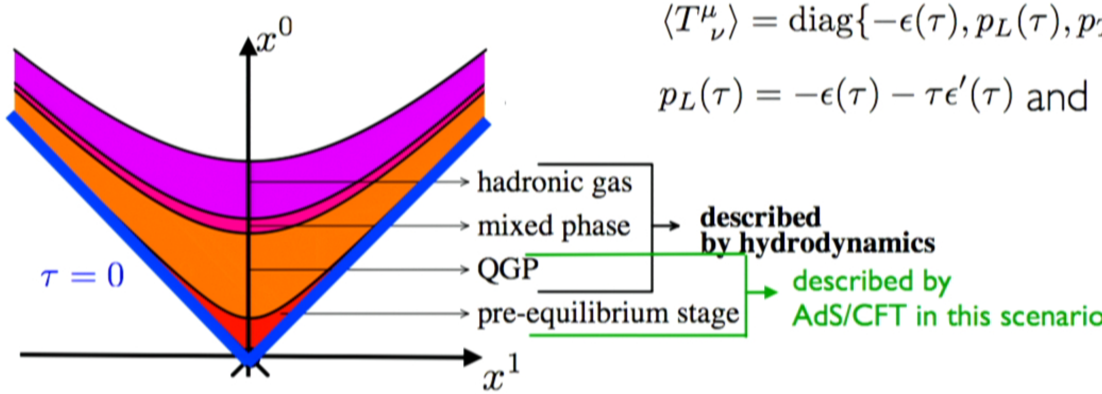
In Bjorken scenario dynamics depends only on proper time $\tau = \sqrt{(x^0)^2 - (x^1)^2}$

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_2^2 + dx_3^2$$

and stress tensor (for a CFT) is entirely expressed in terms of local energy density

$\langle T^{\mu}{}_{\nu} \rangle = \mathrm{diag}\{-\epsilon(\tau), p_L(\tau), p_T(\tau), p_T(\tau)\}$ with

$p_L(\tau) = -\epsilon(\tau) - \tau\epsilon'(\tau)$ and $p_T(\tau) = \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau)$



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Boost-invariant hydrodynamics

In hydrodynamics, the stress tensor is expressed in terms of T, u^μ and their ∇_ν

Key observation: in Bjorken flow u^μ is fixed by the symmetries and takes the form

$$u^\mu \partial_\mu = \partial_\tau$$

Its gradients will come thus from Christoffel symbols $(ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_2^2 + dx_3^2)$

$$\Gamma^y_{\tau y} = \frac{1}{\tau}$$

Lesson: in Bjorken flow hydrodynamic gradient expansion = late time power series

At very late times $p_L = -\epsilon - \tau\epsilon' = p_T = \epsilon + \frac{1}{2}\tau\epsilon' \longrightarrow \epsilon \sim \frac{1}{\tau^{4/3}} \longrightarrow T \sim \frac{1}{\tau^{1/3}}$

In holographic hydrodynamics gradient expansion parameter is $\frac{1}{T}\nabla_\mu u_\nu$

$$T^{\mu\nu} = \epsilon\, u^\mu u^\mu + P(\epsilon)\,\{\eta^{\mu\nu} + u^\mu u^\nu\} - \frac{\eta(\epsilon)}{s(\epsilon)}\,\frac{\epsilon + P(\epsilon)}{T}\sigma^{\mu\nu} + \ldots$$

For Bjorken flow $\frac{1}{T}\nabla_\mu u_\nu$ is $\frac{1}{\frac{1}{\tau^{1/3}}}\,\frac{1}{\tau} = \frac{1}{\tau^{2/3}}$.

Boost-invariant hydrodynamics

The general solution of the boost-invariant hydrodynamics will thus take the form

$$\epsilon = \frac{3}{8} N_c^2 \pi^2 \frac{1}{\tau^{4/3}} \left(\epsilon_2 + \epsilon_3 \frac{1}{\tau^{2/3}} + \epsilon_4 \frac{1}{\tau^{4/3}} + \dots \right)$$

where ϵ_{n+2} comes from n^{th} order of hydrodynamic gradient expansion.

The form of ϵ_{n+2} is known at low orders of gradient expansion

$$\epsilon_2 = \frac{\Lambda^4}{\Lambda^{4/3}} = \Lambda^{8/3}$$ perfect fluid

$$\epsilon_3/\epsilon_2 = -\frac{1}{\Lambda^{2/3}} \frac{2}{3\pi}$$ first order hydro

$$\epsilon_4/\epsilon_2 = \frac{1}{\Lambda^{4/3}} \frac{1 + 2\log 2}{18\pi^2}$$ second order hydro

where Λ $([\Lambda] = \mathrm{length}^{-1})$ encodes the initial conditions for hydrodynamics*.

Task: obtain ϵ_{n+2} for much larger n's (high order hydrodynamics).

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Gravity dual to Bjorken flow

Gravity dual to Bjorken flow

The symmetries of the flow

$$\langle T^{\mu}{}_{\nu} \rangle = \mathrm{diag}\{-\epsilon(\tau), p_L(\tau), p_T(\tau), p_T(\tau)\}$$

lead to the following ansatz for the dual metric [residual diff. $R \to R + f(\tau)$]

$$ds^2 = 2\, d\tau\, dR - A\, d\tau^2 + \Sigma^2 e^{-2B} dy^2 + \Sigma^2 e^{B} \left(dx_2^2 + dx_3^2\right)$$

Einstein's equations give 5 nontrivial equations for 3 functions of 2 variables.

If one solves those equations as the initial value problem, then one needs* to know $A, \quad \Sigma \quad \text{and} \quad B$ as a function of r on the initial „time'' slice.

Such way of formulating the problem typically leads to far from equilibrium physics.

In stark contrast with hydrodynamics: **infinite number of DOFs** vs **4 DOFs**.

Hydrodynamization: (here nonlinear) relaxation to 4 DOFs.

Hydrodynamization…

1103.3452 [hep-th] PRL 108 (2012) 201602:

MPH, R.A. Janik & P.Witaszczyk

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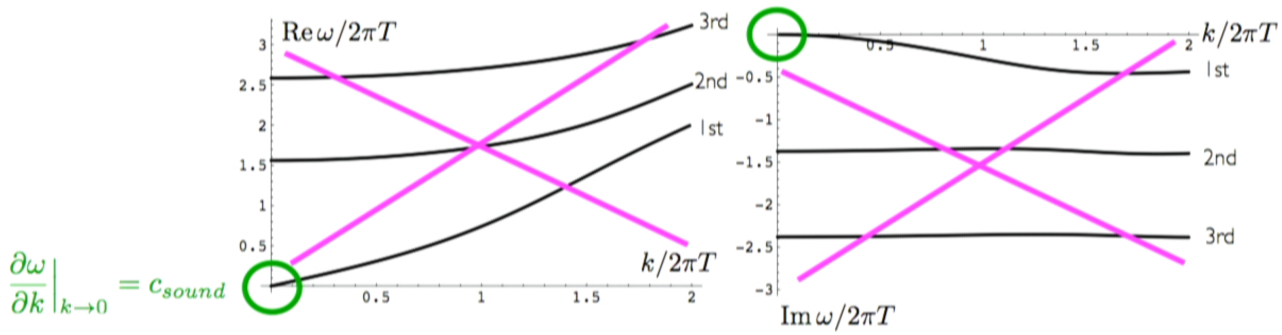
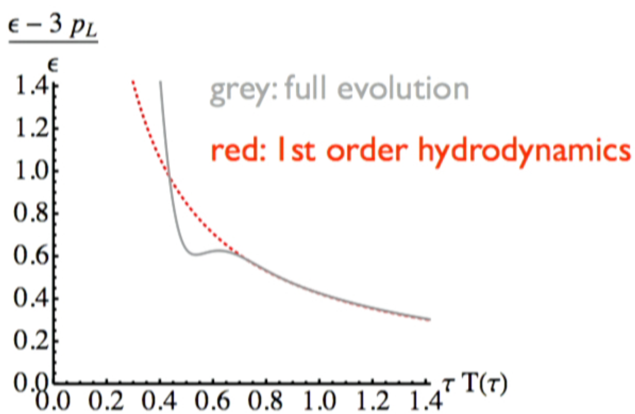


Surprising consequence



Large anisotropy at the onset of hydrodynamics

$$\epsilon - 3\,p_L \approx 0.6\,\epsilon \text{ to } 1.0\,\epsilon$$

Thus

hydrodynamization $\neq$ isotropization, thermalization

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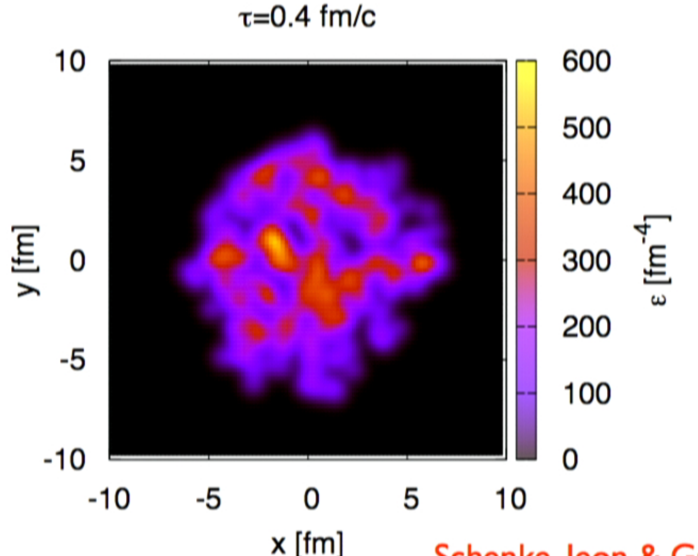
… and its relevance

Event-by-event fluctuations in HIC lead to significant inhomogeneities.

||

large spatial gradients



Schenke, Jeon & Gale
arXiv:1009.3244 [hep-ph]

Nuclear theorists still use hydro and do well in describing the data.

The discovery of hydrodynamization suggests it might not be theoretically outrageous.

Gravity dual to Bjorken hydrodynamics

The form of hydrodynamic energy density

$$\epsilon = \frac{3}{8} N_c^2 \pi^2 \frac{1}{\tau^{4/3}} \left(\epsilon_2 + \epsilon_3 \frac{1}{\tau^{2/3}} + \epsilon_4 \frac{1}{\tau^{4/3}} + \ldots \right)$$

suggests (pre-dating fluid-gravity duality) the form of the dual metric ansatz.

The starting point, of course, is a locally boosted black brane

$$ds^2 = 2d\tau dR - R^2 \left\{ 1 - \frac{\pi^4}{R^4} \left(\frac{\Lambda}{(\Lambda\tau)^{1/3}} \right)^4 \right\} d\tau^2 + (R\tau + 1)^2 dy^2 + R^2 (dx_2^2 + dx_3^2)$$

asymptotic form of temperature

measuring radial position with respect to thermal length scale

Gradient expansion will be thus encoded as $\frac{1}{\tau^{2/3}}$ expansion keeping $R\tau^{1/3}$ fixed

$$A(\tau, R)/R^2\Big|_{hydro} = \left\{ 1 - \frac{\pi^4}{R^4} \left(\frac{\Lambda}{(\Lambda\tau)^{1/3}} \right)^4 \right\} + \frac{1}{\tau^{2/3}} A_1(R\tau^{1/3}) + \frac{1}{\tau^{2/3}} A_2(R\tau^{1/3}) + \ldots$$

$$B(\tau, R)\Big|_{hydro} = -\frac{2}{3} \log\left(\tau + \frac{1}{R}\right) + \frac{1}{\tau^{2/3}} B_1(R\tau^{1/3}) + \frac{1}{\tau^{4/3}} B_2(R\tau^{1/3}) + \ldots$$

$$\Sigma(\tau, R)/R\Big|_{hydro} = \left(\tau + \frac{1}{R}\right)^{1/3} + \frac{1}{\tau^{2/3}} \Sigma_1(R\tau^{1/3}) + \frac{1}{\tau^{4/3}} \Sigma_2(R\tau^{1/3}) + \ldots$$

Numerical implementation in Mathematica

At each order one needs to solve a set of linear inhomogeneous ODEs in 1/v $= R\tau^{1/3}$

Structure is iterative and the difficulty comes in calculating inhomogeneous terms.

We fix the units be setting $\epsilon_2 = \Lambda^{8/3} = \pi^{-4}$ and using residual diff. to set $A_n(1) = 0$.

We solve 3 differential equations using the other 2 to provide 2 v=1 bdry conditions.

We impose regularity at v=1 and demand flat bdry at v =0.

We discretize v direction spectrally and use matrix inversion to solve ODEs.

The energy density is obtained from the near-bdry behavior of A_n .

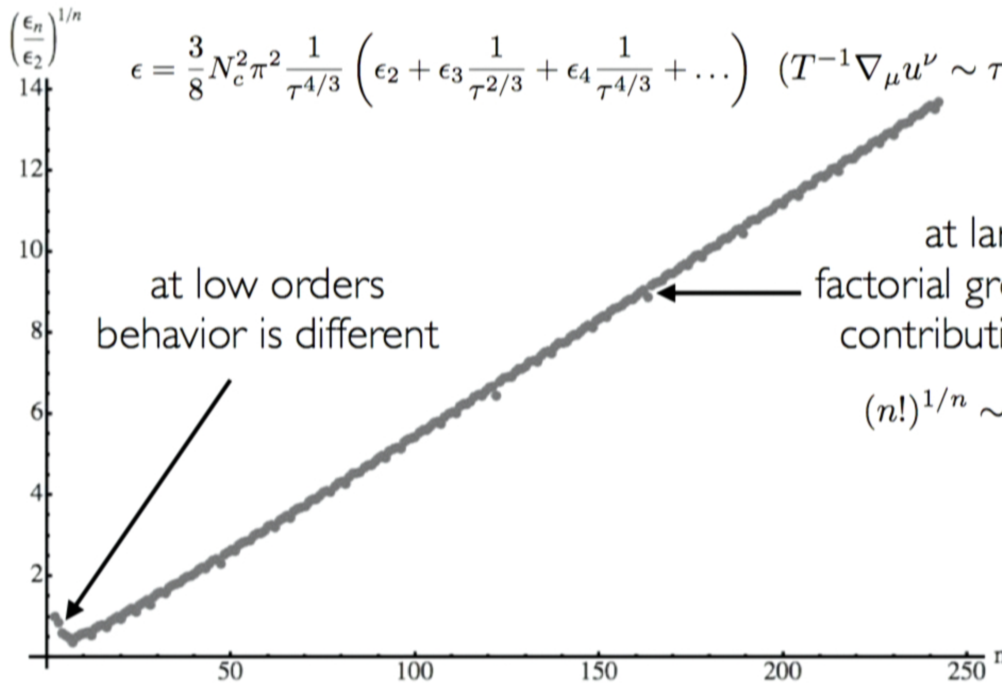
We used 251 grid points and kept 450 digits to resolve high order contributions.

Calculation up to 240 order took 4 weeks on a desktop (3.4 GHz, 16 GB RAM).

Hydrodynamic series at high orders

1302.0697 [hep-th] PRL 110 (2013) 211602: MPH, R. A. Janik & P. Witaszczyk

$$\epsilon = \frac{3}{8} N_c^2 \pi^2 \frac{1}{\tau^{4/3}} \left(\epsilon_2 + \epsilon_3 \frac{1}{\tau^{2/3}} + \epsilon_4 \frac{1}{\tau^{4/3}} + \ldots \right) \quad (T^{-1} \nabla_\mu u^\nu \sim \tau^{-2/3})$$



at large orders
factorial growth of gradient
contributions with order

$$(n!)^{1/n} \sim \frac{(2\pi n)^{1/2n}}{e} \cdot n$$

First evidence that hydrodynamic expansion has zero radius of convergence!

Singularities in the Borel plane

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A standard method for asymptotic series is Borel transform and Borel summation

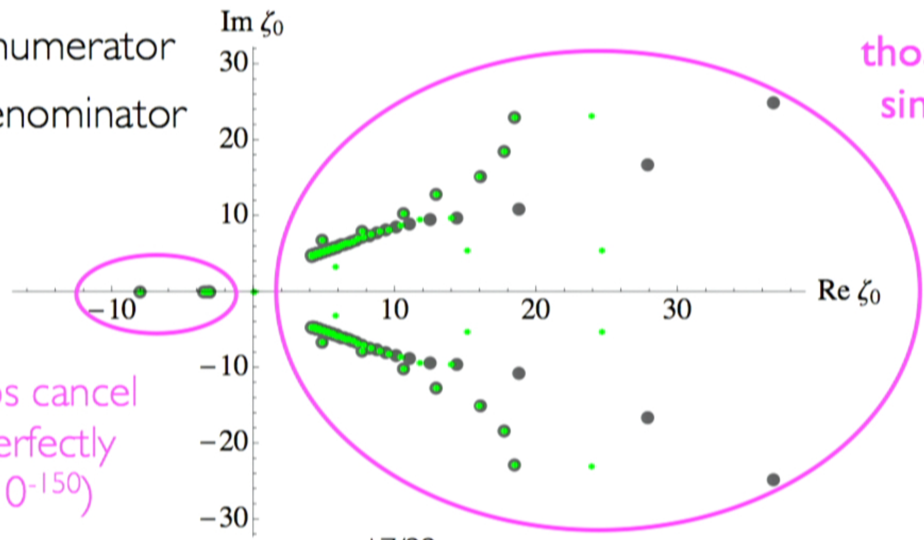
$$\epsilon(u) \sim \sum_{n=2}^{\infty} \epsilon_n u^n \quad (u = \tau^{-2/3}), \quad B\epsilon(\tilde{u}) \sim \sum_{n=2}^{\infty} \frac{1}{n!} \epsilon_n \tilde{u}^n, \quad \text{Borel sum}: \quad \epsilon_{Bs}(u) = \int_0^{\infty} \frac{1}{u} B\epsilon(t) \exp\left(-t/u\right) dt$$

This makes a difference only if we can find analytic continuation of $B\epsilon(\tilde{u})$.

Idea: use Pade approximant $B\epsilon(\tilde{u}) = \frac{\sum_{m=0}^{120} c_m \tilde{u}^m}{\sum_{n=0}^{120} d_n \tilde{u}^n}$ to reveal singularities of $B\epsilon(\tilde{u})$.

green dots: zeros numerator

gray dots: zeros denominator



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Singularities of Borel transform and QNMs

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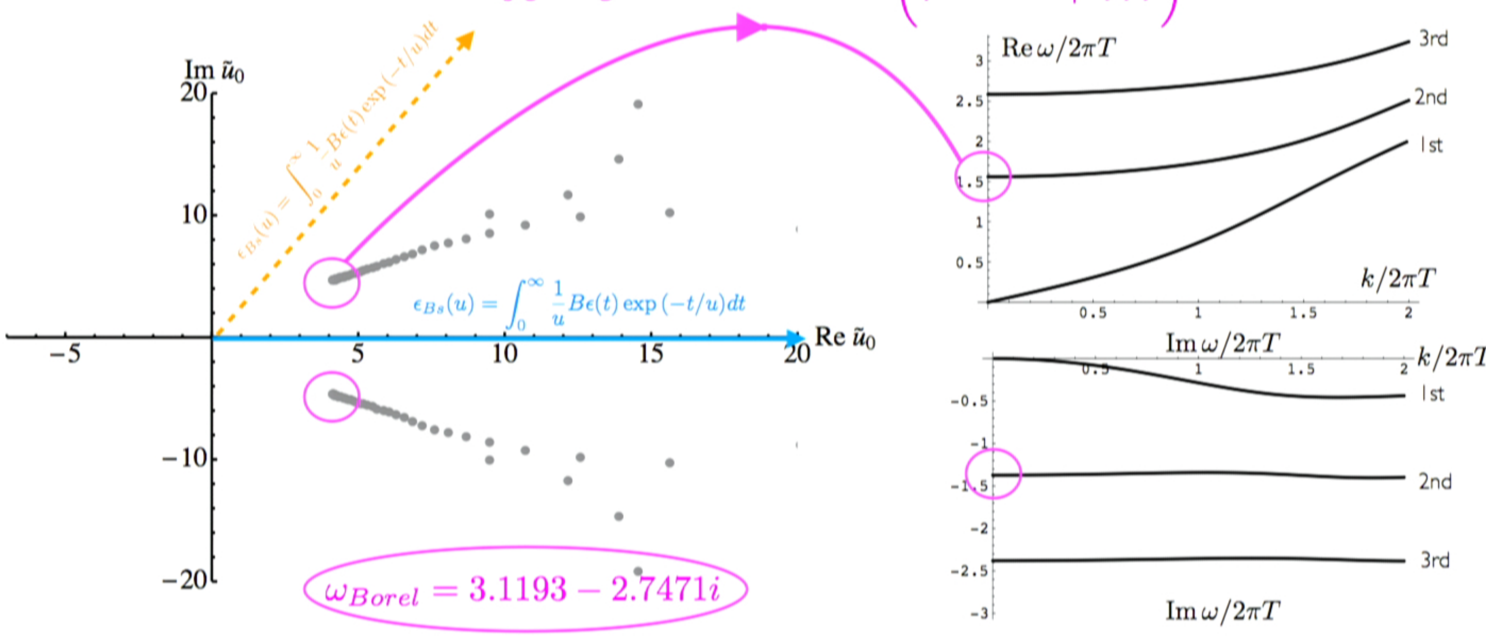
In Borel summation the outcome depends on the contour connect 0 with ∞.
Here there are two inequivalent contours (blue and orange).

$$\delta\epsilon \sim e^{-3/2\, i\, \omega_{Borel} \tau^{2/3}} \left(\tau^{\alpha_{Borel}} + \dots \right)$$



$\omega_{Borel} = 3.1193 - 2.7471i$

$\alpha_{Borel} = -1.5426 + 0.5192i$

ω_{Borel} is the frequency of the lowest non-hydrodynamic metric QNM at $k = 0$!

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Slow and fast modes

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Can we understand 3/2 in the exponent and the pre-exponential term in

$$\delta\epsilon \sim e^{-3/2\, i\, \omega_{Borel} \tau^{2/3}} \left(\tau^{\alpha_{Borel}} + \dots \right) ?$$

Yes, QNMs here are fast evolving modes on top of slowly evolving background.
In local rest frame (as here), at the leading order they only care about T.

But the temperature here is time-dependent. Imagine solving $\ddot{x}(t) = -\omega(t)^2 x(t)$ with slowly varying frequency. The leading order result is $x(t) \sim e^{\pm i \int \omega(t) dt}$

$$-i \int \omega_{QNM}\, 2\pi \frac{\Lambda}{(\Lambda\tau)^{1/3}} d\tau + \dots = -i\, \frac{3}{2} \left(2\pi\, \Lambda^{2/3} \omega_{QNM} \right) \tau^{2/3}$$

← instanton-like dependence on the coupling "$\frac{1}{g^2_{YM}}$" $= \tau^{2/3}$

[hep-th/0606149] Janik & Peschanski

How about pre-exponential term? Schematically $\int \frac{1}{\tau^{1/3}} \left(1 + \frac{1}{\tau^{2/3}} + \dots \right) d\tau \sim \tau^{2/3} + \log \tau$

Indeed $\omega_{\text{Borel}} = 3.1193 - 2.7471\, i$, $\alpha_{\text{Borel}} = -1.5426 + 0.5192\, i$ agrees with $\omega_{qnm} = 3.1195 - 2.7467\, i$, $\alpha_{qnm} = -1.5422 + 0.5199\, i$!!!

Slow and fast modes

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Interpretation and possible relevance

Why hydro series might be asymptotic?

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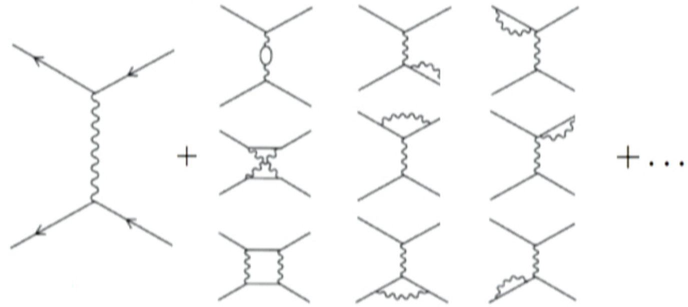
Famous examples of asymptotic expansions arise in pQFTs



There, the number of Feynman graphs grows ~order! at large orders*

We suspect analogous mechanism might work also in the case of hydro series*

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\}$$

$$-\eta\sigma^{\mu\nu}$$

1st order hydro (1 transport coeff)

$$-\tau_\Pi\left[{}^{\langle}D\Pi^{\mu\nu\rangle} + \frac{d}{d-1}\Pi^{\mu\nu}(\nabla\cdot u)\right] + \kappa\left[R^{\langle\mu\nu\rangle} - (d-2)u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta\right]$$
$$+\frac{\lambda_1}{\eta^2}\Pi^{\langle\mu}{}_\lambda\Pi^{\nu\rangle\lambda} - \frac{\lambda_2}{\eta}\Pi^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda} + \lambda_3\Omega^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda}$$

2nd order hydro (5 transport coeffs)

$$+\ldots$$

...

Resummed hydrodynamics?

1103.3452 [hep-th] PRL 108 (2012) 201602: MPH, R. A. Janik & P. Witaszczyk

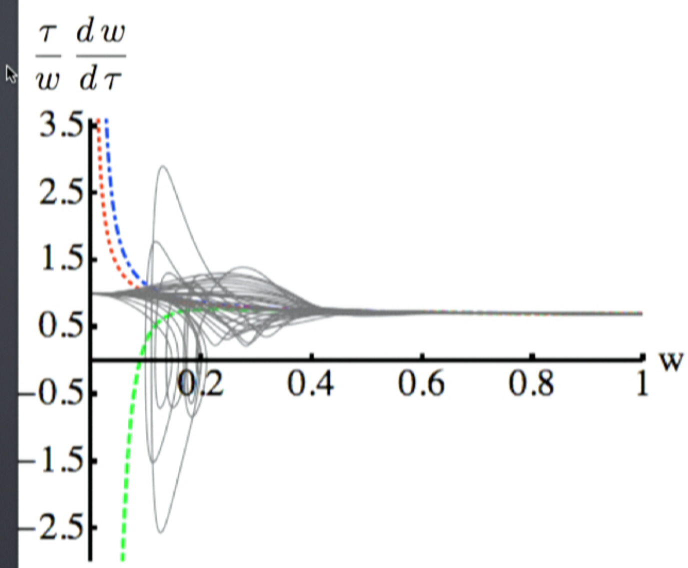
In boost-invariant hydrodynamics, hydro equations can be recast in the form

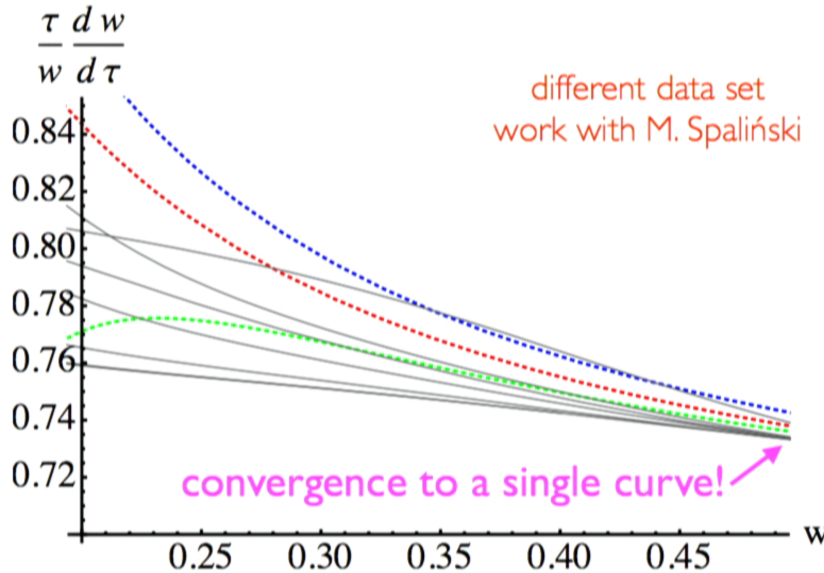
$$\frac{\tau}{w}\frac{d}{d\tau}w = \frac{F_{hydro}(w)}{w} \quad \text{with } \epsilon(\tau) = \frac{3}{8}N_c^2\pi^2 T_{eff}(\tau)^4 \text{ and } w = \tau T_{eff}$$

perfect fluid

$$= \frac{2}{3} + \frac{1}{9\pi w} + \frac{1-\log 2}{27\pi^2 w^2} + \frac{15 - 2\pi^2 - 45\log 2 + 24\log^2 2}{972\pi^3 w^3} + \ldots$$

1st 2nd 3rd order hydro

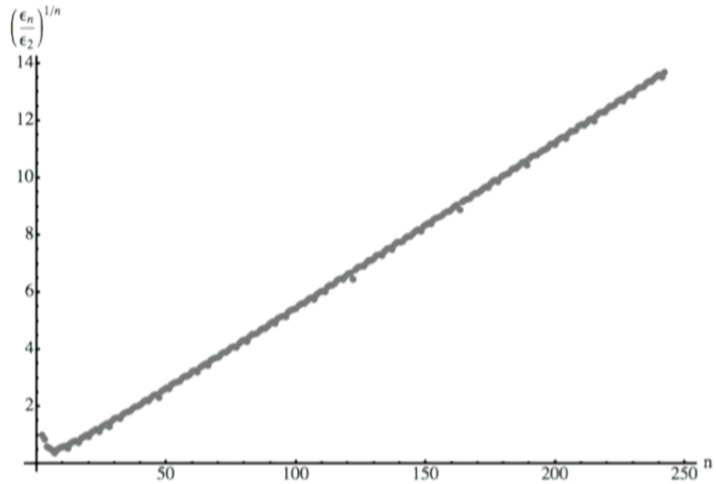




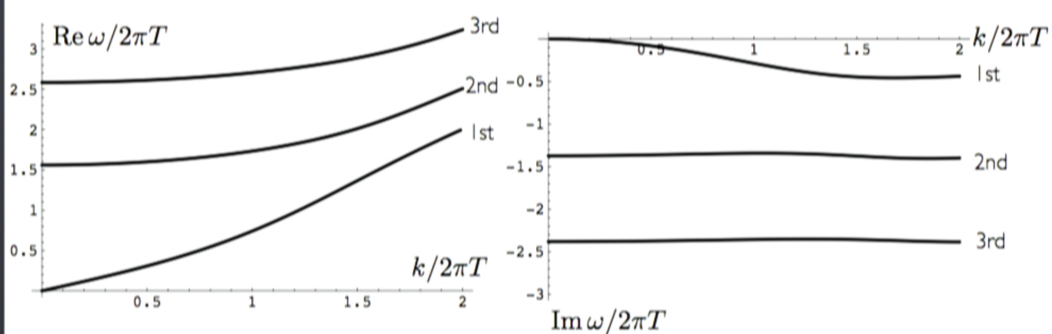
Idea: is it possible to obtain (part of) this curve from Borel resummation?

Summary

1302.0697 [hep-th] PRL 110 (2013) 211602.



Hydrodynamics is an asymptotic series



because in any fluid
there are DOFs
not captured by
hydrodynamic approx.